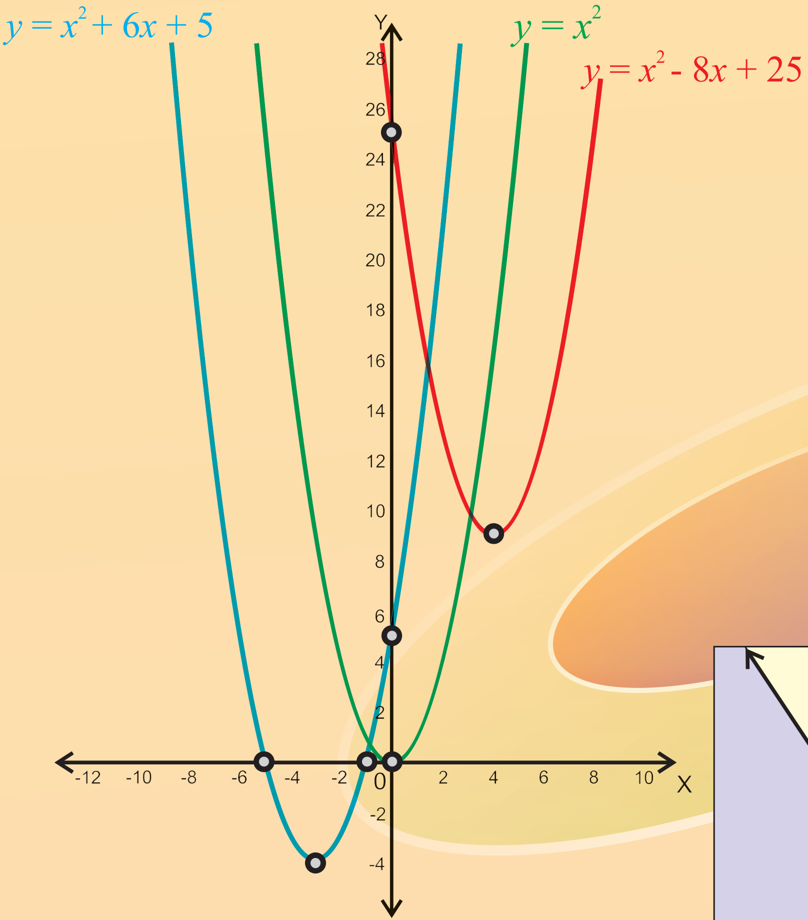
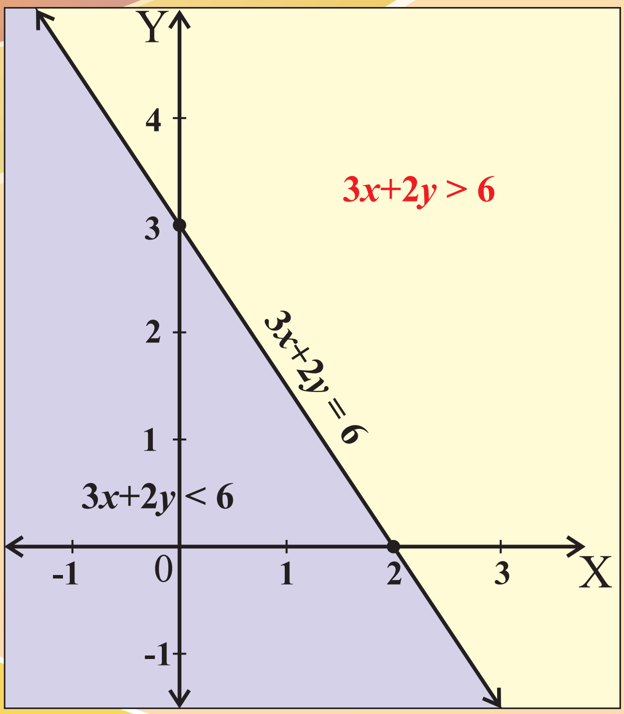
Mathematics & Statistics

Arts & Science Part 1

STANDARD XII





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Mathematics and Statistics (Arts and Science)

Part - I

STANDARD - XII

Maharashtra State Bureau of Textbook Production and Curriculum Research, Pune. - 411 004

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The Constitution of India

Preamble

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC and to secure to all its citizens:

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the unity and integrity of the Nation;

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949, do HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.

NATIONAL ANTHEM

Jana-gana-mana-adhināyaka jaya hē
Bhārata-bhāgya-vidhātā,

Panjāba-Sindhu-Gujarāta-Marāthā
Drāvida-Utkala-Banga

Vindhya-Himāchala-Yamunā-Gangā
uchchala-jaladhi-taranga

Tava subha nāmē jāgē, tava subha āsisa māgē,
gāhē tava jaya-gāthā,

Jana-gana-mangala-dāyaka jaya hē
Bhārata-bhāgya-vidhātā,

Jaya hē, Jaya hē, Jaya hē,
Jaya jaya jaya, jaya hē.

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give my parents, teachers and all elders respect, and treat everyone with courtesy.

To my country and my people, I pledge my devotion. In their well-being and prosperity alone lies my happiness.

PREFACE

Dear Students,

Welcome to Standard XII, an important milestone in your life.

Standard XII or Higher Secondary School Certificate opens the doors of higher education. Alternatively, you can pursue other career paths like joining the workforce. Either way, you will find that mathematics education helps you considerably. Learning mathematics enables you to think logically, consistently, and rationally. The curriculum for Standard XII Mathematics and Statistics for Science and Arts students has been designed and developed keeping both of these possibilities in mind.

The curriculum of Mathematics and Statistics for Standard XII for Science and Arts students is divided in two parts. Part I deals with topics like Mathematical Logic, Matrices, Vectors and Introduction to three dimensional geometry. Part II deals with Differentiation, Integration and their applications, Introduction to random variables and statistical methods.

The new text books have three types of exercises for focussed and comprehensive practice. First, there are exercises on every important topic. Second, there are comprehensive exercises at the end of all chapters. Third, every chapter includes activities that students must attempt after discussion with classmates and teachers. Additional information has been provided on the E-balbharati website (www.ebalbharati.in).

We are living in the age of Internet. You can make use of modern technology with the help of the Q.R. code given on the title page. The Q.R. code will take you to links that provide additional useful information. Your learning will be fruitful if you balance between reading the text books and solving exercises. Solving more problems will make you more confident and efficient.

The text books are prepared by a subject committee and a study group. The books (Paper I and Paper II) are reviewed by experienced teachers and eminent scholars. The Bureau would like to thank all of them for their valuable contribution in the form of creative writing, constructive and useful suggestions for making the text books valuable. The Bureau hopes and wishes that the text books are very useful and well received by students, teachers and parents.

Students, you are now ready to study. All the best wishes for a happy learning experience and a well deserved success. Enjoy learning and be successful.

(Vivek Gosavi)
Director
Maharashtra State Bureau of Textbook
Production and Curriculum Research, Pune.

Pune
Date: 21 February 2020
Bharatiya Saur: 2 Phalguna 1941

Mathematics and Statistics Std. XII Part - I
Competency Statement

Sr.No.	Area	Topic	Competency Statement
1.	Mathematical Logic	Mathematical Logic	The student will be able to • Identify statement in logic and truth value of it. • Combine two or more statements • Construct the truth table and examine logical equivalence of statement patterns • Find dual and negation of statement pattern • Study the applications of logic to switching circuits.
2.	Matrics	Matrics	• Identify orders and types of matrices • Perform basic algebraic operations on matrices. • Find the inverse of a matrix using elementary transformation and adjoint method • Solve the system of linear equations using matrices.
3.	Trigonometric Equations	Trigonometric Equations Solution of a triangle Invers trigonometric function	• Understand and write trigonometric equation • Find the principal and general solution of a trigonometric equation. • Solve triangle by using sine rule, co-sine rule and projection rule and find area of a triangle. • Understand inverse trigonometric functions with domain and range.
4.	Pair of straight lines	Pair of straight lines	• Write and interpret the combined equation of two straight lines in plane. • Find the point of intersection of two lines and calculate the acute angle between them • Study the general second degree equation in x and y with reference to homogeneous part of it

5.	Vectors	Vectors	• Understand scalars and vectors and algebra of vectors. • Write vectors of 2 or 3 dimensions, understand the scalar and vector products • Study applications of vectors to area of triangle, work done by a force, moment of a force. • Interpret scalar triple product and its applications.
6.	Line and Plane	Line and Plane	• Find different forms of equation of line • Find angle between two intersecting planes • Find the angle between a line and a plane • Find condition for perpendicularity and parallelness of planes • Calculate distance of a point from a plane • Find equation of a plane in different forms • Find angle between two intersecting planes • Find the angle between a line and a plane
7.	Linear programming Problem	Linear programming Problem	• Understand linear equations in one and two variables. • Find graphical solution of linear inequation. • Understand meaning and formulation of L.P.P. • Find solution of L.P.P. by graphical methods.

INDEX

1 Mathematical Logic

Let's study.

1 Statement and its truth value.
2. Logical connective, compound statements.
3. Truth tables regation of statements and compound statements.
4. Statement pattern, logical equivalence.
5. Tautology, contradiction and contingency.
6. Quantifiers and quantified statements, Duality.
7. Application of logic to switching circuits, switching table.

Let's recall.

1.1.1 Introduction :

Mathematics is a logical subject and tries to be exact. For exactness, it requires proofs which depend upon proper reasoning. Reasoning requires logic. The word Logic is derived from the Greek word "LOGOS" which means reason. Therefore logic deals with the method of reasoning. In ancient Greece the great philosopher and thinker Aristotle started study of Logic systematically. In mathematics Logic has been developed by English Philosopher and mathematician George Boole (2 November 1815 - 8 December 1864)

Language is the medium of communication of our thoughts. For communication we use sentences. In logic, we use the statements which are special sentences.

1.1.2 Statement :

A statement is a declarative (assertive) sentence which is either true or false, but not both simultaneously. Statements are denoted by $p, q, r,$ .....

1.1.3 Truth value of a statement :

Each statement is either true or false. If a statement is true then its truth value is 'T' and if the statement is false then its truth value is F.

Illustrations :

1) Following sentences are statements.

i) Sun rises in the East.
ii) $5 \times 2 = 11$
iii) Every triangle has three sides.

iv) Mumbai is the capital of Maharashtra.

v) Every equilateral triangle is an equiangular triangle.

vi) A natural number is an integer.

2) Following sentences are not statements.

i) Please, give your Pen.

ii) What is your name ?

iii) What a beautiful place it is !

iv) How are you ?

v) Do you like to play tennis ?

vi) Open the window.

vii) Let us go for tea

viii) Sit dows.

Note : Interrogative, exclamatory, command, order, request, suggestion are not statements.

3) Consider the following.

i) $\frac{3x}{2} - 9 = 0$

ii) He is tall.

iii) Mathematics is an interesting subject.

iv) It is black in colour.

Let us analyse these statements.

i) Fox $x = 6$ it is true but for other than 6 it is not true.

ii) Here, we cannot determine the truth value.

For iii) & iv) the truth value varies from person to person. In all the above sentences, the truth value depends upon the situation. Such sentences are called as open sentences. Open sentence is not a statement.

Solved examples

Q.1. Which of the following sentences are statements in logic ? Write down the truth values of the statements.

i) $6 \times 4 = 25$

ii) $x + 6 = 9$

iii) What are you doing ?

iv) The quadratic equation $x^2 - 5x + 6 = 0$ has 2 real roots.

v) Please, sit down

vi) The Moon revolves around the earth.

vii) Every real number is a complex number.

viii) He is honest.

ix) The square of a prime number is a prime number.

Solution :

i) It is a statement which is false, hence its truth value is F.

ii) It is an open sentence hence it is not a statement.

iii) It is an interrogative hence it is not a statement.

iv) It is a statement which is true hence its truth value is T.

v) It is a request hence it is not a statement.

vi) It is a statement which is true, hence its truth value is T.

vii) It is a statement which is true, hence its truth value is T.

viii) It is open sentence, hence it is not a statement.

ix) It is a statement which is false, hence its truth value is F.

1.1.4 Logical connectives, simple and compound statements :

The words or phrases which are used to connect two statements are called logical connectives. We will study the connectives 'and', 'or', 'if ..... then', 'if and only if ', 'not".

Simple and Compound Statements : A statement which cannot be split further into two or more statements is called a simple statement. If a statement is the combination of two or more simple statements, then it is called a compound statement.

"3 is a prime and 4 is an even number", is a compound statement.

"3 and 5 are twin primes", is a simple statement.

We describe some connectives.

1) Conjunction : If two statements are combined using the connective 'and' then it is called as a conjunction. In other words if p, q are two statements then 'p and q' is called as conjunction. It is denoted by '$p \wedge q$' and it is read as 'p conjunction q' or 'p and q'. The conjunction $p \wedge q$ is said to be true if and only if both p and q are true.

Truth table for conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 1.1

2) Disjunction : If two statements are combined by using the logical connective 'or' then it is called as a disjunction. In other words if p, q are two staements then 'p or q' is called as disjunction. It is denoted by '$p \vee$ q and it is read as 'p or q' or 'p disjunction q'.

Truth table for disconjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 1.2

The disjunction $p \vee$ q is false if and only if both p and q are false.

3) Conditional (Implication) : If two statements are combined by using the connective. 'if .... then', then it is called as conditional or implication. In other words if p, q are two statements then 'if p then q' is called as conditional. It is denoted by $p \to q$ or $p \Rightarrow q$ and it is read as 'p implies q' or 'if p then q'.

Truth table for conditional.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 1.3

The conditional statement $p \to q$ is False only if p is true and q is false. Otherse it is true. Here p is called hypothesis or antecedent and q is called conclusion or consequence.

Note : The following are also conditional statement $p \to q$

i) p is sufficient for q

ii) q is necessary for p

iii) p implies q

iv) q follows from p

v) p only if q.

4) Biconditional (Double implication) :

If two statements are combined using the logical connective 'if and only if ' then it is called as biconditional. In other words if p, q are two statements then 'p if and only if q' is called as biconditional. It is denoted by '$p \leftrightarrow q$' or $p \Leftrightarrow q$. It is read as 'p biconditional q' or 'p if and only if q'.

Truth table for biconditional.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 1.4

Biconditional statement $p \leftrightarrow q$ is true if p and q have same truth values. Otherwise it is False.

5) Negation of a statement : For any given statement p, there is another statement which is defined to be true when p is false, and false when p is true, is called the negation of p and is denoted by $\sim p$.

Truth table for negation.

p	$\sim$ p
T	F
F	T

Table 1.5

Note : Negation of negation of a statement is the statement itself. That is, $\sim(\sim p) = p$.

Solved examples

Ex.1: Express the following compound statements symbolically without examining the truth values.

i) 2 is an even number and 25 is a perfect square.

ii) A school is open or there is a holiday.

iii) Delhi is in India but Dhaka is not in Srilanks.

iv) $3 + 8 \geq 12$ if and only if $5 \times 4 \leq 25$.

Solution :

i) Let p : 2 is an even numder

q : 25 is a perfect square.

The symbolic form is $p \wedge q$.

ii) Let p : The school is open

q : There is a holiday

The symbolic form is $p \vee q$

iii) Let p : Delhi is in India

q : Dhaka is in Srilanka

The symbolic form is $p \wedge \sim q$.

iv) Let $p : 3 + 8 \geq 12$; $q : 5 \times 4 \leq 25$

The symbolic form is $p \leftrightarrow q$

Ex.2. Write the truth values of the following statements.

i) 3 is a prime number and 4 is a rational number.

ii) All flowers are red or all cows are black.

iii) If Mumbai is in Maharashtra then Delhi is the capital of India.

iv) Milk is white if and only if the Sun rises in the West.

Solution :

i) Let p : 3 is a prime number

q : 4 is a rational number.

Truth values of p and q are T and T respectively.

The given statement in symbolic form is $p \wedge q$.

The truth value of given statement is T.

ii) Let p : All flowers are red ; q : All cows are black.

Truth values of p and q are F and F respectively.

The given statement in the symbolic form is $p \vee q$

$\therefore p \vee q \equiv F \vee F$ is F

$\therefore$ Truth value of given statement is F.

iii) Let p : Mumbai is in Maharashtra

q : Delhi is capital of India

Truth values of p and q are T and T respectively.

The given statement in symbolic form is $p \rightarrow q$

$\therefore\ p \rightarrow q \equiv T \rightarrow T$ is T

$\therefore$ Truth value of given statement is T

iv) Let p : Milk is white; q : Sun rises in the West.

Truth values of p and q are T and F respectively.

The given statement in symbolic form is $p \leftrightarrow q$

$\therefore\ p \leftrightarrow q \equiv T \leftrightarrow F$ is F

$\therefore$ Truth value of given statement is F

Ex.3 : If statements p, q are true and r, s are false, determine the truth values of the following.

i) $\sim p \wedge (q \vee \sim r)$ ii) $(p \wedge \sim r) \wedge (\sim q \vee s)$

iii) $\sim (p \rightarrow q) \leftrightarrow (r \wedge s)$ iv) $(\sim p \rightarrow q) \wedge (r \leftrightarrow s)$

Solution :

i) $\sim p \wedge (q \vee \sim r) \equiv \sim T \wedge (T \vee \sim F) \equiv F \wedge (T \vee T) \equiv F \wedge T \equiv F$

Hence truth value is F.

ii) $(p \wedge \sim r) \wedge (\sim q \vee s) \equiv (T \wedge \sim F) \wedge (\sim T \vee F) \equiv (T \wedge T) \wedge (F \vee F) \equiv T \wedge F \equiv F.$

Hence truth value is F.

iii) $[\sim(p \rightarrow q)] \leftrightarrow (r \wedge s) \equiv [\sim(T \rightarrow T)] \leftrightarrow (F \wedge F) \equiv (\sim T) \leftrightarrow (F) \equiv F \leftrightarrow F \equiv T.$

Hence truth value is T

iv) $(\sim p \rightarrow q) \wedge (r \leftrightarrow s) \equiv (\sim T \rightarrow T) \wedge (F \leftrightarrow F) \equiv (F \rightarrow T) \wedge T \equiv T \wedge T \equiv T.$

Hence truth value is T.

Ex.4. Write the negations of the following.

i) Price increases

ii) $0! \neq 1$

iii) $5 + 4 = 9$

Solution :

i) Price does not increase

ii) $0! = 1$

iii) $5 + 4 \neq 9$

Exercise 1.1

Q.1. State which of the following are statements. Justify. In case of statement, state its truth value.

i) $5 + 4 = 13.$

ii) $x - 3 = 14.$

iii) Close the door.

iv) Zero is a complex number.

v) Please get me breafast.

vi) Congruent triangles are similar.

vii) $x^2 = x.$

viii) A quadratic equation cannot have more than two roots.

ix) Do you like Mathematics ?

x) The sun sets in the west

xi) All real numbers are whole numbers

xii) Can you speak in Marathi ?

xiii) $x^2 - 6x - 7 = 0$, when $x = 7$

xiv) The sum of cuberoots of unity is zero.

xv) It rains heavily.

Q.2. Write the following compound statements symbollically.

i) Nagpur is in Maharashtra and Chennai is in Tamilnadu

ii) Triangle is equilateral or isosceles.

iii) The angle is right angle if and only if it is of measure 90º.

iv) Angle is neither acute nor obtuse.

v) If Δ ABC is right angled at B, then m $\angle A$ + m $\angle C = 90^\circ$

iv) Hima Das wins gold medal if and only if she runs fast.

vii) x is not irrational number but is a square of an integer.

Q.3. Write the truth values of the following.

i) 4 is odd or 1 is prime.

ii) 64 is a perfect square and 46 is a prime number.

iii) 5 is a prime number and 7 divides 94.

iv) It is not true that 5-3i is a real number.

v) If $3 \times 5 = 8$ then $3 + 5 = 15$.

vi) Milk is white if and only if sky is blue.

vii) 24 is a composite number or 17 is a prime number.

Q.4. If the statements p, q are true statements and r, s are false statements then determine the truth values of the following.

i) $p \vee (q \wedge r)$

ii) $(p \rightarrow q) \vee (r \rightarrow s)$

iii) $(q \wedge r) \vee (\sim p \wedge s)$

iv) $(p \rightarrow q) \wedge \sim r$

v) $(\sim r \leftrightarrow p) \rightarrow \sim q$

vi) $[\sim p \wedge (\sim q \wedge r)] \vee [(q \wedge r) \vee (p \wedge r)]$

vii) $[(\sim p \wedge q) \wedge \sim r] \vee [(q \rightarrow p) \rightarrow (\sim s \vee r)]$

viii) $\sim[(\sim p \wedge r) \vee (s \rightarrow \sim q)] \leftrightarrow (p \wedge r)$

Q.5. Write the negations of the following.

i) Tirupati is in Andhra Pradesh

ii) 3 is not a root of the equation $x^2 + 3x - 18 = 0$

iii) $\sqrt{2}$ is a rational number.

iv) Polygon ABCDE is a pentagon.

v) $7 + 3 > 5$

1.2 STATEMENT PATTERN, LOGICAL EQUIVALENCE, TAUTOLOGY, CONTRADICTION, CONTINGENCY.

1.2.1 Statement Pattern :

Letters used to denote statements are called statement letters. Proper combination of statement letters and connectives is called a statement pattern. Statement pattern is also called as a proposition. $p \to q, p \wedge q, \sim p \vee q$ are statement patterns. p and q are their prime components.

A table which shows the possible truth values of a statement pattern obtained by considering all possible combinations of truth values of its prime components is called the truth table of the statement pattern.

1.2.2. Logical Equivalence :

Two statement patterns are said to be equivalent if their truth tables are identical. If statement patterns A and B are equivalent, we write it as A ≡ B.

1.2.3 Tautology, Contradiction and Contingency :

Tautology : A statement pattern whose truth value is true for all possible combinations of truth values of its prime components is called a tautology. We denote tautology by t.

Statement pattern $p \vee \sim p$ is a tautology.

Contradiction : A statement pattern whose truth value is false for all possible combinations of truth values of its prime components is called a contradiction. We denote contradiction by c.

Statement pattern $p \wedge \sim p$ is a contradiction.

Contingency : A statement pattern which is neither a tautology nor a contradiction is called a contingency. $p \wedge q$ is a contingency.

Important table for all connectives :

p	q	$\sim p$	$p \wedge q$	$p \vee q$	$p \to q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

☆ **In a statement pattern, different symbols are considered in the following priority**

$\sim, \vee, \wedge, \to, \leftrightarrow$

Solved Examples

Ex.1.: Construct the truth table for each of the following statement patterns.

i) $p \to (q \to p)$

ii) $(\sim p \vee q) \leftrightarrow \sim (p \wedge q)$

iii) $\sim (\sim p \wedge \sim q) \vee q$

iv) $[(p \wedge q) \vee \mathrm{r}] \wedge [\sim \mathrm{r} \vee (p \wedge q)]$

v) $[(\sim p \vee q) \wedge (q \to \mathrm{r})] \to (p \to \mathrm{r})$

Solution :

i) $p \to (q \to p)$

p	q	$q \to p$	$p \to (q \to p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

Table 1.7

ii) $(\sim p \vee q) \leftrightarrow \sim(p \wedge q)$

p	q	$\sim p$	$\sim p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(\sim p \vee q) \leftrightarrow \wedge (p \wedge q)$
T	T	F	T	T	F	F
T	F	F	F	F	T	F
F	T	T	T	F	T	T
F	F	T	T	F	T	T

Table 1.8

iii) $\sim (\sim p \wedge \sim q) \vee q$

p	q	$\sim p$	$\sim q$	$\sim(p \wedge \sim q)$	$\sim(\sim p \wedge \sim q)$	$\sim(\sim p \wedge \sim q)$	$\sim(\sim p \wedge \sim q) \vee q$
T	T	F	F	F	T	T	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	T	F	F	F

Table 1.9

iv) $[(p \wedge q) \vee \mathrm{r}] \wedge [\sim \mathrm{r} \vee (p \wedge q)]$

p	q	r	~r	$p \wedge q$	$(p \wedge q) \vee \mathrm{r}$	$\sim \mathrm{r} \vee (p \wedge q)$	$[p \wedge q) \vee \mathrm{r}] \wedge (\sim \mathrm{r} \vee (p \wedge q)$
T	T	T	F	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	F	T	F	F
T	F	F	T	F	F	T	F
F	T	T	F	F	T	F	F
F	T	F	T	F	F	T	F
F	F	T	F	F	T	F	F
F	F	F	T	F	F	T	F

Table 1.10

v) $[(\sim p \vee q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

p	q	r	$\sim p$	$\sim p \vee q$	$q \rightarrow r$	$p \rightarrow r$	$(\sim p \vee q) \wedge (q \rightarrow r)$	$[(\sim p \vee q) \wedge (q \rightarrow r)] \rightarrow (q \rightarrow r)$
T	T	T	F	T	T	T	T	T
T	T	F	F	T	F	F	F	T
T	F	T	F	F	T	T	F	T
T	F	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	F	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Table 1.11

Ex.2: Using truth tables, prove the following logical equivalences

i) $(p \wedge q) \equiv \sim (p \rightarrow \sim q)$

ii) $(p \leftrightarrow q) \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

iii) $(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$

iv) $p \rightarrow (q \vee r) \equiv (p \rightarrow q) \vee (p \rightarrow r)$

i) **Solution :** (1) $(p \wedge q) \equiv \sim(p \rightarrow \sim q)$

I	II	III	IV	V	VI
p	q	$\sim q$	$p \wedge q$	$p \rightarrow \sim q$	$\sim(p \rightarrow \sim q)$
T	T	F	T	F	T
T	F	T	F	T	F
F	T	F	F	T	F
F	F	T	F	T	F

Table 1.12

Columns (IV) and (VI) are identical $\therefore$ $(p \wedge q) \equiv \sim(p \rightarrow \sim q)$

ii) $(p \leftrightarrow q) \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

I	II	III	IV	V	VI	VII	VIII
p	q	~p	~q	$p \leftrightarrow q$	$(p \wedge q)$	$\sim p \wedge \sim q$	$(p \wedge q) \vee (\sim p \wedge \sim q)$
T	T	F	F	T	T	F	T
T	F	F	T	F	F	F	F
F	T	T	F	F	F	F	F
F	F	T	T	T	F	T	T

Table 1.13

Columns V and VIII are identical

$\therefore$ $(p \leftrightarrow q) \equiv (p \wedge q) \vee (\sim p \vee \sim q)$

(iii) $(p \wedge q) \rightarrow r \equiv p \rightarrow (q \rightarrow r)$

I	II	III	IV	V	VI	VII
p	q	r	$p \wedge q$	$(p \wedge q) \to r$	$q \to r$	$p \to (q \to r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	F	T	T	T
F	T	F	F	T	F	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

Table 1.14

Column (V) and (VII) are identical

$\therefore$ $(p \wedge q) \to r \equiv p \to (q \to r)$

(iv) $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$

I	II	III	IV	V	VI	VII	VIII
p	q	r	$q \vee r$	$p \to (q \vee r)$	$p \to q$	$p \to r$	$(p \to q) \vee (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

Table 1.15

Columns V and VIII are identical

$\therefore$ $p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$

Ex.3. Using truth tables, examine whether each of the following statements is a tautology or a contradiction or contingency.

i) $(p \wedge q) \wedge (\sim p \vee \sim q)$

ii) $[p \wedge (p \to \sim q)] \to p$

iii) $(p \to q) \wedge [(q \to r) \to (p \to r)]$

iv) $[(p \vee q) \vee r] \leftrightarrow [p \vee \leftrightarrow q \vee r)]$

Solution:

i) $(p \wedge q) \wedge (\sim p \vee \sim q)$

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim p \vee \sim q$	$(p \wedge q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	F

Table 1.16

All the truth values in the last column are F. Hence it is contradiction.

ii) $[p \wedge (p \rightarrow \sim q)] \rightarrow q$

p	q	$\sim q$	$p \rightarrow \sim q$	$p \wedge (p \rightarrow \sim q$	$p \wedge (p \rightarrow \sim q) \rightarrow q$
T	T	F	F	F	T
T	F	T	T	T	F
F	T	F	T	F	T
F	F	T	T	F	T

Table 1.17

Truth values in the last column are not identical. Hence it is contingency.

iii) $(p \rightarrow q) \wedge [(q \rightarrow \text{r}) \rightarrow (p \rightarrow \text{r})]$

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$p \rightarrow r$	$(q \rightarrow r) \rightarrow (p \rightarrow \text{r})$	$(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	F	T	T	T	F
T	F	F	F	T	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Table 1.18

Truth values in the last column are not same, hence it is contingency.

iv) $[(p \vee q) \vee \text{r}] \leftrightarrow (p \vee (q \vee r)]$

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$	$[(p \vee q) \vee r] \leftrightarrow [p \vee (q \vee r)]$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	T	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	T

Table 1.19 All the truth values in the last column are T, hence it is tautology.

