JEE-Main-28-07-2022-Shift-1 (Memory Based)

MATHEMATICS

Question: Let $A=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$ & $B_0=A^{49}+2A^{98}$. If $B_n=adj\left(B_{n-1}\right)\forall n\geq 1$ then $|B_4|$

Options:

(a) 3^{28}

(b) 3^{30}

(c) 3^{32}

(d) 3^{36}

Answer: (c)

Solution:

$$A^2=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I$$

$$\therefore B_0=I+2A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+\begin{bmatrix}0&2&0\\2&0&0\\0&0&2\end{bmatrix}=\begin{bmatrix}1&2&0\\2&1&0\\0&0&3\end{bmatrix}$$

As $B_n=adj\left(B_{n-1}\right)$

$\therefore\ B_4=adj\left(B_3\right)=adj\left(adj\,B_2\right)=adj\left(adj\left(adj\,B_1\right)\right)$

$=adj\left(adj\left(adj\left(adj\,B_0\right)\right)\right)$

$\therefore |B_4|=|B_0|^{2^4}=|B_0|^{16}$

$$|B_0|=\begin{bmatrix}1&2&0\\2&1&0\\0&0&3\end{bmatrix}=3(-3)$$

$\therefore\ |B_0|^{16}=9^{16}=3^{32}$

Question: Considering the principal value of ITF, the sum of all solutions of $\cos^{-1}x-2\sin^{-1}x=\cos^{-1}2x$ is

Options:

(a) 0

(b) 1

(c) $\frac{1}{2}$

(d) $-\frac{1}{2}$

Answer: (a)
Solution:

$$\cos^{-1} x - 2\sin^{-1} x = \cos^{-1} 2x$$

$$\cos^{-1} x - \cos^{-1} 2x = 2\sin^{-1} x$$

$$\cos\left(\cos^{-1} x - \cos^{-1} 2x\right) = \cos\left(2\sin^{-1} x\right)$$

$$x(2x) + \sqrt{1-x^2}\sqrt{1-4x^2} = 1-2x^2$$

$$4x^2 - 1 = \sqrt{1-x^2}\sqrt{1-4x^2}$$

$$\left(4x^2-1\right)^2 = \left(1-4x^2-x^2+4x^4\right)$$

$$16x^4 - 8x^2 + 1 = 1 - 5x^2 + 4x^4$$

$$\Rightarrow 12x^4 - 3x^2 = 0$$

$$\Rightarrow x^2\left(4x^2-1\right) = 0$$

$$\Rightarrow x = 0, \frac{1}{2}, -\frac{1}{2}$$

Sum = 0

Question: If minimum value of $f(x) = \dfrac{5x^2}{2} + \dfrac{\alpha}{x^5}, x > 0$ is 14 then $\alpha = ?$

Options:
(a) 32
(b) 64
(c) 128
(d) 256

Answer: (c)
Solution:

$$f(x) = \frac{5x^2}{2} + \frac{\alpha}{x^5}$$

$$f'(x) = 5x - \frac{5\alpha}{x^6} = 0$$

$$\Rightarrow x^7 = \alpha$$

$$\Rightarrow x = \alpha^{\frac{1}{7}}$$

$$f\left(\alpha^{\frac{1}{7}}\right) = 14$$

$$\frac{5\alpha^{\frac{2}{7}}}{2} + \frac{\alpha}{\alpha^{\frac{5}{7}}} = 14$$

$$\frac{7\alpha^{\frac{2}{7}}}{2} = 14$$

$$\alpha^{\frac{2}{7}} = 4$$

$\alpha = 4^{\frac{7}{2}} = 2^7 = 128$

Question: $x\,dy = \left(\sqrt{x^2+y^2}+y\right)dx$ curve passes through $(1,0)$. Find $y(2)=?$

Answer: $\frac{3}{2}$

Solution:

