2017
MATHEMATICS

Full marks: 100 Time: 3 hours

General instructions:

i) *Approximately 15 minutes is allotted to read the question paper and revise the answers.*
ii) *The question paper consists of 26 questions. All questions are compulsory.*
iii) *Marks are indicated against each question.*
iv) *Internal choice has been provided in some questions.*
v) *Use of simple calculators (non-scientific and non-programmable) only is permitted.*

N.B: Check that all pages of the question paper is complete as indicated on the top left side.

Section – A

1. **Choose the correct answer from the given alternatives:**

(a) Consider the set A = {1, 2, 3, 4}. Which one of the following relations R form a reflexive relation? **1**

(i) R = {(1, 1), (1, 2), (2, 2), (3, 4)}
(ii) R = {(1, 1), (2, 2), (2, 3), (3, 3), (3, 4)}
(iii) R = {(1, 1), (2, 2), (2, 3), (3, 3), (3, 4), (4, 4)}
(iv) R = {(1, 1), (2, 1), (2, 3), (3, 3), (3, 4), (4, 4)}

(b) The value of $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$ is **1**

(i) $\frac{4\pi}{3}$ (ii) $\frac{2\pi}{3}$ (iii) 0 (iv) π

(c) Let A = $\begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$, then AB equals **1**

(i) $\begin{bmatrix} 0 & 6 \\ 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 6 \\ 0 & 4 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(d) If $y = \tan^{-1}\frac{x}{2} - \cot^{-1}\frac{x}{2}$, then $\frac{dy}{dx}$ is **1**

(i) $\frac{4}{4+x^2}$ (ii) $\frac{2}{4+x^2}$ (iii) $\frac{1}{4+x^2}$ (iv) $\frac{2}{1+x^2}$

(e) If $y = \log\sqrt{\tan x}$, then the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$ is **1**

(i) 0 (ii) $\frac{1}{2}$ (iii) 1 (iv) ∞

(f) If the rate of change of area of a circle is equal to the rate of change of its diameter, then its radius is **1**

(i) $\frac{1}{\pi}$ (ii) $\frac{2}{\pi}$ (iii) $\frac{\pi}{2}$ (iv) π

(g) $\int (2x+7)^6\, dx$ is equal to **1**

(i) $\frac{(2x+7)^7}{14} + C$ (ii) $\frac{(2x+7)^6}{14} + C$

(iii) $\frac{(2x+7)^7}{7} + C$ (iv) $\frac{-(2x+7)^7}{14} + C$

(h) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ is equal to **1**

(i) $\frac{\pi}{2}$ (ii) $\frac{\pi}{3}$ (iii) $\frac{\pi}{4}$ (iv) $\frac{\pi}{6}$

(i) The degree of the differential equation $\left(1+\frac{dy}{dx}\right)^5 = \left(\frac{d^2y}{dx^2}\right)^2$ is **1**

(i)1 (ii) 2 (iii) 3 (iv) 4

(j) The direction cosines of the vector $\vec{a} = \hat{i} - \hat{j} - 2\hat{k}$ are **1**

(i) $1, -1, -2$ (ii) $\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}$

(iii) $\frac{1}{4}, -\frac{1}{4}, -\frac{2}{4}$ (iv) $\sqrt{\frac{1}{6}}, -\sqrt{\frac{1}{6}}, -\sqrt{\frac{2}{6}}$

Section – B

2. Show that the function $f : \mathbf{N} \to \mathbf{N}$ given by $f(1) = f(2) = 1$ and $f(x) = x - 1$ for every $x > 2$ is onto but not one-one. **2**

3. Express $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$ in the simplest form. **2**

4. Using determinants, show that the points $(a+5,\ a-4)$, $(a-2, a+3)$ and (a, a) do not lie on a straight line. **2**

5. If $y = \sin^{-1}(\cos x) + \cos^{-1}(\sin x)$, prove that $\frac{dy}{dx} = -2$ **2**

6. If $y = \tan x$, prove that $\frac{d^2y}{dx^2} = 2y\frac{dy}{dx}$ **2**

7. Prove that the tangents to the curve $y = x^2 - 5x + 9$ at the points (2, 0) and (3, 0) are at right angles. **2**

8. Evaluate $\int \frac{x^4}{1+x^2}\, dx$ **2**

9. Form a differential equation of the curve $\frac{x}{a} + \frac{y}{b} = 1$, where a and b are arbitrary constants. **2**

10. Find the angle between the vectors $\vec{a}$ and $\vec{b}$ if $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$ **2**

11. If A and B are two events such that $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$, then find (i) $P(A \cup B)$ (ii) $P(\overline{B} \mid \overline{A})$ **2**

Section – C

12. Consider the set of integers **Z**. Define a relation R on **Z** as R = {$(x, y) : x - y = 3k$, where k is some integer}. Prove that R is an equivalence relation. **4**

13. **a.** Using properties of determinants, prove that:

$$\begin{vmatrix} x^2+2x & 2x+1 & 1 \\ 2x+1 & x+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (x-1)^3$$

Or **4**

b. For the matrix $A = \begin{bmatrix} 3 & 1 \\ 7 & 5 \end{bmatrix}$, verify that $A^2 + 8I = 8A$. Hence, find A^{-1}.