Exercie.1.2

Q.1. Construct the truth table for each of the following statement patterns.

i) $[(p \rightarrow q) \wedge q] \rightarrow p$

ii) $(p \wedge \sim q) \leftrightarrow (p \rightarrow q)$

iii) $(p \wedge q) \leftrightarrow (q \vee r)$

iv) $p \rightarrow [\sim (q \wedge r)]$

v) $\sim p \wedge [(p \vee \sim q) \wedge q]$

vi) $(\sim \mathrm{p} \rightarrow \sim \mathrm{q}) \wedge (\sim q \rightarrow \sim p)$

vii) $(q \rightarrow p) \vee (\sim p \leftrightarrow q)$

viii) $[p \rightarrow (q \rightarrow r)] \leftrightarrow [(p \wedge q) \rightarrow r]$

ix) $p \rightarrow [\sim (q \wedge r)]$

x) $(p \vee \sim q) \rightarrow (r \wedge \mathrm{p})$

Q.2. Using truth tables prove the following logical equivalences.

i) $\sim p \wedge q \equiv (p \vee q) \wedge \sim p$

ii) $\sim (p \vee q) \vee (\sim p \wedge q) \equiv \sim p$

iii) $p \leftrightarrow q \equiv \sim [(p \vee q) \wedge \sim (p \wedge q)]$

iv) $p \rightarrow (q \rightarrow p) \equiv \sim p \rightarrow (p \rightarrow q)$

v) $(\mathrm{p} \vee q) \rightarrow \mathrm{r} \equiv (p \rightarrow \mathrm{r}) \wedge (q \rightarrow \mathrm{r})$

vi) $p \rightarrow (q \wedge \mathrm{r}) \equiv (p \rightarrow q) \wedge (p \rightarrow \mathrm{r})$

vii) $p \rightarrow (q \wedge \mathrm{r}) \equiv (p \wedge q) \; (\mathrm{p} \rightarrow \mathrm{r})$

viii) $p \wedge (q \vee \mathrm{r}) \equiv (p \wedge q) \vee (p \wedge \mathrm{r})$

ix) $[\sim (p \vee q) \vee (p \vee q)] \wedge r \equiv r$

x) $\sim (p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

Q.3. Examine whether each of the following statement patterns is a tautology or a contradiction or a contingency.

i) $(p \wedge q) \rightarrow (q \vee p)$

ii) $(p \rightarrow q) \leftrightarrow (\sim p \vee q)$

iii) $[\sim (\sim p \wedge \sim q)] \vee q$

iv) $[(p \rightarrow q) \wedge q)] \rightarrow p$

v) $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$

vi) $(p \leftrightarrow q) \wedge (p \rightarrow \sim q)$

vii) $\sim (\sim q \wedge p) \wedge q$

viii) $(p \wedge \sim q) \leftrightarrow (p \rightarrow q)$

ix) $(\sim p \rightarrow q) \wedge (p \wedge r)$

x) $[p \rightarrow (\sim q \vee r)] \leftrightarrow \sim [p \rightarrow (q \rightarrow r)]$

1.3 QUANTIFIERS, QUANTIFIED STATEMENTS, DUALS, NEGATION OF COMPOUND STATEMENTS, CONVERSE, INVERSE AND CONTRAPOSITIVE OF IMPLICATION.

1.3.1 Quantifiers and quantified statements.

Look at the following statements :

p : "There exists an even prime number in the set of natural numbers".

q : "All natural numbers are positive".

Each of them asserts a *condition* for some or all objects in a *collection*. Words "there exists" and "for all" are called quantifiers. "There exists is called existential quantifier and is denoted by symbol $\exists$. "For all" is called universal quantifier and is denoted by $\forall$. Statements involving quantifiers are called quantified statements. Every quantified statement corresponds to a *collection* and a *condition*. In statement p the collection is 'the set of natural numbers' and the condition is 'being even prime'. What is the condition in the statement *q* ?

A statement quantified by universal quantifier $\forall$ is true if all objects in the collection satisfy the condition. And it is false if at least one object in the collection does not satisfy the conditon.

A statement quantified by existential quantifier $\exists$ is true if at least one object in the collection satisfies the condition. And it is false if no object in the collection satisfies the condition.

Ex.1. If A = {1, 2, 3, 4, 5, 6, 7}, determine the truth value of the following.

i) $\exists\, x \in$ A such that $x - 4 = 3$

ii) $\forall\, x \in$ A , $x + 1 \geq 3$

iii) $\forall\, x \in$ A, $8 - x \leq 7$

iv) $\exists\, x \in$ A, such that $x + 8 = 16$

Solution :

i) For $x = 7$, $x - 4 = 7 - 4 = 3$

$\therefore$ $x = 7$ satisfies the equation $x - 7 = 3$

$\therefore$ The given statement is true and its truth value is T.

ii) For $x = 1$, $x + 1 = 1 + 1 = 2$ which is not greater than or equal to 3

$\therefore$ For $x = 1$ $x^2 + 1 \geq 3$ is not true.

$\therefore$ The truth value of given statement is F.

iii) For each $x \in$ A $8 - x \leq 7$

$\therefore$ The given statement is true.

$\therefore$ Its truth value is T.

iv) There is no x in A which satisfies $x + 8 = 16$.

$\therefore$ The given statement is false. $\therefore$ Its truth value is F.

1.3.2 Dual : We use letters *t* and *c* to denote tautology and contradiction respectively.

If two statements contain logical connectives like $\vee$, $\wedge$ and letters t and c then they are said to be duals of each other if one of them is obtained from the other by interchanging $\vee$ with $\wedge$ and *t* with *c*.

The dual of i) $p \vee$ q is $p \wedge$ q   ii) t $\vee\, p$ is c $\wedge\, p$ iii) t $\wedge\, p$ is c $\vee\, p$

Ex.1. Write the duals of each of the following :

i) $(p \wedge q) \vee r$

ii) $t \vee (p \vee q)$

iii) $p \wedge [\sim q \vee (p \wedge q) \vee \sim r]$

iv) $(p \vee q) \wedge t$

v) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

vi) $p \wedge q \wedge r$

vii) $(p \wedge t) \vee (c \wedge \sim q)$

Solution :

i) $(p \vee q) \wedge r$

ii) $c \wedge (p \wedge q)$

iii) $p \vee [(\sim q \wedge (p \vee q) \wedge \sim r]$

iv) $(p \wedge q) \vee c$

v) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

vi) $p \vee q \vee r$

vii) $(p \vee c) \wedge (t \vee \sim q)$

1.3.3 Negations of compound statements :

Negation of conjunction : When is the statment "6 is even and perfect number" is false? It is so, if 6 is not even or 6 is not perfect number. The negation of $p \wedge q$ *is* $\sim p \vee \sim q$. The negation of "6 is even and perfect number" is "6 not even or not perfect number".

Activity : Using truth table verify that $\sim (p \wedge q) \equiv \sim p \vee \sim q$

Negation of disjunction : When is the statement "x is prime or y is even" is false? It is so, if x is not prime and y is not even. The negation of $p \vee q$ is $\sim p \wedge \sim q$. The negation of "x is prime or y is even" is "x is not prime and y is not even".

Activity : Using truth table verify that $\sim (p \wedge q) \equiv \sim p \vee \sim q$

Note : '$\sim (p \wedge q) \equiv \sim p \vee \sim q$' **and** '$\sim (p \vee q) \equiv \sim p \wedge \sim q$' **are called De'Morgan's Laws**

Negation of implication : Implication $p \rightarrow$ q asserts that "if p is true statement then q is true statement". When is an implication a true statement and when is it false? Consider the statement "If bakery is open then I will buy a cake for you." Clearly statement is false only when the bakery was open and I did not buy a cake for you. The conditional statement "If p then q" is false only in the case "p is true and q is false". In all other cases it is true. The negation of the statement "If p then q" is the statement "p and not q". i.e. p does not imply q

Activity : Using truth table verify that $\sim (p \rightarrow q) \equiv p \wedge \sim q$

Negation of biconditional : The biconditional p $\leftrightarrow$ q is the conjuction of statement p $\rightarrow$ q and q $\rightarrow$ p.

$\therefore\ p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$\therefore$ The conditional statement $p \leftrightarrow q$ is false if $p \rightarrow q$ is false or $q \rightarrow p$ false.

The negation of the statement "p if and only if q " is the statement "p and not q, or q and not p".

$\therefore\ \sim (p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim \text{p})$

Activity : Using truth table verify that $\sim (p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

1.3.4 Converse, inverse and contrapositive

From implication $p \rightarrow q$ we can obtain three implications, called converse, inverse and contrapositive.

$q \rightarrow p$ is called the converse of $p \rightarrow q$

$\sim p \rightarrow \sim q$ is called the inverse of $p \rightarrow q$.

$\sim$ q $\rightarrow \sim p$ is called the contrapositive of $p \rightarrow$ q.

Activity :

Prepare the truth table for $p \to q$, $q \to p$, $\sim p \to \sim q$ and $\sim q \to \sim p$. What is your conclusion from the truth table ?

i) .... ≡ ....

ii) .... ≡ ....

Ex.1) Write the negations of the following.

i) $3 + 3 < 5$ or $5 + 5 = 9$

ii) $7 > 3$ and $4 > 11$

iii) The number is neither odd nor perfect square.

iv) The number is an even number if and only if it is divisible by 2.

Solution :

i) Let $p : 3 + 3 < 5 : q : 5 + 5 = 9$

Given statement is $p \vee q$ and its negation is $\sim(p \vee q)$ and $\sim(p \vee q) \equiv \sim p \wedge \sim q$

$\therefore$ The negation of given statement is $3 + 3 \geq 5$ and $5 + 5 \neq 9$

ii) Let $p : 7 > 3$; $q: 4 > 11$

The given statement is $p \wedge q$

Its negation is $\sim (p \wedge q)$ and $\sim (p \wedge q) \equiv \sim p \vee \sim q$

$\therefore$ The negation of given statement is $7 \leq 3$ or $4 \leq 11$

iii) Let p : The number is odd

q : The number is perfect square

Given statement can be written as 'the number is not odd and not perfect square'

Given statement is $\sim p \wedge \sim q$

Its negation is $\sim (\sim p \wedge \sim q) \equiv p \vee q$

The negation of given statement is 'The number is odd or perfect square'.

iv) Let p : The number is an even number.

q : The number is divisible by 2

Given statement is $p \leftrightarrow q$

Its negation is $\sim (p \leftrightarrow q)$

But $\sim (p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

$\therefore$ The negation of given statement is 'A number is even but not divisible by 2 or a number is divisible by 2 but not even'.

Ex.2. Write the negations of the following statements.

i) All natural numbers are rational.

ii) Some students of class X are sixteen year old.

iii) $\exists\, n \in$ N such that $n + 8 > 11$

iv) $\forall\, x \in$ N, $2x + 1$ is odd

Solution :

i) Some natural numbers are not rationals.

ii) No student of class X is sixteen year old.

iii) $\forall\, n \in N,\ n + 8 \leq 11$

iv) $\exists\, x \in N$ such hat $2x + 1$ is not odd

Ex.3. Write the converse, inverse and contrapositive of the following statements.

i) If a function is differentiable then it is continuous.

ii) If it rains then the match will be cancelled.

Solution :

(1) Let p : A function is differentiable

q : A function is continuous.

$\therefore$ Given statement is $p \rightarrow q$

i) Its converse is $q \rightarrow p$

If a function is continuous then it is differentiable.

ii) Its inverse is $\sim p \rightarrow \sim q$.

If a function not differentiable then it is not continuous.

iii) Its contrapositive is $\sim q \rightarrow \sim p$

If a function is not continuous then it is not differentiable.

(2) Let p : It rains, q : The match gets cancelled.

$\therefore$ Given statement is $p \rightarrow q$

i) Its converse is $q \rightarrow p$

If the match gets cancelled then it rains.

ii) Inverse is $\sim p \rightarrow \sim q$

If it does not rain then the match will not be cancelled.

iii) Its contrapositive is $\sim q \rightarrow \sim p$.

If the match is not cancelled then it does not rain.

Exercise 1.3

Q.1. If A = {3, 5, 7, 9, 11, 12}, determine the truth value of each of the following.

i) $\exists\, x \in A$ such that $x - 8 = 1$

ii) $\forall\, x \in A,\ x^2 + x$ is an even number

iii) $\exists\, x \in A$ such that $x^2 < 0$

iv) $\forall\, x \in A,\ x$ is an even number

v) $\exists\, x \in A$ such that $3x + 8 > 40$

vi) $\forall\, x \in A,\ 2x + 9 > 14$

Q.2. Write the duals of each of the following.

i) $p \vee (q \wedge r)$ ii) $p \wedge (q \wedge r)$ iii) $(p \vee q) \wedge (r \vee s)$ iv) $p \wedge \sim q$

v) $(\sim p \vee q) \wedge (\sim r \wedge s)$ vi) $\sim p \wedge (\sim q \wedge (p \vee q) \wedge \sim r)$

vii) $[\sim (p \vee q)] \wedge [p \vee \sim (q \wedge \sim s)]$ viii) $c \vee \{p \wedge (q \vee r)\}$ ix) $\sim p \vee (q \wedge r) \wedge t$ x) $(p \vee q) \vee c$

Q.3. Write the negations of the following.

i) $x + 8 > 11$ or $y - 3 = 6$

ii) $11 < 15$ and $25 > 20$

iii) Qudrilateral is a square if and only if it is a rhombus.

iv) It is cold and raining.

v) If it is raining then we will go and play football.

vi) $\sqrt{2}$ is a rational number.

vii) All natural numbers are whole numers.

viii) $\forall$ n $\in$ N, $n^2 + n + 2$ is divisible by 4.

ix) $\exists\, x \in$ N such that $x - 17 < 20$

Q.4. Write converse, inverse and contrapositive of the following statements.

i) If $x < y$ then $x^2 < y^2$ (x, y $\in$ R)

ii) A family becomes literate if the woman in it is literate.

iii) If surface area decreases then pressure increases.

iv) If voltage increases then current decreases.

1.4 SOME IMPORTANT RESULTS :

1.4.1.

i) $p \to q \equiv \sim p \vee q$

ii) $p \leftrightarrow q \equiv (p \to q) \wedge (q \to \text{p})$

iii) $p \vee (q \wedge \text{r}) \equiv (p \vee q) \wedge (p \vee \text{r})$

iv) $\text{p} \wedge (\text{q} \vee \text{r}) \equiv (\text{p} \wedge \text{q}) \vee (\text{p} \wedge \text{r})$

I	II	III	IV	V	VI	VII	VIII
p	q	$\sim p$	$p \to q$	$q \to p$	$p \leftrightarrow q$	$\sim p \vee q$	$p \to q \wedge q \to p$
T	T	F	T	T	T	T	T
T	F	F	F	T	F	F	F
F	T	T	T	F	F	T	F
F	F	T	T	T	T	T	T

Table 1.20

Columns (IV, VII) and (VI, VIII) are identical.

$\therefore$ $p \to q \equiv \sim p \vee q$ and $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ are proved.

Activity :

Prove the results (iii) and (iv) by using truth table.

1.4.2. Algebra of statements.

Idempotent Law	$p \wedge p \equiv p, \quad p \vee p \equiv p$
Commutative Law	$p \vee q \equiv q \vee p \quad p \wedge q \equiv q \wedge p$
Associative Law	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r \equiv p \wedge q \wedge r$
	$p \vee (q \vee r) \equiv (p \vee q) \vee r \equiv p \vee q \vee r$
Distributive Law	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
De Morgan's Law	$\sim(p \wedge q) \equiv \sim p \vee \sim q, \sim(p \vee q) \equiv \sim p \wedge \sim q$
Identity Law	$p \wedge T \equiv p, p \wedge F \equiv F, p \vee F \equiv p, p \vee T \equiv T$
Complement Law	$p \wedge \sim p \equiv F, p \vee \sim p \equiv T$
Absorption Law	$p \vee (p \wedge q) \equiv p, p \wedge (p \vee q) \equiv p$
Conditional Law	$p \rightarrow q \equiv \sim p \vee q$
Biconditional Law	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$

Table 1.21

Solved Examples

Ex.1. Write the negations of the following stating the rules used.

i) $(p \vee q) \wedge (q \vee \sim r)$ ii) $(p \rightarrow q) \vee r$

iii) $p \wedge (q \vee r)$ iv) $(\sim p \wedge q) \vee (p \wedge \sim q)$

v) $(p \wedge q) \rightarrow (\sim p \vee r)$

Solution :

i) $\sim[(p \vee q) \wedge (q \vee \sim r)] \equiv \sim(p \vee q) \vee \sim(q \vee \sim r)$ [DeMorgan's law]

$\equiv (\sim p \wedge \sim q) \vee (\sim q \wedge r)$ [DeMorgan's law]

$\equiv (\sim q \wedge \sim p) \vee (\sim q \wedge r)$ [Commutative law]

$\equiv \sim q \wedge (\sim p \vee r)$ [Distributive law]

ii) $\sim[(p \rightarrow q) \vee r] \equiv \sim(p \rightarrow q) \wedge \sim r$ [DeMorgan's law]

$\equiv (p \wedge \sim q) \wedge \sim r$ $[\sim(p \rightarrow q) \equiv p \wedge \sim q]$

iii) $\sim[p \wedge (q \vee r)] \equiv \sim p \vee \sim(q \vee r)$ [DeMorgan's law]

$\equiv \sim p \vee (\sim q \wedge \sim r)$ [DeMorgan's law]

iv) $\sim[(\sim p \wedge q) \vee (p \wedge \sim q)] \equiv \sim(\sim p \wedge q) \wedge \sim(p \wedge \sim q)$ [DeMorgan's law]

$\equiv (p \vee \sim q) \wedge (\sim p \vee q)$ [DeMorgan's law]

v) $\sim[(p \wedge q) \rightarrow (\sim p \vee r)] \equiv (p \wedge q) \wedge \sim(\sim p \vee r)$ $[\sim(p \rightarrow q) \equiv p \wedge \sim q]$

$\equiv (p \wedge q) \wedge [p \wedge \sim r]$ [DeMorgan's law]

$\equiv q \wedge p \wedge p \wedge \sim r$ [Associative law

$\equiv q \wedge p \wedge \sim r$ [Idempotent law]

Ex.2. Rewrite the following statements without using if ...... then.

i) If prices increase then the wages rise.

ii) If it is cold, then we wear woolen clothes.

Solution :

i) Let p : Prices increase

q : The wages rise.

The given statement is $p \to q$

but $p \to q \equiv \sim p \vee q$

The given statement can be written as

'Prices do not increase or the wages rise'.

ii) Let p : It is cold, q : We wear woollen clothes.

The given statement is $p \to q$

but $p \to q \equiv \sim p \vee q$

The given statement can be written as

It is not cold or we wear woollen clothes.

Ex.3. Without using truth table prove that :

i) $p \leftrightarrow q \equiv \sim (p \wedge \sim q) \wedge \sim (q \wedge \sim p)$

ii) $\sim (p \vee q) \vee (\sim p \wedge q) \equiv \sim p$

iii) $\sim p \wedge q \equiv (p \vee q) \wedge \sim p$

Solution :

i) We know that

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$ [Conditional law]

$\equiv \sim (p \wedge \sim q) \wedge \sim (q \wedge \sim p)$ [Demorgan's law]

ii) $\sim (p \vee q) \vee (\sim p \wedge q) \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$ [Demorgan's law]

$\equiv \sim p \wedge (\sim q \vee q)$ [Distributive law]

$\equiv \sim p \wedge T$ [Complement law]

$\equiv \sim p$ [Identity law]

iii) $(p \wedge q) \wedge \sim p \equiv \sim p \wedge (p \vee q)$ [Commutative law]

$\equiv (\sim p \wedge p) \vee (\sim p \wedge q)$ [Distributive law]

$\equiv F \vee (\sim p \wedge q)$ [Complement law]

$\equiv \sim p \wedge q$ [Identity law]

Exercise 1.4

Q.1. Using rules of negation write the negations of the following with justification.

i) $\sim q \rightarrow p$

ii) $p \wedge \sim q$

iii) $p \vee \sim q$

iv) $(p \vee \sim q) \wedge r$

v) $p \rightarrow (p \vee \sim q)$

vi) $\sim (p \wedge q) \vee (p \vee \sim q)$

vii) $(p \vee \sim q) \rightarrow (p \wedge \sim q)$

viii) $(\sim p \vee \sim q) \vee (p \wedge \sim q)$

Q.2. Rewrite the following statements without using if .. then.

i) If a man is a judge then he is honest.

ii) It 2 is a rational number then $\sqrt{2}$ is irrational number.

iii) It $f(2) = 0$ then $f(x)$ is divisible by $(x - 2)$.

Q.3. Without using truth table prove that :

i) $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

ii) $(p \vee q) \wedge (p \vee \sim q) \equiv p$

iii) $(p \wedge q) \vee (\sim p \wedge q) \vee (p \wedge \sim q) \equiv p \vee q$

iv) $\sim [(p \vee \sim q) \rightarrow (p \wedge \sim q)] \equiv (p \vee \sim q) \wedge (\sim p \vee q)$

Application of Logic to switching circuits :

We shall study how the theory of Logic can be applied in switching network. We have seen that a logical statement can be either true or false i.e. it can have truth value either T or F.

A similar situation exists in various electrical devices. For example, an electric switch can be on or off. In 1930 Claude Shannan noticed an analogy between operation of switching circuits and operation of logical connectives.

In an electric circuit, switches are connected by wires. If the switch is 'on', it allows the electric current to pass through, it. If the switch is 'off', it does not allow the electric current to pass through it. We now define the term 'switch' as follows.

Switch : A switch is a two state device used to control the flow of current in a circuit.

We shall denote the switches by letters S, S_1, S_2, S_3 .... etc.

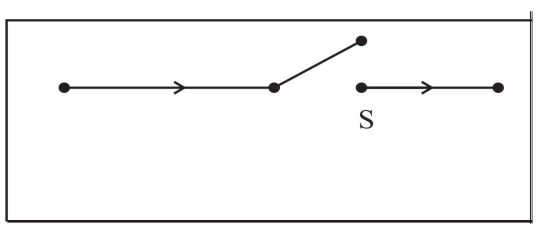


Fig. 1.1

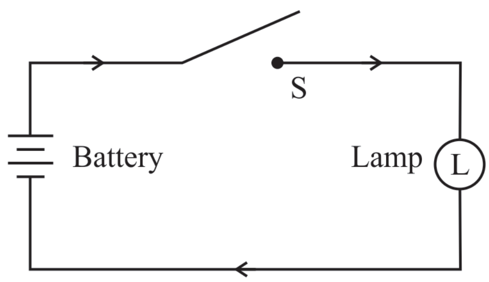


Fig. 1.2

In figure 1.2, we consider a circuit containing an electric lamp L, controlled by a switch S.

When the switch S is closed (i.e. on), then current flows in the circuit and hence the lamp glows. When the switch S is open (i.e. off), then current does not flow in the circuit and subsequently the lamp does not glow.

The theory of symbolic logic can be used to represent a circuit by a statement pattern. Conversely for given statement pattern a circuit can be constructed. Corresponding to each switch in the cirucit we take a statement letter in statement pattern. Switches having the same state will be denoted by the same letter and called equivalent switches. Switches having opposite states are denoted by S and S'. They are called complementary switches. In circuit we don't show whether switch is open or closed. In figure 1.3 switch S_1 corresponds to statement letter p in the corresponding statement pattern.

We write it as p : switch S_1 and $\sim p$: switch S'_1

The correspondence between switch S_2 and statement letter q is shown as q : switch S_2 and $\sim q$: switch S'_2.

We don't know the actual states of switches in the circuit. We consider all possible combinations of states of all switches in the circuit and prepare a table, called "Input Output table", which is similar to truth table of the corresponding statement pattern.

☆ In an Input-output table we represent '1' when the state of the switch is 'on' and '0' when the state of the switch is 'off'.

1.5.1. Two switches in series.

Two switches S_1 and S_2 connected in series and electric lamp 'L' as shown in fig 1.3.

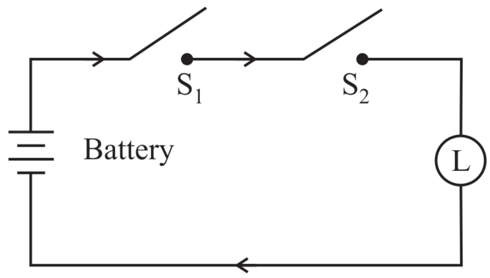


Fig. 1.3

Let p : The switch S_1

q : The switch S_2

L : The lamp L

Input output table (switching table) for $p \wedge q$.