$x\,dy = \left(\sqrt{x^2+y^2}+y\right)dx$

$\frac{dy}{dx} = \sqrt{1+\frac{y^2}{x^2}}+\frac{y}{x}$

$y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

$x\frac{dv}{dx} = \sqrt{1+v^2}$

$\Rightarrow \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$

$= \ln\left|v+\sqrt{1+v^2}\right| = \ln|x| + \ln c$

$\Rightarrow v + \sqrt{1+v^2} = cx$

$\Rightarrow \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = cx$

$x = 1, y = 0$

$\Rightarrow 0 + 1 = c$

$\Rightarrow c = 1$

$\Rightarrow \frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = x$

$\Rightarrow \frac{y}{2} + \sqrt{1+\frac{y^2}{4}} = 2$

$\Rightarrow \sqrt{1+\frac{y^2}{4}} = 2 - \frac{y}{2}$

$\Rightarrow 1 + \frac{y^2}{4} = 4 + \frac{y^2}{4} - 2y$

$\Rightarrow 2y = 3$

$\Rightarrow y = \frac{3}{2}$

Question: Find remainder when $7^{2022} + 3^{2022}$ is divided by 5

Answer: 3.00

Solution:

Given, $7^{2022}+3^{2022}$

$7^{2022}+3^{2022}=(49)^{1011}+9^{1011}$

$=(50-1)^{1011}+(10-1)^{1011}$

$=(5k-1)+(5\lambda-1)$

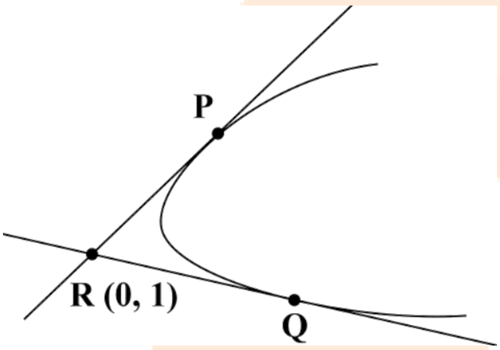
Remainder $=5-2=3$

Question: $y^2=2x+3$. For $A(1,0)$, two tangents are drawn which meets parabola at P & Q. Find orthocentre of APQ.

Answer: (2, -1)

Solution:

$y^2=2x-3$ ....(1)

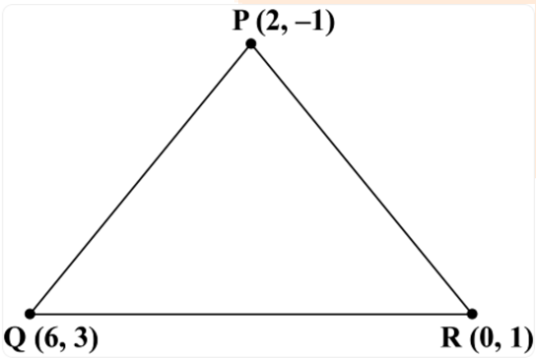


$PQ=T=0$

$y=x-3$ ....(2)

Solving (1) & (2) we get

$(2,-1)$ and $(6,3)$



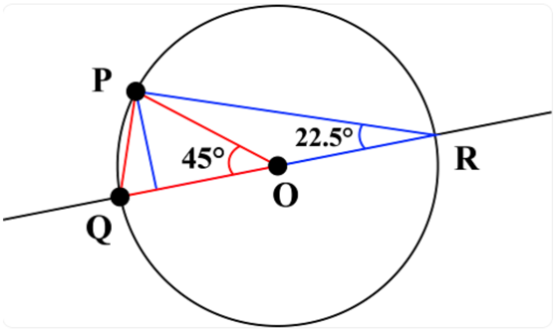
$m_{QR}=-1,\ m_{PQ}=1$

Orthocentre $\equiv(6,3)$

Question: A line passing through centre "O" of circle with radius 4 units, cuts the circle at Q & R. P is a point on the circle such that OP makes $45°$ with QR. Find area of ΔPQR.