14.**a.** Show that the following function $f(x)$ is continuous at $x = -3$ but discontinuous at $x = 3$:

$$f(x) = \begin{cases} |x|+3, & \text{if } x \le -3 \\ -2x, & \text{if } -3 < x < 3 \\ 6x+2, & \text{if } x \ge 3 \end{cases}$$

Or **4**

b. If $y = \left(\tan^{-1} x\right)^2$, show that $\left(1+x^2\right)^2 y_2 + 2x\left(1+x^2\right)y_1 = 2$

15. Evaluate $\int_0^{\pi} \frac{x}{1+\sin x}\, dx$ **4**

16.**a.** Evaluate $\int \left\{ \frac{1}{\log x} - \frac{1}{(\log x)^2} \right\} dx$

Or **4**

b. Evaluate $\int \frac{dx}{2\sin x + \cos x + 3}$

17. Solve the differential equation $2y e^{\frac{x}{y}}\, dx + \left(y - 2x e^{\frac{x}{y}} \right) dy = 0$ **4**

18. **a.** If $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} + \hat{k}$, then find the vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{d}.\vec{c} = 21$.

Or **4**

b. If $\vec{a}$, $\vec{b}$, $\vec{c}$ are three vectors with magnitudes 3, 5, 7 respectively such that $\vec{a} + \vec{b} + \vec{c} = 0$, then find the angle between $\vec{a}$ and $\vec{b}$.

19. Find the Cartesian and vector equations of the plane through the points (–1, 1, 1) and (1, –1, 1) and perpendicular to the plane $x + 2y + 2z = 5$. **4**

20. A random variable X has the following probability distribution: **4**

x_i	0	1	2	3	4	5	6	7
p_i	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Determine: (i) k (ii) P(X < 3) (iii) P(0 < X < 3)

21. **a.** Two cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Find the mean and variance of the number of kings.

Or **4**

b. In a bolt factory, three machines A, B, C manufacture 25%, 35% and 40% of the total production respectively. Of their respective outputs, 5%, 4% and 2% are defective. A bolt is drawn at random from the total product and it is found to be defective. Find the probability that it was manufactured by the machine C.

Section – D

22. **a.** Using elementary row transformations, find the inverse of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Or **6**

b. Using matrix method, solve the following system of linear equations:

$$\begin{aligned} x - y + z &= 1 \\ 2x + y - z &= 2 \\ x - 2y - z &= 4 \end{aligned}$$

23. **a.** Find the point on the curve $y^2 = 4x$ which is nearest to the point (2, –8).

Or **6**

b. An open box with a square base is to be made out of a given quantity of sheet of area a^2. Show that the maximum volume of the box is $\frac{a^3}{6\sqrt{3}}$.

24. **a.** Using integration, find the area of the triangle whose vertices are A(1, 3), B(2, 5) and C(3, 4).

Or **6**

b. Using integration, find the area of the region bounded by the lines:
$y = 1 + |x + 1|$, $x = -2$, $x = 3$ and $y = 0$

25. **a.** Show that the following lines intersect and find the point of their intersection:

$$\vec{r} = \hat{i} + \hat{j} - \hat{k} + \lambda\left(3\hat{i} - \hat{j}\right) \text{ and}$$

$$\vec{r} = 4\hat{i} - \hat{k} + \mu\left(2\hat{i} + 3\hat{k}\right)$$

Or **6**

b. Find the distance of the origin from the plane $x + 2y + z = 4$ measured parallel to the line $\frac{x+1}{2} = \frac{y-1}{2} = \frac{z+2}{1}$

26. **a.** A gardener has a supply of fertilizers of type A which consists of 10% nitrogen and 5% phosphoric acid and of type B which consists of 5% nitrogen and 10% phosphoric acid. After testing the soil, it is found that at least 15 kg of nitrogen and 15 kg of phosphoric acid is required for a good crop. The fertilizer of type A costs Rs 3 per kg and type B costs Rs 4 per kg. How many kilograms of each fertilizer should be used to meet the requirements so that the cost is minimum?

Or **6**

b. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a packet of nuts while it takes 3 hours on machine A and 1 hour on machine B to produce a packet of bolts. He earns a profit of Rs 17.50 per packet on nuts and Rs 7 per packet on bolts. How many packet of each should be produced each day so as to maximize his profit if he operates his machines for at most 12 hours a day? Also find the maximum profit.

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