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Table 1.22

1.5.2 Two switches in parallel :

Two switches S_1 and S_2 are connected in parallel and electric lamp L is as shown in fig. 1.4

Let p : The switch S_1

q : The switch S_2

L : The lamp L

Input - output table. for p ∨ q.

p	q	p ∨ q
1	1	1
1	0	1
0	1	1
0	0	0

Table 1.23

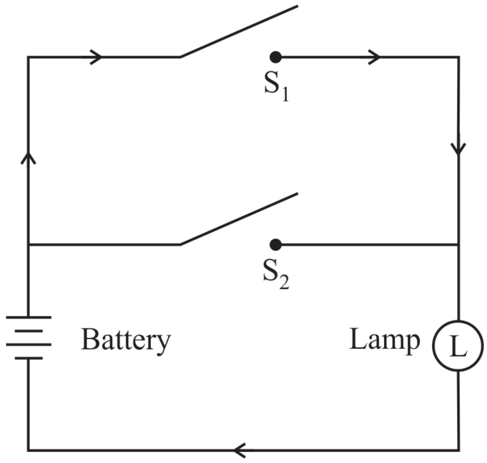


Fig. 1.4

Q.1. Express the following circuits in the symbolic form of logic and write the input-output table.

i)

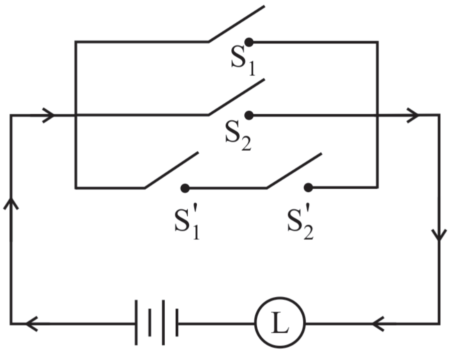


Fig. 1.5

ii)

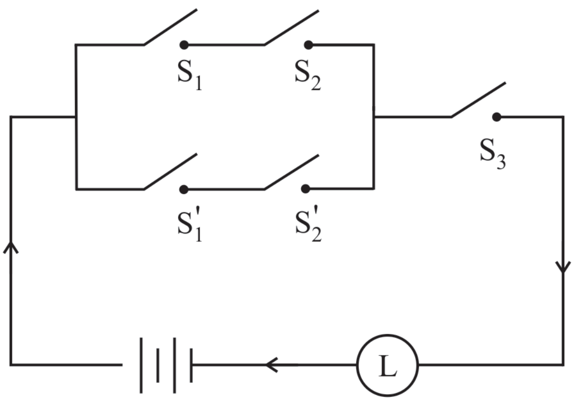


Fig. 1.6

iii)

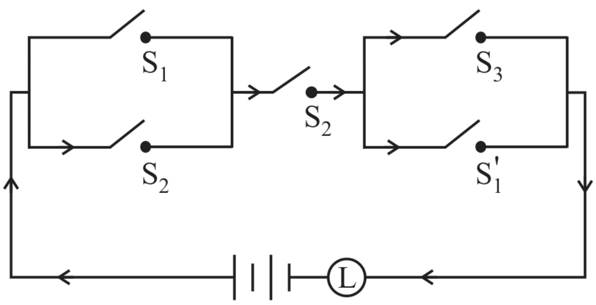


Fig. 1.7

Solution :

i) Let p : The switch S_1 q : The switch S_2 L : The lamp L

Given circuit is expressed as $(p \vee q) \vee (\sim p \wedge \sim q)$

Solution :

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim p \wedge \sim q$	$(p \vee q) \vee (\sim p \sim q)$
1	1	0	0	1	0	1
1	0	0	1	1	0	1
0	1	1	0	1	0	1
0	0	1	1	0	1	1

Table 1.24

ii) Let p : The switch S_1 is closed

q : The switch S_2 r : The switch S_3 L : The lamp L

The symbolic form is $[(p \wedge q) \vee (\sim p \wedge \sim q)] \wedge r$

p	q	r	~p	~q	$p \wedge q$	$(p \wedge q) \vee (\sim p \wedge \sim q)$	$(p \wedge q) \vee (\sim p \wedge \sim q)$	$[(p \wedge q) \vee (\sim p \wedge \sim q)] \wedge r$
1	1	1	0	0	1	0	1	1
1	1	0	0	0	1	0	1	0
1	0	1	0	1	0	0	0	0
1	0	0	0	1	0	0	0	0
0	1	1	1	0	0	0	0	0
0	1	0	1	0	0	0	0	0
0	0	1	1	1	0	1	1	1
0	0	0	1	1	0	1	1	0

Table 1.25

iii) Let p : The switch S_1 q : The switch S_2

r : The switch S_3 L : The lamp L

The symbolic form of given circuit is $(p \vee q) \wedge q \wedge (r \vee \sim p)$

p	q	r	~p	$p \vee q$	$r \vee \sim p$	$(p \vee q) \wedge \sim q$	$(p \vee q) \wedge q \wedge (r \vee \sim p)$
1	1	1	0	1	1	1	1
1	1	0	0	1	0	1	0
1	0	1	0	1	1	0	0
1	0	0	0	1	0	0	0
0	1	1	1	1	1	1	1
0	1	0	1	1	1	1	1
0	0	1	1	0	1	0	0
0	0	0	1	0	1	0	0

Table 1.26

Ex.2. Construct switching circuits of the following.

i) $[(p \vee (\sim p \wedge q)] \vee [(\sim q \wedge r) \vee \sim p]$

ii) $(p \wedge q \wedge r) \vee [p \vee (q \wedge \sim r)]$

iii) $[(p \wedge r) \vee (\sim q \wedge \sim r)] \vee (\sim p \wedge \sim r)$

Solution :

Let p : The Switch S_1

q : The switch S_2

r : The switch S_3

The circuits are as follows.

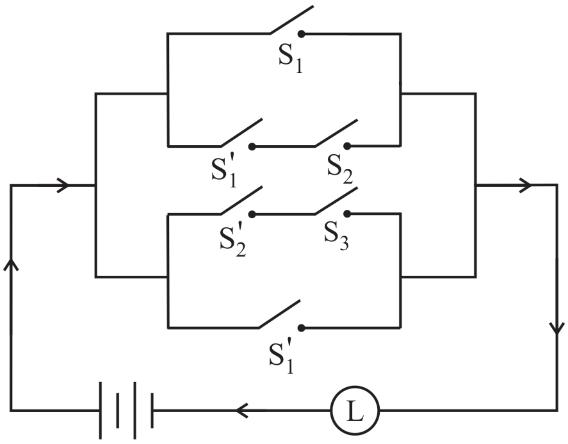


Fig. 1.8

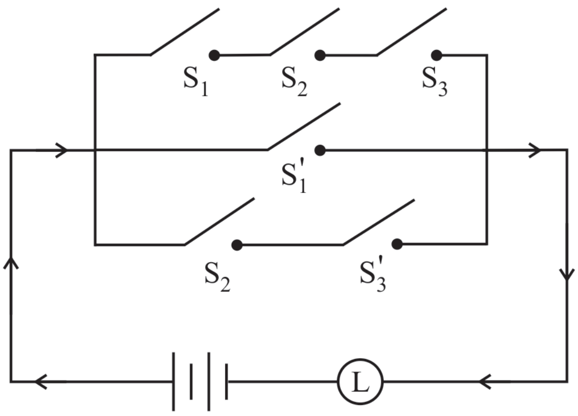


Fig. 1.9

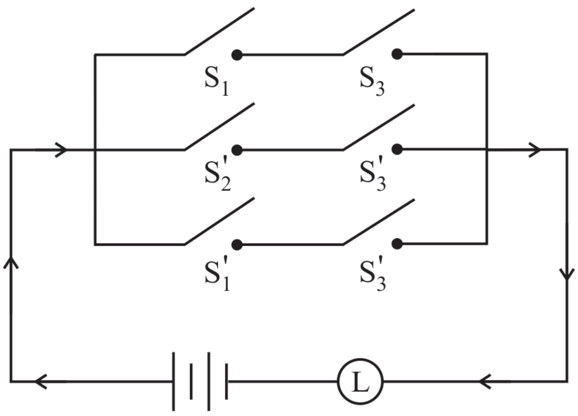


Fig. 1.10

Ex.3. Give an alternative arrangement for the following circuit, so that the new circuit has minimum switches.

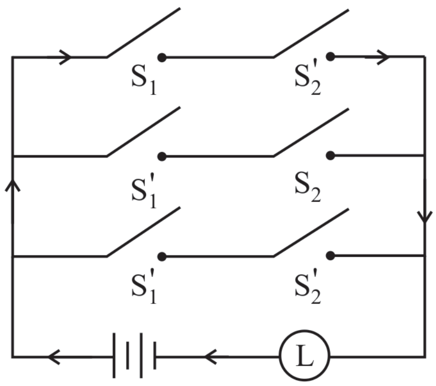


Fig. 1.11

Let p : The switch S_1

q: The switch S_2

The symbolic form is $(p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

Consider $(p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

$\equiv (p \wedge \sim q) \vee [\sim p \wedge (q \vee \sim q)]$	[Distributive Law]
$\equiv (p \wedge \sim q) \vee [\sim p \wedge T]$	[Complement Law]
$\equiv (p \wedge \sim q) \vee \sim p$	[Identity Law]
$\equiv \sim p \vee (p \wedge \sim q)$	[Commutative Law]
$\equiv (\sim p \vee p) \wedge (\sim p \wedge \sim q)$	[Distributive Law]
$\equiv t \wedge (\sim p \vee \sim q)$	[Law of Complement]
$\equiv \sim p \vee \sim q$	[Identity Law]

The alternative arrangement for the given circuit is as follows :

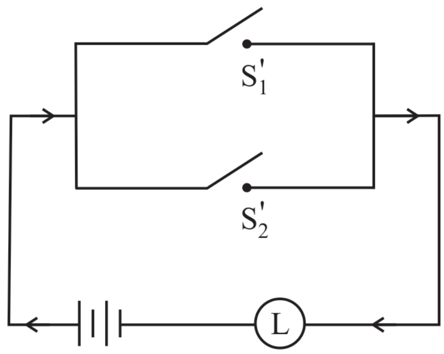


Fig. 1.12

Ex.4. Express the following switching circuit in the symbolic form of Logic. Construct the switching table and interpret it.

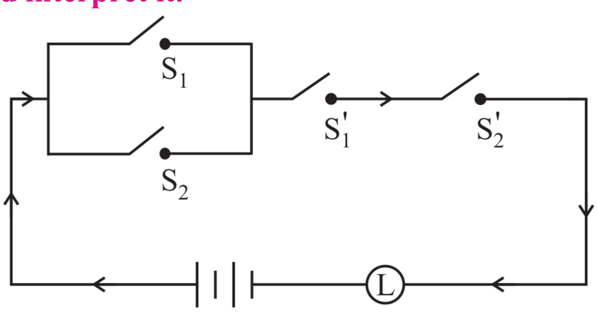


Fig. 1.13

Solution :

Let p : The switch S_1

q : The switch S_2

The symbolic form of the given switching circuit is $(p \vee q) \wedge (\sim p) \wedge (\sim q)$

The switching table.

p	q	~ p	~ q	p ∨ q	(p ∨ q) ∧ (~ p)	(p ∨ q) ∧ (~ p) ∧ (~ q)
1	1	0	0	1	0	0
1	0	0	1	1	0	0
0	1	1	0	1	1	0
0	0	1	1	0	0	0

Table 1.27

Last column contains all 0, lamp will not glow irrespective of the status of the switches.

Ex.5. Simplify the given circuit by writing its logical expression. Also,write your conclusion.

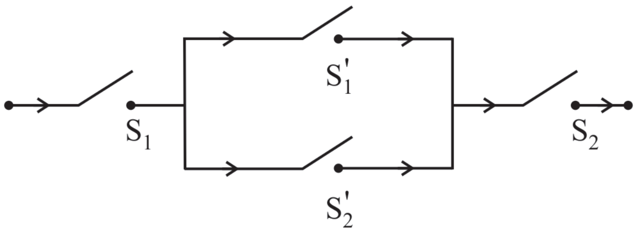


Fig. 1.14

Let p : The switch S_1

q: The The swtch S_2

The logical expression for the given circuit is $p \wedge (\sim p \vee \sim q) \wedge q$

Consider

$p \wedge (\sim p \vee \sim q) \wedge q$

$\equiv$ $[p \wedge (\sim p \vee \sim q] \wedge q$ [Associative Law]

$\equiv$ $[(p \wedge \sim p) \vee (p \vee \sim q)] \wedge q$ [Distributive Law]

$\equiv$ $[F \vee (p \wedge \sim q)] \wedge q$ [Complement Law]

$\equiv$ $(p \wedge \sim q) \wedge q$ [Identity Law]

$\equiv$ $p \wedge (\sim q \wedge q)$ [Associative Law]

$\equiv$ $p \wedge F$ [Complement Law]

$\equiv$ F [Identity Law]

Conclusion : The lamp will not glow irrespective of the status of the switches.

Ex. 6 : In the following switching circuit,

i) Write symbolic form   ii) Construct switching table   iii) Simplify the circuit

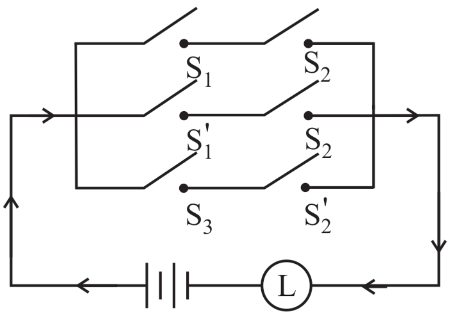


Fig. 1.15

Solution : Let p : The switch S_1. q : The switch S_2. r : The switch S_3

1) The symbolic form of given circuit is $(p \wedge q) \vee (\sim p \wedge q) \vee (r \wedge \sim q)$.

ii) Switching Table :

P	q	r	~ p	~ q	~ p ∧ q	~ p ∧ q	r ∧ ~ q	(p ∧ q) ∨ (~p ∧ q) ∨ (r ∧ ~ q)
1	1	1	0	0	1	0	0	1
1	1	0	0	0	1	0	0	1
1	0	1	0	1	0	0	1	1
1	0	0	0	1	0	0	0	0
0	1	1	1	0	0	1	0	1
0	1	0	1	0	0	1	0	1
0	0	1	1	1	0	0	1	1
0	0	0	1	1	0	0	0	0

Table 1.28

iii) Consider $= (p \wedge q) \vee (\sim p \wedge q) \vee (r \wedge \sim q)$

$= [(p \vee \sim p) \wedge q] \vee [(r \wedge \sim q)]$ [Distributive Law]

$= (T \wedge q) \vee (r \vee \sim q)$ [Complement Law]

$= q \vee (r \wedge \sim q)$ [Identity Law]

$= (q \vee r) \wedge (q \vee \sim q)$ [Distributive Law]

$= (q \vee r) \wedge T$ [Complement Law]

$= (q \vee r)$ [Identify Law]

Simplified circuit is :

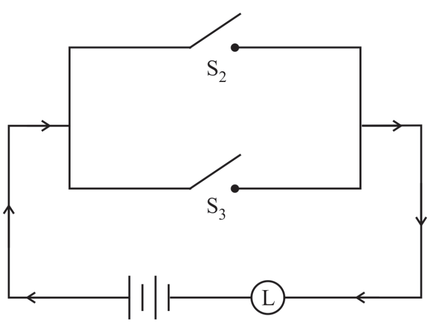


Fig. 1.16

Q.1. Express the following circuits in the symbolic form of logic and writ the input-output table.

(i)

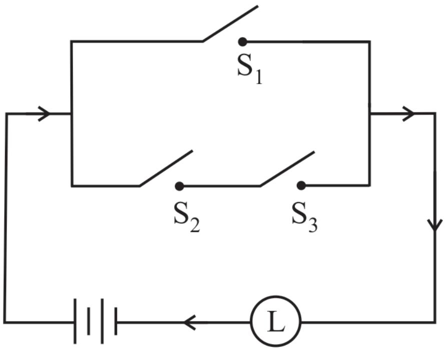


Fig. 1.17

(ii)

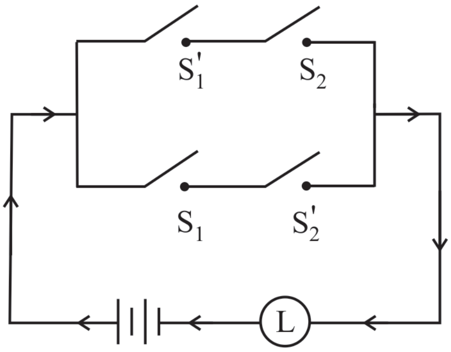


Fig. 1.18

(iii)

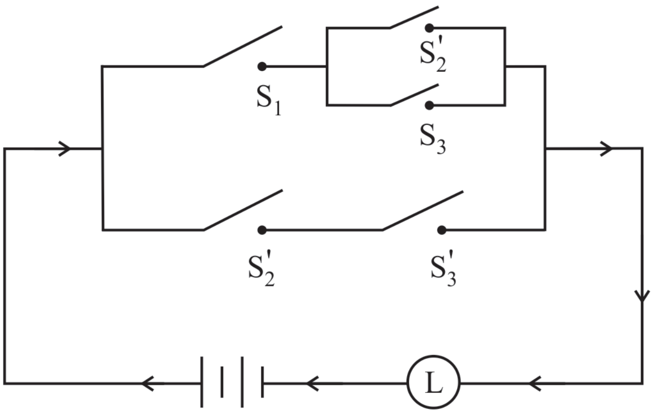


Fig. 1.19

(iv)

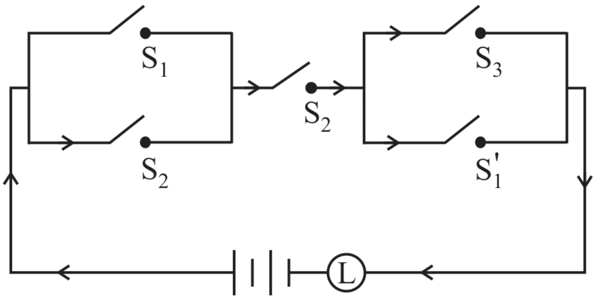


Fig. 1.20

(v)

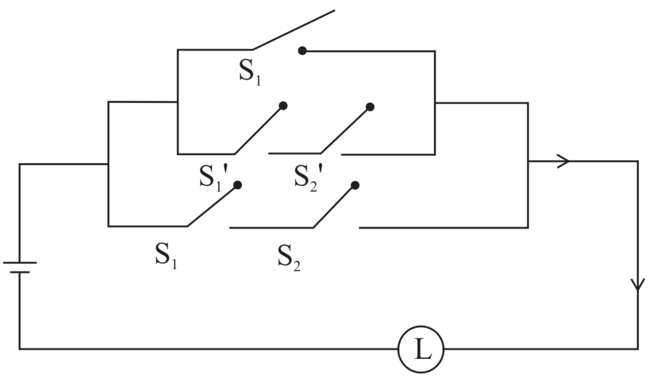


Fig. 1.21

(vi)

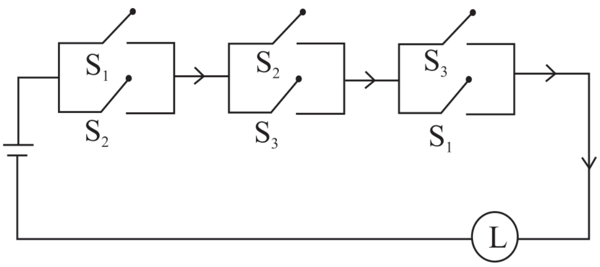


Fig. 1.22

Q.2. Construct the switching circuit of the following :

i) $(\sim p \wedge q) \vee (p \wedge \sim r)$

ii) $(p \wedge q) \vee [\sim p \wedge (\sim q \vee p \vee r)]$

iii) $(p \wedge r) \vee (\sim q \wedge \sim r)] \wedge (\sim p \wedge \sim r)$

iv) $(p \wedge \sim q \wedge r) \vee [\text{p} \wedge (\sim q \vee \sim r)]$

v) $\text{p} \vee (\sim p) \vee (\sim q) \vee (p \wedge q)$

vi) $(p \wedge q) \vee (\sim p) \vee (p \wedge \sim q)$

Q.3. Give an alternative equivalent simple circuits for the following circuits :

(i)

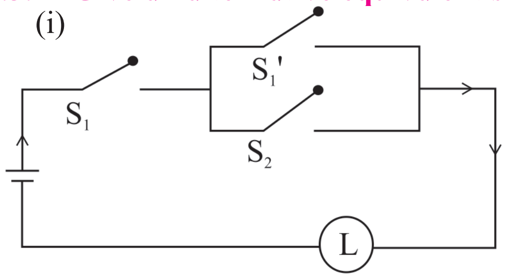


Fig. 1.23

(ii)

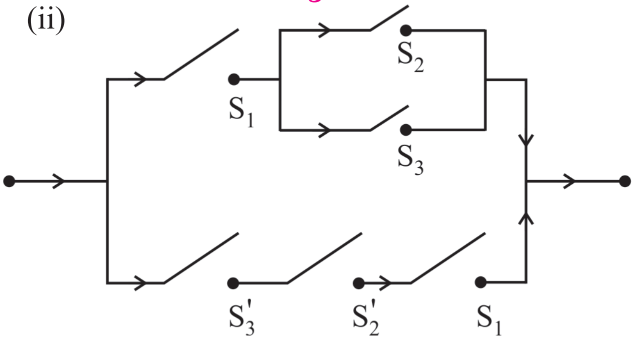


Fig. 1.24

Q.4. Write the symbolic form of the following switching circuits construct its switching table and interpret it.

i)

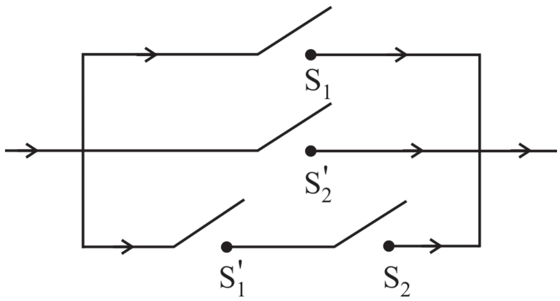


Fig. 1.25

ii)

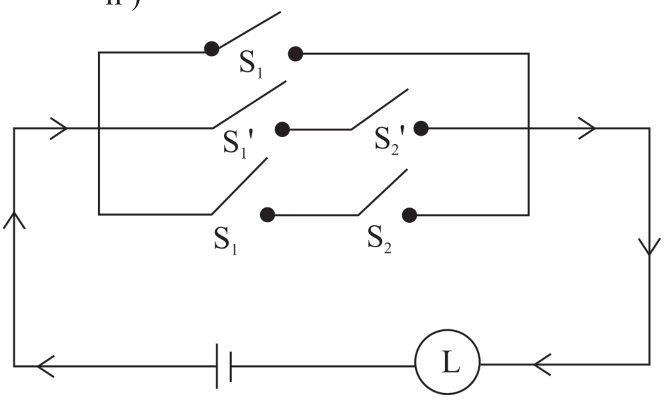


Fig. 1.26

(iii)

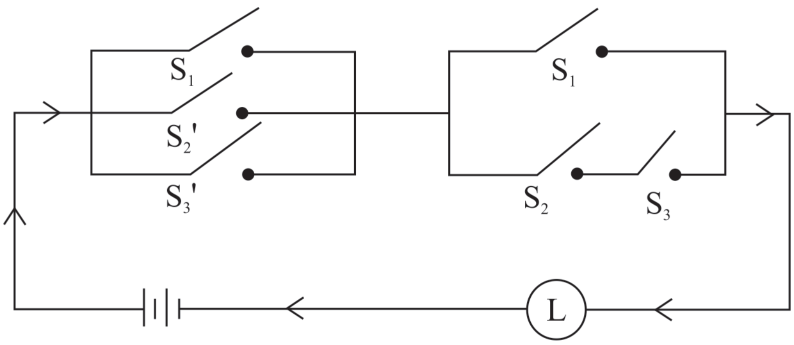


Fig. 1.27

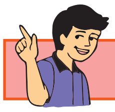
Q.5. Obtain the simple logical expression of the following. Draw the corresponding switching circuit.

i) $p \vee (q \wedge \sim q)$

ii) $(\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge \sim q)]$

iii) $[p (\vee (\sim q) \vee \sim r)] \wedge (p \vee (q \wedge r)$

iv) $(p \wedge q \wedge \sim p) \vee (\sim \text{p} \wedge q \wedge r) \vee (p \wedge \sim q \wedge r)$



Let's remember!

1) A declarative sentence which is either true or false, but not both simultaneously is called a statement.

Sr. No.	Connective	Symbolic Form	Name of Compound Statement	Hint in truth table	Negation
i)	and	$p \wedge q$	Conjunction	T ∧ T ≡ T	$\sim p \vee \sim q$
ii)	or	$p \vee q$	Disjunction	F ∨ F ≡ F	$\sim p \wedge \sim q$
iii)	if... then	$p \rightarrow q$	Conditional	T → F ≡ F	$p \wedge \sim q$
iv)	if and only if	$p \leftrightarrow q$	Biconditional	T ↔ T ≡ T F ↔ F ≡ T	$(p \wedge \sim q) \vee (q \wedge \sim p)$

Table 1.29

3) In the truth table of the statement pattern if all truth values in the last column
   a) are 'T' then it is tautology.
   b) are 'F' then it is contradiction.

4) In the truth table of the statement pattern if some entries are 'T' and some are 'F' then it is called as contingency.

5) The symbol ∀ stands for 'for all' or 'for every'. It is universal quantifier. The symbol ∃ stands for 'for some' or 'for one' or 'there exists at least one'. It is called as existential quantifier.

6) Algebra of statements.

Idempotent Law	$p \wedge p \equiv p, \quad p \vee p \equiv p$
Commutative Law	$p \vee q \equiv q \vee p \quad p \wedge q \equiv q \wedge p$
Associative Law	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r \equiv p \wedge q \wedge r$
	$p \vee (q \vee r) \equiv (p \vee q) \vee r \equiv p \vee q \vee r$
Distributive Law	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
De Morgan's Law	$\sim(p \wedge q) \equiv \sim p \vee \sim q, \sim(p \vee q) \equiv \sim p \wedge \sim q$
Identity Law	$p \wedge T \equiv p, p \wedge F \equiv F, p \vee F \equiv p, p \vee T \equiv T$
Complement Law	$p \wedge \sim p \equiv F, p \vee \sim p \equiv T$
Absorption Law	$p \vee (p \wedge q) \equiv p, p \wedge (p \vee q) \equiv p$
Conditional Law	$p \rightarrow q \equiv \sim p \vee q$
Biconditional Law	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$

7) If $p \rightarrow q$ is conditional then its converse is $q \rightarrow p$, inverse is $\sim p \rightarrow \sim q$ and contrapositive is $\sim q \rightarrow \sim p$.

8) Switching circuits :

i) Switches in series

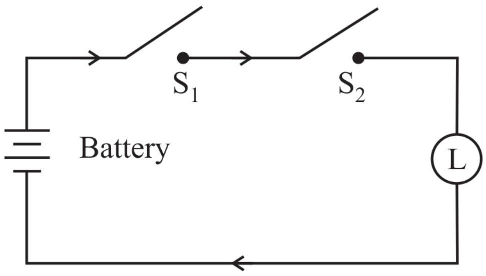


Fig. 1.28

ii) Switches in parallel.

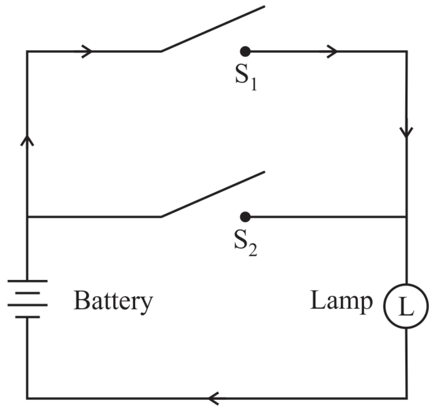


Fig. 1.29

Input-output table

p	q	$p \wedge q$	$p \vee q$
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	0

Table 1.30

Miscellaneous Exercise 1

I] Select and write the correct answer from the given alternatives in each of the following questions :

i) If $p \wedge q$ is false and $p \vee q$ is true, the ________ is not true.

A) $p \vee q$ B) $p \leftrightarrow q$ C) $\sim p \vee \sim q$ D) $q \vee \sim p$

ii) $(p \wedge q) \rightarrow r$ is logically equivalent to ________.

A) $p \rightarrow (q \rightarrow r)$ B) $(p \wedge q) \rightarrow \sim r$ C) $(\sim p \vee \sim q) \rightarrow \sim r$ D) $(p \vee q) \rightarrow r$

iii) Inverse of statement pattern $(p \vee q) \rightarrow (p \wedge q)$ is ________.

A) $(p \wedge q) \rightarrow (p \vee q)$ B) $\sim(p \vee q) \rightarrow (p \wedge q)$

C) $(\sim p \wedge \sim q) \rightarrow (\sim p \vee \sim q)$ D) $(\sim p \vee \sim q) \rightarrow (\sim p \wedge \sim q)$

iv) If $p \wedge q$ is F, $p \rightarrow q$ is F then the truth values of p and q are ________.

A) T, T B) T, F C) F, T D) F, F

v) The negation of inverse of $\sim p \rightarrow q$ is ________.

A) $q \wedge p$ B) $\sim p \wedge \sim q$ C) $p \wedge q$ D) $\sim q \rightarrow \sim p$

vi) The negation of $p \wedge (q \rightarrow r)$ is ________.