Answer: $\frac{16}{\sqrt{2}}$

Solution:



$QR = 8$

$PR = QR\cos 22.5° = 8\cos 22.5°$

$PQ = QR\sin 22.5° = 8\sin 22.5°$

Area $= \frac{1}{2}PR \cdot PQ$

$= \frac{1}{2}8\cos 22.5° \cdot 8\sin 22.5°$

$= 16\sin 22.5°$

$= \frac{16}{\sqrt{2}}$

Question: A 6-8 digit password is to be made using A, B, C, D, E, 1, 2, 3, 4, 5, with repetition allowed. If number of such passwords containing atleast one digit is $x \times 5^6$ then $x = ?$

Answer: 7073.00

Solution:

6 digit _ _ _ _ _ _ : $10^6 - 5^6$

7 digit _ _ _ _ _ _ _ : $10^7 - 5^7$

8 digit _ _ _ _ _ _ _ _ : $10^8 - 5^8$

Total: $10^6 - 5^6 + 10^7 - 5^7 + 10^8 - 5^8$

$= 7073\left(5^6\right)$

$\therefore x = 7073$

Question: Out of total candidate 60% were female, 40% were male. 60% of total passed. Female passed was twice of males passed. A candidate who passes was chosen. Find probability that it was a female.

Answer: $\frac{2}{3}$

Solution:

Given,

Total = 100

Female = 60
Male = 40
Passed = 60
Female passed = 40, Male passes = 20

Probability $= \frac{40}{60} = \frac{2}{3}$

Question: If $f(x) = \int_0^x e^{x-t} f'(t)\,dt + e^x(x^2 - x + 1)$, then find $f(x)\Big|_{\min}$.

Answer: ()

Solution:

$$f(x) = \int_0^x e^{x-t} f'(t)\,dt + e^x(x^2 - x + 1)$$

$$f(x) = e^x \int_0^x e^{-t} f'(t)\,dt + e^x(x^2 - x + 1)$$

$$f'(x) = e^x\left[\int_0^x e^{-t} f'(t)\,dt + e^{-x} f'(x)\right] + e^x(x^2 - x + 1 + 2x - 1)$$

$$f'(x) = e^x\left[\int_0^x e^{-t} f'(x)\,dt\right] + f'(x) + e^x(x^2 + x)$$

$$\Rightarrow 0 = e^x\left[\int_0^x e^{-t} f'(t)\,dt + x^2 + x\right]$$

$$\Rightarrow 0 = \int_0^x e^{-t} f'(x)\,dt + x^2 + x$$

$$\Rightarrow f'(x) = -e^{-x}(2x+1)$$

Put $f'(x) = 0 \Rightarrow x = \frac{-1}{2}$

$$f'(x) = -e^x(2x + 2 - 1)$$

$$f'(x) = -e^x(2x + 2) + e^x$$

$$\Rightarrow f(x) = -e^x(2x) + e^x + C$$

$$f(0) = 1$$

$$\Rightarrow C = 0$$

$$f(x) = -e^x(2x) + e^x$$

$$f\left(\frac{-1}{2}\right) = -e^{\frac{-1}{2}}\left(2\left(\frac{-1}{2}\right)\right) + e^{\frac{-1}{2}} = 0$$

Question: Find principal range of $\cos^{-1}\left(\frac{x^2-4x+2}{x^2+3}\right)$

Answer: ()

Solution:

$$y=\frac{x^2-4x+2}{x^2+3}$$

$$\Rightarrow x^2y+3y=x^2-4x+2$$

$$\Rightarrow x^2(y-1)+4x+3y-2=0$$

$$D\geq 0$$

$$16-4(y-1)(3y-2)\geq 0$$

$$4-(3y^2-2y-3y+2)\geq 0$$

$$\Rightarrow -3y^2+5y+2\geq 0$$

$$\Rightarrow 3y^2-5y-2\leq 0$$

$$\Rightarrow y\in\left[\frac{-1}{3},2\right]$$

Range $\in\left[0,\cos^{-1}\left(\frac{-1}{3}\right)\right]$

Question: Eccentricity of $x^2-y^2=1$ is reciprocal of eccentricity of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. $y=\left(\frac{\sqrt{5}}{2}\right)x+k$ is common tangent. Find $4(a^2+b^2)$.