A) $\sim p \wedge (\sim q \rightarrow \sim r)$ B) $p \vee (\sim q \vee r)$

C) $\sim p \wedge (\sim q \rightarrow \sim r)$ D) $\sim p \vee (\sim q \wedge \sim r)$

vii) If A = {1, 2, 3, 4, 5} then which of the following is not true?

A) $\exists\, x \in$ A such that $x + 3 = 8$ B) $\exists\, x \in$ A such that $x + 2 < 9$

C) $\forall\, x \in$ A, $x + 6 \geq 9$ D) $\exists\, x \in$ A such that $x + 6 < 10$

Q.2. Which of the following sentences are statements in logic? Justify. Write down the truth value of the statements :

i) $4! = 24$.

ii) π is an irrational number.

iii) India is a country and Himalayas is a river.

iv) Please get me a glass of water.

v) $\cos^2\theta - \sin^2\theta = \cos 2\theta$ for all $\theta \in R$.

vi) If x is a whole number the $x + 6 = 0$.

Q.3. Write the truth values of the following statements :

i) $\sqrt{5}$ is an irrational but $3\sqrt{5}$ is a complex number.

ii) $\forall\, n \in N$, $n^2 + n$ is even number while $n^2 - n$ is an odd number.

iii) $\exists\, n \in N$ such that $n + 5 > 10$.

iv) The square of any even number is odd or the cube of any odd number is odd.

v) In ΔABC if all sides are equal then its all angles are equal.

vi) $\forall\, n \in N$, $n + 6 > 8$.

Q.4. If A = {1, 2, 3, 4, 5, 6, 7, 8, 9}, determine the truth value of each of the following statement :

i) $\exists\, x \in A$ such that $x + 8 = 15$.

ii) $\forall\, x \in A$, $x + 5 < 12$.

iii) $\exists\, x \in A$, such that $x + 7 \geq 11$.

iv) $\forall\, x \in A$, $3x \leq 25$.

Q.5. Write the negations of the following :

i) $\forall\, n \in A$, $n + 7 > 6$.

ii) $\exists\, x \in A$, such that $x + 9 \leq 15$.

iii) Some triangles are equilateral triangle.

Q.6. Construct the truth table for each of the following :

i) $p \rightarrow (q \rightarrow p)$

ii) $(\sim p \vee \sim q) \leftrightarrow [\sim(p \wedge q)]$

iii) $\sim(\sim p \wedge \sim q) \vee q$

iv) $[(p \wedge q) \vee r] \wedge [\sim r \vee (p \wedge q)]$

v) $[(\sim p \vee q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

Q.7. Determine whether the following statement patterns are tautologies contradictions or contingencies :

i) $[(p \rightarrow q) \wedge \sim q)] \rightarrow \sim p$

ii) $[(p \vee q) \wedge \sim p] \wedge \sim q$

iii) $(p \rightarrow q) \wedge (p \wedge \sim q)$

iv) $[p \rightarrow (q \rightarrow r)] \leftrightarrow [(p \wedge q) \rightarrow r]$

v) $[(p \wedge (p \rightarrow q)] \rightarrow q$

vi) $(p \wedge q) \vee (\sim p \wedge q) \vee (p \vee \sim q) \vee (\sim p \wedge \sim q)$

vii) $[(p \vee \sim q) \vee (\sim p \wedge q)] \wedge r$

viii) $(p \rightarrow q) \vee (q \rightarrow p)$

Q.8. Determine the truth values of p and q in the following cases :

i) $(p \vee q)$ is T and $(p \wedge q)$ is T

ii) $(p \vee q)$ is T and $(p \vee q) \to q$ is F

iii) $(p \wedge q)$ is F and $(p \wedge q) \to q$ is T

Q.9. Using truth tables prove the following logical equivalences :

i) $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

ii) $(p \wedge q) \to r \equiv p \to (q \to r)$

Q.10. Using rules in logic, prove the following :

i) $p \leftrightarrow q \equiv \sim (p \wedge \sim q) \vee \sim (q \wedge \sim p)$

ii) $\sim p \wedge q \equiv (p \vee q) \wedge \sim p$

iii) $\sim (p \vee q) \vee (\sim p \wedge q) \equiv \sim p$

Q.11. Using the rules in logic, write the negations of the following :

i) $(p \vee q) \wedge (q \vee \sim r)$

ii) $p \wedge (q \vee r)$

iii) $(p \to q) \wedge r$

iv) $(\sim p \wedge q) \vee (p \wedge \sim q)$

Q.12. Express the following circuits in the symbolic form. Prepare the switching table :

(i)

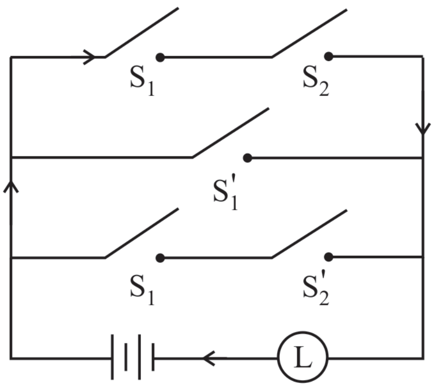


Fig. 1.30

(ii)

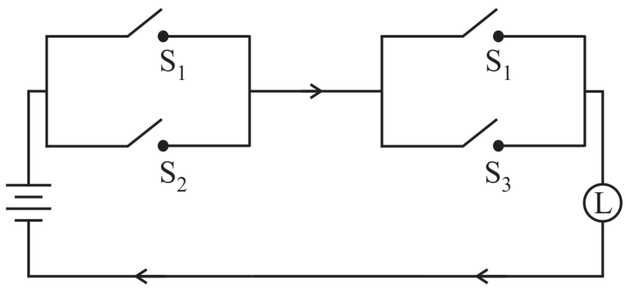


Fig. 1.31

Q.13. Simplify the following so that the new circuit has minimum number of switches. Also, draw the simplified circuit.

(i)

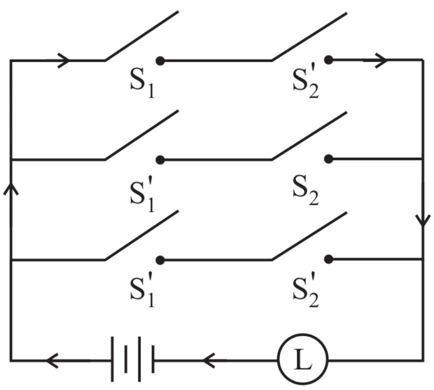


Fig. 1.32

(ii)

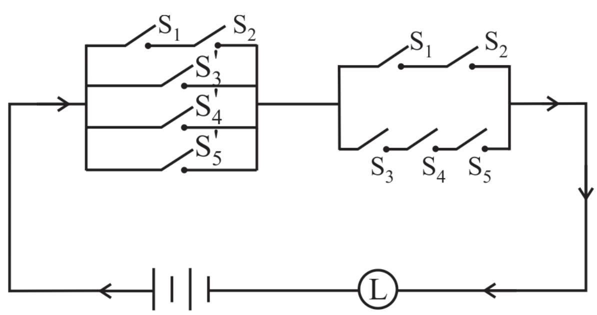


Fig. 1.33

Q.14. Check whether the following switching circuits are logically equivalent - Justify.

(A) i)

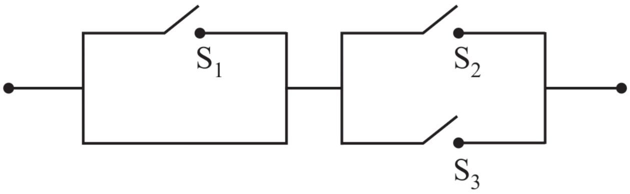


Fig. 1.34

ii)

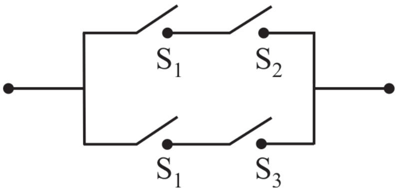


Fig. 1.35

(B) i)

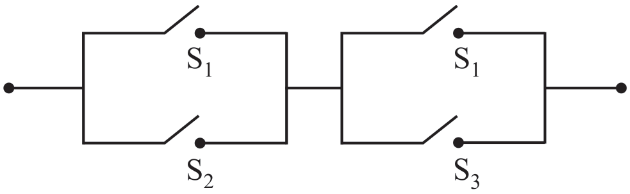


Fig. 1.36

ii)

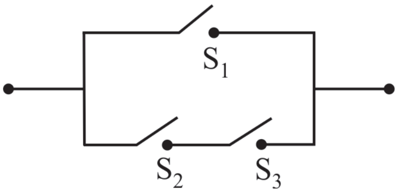


Fig. 1.37

Q.15. Give alternative arrangement of the switching following circuit, has minimum switches.

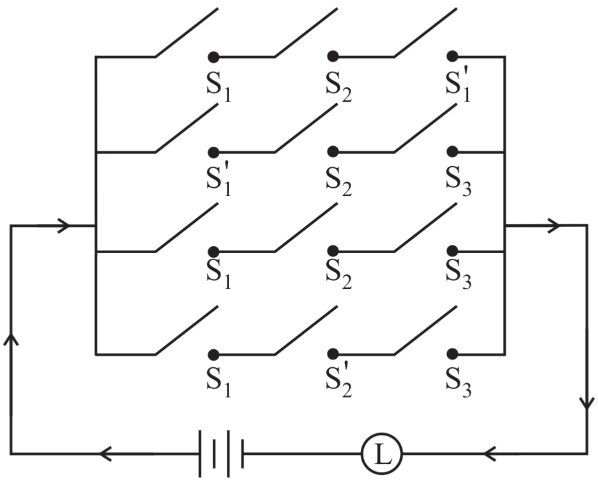


Fig. 1.38

Q.16 Simplify the following so that the new circuit circuit.

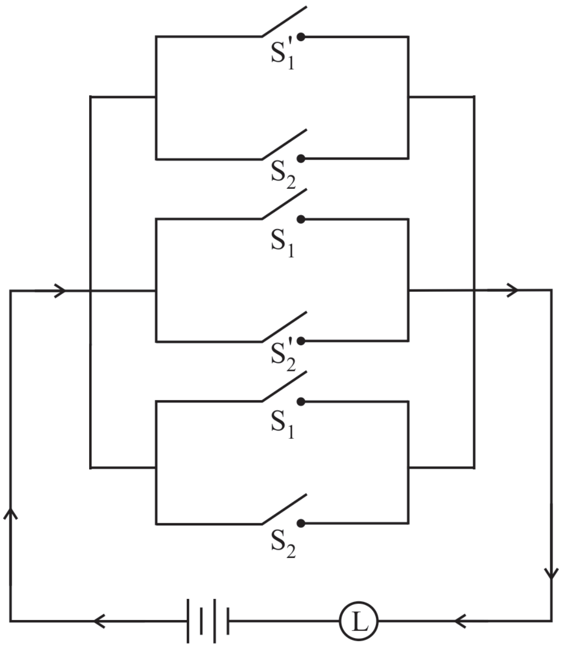


Fig. 1.39

Q.17. Represent the following switching circuit in symbolic form and construct its switching table. Write your conclusion from the switching table.

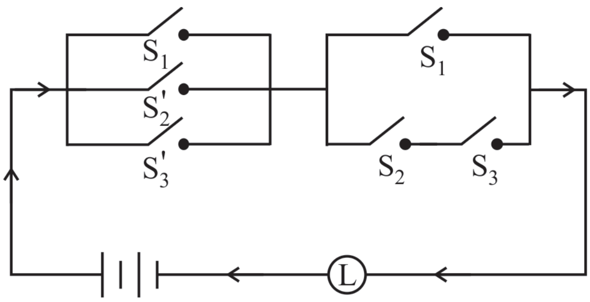


Fig. 1.40

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2 Matrices

Let's Study.

2.1 Elementry transformations.

2.2 Inverse of a matrix

2.2.1 Elementary transformation Method

2.2.2 Adjoint method

2.3 Application of matrices .

Solution of a system of linear equations

2.3.1 Method of Inversion

2.3.2 Method of Reduction

Let's learn.

A matrix of order mxm is a square arrangement of m^2 elements. The corresponding determinant of the same elements, after expansion is seen to be a value which is an element itself.

In standard XI, we have studies the types of matrices and algebra of matrices namely addition, subtraction, multiplication of two matrices.

The matrices are useful in almost every branch of science. Many problems in Statistics are expressed in terms of matrices. Matrices are also useful in Economics, Operation Research. It would not be an exaggeration to say that the matrices are the language of atomic Physics.

Hence, it is necessary to learn the uses of matrices with the help of **elementary transformations** and the **inverse of a matrix**.

2.1 Elementary Transformation :

Let us first understand the meaning and applications of elementary transformations.

The elementary transformation of a matrix are the six operations, three of which are due to row and three are due to column.

They are as follows :

(a) Interchange of any two rows or any two columns. If we interchange the i^{th} row and the j^{th} row of a matrix then after this interchange the original matrix is transformed to a new matrix.

This transformation is symbolically denoted as $R_i \leftrightarrow R_j$ or R_{ij}.

The similar transformation can be due to two columns say $C_k \leftrightarrow C_i$ or C_{ki}.

(Recall that R and C symbolically represent the rows and columns of a matrix.)

For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ then $R_1 \leftrightarrow R_2$ gives the new matrix $\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $C_1 \leftrightarrow C_2$ gives the new matrix $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$.

Note that $A \neq \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $\neq \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ but we write $A \sim \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $A \sim \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

Note : The symbol ~ is read as equivalent to.

(b) Multiplication of the elements of any row or column by a non-zero scalar :

If k is a non-zero scalar and the row R_i is to be multiplied by constant k then we multiply every element of R_i by the constant k and symbolically the transformation is denoted by kR_i or $R_i \to kR_i$.

For example, if $A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$ then $R_2 \to 4R_2$ gives $A \sim \begin{bmatrix} 0 & 2 \\ 12 & 16 \end{bmatrix}$

Similarly, if any column of a matrix is to be multiplied by a constant then we multiply every element of the column by the constant. It is denoted as kC_l or $C_l \to kC_l$.

For example, if $A = \begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}$ then $C_l \to -3\, C_l$ gives $A \sim \begin{bmatrix} 0 & 2 \\ -9 & 4 \end{bmatrix}$

Can you say that $A = \begin{bmatrix} 0 & 2 \\ 12 & 16 \end{bmatrix}$ or $A = \begin{bmatrix} 0 & 2 \\ -9 & 4 \end{bmatrix}$

(c) Adding the scalar multiples of all the elements of any row (column) to corresponding elements of any other row (column).

If k is a non-zero scalar and the k-multiples of the elements of R_i (C_i) are to be added to the elements of R_j (C_j) then the transformation is symbolically denoted as $R_j \to R_j + kR_i$, $C_j \to C_j + kC_i$

For example, if $A = \begin{bmatrix} -1 & 4 \\ 2 & 5 \end{bmatrix}$ and k = 2 then $R_l \to R_l + 2R_2$ gives

$$A \sim \begin{bmatrix} -1+2(2) & 4+2(5) \\ 2 & 5 \end{bmatrix}$$

i.e. $$A \sim \begin{bmatrix} 3 & 14 \\ 2 & 5 \end{bmatrix}$$

(Can you find the transformation of A using $C_2 \to C_2 + (-3)\, C_1$?)

Note (1) : After the transformation , $R_j \to R_j + kR_i$, R_i remains the same as in the original matrix. Similarly, with the transformation , $C_j \to C_j + kC_i$, C_i remains the same as in the original matrix.

Note (2) : After the elementary transformation, the matrix obtained is said to be equivalent to the original matrix.

Ex. 1 : If $A = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$, apply the transformation $R_1 \leftrightarrow R_2$ on A.

Solution :

As $A = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$

$R_1 \leftrightarrow R_2$ gives

$$A \sim \begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}$$

Ex. 2 : If $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \end{bmatrix}$, apply the transformation $C_1 \to C_1 + 2C_3$.

Solution :

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

$C_1 \to C_1 + 2C_3$ gives

$$A \sim \begin{bmatrix} 1+2(2) & 0 & 2 \\ 2+2(4) & 3 & 4 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 5 & 0 & 2 \\ 10 & 3 & 4 \end{bmatrix}$$

Ex. 3 : If $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 5 \end{bmatrix}$, apply $R_1 \leftrightarrow R_2$ and then $C_1 \to C_1 + 2C_3$ on A.

Solution :

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 5 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$ gives

$$A \sim \begin{bmatrix} 3 & -2 & 5 \\ 1 & 2 & -1 \end{bmatrix}$$

Now $C_1 \to C_1 + 2C_3$ gives

$$A \sim \begin{bmatrix} 3+2(5) & -2 & 5 \\ 1+2(-1) & 2 & -1 \end{bmatrix} \quad \therefore \quad A \sim \begin{bmatrix} 13 & -2 & 5 \\ -1 & 2 & -1 \end{bmatrix}$$

Exercise

Apply the given elementary transformation on each of the following matrices.

1. $A = \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}$, $R_1 \leftrightarrow R_2$. 2. $B = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 5 & 4 \end{bmatrix}$ $R_1 \to R_1 - R_2$.

3. $A = \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix}$, $C_1 \leftrightarrow C_2$; $B = \begin{bmatrix} 3 & 1 \\ 4 & 5 \end{bmatrix}$ $R_1 \leftrightarrow R_2$.

   What do you observe?

4. $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}$, $2C_2$ $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 5 \end{bmatrix}$, $-3R_1$.

   Find the addition of the two new matrices.

5. $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$, $3R_3$ and then $C_3 + 2C_2$.

6. $A = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$, $C_3 + 2C_2$ and then $3R_3$.

   What do you conclude from ex. 5 and ex. 6?

7. Use suitable transformation on $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ to convert it into an upper triangular matrix.

8. Convert $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ into an identity matrix by suitable row transformations.

9. Transform $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{bmatrix}$ into an upper triangular matrix by suitable column transformations.

2.2 Inverse of a matrix :

Definition : If A is a square matrix of order m and if there exists another square matrix B of the same order such that AB = BA = I, where I is the identity matrix of order m, then B is called as the inverse of A and is denoted by A^{-1}.

Using the notation A^{-1} for B we get the above equation as $AA^{-1} = A^{-1}A = I$. Hence, using the same definition we can say that A is also the inverse of B.

$$\therefore \quad B^{-1} = A$$

For example, if $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ then $AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$$\therefore \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 = I_2$$

$$\text{and} \quad BA = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$\therefore \quad B = A^{-1} \quad \text{and} \quad A = B^{-1}$$

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, can you find a matrix X such that AX = I ? Justify the answer.

This example illustrates that for the existence of such a matrix X, the necessary condition is $|A| \neq 0$, i.e. A is a non-singular matrix.

Note that -

(1) Every square matrix A of order m × m has its corresponding determinant; det A = |A|

(2) A matrix is said to be invertible if its inverse exists.

(3) A square matrix A has inverse if and only if $|A| \neq 0$

Uniqueness of inverse of a matrix

It can be proved that if A is a square matrix where $|A| \neq 0$ then its inverse, say A^{-1}, is unique.

Theorem : Prove that if A is a square matrix and its inverse exists then it is unique.

Proof : Let, 'A' be a square matrix of order 'm' and let its inverse exist.

Let, if possible, B and C be the two inverses of A.

Therefore, by definition of inverse AB = BA = I and AC = CA = I.

Now consider $B = BI = B(AC)$

$\therefore \quad B = (BA)C = IC$

$\therefore \quad B = C$

Hence $B = C$ i.e. the inverse is unique.

The inverse of a matrix (if it exists) can be obtained by using two methods.

(i) Elementary row or column transformation

(ii) Adjoint method

We now study these methods.

2.2.1 Inverse of a nonsingular matrix by elementary transformation :

By definition of inverse of A, if A^{-1} exists then $AA^{-1} = A^{-1}A = I$.

Let us consider the equation $AA^{-1} = I$. Here A is the given matrix of order m and I is the identity matrix of order 'm'. Hence the only unknown matrix is A^{-1}. Therefore, to find A^{-1}, we have to first convert A into I. This can be done by using elementary transformations.

Here we note that whenever any elementary row transformation is to be applied on the product $AB = C$ of two matrices A and B, it is enough to apply it only on the prefactor, A. B remains unchanged. And apply the same row transformation to C.

For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix}$ then $AB = \begin{bmatrix} 1 & 10 \\ 1 & 20 \end{bmatrix} = C$ (say)

Now if we require C to be transformed to a new matrix by $R_1 \leftrightarrow R_2$ then $C \sim \begin{bmatrix} 1 & 20 \\ 1 & 10 \end{bmatrix}$

If the same transformation is used for A then $A \sim \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and B remains unchanged,

then the product $AB = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 20 \\ 1 & 10 \end{bmatrix}$ = as required. (Verify the product.)

Hence, the equation $AA^{-1} = I$ can be transformed into an equation of the type $IA^{-1} = B$, by applying same series of row transformations on both the sides of the equation.

However, if we start with the equation $A^{-1}A = I$ (which is also true by the definition of inverse) then the transformation of A should be due to the column transformation. Apply column transformation to post factor and other side, where as prefactor remains unchanged.

Thus, starting with the equation $AA^{-1} = I$, we perform a series of row transformations on both sides of the equation, so that 'A' gets transformed to I. Thus,

$$A \quad A^{-1} = I$$

$\downarrow$ Row Transformations $\qquad \downarrow$ Row Transformations

$$I \quad A^{-1} = B$$

$$\therefore \quad A^{-1} = B$$

and for the equation $A^{-1}A = I$, we use a series of column transformations. Thus

$$\begin{array}{lcl} A^{-1}\,A & = & I \\ \downarrow \text{Column Transformation} & & \downarrow \text{Column Transformation} \\ A^{-1}\,I & = & B \\ \therefore\; A^{-1} & = & B \end{array}$$

Now if A is a given matrix of order '3' and it is nonsingular then we consider

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

For reducing the above matrix to

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$, the suitable row transformations are as follows :

(1) Reduce a_{11} to '1'.

(2) Then, reduce a_{21} and a_{31} to '0'.

(3) Reduce a_{22} to '1'.

(4) Then, reduce a_{12} and a_{32} to '0'.

(5) Reduce a_{33} to '1'.

(6) Then, reduce a_{13} and a_{23} to '0'.

Remember that a similar working rule (but not the same) can be used if you are using column transformations.

Solved Examples

Ex. 1 : Find which of the following matrices are invertible

(i) $A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ (ii) $B = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (iii) $C = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

Solution :

(i) As $|A| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 0$

$\therefore$ A is singular and hence A is not invertible.

(ii) As $|B| = \begin{vmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{vmatrix}$

$= \cos^2\theta + \sin^2\theta = 1 \neq 0$

$\therefore$ B is nonsingular,

$\therefore$ B is invertible.

(iii) $C = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

$\therefore \quad |C| = \begin{vmatrix} 1 & 3 & 2 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -12 \neq 0$

$\therefore$ C is nonsingular and hence C is invertible.

Ex. 2 : Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Solution :

As $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = -2$

$\therefore \quad |A| \neq 0 \qquad \therefore$ A^{-1} exists.

Let $AA^{-1} = I$ (Here we can use only row transformation)

Using $R_2 \to R_2 - 3R_1$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ becomes

$\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

Using $-\frac{1}{2} R_2$ we get

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

Using $R_1 \to R_1 - 2R_2$

We get $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

$\therefore \quad A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ (Verify the answer.)

Ex. 3 : Find the inverse of $A = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ by using elementary row transformations.

Solution : Consider $|A| = 1 \neq 0 \quad \therefore A^{-1}$ exists.

Now as row transformations are to be used we have to consider the equation $AA^{-1} = I$ and have to perform row transformations on A.

Consider $\quad AA^{-1} = I$

i.e. $$\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use $R_1 \leftrightarrow R_2$

i.e. $$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 6 \\ 2 & 2 & 5 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$

$$\therefore \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Now use $R_2 \to -R_2$

$$\therefore \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Use $R_1 \to R_1 - R_2$

$$\therefore \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Use $R_1 \to R_1 - 2R_3$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\therefore IA^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad \therefore A^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

(You can verify that $AA^{-1} = I$)

Ex. 4 : Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ by elementary column transformation.

Solution :

As A^{-1} is required by column transformations therefore we have to consider $A^{-1}A = I$ and have to perform column transformations on A.

Consider

$$A^{-1}A = I$$

$$\therefore \quad A^{-1}\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using $C_2 \to C_2 - 3C_1$ and $C_3 \to C_3 - 3C_1$

$$\therefore \quad A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use $C_1 \to C_1 - C_2$

$$\therefore \quad A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -3 & -3 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Use $C_1 \to C_1 - C_3$

$$\therefore \quad A^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1}I = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

2.2.2 Inverse of a square matrix by adjoint method :

From the previous discussion about finding the inverse of a square matrix by elementary transformation it is clear that the method is elaborate and requires a series of transformations.

There is another method for finding the inverse and it is called as the inverse by the adjoint method. This method can be directly used for finding the inverse. However, for understanding this method you should know the definition of a minor, a co-factor and adjoint of the given matrix.

Let us first recall the definition of minor and co-factor of an element of a determinant.

Definition : Minor of an element a_{ij} of a determinant is the determinant obtained by deleting i th row and j th column in which the element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

(Can you find the order of the minor of any element of a determinant of order 'n'?)

Definition : Co-factor of an element a_{ij} of a determinant is given by $(-1)^{i+j} M_{ij}$, where M_{ij} is minor of the element a_{ij}. Co-factor of an element a_{ij} is denoted by A_{ij}.

Now for defining the adjoint of a matrix, we require the co-factors of the elements of the matrix.

Consider a matrix $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}$. Its corresponding determinant is $|A| = \begin{vmatrix} 1 & -2 & 3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{vmatrix}$

Here if we require the minor of the element '4', then it is $\begin{vmatrix} -2 & 3 \\ -8 & 9 \end{vmatrix} = -18+24 = 6$

Now as the element '4' belongs to 2nd row and 1st column,

using the notation we get $M_{21} = 6$

If further we require the co-factor of '4' then it is

$$= (-1)^{2+1} M_{21}$$
$$= (-1)(6)$$
$$= -6$$

Hence using notation, $A_{21} = -6$

Thus for any given matrix A, which is a square matrix, we can find the co-factors of all of its elements.

Definition :

The adjoint of a square matrix $A = [a_{ij}]_{m\times m}$ is defined as the transpose of the matrix $[A_{ij}]_{m\times m}$ where A_{ij} is the co-factor of the element a_{ij} of A, for all *i* and *j*, where *i*, *j* = 1, 2, ….., m.

The adjoint of the matrix A is denoted by adj A.

For example, if A is a square matrix of order 3 × 3 then the matrix of its co-factors is

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

and the required adjoint of A is the transpose of the above matrix. Hence

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Ex. 1 : Find the co-factors of the elements of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution :

Here $a_{11} = 1 \quad \therefore \quad M_{11} = 4 \quad$ and $\quad A_{11} = (-1)^{1+1}(4) \quad = 4$

$a_{12} = 2 \quad \therefore \quad M_{12} = 3 \quad$ and $\quad A_{12} = (-1)^{1+2}(3) \quad = -3$

$a_{21} = 3 \quad \therefore \quad M_{21} = 2 \quad$ and $\quad A_{21} = (-1)^{2+1}(2) \quad = -2$

$a_{22} = 4 \quad \therefore \quad M_{22} = 1 \quad$ and $\quad A_{22} = (-1)^{2+2}(1) \quad = 1$

$\therefore$ the required co-factors are 4, -3, -2, 1.

Ex. 2 : Find the adjoint of matrix $A = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}$

Solution : Here $a_{11} = 2 \quad \therefore \quad M_{11} = 1$

$\therefore \quad A_{11} = (-1)^{1+1}(1) = 1$

$a_{12} = -3 \quad \therefore \quad M_{12} = 4$

$\therefore \quad A_{12} = (-1)^{1+2}(4) = -4$

$a_{21} = 4 \quad \therefore \quad M_{21} = -3$

$\therefore \quad A_{21} = (-1)^{2+1}(-3) = 3$

$a_{22} = 1 \quad \therefore \quad M_{22} = 2$

$\therefore \quad A_{22} = (-1)^{2+2}(2) = 2$

$\therefore$ the matrix $[A_{ij}]_{2\times2} = \begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix}$

$\therefore \quad [A_{ij}]^T_{2\times2} = \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$

$\therefore \quad \text{adj } A = \begin{bmatrix} 1 & 3 \\ -4 & 2 \end{bmatrix}$

Ex. 3 : Find the adjoint of matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 3 & 1 & 2 \\ -1 & 1 & 2 \end{bmatrix}$

Solution : Here $a_{11} = 2 \quad \therefore \quad M_{11} = 0$

$\therefore \quad A_{11} = (-1)^{1+1} M_{11} = 0$

$a_{12} = 0 \quad \therefore \quad M_{12} = 8$

$\therefore \quad A_{12} = (-1)^{1+2} M_{12} = -8$

$a_{13} = -1 \quad \therefore \quad M_{13} = 4$

$\therefore \quad A_{13} = (-1)^{1+3} M_{13} = 4$

$a_{21} = 3 \quad \therefore \quad M_{21} = 1$

$\therefore \quad A_{21} = (-1)^{2+1} M_{21} = -1$

$a_{22} = 1 \quad \therefore \quad M_{22} = 3$

$\therefore \quad A_{22} = (-1)^{2+2} M_{22} = 3$

$a_{23} = 2 \quad \therefore \quad M_{23} = 2$

$\therefore \quad A_{23} = (-1)^{2+3} M_{23} = -2$

$a_{31} = -1 \quad \therefore \quad M_{31} = 1$

$\therefore \quad A_{31} = (-1)^{3+1} M_{31} = 1$

$a_{32} = 1 \quad \therefore \quad M_{32} = 7$

$\therefore \quad A_{32} = (-1)^{3+2} M_{32} = -7$

$a_{33} = 2 \quad \therefore \quad M_{33} = 2$

$\therefore \quad A_{33} = (-1)^{3+3} M_{33} = 2$

$\therefore$ the matrix of co-factors is $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 0 & -8 & 4 \\ -1 & 3 & -2 \\ 1 & -7 & 2 \end{bmatrix}$

$\therefore \quad \text{adj } A = [A_{ij}]^T_{3\times3} = \begin{bmatrix} 0 & -1 & 1 \\ -8 & 3 & -7 \\ 4 & -2 & 2 \end{bmatrix}$

We know that a determinant can be expanded with the help of any row. For example, expansion by 2nd row $a_{21} A_{21} + a_{22} A_{22} + ... + a_{2n} A_{2n} = |A|$.

But if we multiply the row by a different row of cofactors, then the sum is zero.

For example, $a_{21} A_{31} + a_{22} A_{32} + ... a_{2n} A_{3n} = 0$

This helps us to prove that $A^{-1} = \dfrac{\text{adj} A}{|A|}$

$$\therefore A.\text{adj } A = \begin{vmatrix} |A| & 0 & 0 & ... & 0 \\ 0 & |A| & 0 & ... & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & ... & |A| \end{vmatrix} = |A| \cdot I$$

$$\therefore \quad A^{-1} = \frac{\text{adj} A}{|A|}$$

Thus, if $A = [a_{ij}]_{m\times m}$ is a non-singular square matrix then its inverse exists and it is given by $A^{-1} = \dfrac{1}{|A|}(\text{adj} A)$

Think why A^{-1} does not exist if A is singular.

Ex.1. : If $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$, then find A^{-1} by the adjoint method.

Solution : For given matrix A, we get,

$M_{11} = 3, \quad A_{11} = (-1)^{1+1}(3) = 3$

$M_{12} = 4, \quad A_{12} = (-1)^{1+2}(4) = -4$

$M_{21} = -2, \quad A_{21} = (-1)^{2+1}(-2) = 2$

$M_{22} = 2, \quad A_{22} = (-1)^{2+2}(2) = 2$

$\therefore \text{adj } A = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

and $|A| = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6 + 8 = 14 \neq 0$

$\therefore$ using $A^{-1} = \dfrac{1}{|A|}(\text{adj} A)$

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

Ex. 2 : If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, find A^{-1} by the adjoint method.

Solution : For the given matrix A

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} = 1$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = -1$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix} = 1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = 1$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} = -1$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} = 1$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\text{Now} \quad |A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= 2(4-1) + 1(-2+1) + 1(1-2)$$

$$= 6 - 1 - 1$$

$$= 4$$

Therefore by using the formula for A^{-1}

$$A^{-1} = \frac{1}{|A|}(adj\, A)$$

$$\therefore \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Ex. 3 : If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, verify that A (adj A) = (adj A) A = |A| I.

Solution : For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A_{11} = (-1)^{1+1}(4) = 4$$

$$A_{12} = (-1)^{1+2}(3) = -3$$

$$A_{21} = (-1)^{2+1}(2) = -2$$

$$A_{22} = (-1)^{2+2}(1) = 1$$

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$\therefore \quad A(\text{adj } A) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{... (i)}$$

$$(\text{adj } A) \,.\, A = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-6 & 8-8 \\ -3+3 & -6+4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{... (ii)}$$

$$\text{and } \ |A|\, I = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (-2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \quad \text{... (iii)}$$

From (i), (ii) and (iii) we get, $A(\text{adj}A) = (\text{adj } A)\, A = |A|\, I$

(Note that this equation is valid for every nonsingular square matrix A)

Exercise

1. Find the co-factors of the elements of the following matrices

   (i) $\begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & 5 \\ -2 & 0 & -1 \end{bmatrix}$

2. Find the matrix of co-factors for the following matrices

   (i) $\begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 3 \\ 0 & 3 & -5 \end{bmatrix}$

3. Find the adjoint of the following matrices.

   (i) $\begin{bmatrix} 2 & -3 \\ 3 & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & 5 \\ -2 & 0 & -1 \end{bmatrix}$

4. If $A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

verify that $A\,(\text{adj } A) = (\text{adj } A)\, A = |\,A\,|\, I$

5. Find the inverse of the following matrices by the adjoint method.

(i) $\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

6. Find the inverse of the following matrices

(i) $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Miscellaneous exercise 2 (A)

1. If A = $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$ then reduce it to I_3 by using column transformations.

2. If A = $\begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ then reduce it to I_3 by using row transformations.

3. Check whether the following matrices are invertible or not

(i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & 3 \\ 10 & 15 \end{bmatrix}$

(v) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ (vi) $\begin{bmatrix} \sec\theta & \tan\theta \\ \tan\theta & \sec\theta \end{bmatrix}$ (vii) $\begin{bmatrix} 3 & 4 & 3 \\ 1 & 1 & 0 \\ 1 & 4 & 5 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 2 & 3 \end{bmatrix}$

(ix) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 6 & 8 \end{bmatrix}$

4. Find AB, if A = $\begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}$ and B = $\begin{bmatrix} 1 & -1 \\ 1 & 2 \\ 1 & -2 \end{bmatrix}$ Examine whether AB has inverse or not.

5. If A = $\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is a nonsingular matrix then find A^{-1} by elementary row transformations.

Hence, find the inverse of $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

6. If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and X is a 2×2 matrix such that $AX = I$, then find X.

7. Find the inverse of each of the following matrices (if they exist).

(i) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$ (iv) $\begin{bmatrix} 2 & -3 \\ 5 & 7 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$ (vi) $\begin{bmatrix} 3 & -10 \\ 2 & -7 \end{bmatrix}$ (vii) $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$ (viii) $\begin{bmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{bmatrix}$

(ix) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ (x) $\begin{bmatrix} 1 & 2 & -2 \\ 0 & -2 & 1 \\ -1 & 3 & 0 \end{bmatrix}$

8. Find the inverse of $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

by (i) elementary row transformations
(ii) elementary column transformations

9. If $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ find AB and $(AB)^{-1}$. Verify that $(AB)^{-1} = B^{-1}A^{-1}$

10. If $A = \begin{bmatrix} 4 & 5 \\ 2 & 1 \end{bmatrix}$, then show that $A^{-1} = \frac{1}{6}(A - 5I)$

11. Find matrix X such that AX = B, where

$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 2 & 4 \end{bmatrix}$

.

12. Find X, if AX = B where

$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

13. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 24 & 7 \\ 31 & 9 \end{bmatrix}$ then find matrix X such that AXB = C.

14. Find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$ by adjoint method.

15. Find the inverse of $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ by adjoint method.

16. Find A^{-1} by adjoint method and by elementary transformations if $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$.

17. Find the inverse of $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ by elementary column transformations.

18. Find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$ by elementary row transformations.

19. Show with usual notations that for any matrix $A = [a_{ij}]_{3 \times 3}$

(i) $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$

(ii) $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = |A|$

20. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$ then, find a matrix X such that XA= B.

2.3 Application of matrices :

In the previous discussion you have learnt the concept of inverse of a matrix. Now we intend to discuss the application of matrices for solving a system of linear equations.

For this we first learn to convert the given system of equations in the form of a matrix equation.

Consider the two linear equations, $2x + 3y = 5$ and $x - 4y = 9$. These equations can be written as shown below

$$\begin{bmatrix} 2x+3y \\ x-4y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

(Recall the meaning of equality of two matrices.)

Now using the definition of multiplication of matrices we can consider the above equation as

$$\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

Now if we denote $\begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix} = A$, $\begin{bmatrix} x \\ y \end{bmatrix} = X$ and $\begin{bmatrix} 5 \\ 9 \end{bmatrix} = B$

then the above equation can be written as AX = B

In the equation AX = B, X is the column matrix of variables, A is the matrix of coefficients of variables and B is the column matrix of constants.

Note that if A is of order 2×2, X is of order 2×1, then B is of order 2×1.

Similarly, if there are three linear equations in three variables then as shown above they can be expressed as AX = B.

Find the respective orders of the matrices A, X and B in case of three equations in three variables.

This matrix equation AX = B (in both the cases) can be used to find the values of the variables x and y or x, y and z as the case may be. There are two methods for this application which are namely

(i) method of inversion (ii) method of reduction

2.3.1 Method of inversion :

From the name of this method you can guess that here we are going to use the inverse of a matrix.

This can be done as follows :

Consider the three equations as

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

As explained in the beginning, they can be expressed as

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \text{i.e. } AX = B.$$

Observe that the respective orders of A, X and B are 3×3, 3×1 and 3×1.

Now, if the solution of the three equations exists, then the matrix A must be nonsingular. Hence, A^{-1} exists. Therefore, A^{-1} can be found out either by transformation method or by adjoint method.

After finding A^{-1}, pre-multiply the matrix equation AX = B by A^{-1}

Thus we get,

$$A^{-1}(AX) = A^{-1}(B)$$
i.e. $$(A^{-1}A)X = A^{-1}B$$
i.e. $$IX = A^{-1}B$$
i.e. $$X = A^{-1}B$$ which gives the required solution.

Solved Examples

Ex. 1 : Solve the equations 2x + 5y = 1 and 3x + 2y = 7 by the method of inversion.

Solution : Using the given equations we get the corresponding matrix equation as

$$\begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

i.e. $AX = B$, where $A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

Hence, premultiplying the above matrix equation by A^{-1}, we get

$$(A^{-1}A)X = A^{-1}B$$

i.e. $IX = A^{-1}B$

i.e. $X = A^{-1}B$ …… (i)

Now as $A = \begin{bmatrix} 2 & 5 \\ 3 & 2 \end{bmatrix}$, $|A| = -11$ and adj $A = \begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$\text{i.e.} \quad A^{-1} = \frac{1}{-11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}$$

Hence using (i) we get

$$X = \frac{1}{-11}\begin{bmatrix} 2 & -5 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$X = \frac{1}{11}\begin{bmatrix} -2 & 5 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

$$\text{i.e.} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 33 \\ -11 \end{bmatrix}$$

$$\text{i.e.} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Hence by equality of matrices we get $x = 3$ and $y = -1$.

Ex. 2 : Solve the following equations by the method of inversion

$x - y + z = 4,\ 2x + y - 3z = 0,\ x + y + z = 2.$

Solution : The required matrix equation is $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ i.e. $AX = B$

Hence, by premultiplying the equation by A^{-1}, we get,

i.e. $(A^{-1}A)X = A^{-1}B$

i.e. $IX = A^{-1}B$

i.e. $X = A^{-1}B$ …… (i)

Now as $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$, By definition, $\text{adj } A = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$ and $|A| = 10$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

i.e. $$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

Hence using (i)

$$X = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$X = \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Hence, by equality of matrices we get $x = 2$, $y = -1$ and $z = 1$

2.3.2 Method of reduction :

From the name of the method, it can be guessed that, the given equations are reduced to a certain form to get the solution.

Here also, we start by converting the given linear equation into matrix equation AX = B.

Then we perform the suitable row transformations on the matrix A.

Using the row transformations on A reduce it to an upper triangular matrix or lower triangular matrix or diagonal matrix.

The same row transformations are performed simultaneously on matrix B.

After this step we rewrite the equation in the form of system of linear equations. Now they are in such a form that they can be easily solved by elimination method. Thus, the required solution is obtained.

Solved Examples

Ex. 1 : Solve the equation $2x + 3y = 9$ and $y - x = -2$ using the method of reduction.

Solution : The given equations can be written as

$$2x + 3y = 9$$

and $$-x + y = -2$$

Hence the matrix equation is $\begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$ (i.e. AX = B)

Now use $R_2 \to 2R_2 + R_1$

$\therefore$ We get $\begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$

We rewrite the equations as

$$2x + 3y = 9 \quad \ldots\ldots \text{(i)}$$
$$5y = 5 \quad \ldots\ldots \text{(ii)}$$

From (ii) $y = 1$ and using (i) we get $x = 3$

$\therefore$ $x = 3, y = 1$ is the required solution.

Ex. 2 : Solve the following equations by the method of reduction.

$x + 3y + 3z = 12$, $x + 4y + 4z = 15$ and $x + 3y + 4z = 13$.

Solution : The above equations can be written in the form AX = B

i.e. $$\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 4 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 15 \\ 13 \end{bmatrix}$$

using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

we get $$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 3 \\ 1 \end{bmatrix}$$

Again using $R_1 \to R_1 - 3R_2$ and $R_2 \to R_2 - R_3$ We get $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Hence the required solution is $x = 3, y = 2, z = 1$. (verify)

Ex. 3 : Solve the following equations by the method of reduction.

$x + y + z = 1$, $2x + 3y + 2z = 2$ and $x + y + 2z = 4$.

Solution : The above equation can be written in the form AX = B as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

using $R_2 \to R_2 - R_3$ and $R_1 \to R_1 - \frac{1}{2}R_3$

we get $$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 4 \end{bmatrix}$$

Now using $R_1 \to R_1 - \frac{1}{4} R_2$ we get $\begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -2 \\ 4 \end{bmatrix}$

Note that here we have reduced the original matrix A to a lower triangular matrix. Hence we can rewrite the equations in their original form as

$$\frac{x}{4} = -\frac{1}{2} \quad \ldots\ldots\ldots(i) \quad \text{i.e.} \quad x = -2$$

$$x + 2y = -2 \quad \ldots\ldots\ldots(ii)$$

$$\therefore \quad 2y = -2 + 2 = 0 \quad \therefore \quad y = 0$$

$$\text{and} \quad x + y + 2z = 4$$

$$\therefore \quad 2z = 4 + 2 + 0$$

$$\therefore \quad 2z = 6$$

$$\therefore \quad z = 3$$

$\therefore$ $x = -2$, $y = 0$, $z = 3$ is the required solution.

Ex. 4 : The cost of 2 books and 6 note books is Rs. 34 and the cost of 3 books and 4 notebooks is Rs. 31.

Using matrices, find the cost of one book and one note-book.

Solution : Let Rs. 'x' and ` Rs. 'y' be the costs of one book and one notebook respectively.

Hence, using the above information we get the following equations

$$2x + 6y = 34$$

$$\text{and} \quad 3x + 4y = 31$$

The above equations can be expressed in the form

$$\begin{bmatrix} 2 & 6 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 34 \\ 31 \end{bmatrix} \quad \text{i.e. } AX = B$$

Now using $R_2 \to R_2 - \frac{3}{2} R_1$ we get

$$\begin{bmatrix} 2 & 6 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 34 \\ -20 \end{bmatrix}$$

As the above matrix 'A' is reduced to an upper triangular matrix, we can write the equations in their original form as $2x + 6y = 34$

$$\text{and} \quad -5y = -20 \quad \therefore y = 4$$

$$\text{and} \quad 2x = 34 - 6y = 34 - 24 \quad \therefore 2x = 10 \quad \therefore x = 5$$

$\therefore$ the cost of a book is Rs. 5 and that of a note book is Rs. 4.

Exercise 2.3

1. Solve the following equations by inversion method.

(i) $x + 2y = 2$, $2x + 3y = 3$

(ii) $x + y = 4$, $2x - y = 5$

(iii) $2x + 6y = 8$, $x + 3y = 5$

2. Solve the following equations by reduction method.

(i) $2x + y = 5, \quad 3x + 5y = -3$

(ii) $x + 3y = 2, \quad 3x + 5y = 4$

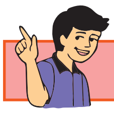
(iii) $3x - y = 1, \quad 4x + y = 6$

(iv) $5x + 2y = 4, \quad 7x + 3y = 5$

3. The cost of 4 pencils, 3 pens and 2 erasers is Rs. 60. The cost of 2 pencils, 4 pens and 6 erasers is Rs. 90, whereas the cost of 6 pencils, 2 pens and 3 erasers is Rs.70. Find the cost of each item by using matrices.

4. If three numbers are added, their sum is '2'. If 2 times the second number is subtracted from the sum of first and third number we get '8' and if three times the first number is added to the sum of second and third number we get '4'. Find the numbers using matrices.

5. The total cost of 3 T.V. sets and 2 V.C.R.s is Rs. 35000. The shop-keeper wants profit of 1000 per television and Rs. 500 per V.C.R. He can sell 2 T. V. sets and 1 V.C.R. and get the total revenue as Rs. 21,500. Find the cost price and the selling price of a T.V. sets and a V.C.R.



Let's Remember :

- If $A = [a_{ij}]_{m \times n}$ then A' or $A^T = [a_{ji}]_{n \times m}$
- If (i) A is symmetric then $A = A^T$ and (ii) if A is skew-symmetric then $-A = A^T$
- If A is a non singular matrix then $A^{-1} = \dfrac{1}{|A|}(\text{adj}\,A)$
- If A, B and C and three matrtices of the same order then

  (i) $A + B = B + A$ (Commutative law of addition)

  (ii) $(A + B) + C = A + (B + C)$ (Associative law for addition)

- If A, B and C are three matrices of appropriate orders so that the following products are defined then

  (i) $(AB)\,C = A\,(BC)$ (Associative Law of multiplication)

  (ii) $A(B + C) = AB + AC$ (Left Distributive Law)

  (iii) $(A + B)\,C = AC + BC$ (Right Distributive Law)

- The three types of elementary transformations are denoted as

  (i) $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

  (ii) $R_i \to kR_j$ or $C_i \to kC_j$ (k is a scalar), $k \neq 0$

  (iii) $R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$ (k is a scalar), $k \neq 0$

- If A and B are two square matrices of the same order such that $AB = BA = I$, then A and B are inverses of each other. A is denoted as B^{-1} and B is denoted as A^{-1}.
- For finding the inverse of A, if row transformations are to be used then we consider $AA^{-1} = I$ and if column transformations are to be used then we consider $A^{-1}A = 1$.

- For finding the inverse of any nonsingular square matrix, two methods can be used
  (i) Elementary transformation method (ii) Adjoint method.
- Using matrices, a system of linear equations can be solved. The two methods are
  (i) Method of inversion (ii) Method of reduction.
- If A is a non singular matrix then $A^{-1} = \frac{1}{|A|}(\text{adj } A)$
- A diagonal matrix is invertible iff all the elements in the diagonal are non - zero.
- The inverse of a diagonal matrix is a diagonal matrix.
- If A is a non singular matrix then det $A^{-1} = \frac{1}{|A|}$
- The adjoint of a square matrix $A = [a_{ij}]_{m\times m}$ is the transpose of the matrix $[A_{ij}]_{m\times m}$, where A_{ij} is the co-factor of the element a_{ij} of A.
- Every square matrix A can be expressed as the sum of symmetric and skew-symmetric matrix. i.e. $A=\frac{1}{2}[A+A^T] + \frac{1}{2}[A - A^T]$

Miscellaneous Exercise 2(B)

I) 1) Choose the correct alternative.

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, adj $A = \begin{bmatrix} 4 & a \\ -3 & b \end{bmatrix}$ then the values of a and b are,

A) a = – 2, b = 1   B) a = 2, b = 4

C) a = 2, b = –1   D) a = 1, b = –2

2) The inverse of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is

A) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ B) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

C) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ D) None of these

3) If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and A(adjA) = k I then the value of k is .....

A) 1   B) –1

C) 0   D) –3

4) If $A = \begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix}$ then the adjoint of matrix A is

A) $\begin{bmatrix} -1 & 3 \\ -4 & 1 \end{bmatrix}$ B) $\begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$

C) $\begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix}$ D) $\begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix}$

5) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and A(adj A) = k I then the value of k is

A) 2 B) –2

C) 10 D) –10

6) If $A = \begin{bmatrix} \lambda & 1 \\ -1 & -\lambda \end{bmatrix}$ then A^{-1} does not exist if λ =

A) 0 B) ± 1

C) 2 D) 3

7) If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ then A^{-1} =

A) $\begin{bmatrix} 1/\cos\alpha & -1/\sin\alpha \\ 1/\sin\alpha & 1/\cos\alpha \end{bmatrix}$ B) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

C) $\begin{bmatrix} -\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ D) $\begin{bmatrix} -\cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$

8) If $F(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where $\alpha \in R$ then $[F(\alpha)]^{-1}$ is =

A) $F(-\alpha)$ B) $F(\alpha^{-1})$

C) $F(2\alpha)$ D) None of these

9) The inverse of $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is

A) I B) A C) A' D) – I

10) The inverse of a symmetric matrix is -

A) Symmetric B) Non-symmetric

C) Null matrix D) Diagonal matrix

11) For a 2 × 2 matrix A, if $A(adjA) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ then determinant A equals

A) 20 B) 10

C) 30 D) 40

12) If $A' = -\frac{1}{2}\begin{bmatrix} 1 & -4 \\ -1 & 2 \end{bmatrix}$ then A =

a) $\begin{bmatrix} 2 & 4 \\ -1 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$

c) $\begin{bmatrix} 2 & -4 \\ 1 & 1 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$

II) 1) Solve the following equations by the methods of inversion.

(i) $2x–y = –2$

(ii) $x + y + z = 1, 2x + 3y + 2z = 2$ and $ax + ay + 2az = 4$

(iii) $5x – y + 4z = 5, 2x + 3y + 5z = 2$ and $5z – 2y + 6z = –1$

(iv) $2x + 3y = –5, 3d + y = 3$

(v) $x + y + z = –1, y + z = 2$ *and* $x + y – z = 3$

2) Express the following equation in matrix from and solve them by the method of reduction.

(i) $x – y + z = 1, 2x - y = 1, 3x + 3y – 4z = 2$

(ii) $x + y = 1, y + x =$

(iii) $2x– + z = 1, x + 2y + 3z = 8$ and $3x + y – 4z = 1$

(iv) $x + y + z = 6, 3x – y + 3z = 10$ and $5z + 5y – 4z = 3$

(v) $x + 2y + z = 8, 2x + 3y – z = 1$ and $3x – y – 2z = 5$

(vi) $x + 3y + 2z = 6, 3x – 2y + 5z = 5$ and $2x – 3y + 6z = 7$

3) The sum of three numbers is 6. If we multiply third number by 3 and add it to the second number we get 11.

By adding first and the third numbers we get a number which is double the second number. Use this information and find a system of linear equations. Find the three numbers using matrices.

4) The cost of 4 pencils, 3 pens and 2 books is Rs.150. The cost of 1 pencil, 2 pens and 3 books is Rs.125. The cos of 6 pencils, 2 pens and 3 books is Rs.175. Fild the cost of each item by using Matrices.

5) The sum of three numbers is 6. Thrice the third number when added to the first number gives 7.

On adding three times first number to the sum of second and third number we get 12. Find the three numbers by using Matrices.

6) The sum of three numbers is 2. If twice the second number is added to the sum of first and third number, we get o adding five times the first number to the sum of second and third we get 6. Find the three numbers by using matrices.

7) An amoun of Rs.5000 is invested in three types of investments, at interest rates 6.7, 7.7, 8% per annum respectively. The total annual income from these investimest is Rs.350/- If the total annual income from first two investment is Rs.70 more than the income from the third, find the amount of each investment using matrix method.

8) The sum of the costs of one ook each of Mathematics, Physics and Chemistry is Rs.210. Total cost of a mathematics book, 2 physics books, and a chemistry book is Rs. 240/- Also the total cost of a Mathematics book, 3 physics book and chemistry books is Rs. 300/-. Find the cost of each book, using Matrices.

✦✦✦✦✦

3 Trigonometric Functions

Let's Study

3.1 **Trigonometric Equations and their solutions**

3.2 **Solutions of triangle**

3.2.1 Polar co-ordinates

3.2.2 Relation between the polar co-ordinates and the Cartesian co-ordinates

3.2.3 Solving a Triangle

3.2.4 The Sine rule

3.2.5 The Cosine rule

3.2.6 The Projection rule

3.2.7 Applications of the Sine rule, the Cosine rule and the Projection rule.

3.3 **Inverse Trigonometric Functions**

Properties, Principal values of inverse trigonometric functions

INTRODUCTION :

We are familiar with algebraic equations. In this chapter we will learn how to solve trigonometric equations, their principal and general solutions, their properties. Trigonometric functions play an important role in integral calculus.

Let's learn.

3.1 Trigonometric Equations and their solutions:

Trigonometric equation :

Definition : An equation involving trignometric function (or functions) is called trigonometric equation.

For example : $\sin\theta = \frac{1}{2}$, $\tan\theta = 2$, $\cos 3\theta = \cos 5\theta$ are all trignometric equations, $x = a\sin\left(\omega t + \alpha\right)$ is also a trigonometric equation.

Solution of Trigonometric equation :

Definition : A value of a variable in a trigonometric equation which satisfies the equation is called a *solution of the trigonometric equation*.

A trigonometric equation can have more than one solutions.

For example, $\theta = \frac{\pi}{6}$ satisfies the equation, $\sin\theta = \frac{1}{2}$, Therefore $\frac{\pi}{6}$ is a solution of the trigonometric equation $\sin\theta = \frac{1}{2}$,

$\frac{\pi}{4}$ is a solution of the trigonometric equation $\cos\,\theta = \frac{1}{\sqrt{2}}$.

$\frac{7\pi}{4}$ is a solution of the trigonometric equation $\cos\theta = \frac{1}{\sqrt{2}}$.

Is π a solution of equation $\sin\theta - \cos\theta = 1$? Can you write one more solution of this equation? Equation $\sin\theta = 3$ has no solution. Can you justify it?

Because of periodicity of trigonometric functions, trigonometric equation may have infinite number of solutions. Our interest is in finding solutions in the interval $[0, 2\pi)$.

Principal Solutions :

Definition : A solution α of a trigonometric equation is called a principal solution if $0 \le \alpha < 2\pi$.