Answer: $\frac{6}{7}$

Solution:

$$x^2-y^2=1$$

$$e=\sqrt{2}$$

$$b^2=a^2\left(1-\frac{1}{2}\right)$$

$$b^2=\frac{a^2}{2}$$

$$y=\frac{\sqrt{5}}{2}x+R$$

$$R^2=\frac{1\times 5}{4}-1=\frac{1}{4}$$

$$R^2=a^2m^2+b^2$$

$$\frac{1}{4}=\frac{a^25}{4}+b^2$$

$1=5a^2+4b^2$

$1=5a^2+2a^2$

$a^2=\frac{1}{7}, b^2=\frac{1}{14}$

$4\left(a^2+b^2\right)=4\left(\frac{1}{7}+\frac{1}{14}\right)$

$=4\left(\frac{3}{14}\right)$

$=\frac{6}{7}$

Question: If $a_{n+2}=\frac{2}{a_{n+1}}+a_n, a_1=1, a_2=2$ & $\left(\frac{a_1+\frac{1}{a_2}}{a_3}\right)\left(\frac{a_2+\frac{1}{a_3}}{a_4}\right)\times\ldots\times\left(\frac{a_{30}+\frac{1}{a_{31}}}{a_{32}}\right)=2^{\alpha}\times {}^{61}C_{31}$

then $\alpha=?$

Answer: ()

Solution:

$\because a_{n+2}\cdot a_{n+1}-a_{n+1}\cdot a_n=2$ where $a_1=1, a_2=2$ and $a_3=2$

Let $T_r=a_{r+1}\cdot a_r$

So, T_r is A.P. with common difference 2 and first term 2.

Clearly $T_r=2r$

Now $\prod_{i=1}^{30}\left(\frac{a_i+\frac{1}{a_{i+1}}}{a_{i+2}}\right)=\prod_{i=1}^{30}\left(\frac{T_r+1}{T_{r+1}}\right)$

$=\prod_{i=1}^{30}\left(\frac{2r+1}{2r+2}\right)=\frac{3\cdot5\cdot7....61}{4\cdot6\cdot8...62}=\frac{62!}{2(4\cdot6\cdot8....62)^2}$

$=\frac{62!}{2^{61}\cdot(31!)^2}$

$=2^{-61}\cdot {}^{62}C_{31}$

$=2^{-60}\cdot {}^{61}C_{30}$

$2^{-60}\cdot {}^{61}C_{31}$

Question: Let $z_1\in C, |z_1-3|=\frac{1}{2}, z_2\in C, \left|z_2+|z_2-1|\right|=\left|z_2-|z_2+1|\right|$, then least value of $|z_1-z_2|$ is:

Answer: $\frac{3}{2}$

Solution:

$$\left|z_2+\left|z_2-1\right|\right|^2=\left|z_2-\left|z_2+1\right|\right|^2$$

$$\Rightarrow\left(z_2+\left|z_2-1\right|\right)\left(\overline{z}_2+\left|z_2-1\right|\right)=\left(z_2-\left|z_2+1\right|\right)\left(\overline{z}_2-\left|z_2+1\right|\right)$$

$$\Rightarrow z_2\left(\left|z_2-1\right|+\left|z_2+1\right|\right)+\overline{z}_2\left(\left|z_2-1\right|+\left|z_2+1\right|\right)=\left|z_2+1\right|^2-\left|z_2-1\right|^2$$

$$\Rightarrow\left(z_2+\overline{z}_2\right)\left(\left|z_2+1\right|+\left|z_2-1\right|\right)=2\left(z_2+\overline{z}_2\right)$$

Either $z_2+\overline{z}_2=0$ or $\left|z_2+1\right|+\left|z_2-1\right|=2$

So, z_2 lies on imaginary axis or on real axis with in $[-1,1]$. Also $\left|z_1-3\right|=\frac{1}{2}$, lies on the circle having centre 3 and radius $\frac{1}{2}$.

Clearly $\left|z_1-z_2\right|_{\min}=\frac{3}{2}$