$\frac{\pi}{6}$ and $\frac{5\pi}{6}$ are the principal solutions of trigonometric equation $\sin\,\theta = \frac{1}{2}$.

Note that $\frac{13\pi}{6}$ is a solution but not principal solution of $\sin\theta = \frac{1}{2}$, $\left\{\because \frac{13\pi}{6} \notin [0, 2\pi)\right\}$

0 is the principal solution of equation $\sin\theta = 0$ but 2π is not a principal solution.

Trigonometric equation $\cos\theta = -1$ has only one principal solution. $\theta = \pi$ is the only principal solution of this equation.

Solved Examples

Ex. (1) Find the principal solutions of $\sin\theta = \frac{1}{\sqrt{2}}$.

Solution :

As $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ *and* $0 \le \frac{\pi}{4} < 2\pi$, $\frac{\pi}{4}$ is a principal solution.

By allied angle formula, $\sin\theta = \sin(\pi - \theta)$.

$\therefore \sin\frac{\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{3\pi}{4}$ *and* $0 \le \frac{3\pi}{4} < 2\pi$

$\therefore \frac{3\pi}{4}$ is also a principal solution.

$\therefore \frac{\pi}{4}$ *and* $\frac{3\pi}{4}$ are the principal solutions of $\sin\theta = \frac{1}{\sqrt{2}}$.

Ex.(2) Find the principal solutions of $\cos\theta = \frac{1}{2}$.

Solution : As $\cos\frac{\pi}{3} = \frac{1}{2}$ and $0 \le \frac{\pi}{3} < 2\pi$, $\frac{\pi}{3}$ is a principal solution.

By allied angle formula, cos θ = cos(2π−θ).

$$\therefore \cos\frac{\pi}{3} = \cos\left(2\pi - \frac{\pi}{3}\right) = \cos\frac{5\pi}{3} \text{ and } 0 \le \frac{5\pi}{3} < 2\pi$$

$\therefore \frac{5\pi}{3}$ is also a principal solution.

$\therefore \frac{\pi}{3}$ *and* $\frac{5\pi}{3}$ are the principal solutions of $\cos\theta = \frac{1}{2}$.

Ex. (3) Find the principal solutions of $\cos\theta = -\frac{1}{2}$

Solution : We known that $\cos\frac{\pi}{3} = \frac{1}{2}$

As cos(π−θ) = cos(π+θ) = − cos θ,

$$\cos\left(\pi - \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2} \text{ and } \cos\left(\pi + \frac{\pi}{3}\right) = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

$$\therefore \cos\frac{2\pi}{3} = -\frac{1}{2} \text{ and } \cos\frac{4\pi}{3} = -\frac{1}{2}$$

Also $0 \le \frac{2\pi}{3} \le 2\pi$ *and* $0 \le \frac{4\pi}{3} < 2\pi$. Therefore $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ are principal solutions of $\cos\theta = -\frac{1}{2}$

Ex.(4) Find the principal solutions of cot $\theta = -\sqrt{3}$

Solution : We know that cot $\theta = -\sqrt{3}$ if and only if tan $\theta = -\frac{1}{\sqrt{3}}$

We know that tan $\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

Using identities, tan (π−θ) = − tan θ and tan (2π − θ) = − tan θ, we get

$$\tan\frac{5\pi}{6} = -\frac{1}{\sqrt{3}} \text{ and } \tan\frac{11\pi}{6} = -\frac{1}{\sqrt{3}}$$

An $0 \le \frac{5\pi}{6} < 2\pi$ and $0 \le \frac{11\pi}{6} < 2\pi$

$\therefore \frac{5\pi}{6}$ and $\frac{11\pi}{6}$ are required principal solutions.

The General Solution :

Definition : The solution of a trigonometric equation which is generalized by using its periodicity is called the general solution

For example : All solutions of the equation $\sin\theta = \frac{1}{2}$, are $\left\{....., -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, ...\right\}$. We can generate all these solutions from the expression $n\pi + (-1)^n \frac{\pi}{6}$, $n \in Z$. The solution $n\pi + (-1)^n \frac{\pi}{6}, n \in Z$ is called the general solution of $\sin\theta = \frac{1}{2}$.

Theorem 3.1 : The general solution of $\sin\theta = \sin\alpha$ is $\theta = n\pi + (-1)^n\alpha$, where $n \in Z$.

Proof : As $\sin\theta = \sin\alpha$, α is a solution.

As $\sin(\pi-\alpha) = \sin\alpha$, $\pi-\alpha$ is also a solution. Using periodically, we get

$\sin\theta = \sin\alpha = \sin(2\pi+\alpha) = \sin(4\pi+\alpha) = ...$ and

$\sin\theta = \sin(\pi-\alpha) = \sin(3\pi-\alpha) = \sin(5\pi-\alpha) = ...$

$\therefore$ $\sin\theta = \sin\alpha$ if and only if $\theta = \alpha, 2\pi+\alpha, 4\pi+\alpha, ...$ or $\theta = \pi-\alpha, 3\pi-\alpha, 5\pi-\alpha, ...$

$\therefore$ $\theta = ..., \alpha, \pi-\alpha, 2\pi+\alpha, 3\pi-\alpha, 4\pi+\alpha, 5\pi-\alpha, ...$

$\therefore$ The general solution of $\sin\theta = \sin\alpha$ is $\theta = n\pi + (-1)^n\alpha$, where $n \in Z$.

Theorem 3.2 : The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.

Proof : As $\cos\theta = \cos\alpha$, α is a solution.

As $\cos(-\alpha) = \cos\alpha$, $-\alpha$ is also a solution.

Using periodically, we get

$\cos\theta = \cos\alpha = \cos(2\pi+\alpha) = \cos(4\pi+\alpha) = ...$ and

$\cos\theta = \cos(-\alpha) = \cos(2\pi-\alpha) = \cos(4\pi-\alpha) = ...$

$\therefore \cos\theta = \cos\alpha$ if and only if $\theta = \alpha, 2\pi+\alpha, 4\pi+\alpha, ...$ or $\theta = -\alpha, 2\pi-\alpha, 4\pi-\alpha, 6\pi-\alpha, ...$

$\therefore$ The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.

Theorem 3.3 : The general solution of $\tan\theta = \tan\alpha$ is $\theta = n\pi+\alpha$, where $n \in Z$.

Proof : We know that $\tan\theta = \tan\alpha$ if and only if $\frac{\sin\theta}{\cos\theta} = \frac{\sin\alpha}{\cos\alpha}$

If and only if $\sin\theta\cos\alpha = \cos\theta\sin\alpha$

If and only if $\sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$

If and only if $\sin(\theta-\alpha) = \sin 0$

If and only if $\theta-\alpha = n\pi + (-1)^n \times 0 = n\pi$, where $n \in Z$.

If and only if $\theta = n\pi + \alpha$, where $n \in Z$.

The general solution of $\tan\theta = \tan\alpha$ is $\theta = n\pi + \alpha$, where $n \in Z$.

Remark : For $\theta \in R$, we have the following :

(i) $\sin\theta = 0$ if and only if $\theta = n\pi$, where $n \in Z$.

(ii) $\cos\theta = 0$ if and only if $\theta = (2n+1)\frac{\pi}{2}$, where $n \in Z$.

(iii) $\tan\theta = 0$ if and only if $\theta = n\pi$, where $n \in Z$.

Theorem 3.4 : The general solution of $\sin^2\theta = \sin^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

Proof : $\sin^2\theta = \sin^2\alpha$

$\therefore \quad \sin\theta = \pm \sin\alpha$

$\therefore \quad \sin\theta = \sin\alpha$ or $\sin\theta = -\sin\alpha$

$\therefore \quad \sin\theta = \sin\alpha$ or $\sin\theta = \sin(-\alpha)$

$\therefore \quad \theta = n\pi + (-1)^n\alpha$ or $\theta = n\pi + (-1)^n(-\alpha)$, where $n \in Z$.

$\therefore \quad \theta = n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\sin^2\theta = \sin^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

Alternative Proof : $\sin^2\theta = \sin^2\alpha$

$$\therefore \frac{1-\cos 2\theta}{2} = \frac{1-\cos 2\alpha}{2}$$

$\therefore \quad \cos 2\theta = \cos 2\alpha$

$\therefore \quad 2\theta = 2\pi\pi \pm 2\alpha$, where $n \in Z$.

$\therefore \quad \theta = n\pi \pm \alpha$, where $n \in Z$.

Theorem 3.5 : The general solution of $\cos^2\theta = \cos^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

Proof : $\cos^2\theta = \cos^2\alpha$

$$\therefore \quad \frac{1+\cos 2\theta}{2} = \frac{1+\cos 2\alpha}{2}$$

$\therefore \quad \cos 2\theta = \cos 2\alpha$

$\therefore \quad 2\theta = 2n\pi = \pm 2\alpha$, where $n \in Z$.

$\therefore \quad \theta = n\pi \pm \alpha$, where $n \in Z$.

Theorem 3.6 : The general solution of $\tan^2\theta = \tan^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

Proof : $\tan^2\theta = \tan^2\alpha$

$$\therefore \quad \frac{1+\tan^2\theta}{1-\tan^2\theta} = \frac{1+\tan^2\alpha}{1-\tan^2\alpha} \text{ by componendo and dividendo}$$

$$\therefore \quad \therefore \frac{1-\tan^2\theta}{1+\tan^2\theta} = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} \text{ by intervened}$$

$\therefore \quad \cos 2\theta = \cos 2\alpha$

$\therefore \quad 2\theta = 2n\pi \pm 2\alpha$, where $n \in Z$.

$\therefore \quad \theta = n\pi \pm \alpha$, where $n \in Z$.

Solved Examples

Ex.(1) Find the general solution of

$(i) \sin\theta = \frac{\sqrt{3}}{2}$ $\quad (ii) \cos\theta = \frac{1}{\sqrt{2}}$ $\quad (iii) \tan\theta = \sqrt{3}$

Solution : (i) We have $\sin\theta = \frac{\sqrt{3}}{2}$

$\therefore \quad \sin\theta = \sin\frac{\pi}{3}$

The general solution of $\sin\theta = \sin\alpha$ is $\theta = n\pi + (-1)^n \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\sin\theta = \sin\frac{\pi}{3}$ is $\theta = n\pi + (-1)^n \frac{\pi}{3}$, where $n \in Z$.

$\therefore$ The general solution of $\sin\theta = \frac{\sqrt{3}}{2}$ is $\theta = n\pi + (-1)^n \frac{\pi}{3}$, where $n \in Z$.

(ii) We have $\cos\theta = \frac{1}{\sqrt{2}}$

$\therefore \quad \cos\theta = \cos\frac{\pi}{4}$

The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\cos\theta = \cos\frac{\pi}{4}$ is $\theta = 2n\pi \pm \frac{\pi}{4}$, where $n \in Z$.

$\therefore$ The general solution of $\cos\theta = \frac{1}{\sqrt{2}}$ is $\theta = 2n\pi \pm \frac{\pi}{4}$, where $n \in Z$.

(iii) $\tan\theta = \sqrt{3}$

$\therefore \quad \tan\theta = \tan\frac{\pi}{3}$

The general solution of $\tan\theta = \tan\alpha$ is $\theta = n\pi + \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\tan\theta = \tan\frac{\pi}{3}$ is $\theta = n\pi + \frac{\pi}{3}$ where $n \in Z$.

$\therefore$ The general solution of $\tan\theta = \sqrt{3}$ is $\theta = n\pi + \frac{\pi}{3}$, where $n \in Z$.

Ex. (2) Find the general solution of

(i) $\sin\theta = -\frac{\sqrt{3}}{2}$ (ii) $\cos\theta = -\frac{1}{2}$ (iii) $\cot\theta = -\sqrt{3}$

Solution : (i) $\sin\theta = -\frac{\sqrt{3}}{2}$

$\therefore \quad \sin\theta = \sin\frac{4\pi}{3}$ (As $\sin\frac{4\pi}{3} = \frac{\sqrt{3}}{2}$ and $\sin(\pi+A) = -\sin A$)

The general solution of $\sin\theta = \sin\alpha$ is $\alpha = n\pi + (-1)^n\alpha$, where $n \in Z$.

$\therefore$ The general solution of $\sin\theta = \sin\frac{4\pi}{3}$ is $\theta = n\pi + (-1)^n \frac{4\pi}{3}$, where $n \in Z$.

$\therefore$ The general solution of $\sin\theta = -\frac{\sqrt{3}}{2}$ is $\theta = n\pi + (-1)^n \frac{4\pi}{3}$ where $n \in Z$.

(ii) $\cos\theta = -\frac{1}{2}$

$\therefore$ $\cos\theta = \cos\frac{2\pi}{3}$ $\left(As \cos\frac{\pi}{3} = \frac{1}{2} \text{ and } \cos(\pi - A) = -\cos A\right)$

The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\cos\theta = \cos\frac{2\pi}{3}$ is $\theta = 2n\pi \pm \frac{2\pi}{3}$ where $n \in Z$.

$\therefore$ The general solution of $\cos\theta = -\frac{1}{2}$ is $\theta = 2n\pi \pm \frac{2\pi}{3}$, where $n \in Z$.

(iii) $\cot\theta = -\sqrt{3}$ $\therefore$ $\tan\theta = -\frac{1}{\sqrt{3}}$

$\therefore$ $\tan\theta = \tan\frac{5\pi}{6}$ $\left(As \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \text{ and } \tan(\pi - A) = -\tan A\right)$

The general solution of $\tan\theta = \tan\alpha$ is $\theta = n\pi + \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\tan\theta = \tan\frac{5\pi}{6}$ is $\theta = n\pi + \frac{5\pi}{6}$, where $n \in Z$.

$\therefore$ The general solution of $\cot\theta = -\sqrt{3}$ is $\theta = n\pi + \frac{5\pi}{6}$, where $n \in Z$.

Ex. (3) Find the general solution of

(i) $\operatorname{cosec}\theta = 2$ (ii) $\sec\theta + \sqrt{2} = 0$

Solultion : (i) We have $\operatorname{cosec}\theta = 2$ $\therefore \sin\theta = \frac{1}{2}$

$\therefore$ $\sin\theta = \sin\frac{\pi}{6}$

The general solution of $\sin\theta = \sin\alpha$ is $\theta = n\pi + (-1)^n \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\sin\theta = \sin\frac{\pi}{6}$ is $\theta = n\pi + (-1)^n \frac{\pi}{6}$, where $n \in Z$.

$\therefore$ The general solution of $\operatorname{cosec}\theta = 2$ is $\theta = n\pi + (-1)^n \frac{\pi}{6}$, where $n \in Z$.

(ii) We have $\sec\theta + \sqrt{2} = 0$ $\therefore$ $\cos\theta = -\frac{1}{\sqrt{2}}$

$\therefore$ $\cos\theta = \cos\frac{3\pi}{4}$ $\left(As \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \text{ and } \cos(\pi - A) = -\cos A\right)$

The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\cos\theta = \cos\frac{3\pi}{4}$ is $\theta = 2n\pi \pm \frac{3\pi}{4}$, where $n \in Z$.

$\therefore$ The general solution of $\sec\theta = \sqrt{2}$ is $\theta = 2n\pi \pm \frac{3\pi}{4}$, where $n \in Z$.

Ex. (4) Find the general solution of

(i) $\cos 2\theta = -\dfrac{1}{\sqrt{2}}$ (ii) $\tan 3\theta = -1$ (iii) $\sin 4\theta = \dfrac{\sqrt{3}}{2}$

Solution : (i) We have $\cos 2\theta = -\dfrac{1}{\sqrt{2}}$

$\therefore$ $\cos 2\theta = \cos \dfrac{3\pi}{4}$ $\left(As \cos \dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}} \text{ and } \cos(\pi - A) = \cos A\right)$

The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\cos 2\theta = \cos \dfrac{3\pi}{4}$ is $2\theta = 2n\pi \pm \dfrac{3\pi}{4}$, where $n \in Z$.

$\therefore$ The general solution of $\cos 2\theta = -\dfrac{1}{\sqrt{2}}$ is $\theta = n\pi \pm \dfrac{3\pi}{8}$, where $n \in Z$.

(ii) We have $\tan 3\theta = -1$

$\therefore$ $\tan 3\theta = \tan \dfrac{3\pi}{4}$ $\left(As \tan \dfrac{\pi}{4} = 1 \text{ and } \tan(\pi - A) = -\tan A\right)$

The general solution of $\tan\theta = \tan\alpha$ is $\theta = n\pi + \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\tan 3\theta = \tan \dfrac{3\pi}{4}$ is $3\theta = n\pi + \dfrac{3\pi}{4}$ where $n \in Z$.

$\therefore$ The general solution of $\tan 3\theta = -1$ is $\theta = \dfrac{n\pi}{3} + \dfrac{\pi}{4}$, where $n \in Z$.

(iii) $\sin 4\theta = \dfrac{\sqrt{3}}{2}$

$\therefore$ $\sin 4\theta = \sin \dfrac{\pi}{3}$

The general solution of $\sin\theta = \sin\alpha$ is $\theta = n\pi + (-1)^n \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\sin 4\theta = \sin \dfrac{\pi}{3}$ is $4\theta = n\pi + (-1)^n \dfrac{\pi}{3}$, where $n \in Z$.

$\therefore$ The general solution of $\sin 4\theta = \dfrac{\sqrt{3}}{2}$ is $\theta = \dfrac{n\pi}{4} + (-1)^n \dfrac{\pi}{12}$ where $n \in Z$.

Ex. (5) Find the general solution of

(i) $4\cos^2\theta = 1$ (ii) $4\sin^2\theta = 3$ (iii) $\tan^2\theta = 1$

Solution : (i) We have $4\cos^2\theta = 1$

$\therefore$ $\cos^2\theta = \dfrac{1}{4} = \left(\dfrac{1}{2}\right)^2$

$\therefore$ $\cos^2\theta = \cos^2 \dfrac{\pi}{3}$

The general solution of $\cos^2\theta = \cos^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\cos^2\theta = \cos^2\frac{\pi}{3}$ is $\theta = n\pi \pm \frac{\pi}{3}$, where $n \in Z$.

$\therefore$ The general solution of $4\cos^2\theta = 1$ is $\theta = n\pi \pm \frac{\pi}{3}$, where $n \in Z$.

(ii) We have $4\sin^2\theta = 3$

$\therefore$ $\sin^2\theta = \frac{3}{4} = \left(\frac{\sqrt{3}}{2}\right)^2$

$\therefore$ $\sin^2\theta = \sin^2\frac{\pi}{3}$

The general solution of $\sin^2\theta = \sin^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\sin^2\theta = \sin^2\frac{\pi}{3}$ is $\theta = n\pi \pm \frac{\pi}{3}$, where $n \in Z$.

$\therefore$ The general solution of $4\sin^2\theta = 3$ is $\theta = n\pi \pm \frac{\pi}{3}$, where $n \in Z$.

(iii) We have $\tan^2\theta = 1$

$\therefore$ $\tan^2\theta = \tan^2\frac{\pi}{4}$

The general solution of $\tan^2\theta = \alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

$\therefore$ The general solution of $\tan^2\theta = \tan^2\frac{\pi}{4}$ is $\theta = n\pi \pm \frac{\pi}{4}$, where $n \in Z$.

$\therefore$ The general solution of $\tan^2\theta = 1$ is $\theta = n\pi \pm \frac{\pi}{4}$, where $n \in Z$.

Ex. (6) Find the general solution of $\cos 3\theta = \cos 2\theta$

Solution : We have $\cos 3\theta = \cos 2\theta$

$\therefore$ $\cos 3\theta - \cos 2\theta = 0$

$\therefore$ $-2\sin\frac{5\theta}{2}\sin\frac{\theta}{2} = 0$

$\therefore$ $\sin\frac{5\theta}{2} = 0$ or $\sin\frac{\theta}{2} = 0$

$\therefore$ $\frac{5\theta}{2} = n\pi$ or $\frac{\theta}{2} = n\pi$ where $n \in Z$.

$\therefore$ $\theta = \frac{2n\pi}{5}$, $n \in Z$.

$\therefore$ $\theta = \frac{2n\pi}{5}$ where $n \in Z$ is the required general solution.

Alternative Method : We know that the general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi \pm \alpha$,where $n \in Z$.

$\therefore$ The general solution of $\cos 3\theta = \cos 2\theta$ is $3\theta = 2n\pi \pm 2\theta$, where $n \in Z$.

$\therefore$ $3\theta = 2n\pi - 2\theta$ or $3\theta = 2n\pi + 2\theta$, where $n \in Z$.

$\therefore$ $5\theta = 2n\pi$ or $\theta = 2n\pi$,where $n \in Z$.

$\therefore$ $\theta = \dfrac{2n\pi}{5}$, $n \in Z$ where $n \in Z$ is the required general solution.

Ex. (7) Find the general solution of $\cos 5\theta = \sin 3\theta$

Solution : We have $\cos 5\theta = \sin 3\theta$

$\therefore$ $\cos 5\theta = \cos\left(\dfrac{\pi}{2} - 3\theta\right)$

$\therefore$ $5\theta = 2n\pi \pm \left(\dfrac{\pi}{2} - 3\theta\right)$

$\therefore$ $5\theta = 2n\pi - \left(\dfrac{\pi}{2} - 3\theta\right)$ or $5\theta = 2n\pi + \left(\dfrac{\pi}{2} - 3\theta\right)$

$\therefore$ $\theta = n\pi - \dfrac{\pi}{4}$ or $\theta = \dfrac{n\pi}{4} + \dfrac{\pi}{16}$, where $n \in Z$ are the required general solutions.

Ex. (8) Find the general solution of $\sec^2 2\theta = 1 - \tan 2\theta$

Solution : Given equation is $\sec^2 2\theta = 1 - \tan 2\theta$

$\therefore$ $1 + \tan^2 2\theta = 1 - \tan 2\theta$

$\therefore$ $\tan^2 2\theta + \tan 2\theta = 0$

$\therefore$ $\tan 2\theta\ (\tan 2\theta + 1) = 0$

$\therefore$ $\tan 2\theta = 0$ or $\tan 2\theta + 1 = 0$

$\therefore$ $\tan 2\theta = \tan 0$ or $\tan 2\theta = \tan\dfrac{3\pi}{4}$

$\therefore$ $2\theta = n\pi$ or $2\theta = n\pi + \dfrac{3\pi}{4}$, where $n \in Z$.

$\therefore$ $\theta = \dfrac{n\pi}{2}$ or $\theta = \dfrac{n\pi}{2} + \dfrac{3\pi}{8}$, where $n \in Z$ is the required general solution.

Ex. (9) Find the general solution of $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

Solution : We have $\sin\theta + \sin 3\theta + \sin 5\theta = 0$

$\therefore$ $(\sin\theta + \sin 5\theta) + \sin 3\theta = 0$

$\therefore$ $2\sin 3\theta\cos 2\theta + \sin 3\theta = 0$

$\therefore$ $(2\cos 2\theta + 1)\sin 3\theta = 0$

$\therefore$ $\sin 3\theta = 0$ or $\cos 2\theta = -\dfrac{1}{2}$

$\therefore$ $\sin 3\theta = 0$ or $\cos 2\theta = \cos\dfrac{2\pi}{3}$

$\therefore \quad 3\theta = n\pi$ or $2\theta = 2n\pi \pm \dfrac{2\pi}{3}$, where $n \in Z$.

$\therefore \quad \theta = \dfrac{n\pi}{3}$ or $\theta = n\pi \pm \dfrac{\pi}{3}$, where $n \in Z$ is the required general solution.

Ex. (10) Find the general solution of $\cos\theta - \sin\theta = 1$

Solution : We have $\cos\theta - \sin\theta = 1$

$$\therefore \quad \frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \cos\theta\cos\frac{\pi}{4} - \sin\theta\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \cos\left(\theta + \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$$

$$\therefore \quad \theta + \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$$

$$\therefore \quad \theta + \frac{\pi}{4} = 2n\pi - \frac{\pi}{4} \text{ or } \theta + \frac{\pi}{4} = 2n\pi + \frac{\pi}{4}$$

$\therefore \quad \theta = 2n\pi - \dfrac{\pi}{2}$ or $\theta = 2n\pi$, where $n \in Z$ is the required general solution.

Exercise 3.1

1) Find the principal solutions of the following equations :

(i) $\cos\theta = \dfrac{1}{2}$ (ii) $\sec\theta = \dfrac{2}{\sqrt{3}}$ (iii) $\cot\theta = \sqrt{3}$ (iv) $\cot\theta = 0$

2) Find the principal solutions of the following equations:

(i) $\sin\theta = -\dfrac{1}{2}$ (ii) $\tan\theta = -1$ (iii) $\sqrt{3}\operatorname{cosec}\theta + 2 = 0$

Find the general solutions of the following equations :

3) (i) $\sin\theta = \dfrac{1}{2}$ (ii) $\cos\theta = \dfrac{\sqrt{3}}{2}$ (iii) $\tan\theta = \dfrac{1}{\sqrt{3}}$ (iv) $\cot\theta = 0$

4) (i) $\sec\theta = \sqrt{2}$ (ii) $\operatorname{cosec}\theta = -\sqrt{2}$ (iii) $\tan\theta = -1$

5) (i) $\sin 2\theta = \dfrac{1}{2}$ (ii) $\tan\dfrac{2\theta}{3} = \sqrt{3}$ (iii) $\cot 4\theta = -1$

6) (i) $4\cos^2\theta = 3$ (ii) $4\sin^2\theta = 1$ (iii) $\cos 4\theta = \cos 2\theta$

7) (i) $\sin\theta = \tan\theta$ (ii) $\tan^3\theta = 3\tan\theta$ (iii) $\cos\theta + \sin\theta = 1$

8) Which of the following equations have solutions ?

(i) $\cos 2\theta = -1$ (ii) $\cos^2\theta = -1$ (iii) $2\sin\theta = 3$ (iv) $3\tan\theta = 5$

3.2 Solution of triangle

3.2.1 Polar co-ordinates : Let O be a fixed point in a plane. Let OX be a fixed ray in the plane. O is called the pole and ray OX is called the polar axis. Let P be a point in the plane other than pole O.

Let OP = r and m $\angle$XOP = θ. The ordered pair (r, θ) determines the position of P in the plane. They are called the polar co-ordinates of P . 'r' is called the radius vector and θ is called the vectorial angle of point P.

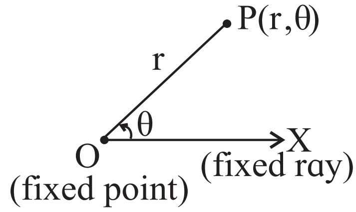


Fig 3.1

Remarks :

i) Vectorial angle θ is the smallest non-negative angle made by OP with the ray OX.

ii) $0 \leq \theta < 2\pi$

iii) Pole has no polar co-ordinates.

3.2.2 Relation between the Cartesian and the Polar co-ordinates: Let O be the pole and OX be the polar axis of polar co-ordinates system. We take line along OX as the X - axis and line perpendicular to OX through O as the Y - axis.

Let P be any point in the plane other than origin. Let (x , y) and (r , θ) be Cartsian and polar co-ordinates of P. To find the relation between them.

By definition of trigonometric functions, we have $\sin\theta = \frac{y}{r}$ and

$\cos\theta = \frac{x}{r}$

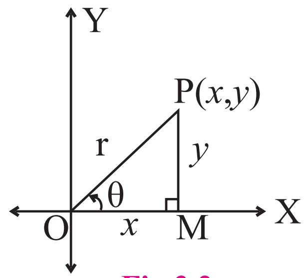


Fig 3.2

$\therefore$ $x = r\cos\theta$ and $y = r\sin\theta$

This is the relation between Cartesian and polar co-ordinates.

Ex. (1) Find the Cartesian co-ordinates of the point whose polar co-ordinates are $\left(2, \frac{\pi}{4}\right)$

Solution : Given $r = 2$ and $\theta = \frac{\pi}{4}$

Using $x = \text{r}\cos\theta$ and $y = \text{r}\sin\theta$, we get

$$x = 2\cos\frac{\pi}{4} = 2\times\frac{1}{\sqrt{2}} = \sqrt{2}$$

$$y = 2\sin\frac{\pi}{4} = 2\times\frac{1}{\sqrt{2}} = \sqrt{2}$$

The required Cartesian co-ordinates are $\left(\sqrt{2}, \sqrt{2}\right)$.

Ex. (2) Find the polar co-ordinates of point whose Cartesian co-ordinates are $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

Solution : From the co-ordinates of the given point we observe that point lies in the fourth quadrant.

$$r^2 = x^2 + y^2$$

$$\therefore\ r^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

$\therefore \quad r = 1$

$x = r\cos\theta \ , y = r\sin\theta$

$\therefore \quad \frac{1}{\sqrt{2}} = 1 \times \cos\theta$ and $-\frac{1}{\sqrt{2}} = 1 \times \sin\theta$

$\therefore \quad \cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$

$\therefore \quad \theta = \frac{7\pi}{4}$

$\therefore$ The required polar co-ordinates are $\left(1, \frac{7\pi}{4}\right)$.

3.2.3 Solving a Triangle :

Three sides and three angles of a triangle are called the elements of the triangle. If we have a certain set of three elements of a triangle, in which at least one element is a side, then we can determine other three elements of the triangle. To solve a triangle means to find unkown elements of the triangle. Using three angles of a triangle we can't solve it. At least one side should be known. In ΔABC, we use the following notations : $l(BC) = BC = a$, $l(CA) = AC = b$, $l(AB) = AB = c$. This notation is called as the usual notation. Following are some standard relations between elements of triangle.

3.2.4 The Sine Rule : In ΔABC, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where R is the circumradius of ΔABC.

Proof : Let AD be perpendicular to BC.

$AD = b\sin C$

$\therefore \quad A(\Delta ABC) = \frac{1}{2} BC \times AD$

$= \frac{1}{2} a \times b\sin C$

$\therefore \quad A(\Delta ABC) = \frac{1}{2} ab\sin C$

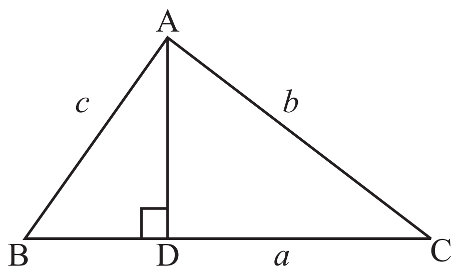


Fig 3.3

$\therefore \quad 2A(\Delta ABC) = ab\sin C$

Similarly $2A(\Delta ABC) = ac\sin B$ and $2A(\Delta ABC) = bc\sin A$

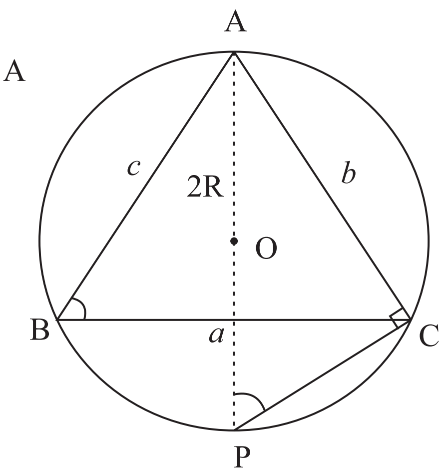
$\therefore \quad bc\sin A = ac\sin B = ab\sin C$

Divide by abc,

$\therefore \quad \frac{bc\sin A}{abc} = \frac{ac\sin B}{abc} = \frac{ab\sin C}{abc}$

$\therefore \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$\therefore \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \ldots (1)$



To prove that each ratio is equal to 2R.

As the sum of three angles is 180^0, at least one of the angle of the triangle is not right angle. Suppose A is not right angle.

Draw diameter through A. Let it meet circle in P.

$\therefore$ AP = 2R and ΔACP is a right angled triangle. $\angle$ ABC and $\angle$ APC are inscribed in the same arc.

$\therefore$ m $\angle$ ABC = m $\angle$ APC

$\therefore$ $\sin B = \sin P = \frac{b}{AP} = \frac{b}{2R}$

$\therefore$ $\sin B = \frac{b}{2R}$

$\therefore$ $\frac{b}{\sin B} = 2R$ ... (2)

From (1) and (2), we get

$\therefore$ $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

Different forms of Sine rule : Following are the different forms of the Sine rule.

In ΔABC.

(i) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

(ii) a = 2R sin A, b = 2R sinB, c = 2R sin C

(iii) $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = k$

(iv) b sin A = a sin B, c sin B = b sin C, c sin A = a sin C

(v) $\frac{a}{b} = \frac{\sin A}{\sin B}, \frac{b}{c} = \frac{\sin B}{\sin C}$

Ex.(1) In ΔABC if A = 30º, B = 60º then find the ratio of sides.

Solution : To find a : b : c

Given A = 30º, B = 60º.

As A, B, C are angles of the triangle, A + B + C = 180º

$\therefore$ C = 90º

By Sine rule,

$\therefore$ $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

$\therefore$ $\frac{a}{\sin 30^0} = \frac{b}{\sin 60^0} = \frac{c}{\sin 90^0}$

$\therefore$ $\frac{a}{\frac{1}{2}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{1}$

$\therefore \quad a : b : c = \frac{1}{2} : \frac{\sqrt{3}}{2} : 1$

$\therefore \quad a : b : c = 1 : \sqrt{3} : 2$

Ex.(2) In ΔABC if $a = 2$, $b = 3$ and $\sin A = \frac{2}{3}$ then find B.

Solution : By sine rule, $\frac{a}{\sin A} = \frac{b}{\sin B}$

$\therefore \quad \frac{2}{\frac{2}{3}} = \frac{3}{\sin B}$

$\therefore \quad \sin B = 1$

$\therefore \quad B = 90^\circ = \frac{\pi}{2}$

Ex. (3) In ΔABC, prove that $a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0$

Solution : L.H.S. $= a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B)$

$= \quad a \sin B - a \sin C + b \sin C - b \sin A + c \sin A - c \sin B$

$= \quad (a \sin B - b \sin A) + (b \sin C - c \sin B) + (c \sin A - a \sin C)$

$= \quad 0 + 0 + 0$

$= \quad 0 =$ R.H.S.

Ex.(4) In ΔABC, prove that $(a - b) \sin C + (b - c) \sin A + (c - a) \sin B = 0$

Solution : L.H.S. $= (a - b) \sin C + (b - c) \sin A + (c - a) \sin B$

$= \quad (a \sin C - b \sin C) + (b \sin A - c \sin A) + (c \sin B - a \sin B)$

$= \quad (a \sin C - c \sin A) + (b \sin A - a \sin B) + (c \sin B - b \sin C)$

$= \quad 0 + 0 + 0 = 0 =$ R.H.S.

3.3.5 The Cosine Rule : In ABC,

(i) $a^2 = b^2 + c^2 - 2bc \cos A$ (ii) $b^2 = c^2 + a^2 - 2ca \cos B$ (iii) $c^2 = a^2 + b^2 - 2ab \cos C$

Proof : Take A as the origin, X - axis along AB and the line perpendicular to AB through A as the Y - axis. The co-ordinates of A, B and C are (0,0). (c, 0) and (b cos A, b sin A) respectively.

To prove that $a^2 = b^2 + c^2 - 2bc \cos A$

L.H.S. $= a^2 = BC^2$

$= \quad (c - b \cos A)^2 + (0 - b \sin A)^2$ (by distance formula)

$= \quad c^2 + b^2 \cos^2 A - 2 bc \cos A + b^2 \sin^2 A$

$= \quad c^2 + b^2 \cos^2 A + b^2 \sin^2 A - 2bc \cos A$

$= \quad c^2 + b^2 - 2 bc \cos A$

$=$ R.H.S.

$\therefore \quad a^2 = b^2 + c^2 - 2bc \cos A$

Similarly, we can prove that

$b^2 = c^2 + a^2 - 2ca \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

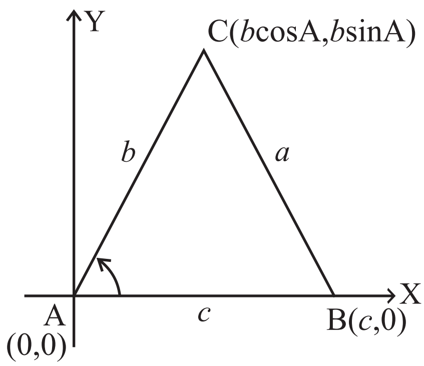


Fig 3.5

Remark : The cosine rule can be stated as : In ΔABC,

$$\cos A = \frac{b^2+c^2-a^2}{2bc}, \quad \cos B = \frac{c^2+a^2-b^2}{2ca}, \quad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

Ex.(5) In ΔABC, if $a = 2$, $b = 3$, $c = 4$ then prove that the triangle is obtuse angled.

Solution : We know that the angle opposite to largest side of a triangle is the largest angle of the triangle.

Here side AB is the largest side. C is the largest angle of ΔABC. To show that C is obtuse angle.

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{2^2+3^2-4^2}{2(3)(4)} = -\frac{3}{24} = -\frac{1}{8}$$

As cos C is negative, C is obtuse angle.

$\therefore$ ΔABC is obtuse angled triangle.

Ex.(6) In ΔABC, if $A = 60^0$, $b = 3$ and $c = 8$ then find a. Also find the circumradius of the triangle.

Solution : By Cosine rule, $a^2 = b^2 + c^2 - 2bc \cos A$

$\therefore$ $a^2 = 3^2 + 8^2 - 2(3)(8)\cos(60^0)$

$= 9 + 64 - 48 \times \frac{1}{2}$

$= 73 - 24 = 49$

$\therefore$ $a^2 = 49$

$\therefore$ $a = 7$

Now by sine rule $\frac{a}{\sin A} = 2R$

$\therefore$ $\frac{7}{\sin 60^\circ} = 2R$

$\therefore$ $\frac{7}{\frac{\sqrt{3}}{2}} = 2R$

$\therefore$ $R = \frac{7}{\sqrt{3}} = \frac{7\sqrt{3}}{3}$

The circumradius of the ΔABC is $\frac{7\sqrt{3}}{3}$

Ex. (7) In ΔABC prove that $a(b\cos C - c\cos B) = b^2 - c^2$

Solution :

L.H.S. $= a(b\cos C - c\cos B)$

$= ab\cos C - ac\cos B$

$= \frac{1}{2}(2ab\cos C - 2ac\cos B)$

$= \frac{1}{2}\{(a^2+b^2-c^2) - (c^2+a^2-b^2)\}$

$= \frac{1}{2}\{ a^2 + b^2 - c^2 - c^2 - a^2 + b^2\}$

$= \frac{1}{2}\{ 2b^2 - 2c^2\}$

$= b^2 - c^2 = \text{R.H.S.}$

3.3.6 The projection Rule : In ΔABC,

(i) $a = b \cos C + c \cos B$

(ii) $b = c \cos A + a \cos C$

(iii) $c = a \cos B + b \cos A$

Proof : Here we give proof of one of these three statements, by considering all possible cases.

To prove that $a = b \cos C + c \cos B$

Let altitude drawn from A meets BC in D.

BD is called the peojection of AB on BC.

DC is called the projection of AC on BC.

$\therefore$ Projection of AB on BC = $c \cos B$

And projection of AC on BC = DC = $b \cos C$

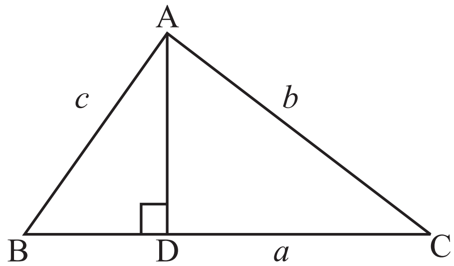


Fig 3.6

Case (i) B and C are acute angles.

$\therefore$ Projection of AB on BC = BD = $c \cos B$

And projection of AC on BC = DC = $b \cos C$

From figure we have,

$a = BC = BD + DC$

$= c \cos B + b \cos C$

$= b \cos C + c \cos B$

$\therefore a = b \cos C + c \cos B$

Case (ii) B is obtuse angle.

$\therefore$ Projection AB on BC = BD = $c \cos(\pi - B) = -c \cos B$

And projection of AC on BC = DC = $b \cos C$

From figure we have,

$a = BC = DC - BD$

$= b \cos C - (-c \cos B)$

$= b \cos C + c \cos B$

$\therefore a = b \cos C + c \cos B$

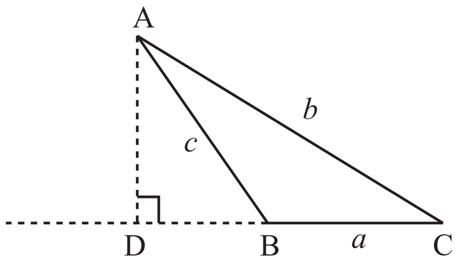


Fig 3.7

Case (iii) B is right angle. In this case D coincides with B.

R.H.S. = $b \cos C + c \cos B$

$= BC + 0$

$= a = \text{L.H.S.}$

$\therefore a = b \cos C + c \cos B$

Similarly we can prove the cases where C is obtuse angle and C right angle.

Therefore in all possible cases, $a = b \cos C + c \cos B$

Similarly we can prove other statements.

Ex.(8) In ΔABC, prove that $(a + b) \cos C + (b + c) \cos A + (c + a) \cos B = a + b + c$

Solution :

L.H.S. $= (a + b) \cos C + (b + c) \cos A + (c + a) \cos B$

$= (a \cos C + b \cos C) + (b \cos A + c \cos A) + (c \cos B + a \cos B)$

$= (a \cos C + c \cos A) + (b \cos A + a \cos B) + (c \cos B + b \cos C)$

$= a + b + c =$ R.H.S.

Ex.(9) In ΔABC , prove that $a(\cos C - \cos B) = 2(b - c) \cos^2 \left(\frac{A}{2}\right)$

Solution : By Projection rule, we have $a \cos C + c \cos A = b$ and $a \cos B + b \cos A = c$

$\therefore$ $a \cos C = b - c \cos A$ and $a \cos B = c - b \cos A$

L.H.S. $= a(\cos C - \cos B)$

$= a \cos C - a \cos B$

$= (b - c \cos A) - (c - b \cos A)$

$= b - c \cos A - c + b \cos A$

$= (b - c) + (b - c) \cos A$

$= (b - c)(1 + \cos A)$

$= (b - c) \times 2\cos^2 \frac{A}{2}$

$= 2(b - c) \cos^2 \frac{A}{2}$

$=$ R.H.S.

Ex.(10) Prove the Cosine rule using the Projection rule.

Solution : Given: In ΔABC, $a = b \cos C + c \cos B$

$b = c \cos A + a \cos C$

$c = a \cos B + b \cos A$

Mulitiply these equations by a,b,c respectively.

$a^2 = ab \cos C + ac \cos B$

$b^2 = bc \cos A + ab \cos C$

$c^2 = ac \cos B + bc \cos A$

$a^2 + b^2 - c^2 = (ab \cos C + ac \cos B) + (bc \cos A + ab \cos C) - (ac \cos B + bc \cos A)$

$= ab \cos C + ac \cos B + bc \cos A + ab \cos C - ac \cos B - bc \cos A$

$= 2ab \cos C$

$\therefore$ $a^2 + b^2 - c^2 = 2ab \cos C$ $\therefore$ $c^2 = a^2 + b^2 - 2ab \cos C$.

Similarly we can prove that

$a^2 = b^2 + c^2 - 2bc \cos A$ and $b^2 = c^2 + a^2 - 2ca \cos B$.

3.3.7 Applications of Sine rule, Cosine rule and Projection rule:

(1) Half angle formulae : In ΔABC, if $a + b + c = 2s$ then

(i) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$ (iii) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

Proof : (i) We have, $1 \therefore \cos A = 2\sin^2 \dfrac{A}{2}$

$$\therefore \quad 1 \therefore \left(\frac{b^2+c^2-a^2}{2bc}\right) = 2\sin^2 \frac{A}{2} \quad \text{by cosine rule}$$

$$\therefore \quad \frac{2bc-b^2-c^2+a^2}{2bc} = 2\sin^2 \frac{A}{2}$$

$$\therefore \quad \frac{a^2-(b^2+c^2-2bc)}{2bc} = 2\sin^2 \frac{A}{2}$$

$$\therefore \quad \frac{a^2-(b-c)^2}{2bc} = 2\sin^2 \frac{A}{2}$$

$$\therefore \quad \frac{\{a-(b-c)\}\{a+(b-c)\}}{2bc} = 2\sin^2 \frac{A}{2}$$

$$\therefore \quad \frac{\{a+b+c-2b\}\{a+b+c-2c\}}{2bc} = 2\sin^2 \frac{A}{2}$$

$$\therefore \quad \frac{\{2s-2b\}\{2s-2c\}}{2bc} = 2\sin^2 \frac{A}{2}$$

$$\therefore \quad \frac{(s-b)(s-c)}{bc} = \sin^2 \frac{A}{2}$$

$$\therefore \quad \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

(ii) We have, $1 + \cos A = 2\cos^2 \dfrac{A}{2}$

$$\therefore \quad 1 + \left(\frac{b^2+c^2-a^2}{2bc}\right) = 2\cos^2 \frac{A}{2} \quad \text{by cosine rule}$$

$$\therefore \quad \frac{2bc+b^2+c^2-a^2}{2bc} = 2\cos^2 \frac{A}{2}$$

$$\therefore \quad \frac{(b^2+c^2+2bc)-a^2}{2bc} = 2\cos^2 \frac{A}{2}$$

$$\therefore \quad \frac{(b+c)^2-a^2}{2bc} = 2\cos^2 \frac{A}{2}$$

$$\therefore \quad \frac{(b+c+a)(b+c-a)}{2bc} = 2\cos^2 \frac{A}{2}$$

$$\therefore \quad \frac{(b+c+a)(b+c+a-2a)}{2bc} = 2\cos^2\frac{A}{2}$$

$$\therefore \quad \frac{(2s)(2s-2a)}{2bc} = 2\cos^2\frac{A}{2}$$

$$\therefore \quad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

(iii) $\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \frac{\sqrt{\frac{(s-b)(s-c)}{bc}}}{\sqrt{\frac{s(s-a)}{bc}}}$

$$= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\therefore \quad \tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

Similarly we can prove that

$$\sin\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}}, \ \sin\frac{C}{2} = \sqrt{\frac{(s-b)(s-a)}{ab}}$$

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}, \ \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \ \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

(2) Heron's Formula : If a,b,c are sides of ΔABC and $a + b + c = 2s$ then

$A(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}$

Proof : We know that $A(\Delta ABC) = \frac{1}{2}ab\sin C$

$\therefore \quad A(\Delta ABC) = \frac{1}{2}ab\,2\sin\frac{C}{2}\cos\frac{C}{2}$

$$= ab\sqrt{\frac{(s-b)(s-a)}{ab}}\sqrt{\frac{s(s-c)}{ab}} = \sqrt{s(s-a)(s-b)(s-c)}$$

(3) Napier's Analogy : In ΔABC, $\tan\left(\frac{B-C}{2}\right) = \frac{(b-c)}{(b+c)}\cot\frac{A}{2}$

Proof : By sine rule b = 2R sin B and c =2Rsin C

$$\therefore \quad \frac{(b-c)}{(b+c)} = \frac{2R\sin B - 2R\sin C}{2R\sin B + 2R\sin C}$$

$$\therefore \quad \frac{(b-c)}{(b+c)} = \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$\therefore \quad \frac{(b-c)}{(b+c)} = \frac{2\cos\left(\frac{B+C}{2}\right)\sin\left(\frac{B-C}{2}\right)}{2\sin\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}$$

$$\therefore \quad \frac{(b-c)}{(b+c)} = \cot\left(\frac{B+C}{2}\right)\tan\left(\frac{B-C}{2}\right)$$

$$\therefore \quad \frac{(b-c)}{(b+c)} = \cot\left(\frac{\pi}{2} - \frac{A}{2}\right)\tan\left(\frac{B-C}{2}\right)$$

$$\therefore \quad \frac{(b-c)}{(b+c)} = \tan\left(\frac{A}{2}\right)\tan\left(\frac{B-C}{2}\right)$$

$$\therefore \quad \tan\left(\frac{B-C}{2}\right) = \frac{(b-c)}{(b+c)}\cot\frac{A}{2}$$

Similarly we can prove that

$$\tan\left(\frac{C-A}{2}\right) = \frac{(c-a)}{(c+a)}\cot\frac{B}{2}$$

$$\tan\left(\frac{A-B}{2}\right) = \frac{(a-b)}{(a+b)}\cot\frac{C}{2}$$

Solved Examples

Ex.(1) In ΔABC if a = 13, b = 14, c = 15 then find the values of

(i) cosA (ii) $\sin\frac{A}{2}$ (iii) $\cos\frac{A}{2}$ (iv) $\tan\frac{A}{2}$ (v) A(ΔABC) (vi) sinA

Solution :

$$s = \frac{a+b+c}{2} = \frac{13+14+15}{2} = 21$$

$(s - a) = 21 - 13 = 8$
$(s - b) = 21 - 14 = 7$
$(s - c) = 21 - 15 = 6$

(i) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

$$= \frac{13^2 + 15^2 - 14^2}{2(13)(15)} = \frac{198}{390} = \frac{33}{65}$$

(ii) $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b(s-c)}{bc}}$

$$= \sqrt{\frac{7 \times 6}{14 \times 15}} = \frac{1}{\sqrt{5}}$$

(iii) $\cos \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$

$$= \sqrt{\frac{21 \times 8}{14 \times 15}} = \frac{2}{\sqrt{5}}$$

(iv) $\tan \dfrac{A}{2} = \dfrac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \dfrac{\frac{1}{\sqrt{5}}}{\frac{2}{\sqrt{5}}} = \dfrac{1}{2}$

(v) $A(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{21 \times 8 \times 7 \times 6} = 84 \text{ sq. unit}$$

(vi) $\sin A = 2 \sin \dfrac{A}{2} \cos \dfrac{A}{2}$

$$= 2 \text{ x } \frac{1}{\sqrt{5}} \times \frac{2}{\sqrt{5}} = \frac{4}{5}$$

Ex.(2) In ΔABC prove that $\cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2} = \dfrac{a+b+c}{b+c-a} \cot \dfrac{A}{2}$

Solution : L.H.S. $= \cot \dfrac{A}{2} + \cot \dfrac{B}{2} + \cot \dfrac{C}{2}$

$$= \frac{1}{\tan \frac{A}{2}} + \frac{1}{\tan \frac{B}{2}} + \frac{1}{\tan \frac{C}{2}}$$

$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \sqrt{\frac{s(s-a)^2}{(s-b)(s-c)(s-a)}} + \sqrt{\frac{s(s-b)^2}{(s-a)(s-c)(s-b)}} + \sqrt{\frac{s(s-c)^2}{(s-b)(s-a)(s-c)}}$$

$$= \sqrt{\frac{s}{(s-b)(s-a)(s-c)}} \left\{\sqrt{(s-a)^2} + \sqrt{(s-b)^2} + \sqrt{(s-c)^2}\right\}$$

$$= \sqrt{\frac{s}{(s-b)(s-a)(s-c)}} \{(s-a)+(s-c)+(s-b)\}$$

$$= \sqrt{\frac{s}{(s-b)(s-a)(s-c)}} \{3s-(a+b+c)\}$$

$$= \sqrt{\frac{s}{(s-b)(s-a)(s-c)}} \{3s-2s\}$$

$$= \sqrt{\frac{s}{(s-b)(s-a)(s-c)}} \times \text{s}$$

$$= \sqrt{\frac{s}{(s-b)(s-c)}} \times \frac{s}{\sqrt{(s-a)}}$$

$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} \times \frac{s}{(s-a)}$$

$$= \frac{2s}{(2s-2a)} \times \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= \frac{a+b+c}{(a+b+c-2a)} \times \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$= \frac{a+b+c}{b+c-a} \cot \frac{A}{2} = \text{R.H.S.}$$

Exerecise 3.2

1) Find the Cartesian co-ordinates of the point whose polar co-ordinates are :

(i) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ (ii) $\left(4, \frac{\pi}{2}\right)$ (iii) $\left(\frac{3}{4}, \frac{3\pi}{4}\right)$ (iv) $\left(\frac{1}{2}, \frac{7\pi}{3}\right)$

2) Find the of the polar co-ordinates point whose Cartesian co-ordinates are.

(i) $\left(\sqrt{2}, \sqrt{2}\right)$ (ii) $\left(0, \frac{1}{2}\right)$ (iii) $\left(1, -\sqrt{3}\right)$ (iv) $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$

3) In ΔABC, if $A = 45^\circ$, $B = 60^\circ$ then find the ratio of its sides.

4) In ΔABC, prove that $\sin\left(\frac{B-C}{2}\right) = \left(\frac{b-c}{a}\right) \cos\frac{A}{2}$.

5) With usual notations prove that $2\left\{a\sin^2\frac{C}{2} + c\sin^2\frac{A}{2}\right\} = a - b + c$.

6) In ΔABC, prove that $a^3 \sin(B-C) + b^3\sin(C-A) + c^3\sin(A-B) = 0$

7) In ΔABC, if cot A, cot B, cot C are in A.P. then show that a^2, b^2, c^2 are also in A.P

8) In ΔABC, if $a \cos A = b \cos B$ then prove that the triangle is right angled or an isosceles traingle.

9) With usual notations prove that $2(bc \cos A + ac \cos B + ab \cos C) = a^2 + b^2 + c^2$

10) In ΔABC, if $a = 18$, $b = 24$, $c = 30$ then find the values of

(i) cosA (ii) $\sin\frac{A}{2}$ (iii) $\cos\frac{A}{2}$ (iv) $\tan\frac{A}{2}$ (v) $A(\Delta ABC)$ (vi) sinA

11) In ΔABC prove that $(b + c - a) \tan\frac{A}{2} = (c + a - b) \tan\frac{B}{2} = (a + b - c) \tan\frac{C}{2}$

12) In ΔABC prove that $\sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2} = \frac{[A(\Delta ABC)]^2}{abcs}$

3.3 Inverse Trigonometric Function :

We know that if a function $f : A \rightarrow B$ is one - one and onto then its inverse function, denoted by $f^{-1} : B \rightarrow A$, exists. For $x \in A$ and $y \in B$ if $y = f(x)$ then $x = f^{-1}(y)$.

Clearly, the domain of f^{-1} = the range of f and the range of f^{-1} = the domain of f. Trigonometric ratios defines functions, called trigonometric functions or circular dunctions. Their inverse functions are called inverse trigonometric functions or inverse cicular functions. Before finding inverse of trigonometric (circular) function , let us revise domain , range and period of the trigonometric function. We summarise them in the following table .

No trigonometric fuction is one-one. An equation of the type $\sin\theta = k$, $(|k| \leq 1)$ has infinitely many solutions given by $\theta = n\pi + (-1)^n \alpha$, where $\sin\alpha = k$, $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$.

Function	Domain	Range	Period
sin	R	$[-1, 1]$	2π
cos	R	$[-1, 1]$	2π
tan	$R-\left\{\frac{(2n+1)\pi}{2} : n\in Z\right\}$	R	π
cot	$R-\{n\pi : n\in Z\}$	R	π
sec	$R-\left\{\frac{(2n+1)\pi}{2} : n\in Z\right\}$	$R-(-1, 1)$	2π
cosec	$R-\{n\pi : n\in Z\}$	$R-(-1, 1)$	2π

There are infinitely many elements in the domain for which the sine function takes the same value. This is true for other trigonometric functions also.

We therefore arrive at the conclusion that inverse of trigonometric functions do not exist. However from the graphs of these functions we see that there are some intervals of their domain, on which they are one-one and onto. Therefore, on these intervals we can define their inverses.

3.3.1 Inverse sine function: Consider the function sin : $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to[-1,1]$. It can be verifide from the graph that with this domain and range it is one-one and onto function. Therefore inverse sine function exists. It is denoted by

$\sin^{-1}: [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

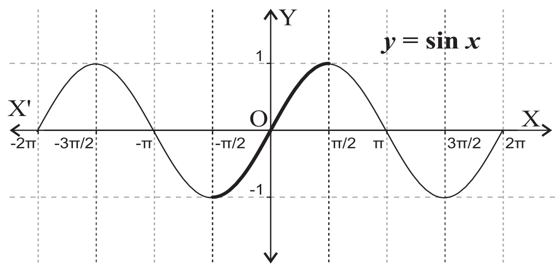


Fig 3.8(a)

For $x \in [-1,1]$ $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$,

we write $\sin^{-1}x = \theta$ if $\sin\theta = x$.

Here θ is known as the principal value of $\sin^{-1} x$.

For example:

1) $\sin\frac{\pi}{6} = \frac{1}{2}$, where $\frac{1}{2} \in [-1, 1]$ and $\frac{\pi}{6} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\therefore$ $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$,

The principal value of $\sin^{-1}\frac{1}{2}$ is $\frac{\pi}{6}$,

However, though $\sin\frac{5\pi}{6} = \frac{1}{2}$,

we cannot write $\sin^{-1}\frac{1}{2} = \frac{5\pi}{6}$ as $\frac{5\pi}{6} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

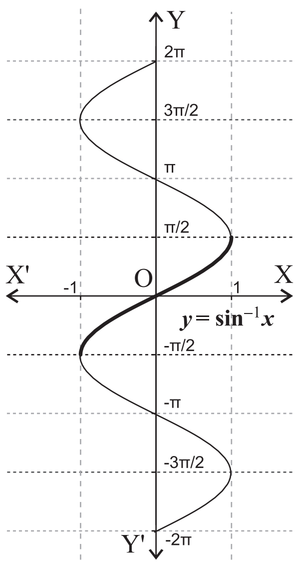


Fig 3.8(b)

2) $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$, where $-\frac{1}{\sqrt{2}} \in [-1, 1]$ and $-\frac{\pi}{4} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \quad \sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

The principal value of $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $-\frac{\pi}{4}$.

Note:

1. $\sin(\sin^{-1}x) = x$, for $x \in [-1, 1]$
2. $\sin^{-1}(\sin y) = y$, for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

3.3.2 Inverse cosine function: Consider the function $\cos : [0, \pi] \to [-1, 1]$. It can be verified from the graph that it is a one-one and onto function. Therefore its inverse fuctions exsit. It is denoted by $\cos^{-1}$.

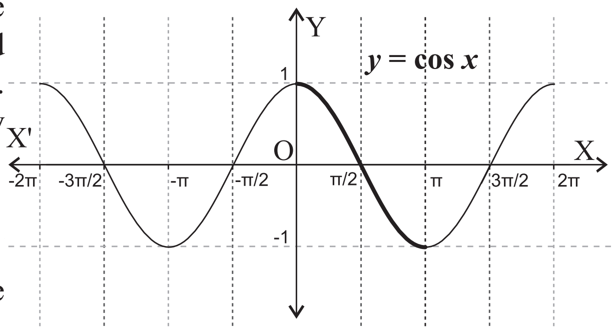


Fig 3.9(a)

Thus, $\cos^{-1}: [-1, 1] \to [0, \pi]$.

For $x \in [-1, 1]$ and $\theta \in [0, \pi]$, we write $\cos^{-1} x = \theta$ is $\cos\theta = x$. Here θ is known as the principal value of $\cos^{-1}x$.

For example : $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, where $\frac{1}{\sqrt{2}} \in [-1, 1]$ and $\frac{\pi}{4} \in [0, \pi]$

$\therefore \quad \cos^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$.

The principal value of $\cos^{-1}\frac{1}{\sqrt{2}}$ is $\frac{\pi}{4}$

Though, $\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$,

We cannot write $\cos^{-1}\frac{1}{\sqrt{2}} = -\frac{\pi}{4}$ as $\frac{-\pi}{4} \notin [0, \pi]$

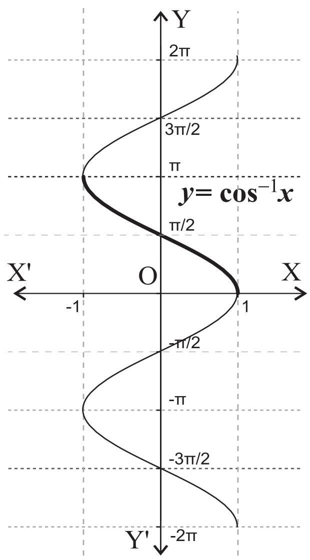


Fig 3.9(b)

Note: 1. $\cos(\cos^{-1} x) = x$ for $x \in [-1, 1]$
2. $\cos^{-1}(\cos y) = y$, for $y \in [0, \pi]$

3.3.3 Inverse tangent function : Consider the function tan $\left(-\frac{\pi}{2},\frac{\pi}{2}\right) \to R$. It can be verified from the graph that it is a one-one and onto function .

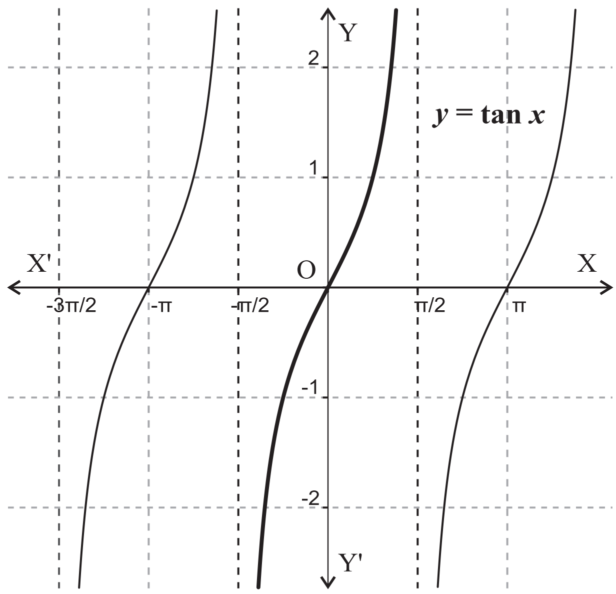


Fig 3.10(a)

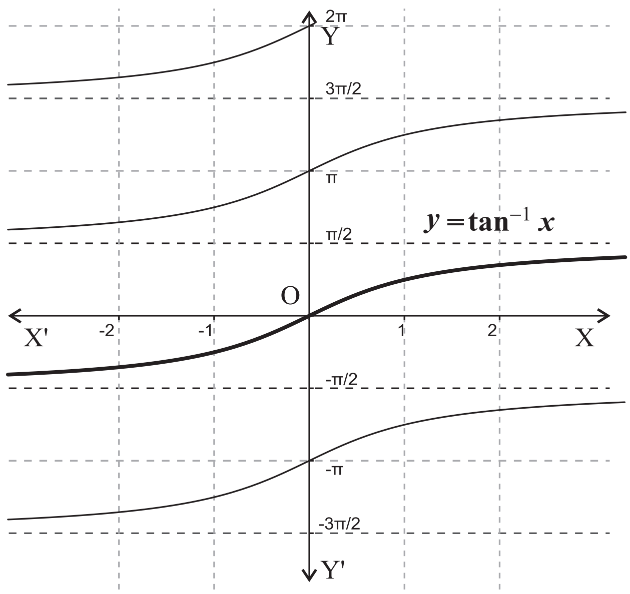


Fig 3.10(b)

Therefore, its inverse function exist. It is denoted by $\tan^{-1}$

Thus, $\tan^{-1}: R \to \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

For $x \in R$ and $\theta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, we write $\tan^{-1} x = \theta$ if $\tan\theta = x$

For example $\tan\frac{\pi}{4} = 1$, where $1 \in R$ and $\frac{\pi}{4} \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

$\therefore \quad \tan^{-1}(1) = \frac{\pi}{4}$.

The principal value of $\tan^{-1} 1$ = is $\frac{\pi}{4}$.

Note: 1. $\tan(\tan^{-1} x) = x$, for $x \in R$ 2. $\tan^{-1}(\tan y) = y$, for $y \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

3.3.4 Inverse cosecant function : Consider the function cosec : $\left(-\frac{\pi}{2},\frac{\pi}{2}\right) - \{0\} \to R - (-1, 1)$. It can be verified from the graph that it is a one-one and onto function. Therefore, its inverse function exists. It is denoted by cosec^{-1}

Thus, $\text{cosec}^{-1}: R-(-1,1) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

For $x \in R-(-1,1)$ and

$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$,

we write $\text{cosec}^{-1}x = \theta$ if $\text{cosec}\,\theta = x$

For example $\text{cosec}\,\frac{\pi}{6} = 2$,

where $2 \in R-(-1,1)$ and $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$\text{cosec}^{-1}(2) = \frac{\pi}{6}$

The principal value of $\text{cosec}^{-1}\,2$ is $\frac{\pi}{6}$.

Note: 1. $\text{cosec}(\text{cosec}^{-1}x) = x$, for $x \in R-(-1,1)$

2. $\text{cosec}^{-1}(\text{cosec}\,y) = y$, for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

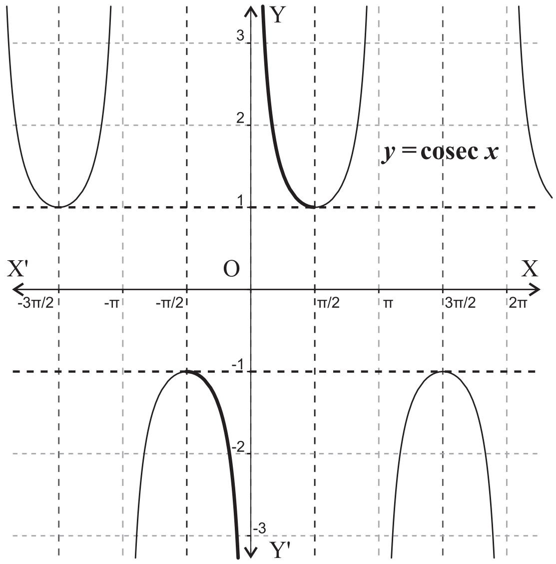


Fig 3.11(a)

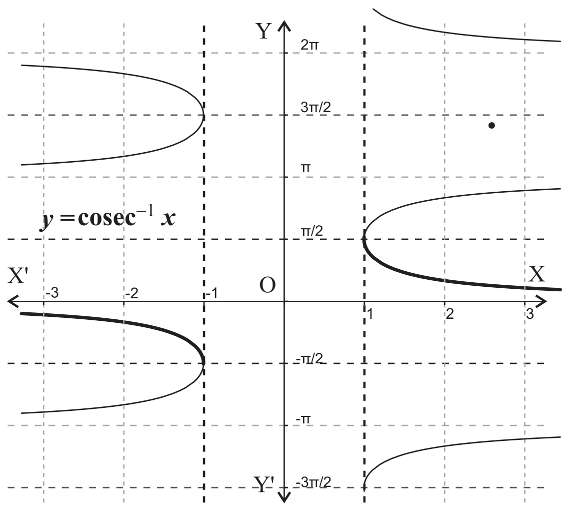


Fig 3.11(b)

3.3.5 Inverse secant function : consider the function

$\sec : [0, \pi] - \left\{\frac{\pi}{2}\right\} \to R-(-1,1)$

It can be verified from the graph that it is a one-one and onto function. Therefore its inverse function exists. It is denoted by $\sec^{-1}$

Thus, $\sec^{-1} : R-(-1,1) \to [0,\pi] - \left\{\frac{\pi}{2}\right\}$.

For $x \in R-(-1,1)$ and $\theta \in [0,\pi] - \left\{\frac{\pi}{2}\right\}$,

we write $\sec^{-1}x = \theta$ if $\sec\theta = x$

For example $\sec\pi = -1$, where $-1 \in R-(-1,1)$

and $\pi \in [0,\pi] - \left\{\frac{\pi}{2}\right\}$.

$\therefore$ $\sec^{-1}(-1) = \pi$

The principal value of $\sec^{-1}(-1)$ is π .

Note: 1. $\sec(\sec^{-1}x) = x$, for $x \in R-(-1,1)$

2. $\sec^{-1}(\sec^{-1} y) = y$, for $y \in [0,\pi] - \left\{\frac{\pi}{2}\right\}$

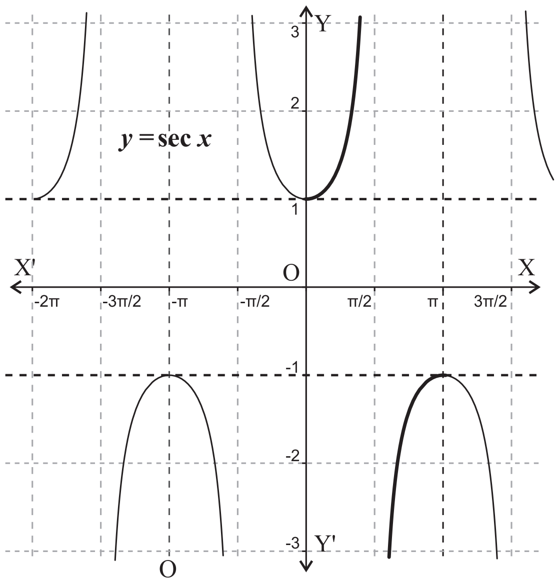


Fig 3.12(a)

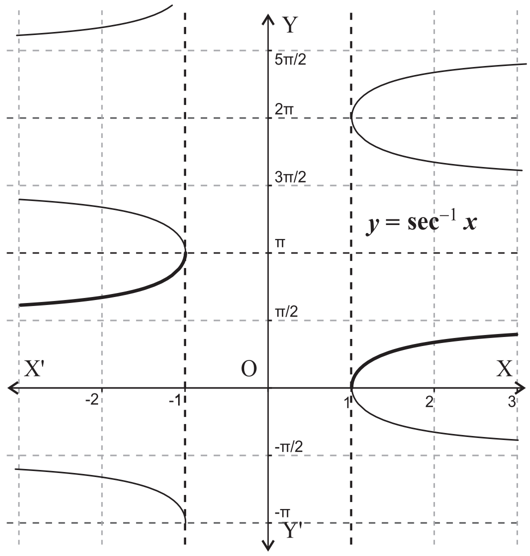


Fig 3.12(b)

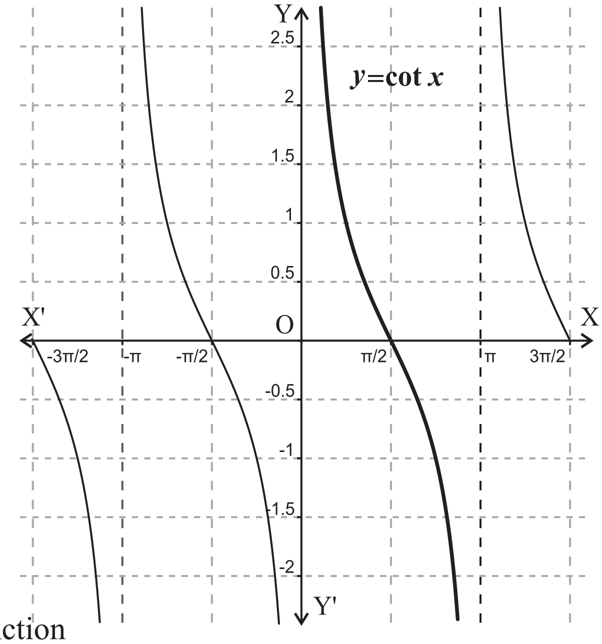


Fig 3.13(a)

3.4.6 Inverse cotangent function :

Consider the function cot : $(0, \pi) \rightarrow R$. It can be verifide from the graph that it is a one-one and onto function. Therefore, its inverse exists. It is denoted by $\cot^{-1}$.

Thus, $\cot^{-1} : R \rightarrow (0, \pi)$

For $x \in R$ and $\theta \in (0, \pi)$, we write

$\cot^{-1} x = \theta$ if $\cot \theta = x$

For example: $\cot \frac{\pi}{3} = \frac{1}{\sqrt{3}}$, where $\frac{1}{\sqrt{3}} \in R$

and $\frac{\pi}{3} \in (0, \pi)$

$$\therefore \quad \cot^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{3}$$

The principal value of $\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$ is $\frac{\pi}{3}$.

Note: 1. $\cot(\cot^{-1}x) = x$, for $x \in R$

2. $\cot^{-1}(\cot y) = y$, for $y \in (0, \pi)$

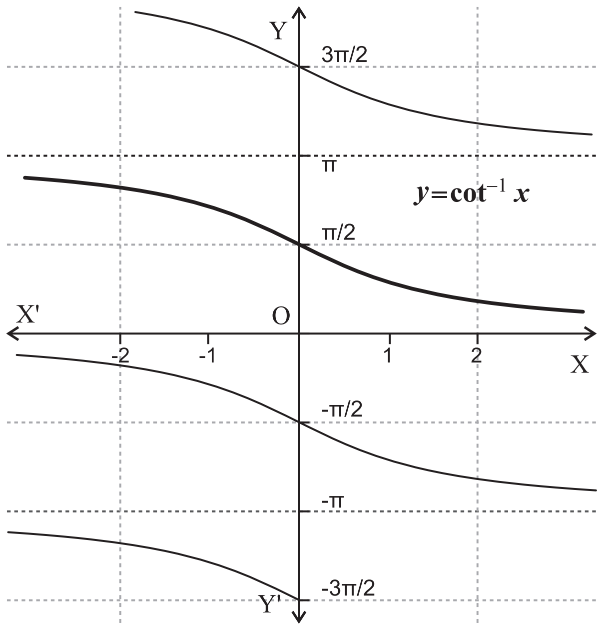


Fig 3.13(b)

3.4.7 Principal Values of Inverse Trigonometric Functions :

The following table shows domain and range of all inverse trigonometic functions. The value of function in the range is called the principal value of the function.

$\sin^{-1} : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1} : [-1,1] \to [0, \pi]$
$\tan^{-1} : R \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\text{cosec}^{-1} : R - (-1,1) \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
$\sec^{-1} : R - (-1,1) \to [0, \pi] - \left\{\frac{\pi}{2}\right\}$

3.4.8 Properties of inverse trigonometric functions :

i) If $-1 \le x \le 1$ and $x \ne 0$ then $\sin^{-1} x = \text{cosec}^{-1}\left(\frac{1}{x}\right)$

Proof : By the conditions on x, $\sin^{-1} x$ and $\text{cosec}^{-1}\left(\frac{1}{x}\right)$ are defined.

As $-1 \le x \le 1$ and $x \ne 0$, $\frac{1}{x} \in R - (-1, 1)$ ...(1)

Let $\sin^{-1} x = \theta$

$\therefore \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\theta \ne 0$ (as $x \ne 0$)

$\therefore \quad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ ...(2)

$\therefore \quad \sin\theta = x$

$\therefore \quad \text{cosec}\,\theta = \frac{1}{x}$ ...(3)

From (1), (2) and (3) we get

$\text{cosec}^{-1}\left(\frac{1}{x}\right) = \theta$

$\therefore \quad \theta = \text{cosec}^{-1}\left(\frac{1}{x}\right)$

$\therefore \quad \sin^{-1} x = \text{cosec}^{-1}\left(\frac{1}{x}\right)$

Similarly we can prove the following result.

(i) $\cos^{-1} x = \sec^{-1}\left(\frac{1}{x}\right)$ if $-1 \le x \le 1$ and $x \ne 0$

(ii) $\tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right)$ if $x > 0$

Proof : Let $\tan^{-1} x = \theta$

$\therefore \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. As $x > 0$, $\theta \in \left(0, \frac{\pi}{2}\right)$

$\therefore \tan\theta = x$

$\therefore \cot\theta = \frac{1}{x}$, where $\frac{1}{x} \in R$ ....(1)

As $\theta \in \left(0, \frac{\pi}{2}\right)$ and $\left(0, \frac{\pi}{2}\right) \subset (0, \pi)$ ...(2)

From (1) and (2) we get $\cot^{-1}\left(\frac{1}{x}\right) = \theta$

$\therefore \theta = \cot^{-1}\left(\frac{1}{x}\right)$

$\therefore \tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right)$

(iii) Similarly we can prove that : $\tan^{-1} x = -\pi + \cot^{-1}\left(\frac{1}{x}\right)$ if $x < 0$.

Activity : Verify the above result for $x = -\sqrt{3}$

iv) if $-1 \le x \le 1$ then $\sin^{-1}(-x) = -\sin^{-1}(x)$

Proof : As $-1 \le x \le 1$, $x \in [-1, 1]$

$\therefore -x \in [-1,1]$ ...(1)

Let $\sin^{-1}(-x) = \theta$

$\therefore \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $\sin\theta = x$

Now $\sin(-\theta) = -\sin\theta = -x$

$\therefore \sin(-\theta) = -x$ ...(2)

Also from (1) $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\therefore \frac{\pi}{2} \ge -\theta \ge -\frac{\pi}{2}$

$\therefore -\frac{\pi}{2} \le -\theta \le \frac{\pi}{2}$

$\therefore -\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ...(3)

From (1), (2) and (3) we can write

$\therefore \quad \sin^{-1}(-x) = -\theta$

$\therefore \quad \sin^{-1}(-x) = -\sin^{-1}(x)$

Similarly we can prove the following results.

v) If $-1 \le x \le 1$ then $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

vi) For all $x \in R$, $\tan^{-1}(-x) = -\tan^{-1}(x)$

vii) If $|x| \ge 1$ then $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x)$

viii) If $|x| \ge 1$ then $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$

ix) For all $x \in R$, $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

x) If $-1 \le x \le 1$ then $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

Proof : Let $\sin^{-1} x = \theta$, where $x \in [-1, 1]$ and $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \quad -\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore \quad \frac{\pi}{2} - \theta \in [0,\pi]$, the principal domain of the cosine fuction.

$\therefore \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

$\cos\left(\frac{\pi}{2} - \theta\right) = x$

$\therefore \quad \cos^{-1} x = \frac{\pi}{2} - \theta$

$\therefore \quad \theta + \cos^{-1} x = \frac{\pi}{2}$

$\therefore \quad \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

Similarly we can prove the following results.

xi) For $x \in R$, $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

xii) For $x \ge 1$, $\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

xiii) If $x > 0$, $y > 0$ and $xy < 1$ then

$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

Proof : Let $\tan^{-1} x = \theta$ and $\tan^{-1} y = \phi$

$\therefore \quad \tan\theta = x$ and $\tan\phi = y$

As $x > 0$ and $y > 0$, we have $0 < \theta < \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$

$\therefore \quad 0 < \theta + \phi < \pi \quad \ldots(1)$

Also $\tan(\theta+\phi) = \dfrac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi} = \dfrac{x+y}{1-xy}$

As $x > 0$, $y > 0$ and $xy < 1$, x, y and $1-xy$ are all positive.

$\therefore \quad \dfrac{x+y}{1-xy}$ is positive.

$\tan(\theta+\phi)$ is positive. ....(2)

From (1) and (2) we get $(\theta+\phi) \in \left(0, \dfrac{\pi}{2}\right)$, the part of the principal domain of the tangent function.

$$\therefore \quad \theta + \phi = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\therefore \quad \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

Similarly we can prove the following results.

xiv) If $x > 0$, $y > 0$ and $xy > 1$ then

$$\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

xv) If $x > 0$, $y > 0$ and $xy = 1$ then

$$\tan^{-1} x + \tan^{-1} y = \frac{\pi}{2}$$

xvi) If $x > 0$, $y > 0$ then

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

Ex.(1) Find the principal values of the following :

(i) $\sin^{-1}\left(-\dfrac{1}{2}\right)$ (ii) $\cos^{-1}\left(\dfrac{\sqrt{3}}{2}\right)$ (iii) $\cot^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$

Solution : (i) $\sin^{-1}\left(-\dfrac{1}{2}\right)$

We have, $\sin\left(-\dfrac{1}{2}\right) = -\dfrac{1}{2}$, where $-\dfrac{1}{2} \in [-1,1]$ and $-\dfrac{\pi}{6} \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$$\therefore \quad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

The principal value of $\sin^{-1}\left(-\dfrac{1}{2}\right)$ is $-\dfrac{\pi}{6}$.

(ii) $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

We have, $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, where $\frac{\sqrt{3}}{2} \in [-1,1]$ and $\frac{\pi}{6} \in [0, \pi]$

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

The principal value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$ is $\frac{\pi}{6}$.

(iii) $\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

We have, $\cot\frac{2\pi}{3} = \frac{-1}{\sqrt{3}}$, where $\frac{-1}{\sqrt{3}} \in R$ and $\frac{\pi}{6} \in (0, \pi)$

$\therefore$ The principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is $\frac{2\pi}{3}$

Ex.(2) Find the values of the following

(i) $\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$ (ii) $\tan^{-1}\left(\tan\frac{\pi}{4}\right)$

(iii) $\sin\left(\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right)$

(iv) $\sin\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{5}{12}\right)$

Solution : (i) $\sin^{-1}\left(\sin\left(\frac{5\pi}{3}\right)\right) = \sin^{-1}\left(\sin\left(2\pi-\frac{\pi}{3}\right)\right)$

$= \sin^{-1}\left(\sin\left(-\frac{\pi}{3}\right)\right) = -\frac{\pi}{3}$, as $-\frac{\pi}{3} \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(\sin\frac{5\pi}{3}\right) = -\frac{\pi}{3}$

(ii) $\tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}$, as $\frac{\pi}{4} \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(iii) We have $\cos^{-1}(-x) = \pi - \cos^{-1} x$

$$\therefore \quad \sin\left(\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right) = \sin\left(\pi - \cos^{1}\frac{1}{\sqrt{2}}\right)$$

$$= \quad \sin\left(\pi - \frac{\pi}{4}\right) = \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\therefore \quad \sin\left(\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right) = \frac{1}{\sqrt{2}}$$

(iv) Let $\cos^{-1}\dfrac{4}{5} = \theta$ and $\tan^{-1}\dfrac{5}{12} = \phi$

$$\therefore \quad \cos\theta = \frac{4}{5} \text{ and } \tan\phi = \frac{5}{12}$$

$$\therefore \quad \sin\theta = \frac{3}{5} \text{ and } \sin\phi = \frac{5}{13}, \quad \cos\phi = \frac{12}{13}$$

$$\sin\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12}\right) = \sin(\theta + \phi)$$

$$= \quad \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$= \quad \frac{3}{5}\ \frac{12}{13} + \frac{4}{5}\ \frac{5}{13} = \frac{56}{65}$$

$$\therefore \quad \sin\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{5}{12}\right) = \frac{56}{65}$$

Ex.(3) Find the values of the following :

(i) $\sin\left[\sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right)\right]$

(ii) $\cos\left[\cos^{-1}\left(-\frac{1}{2}\right) + \tan^{-1}\sqrt{3}\right]$

Solution : (i) We have if $-1 \le x \le 1$ then

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

Here $-1 < \dfrac{3}{5} < 1$

$$\therefore \quad \sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$$

$$\sin\left[\sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right)\right] = \sin\left(\frac{\pi}{2}\right) = 1.$$

(ii) $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$ *and* $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$.

$\therefore \quad \cos\left[\cos^{-1}\left(-\frac{1}{2}\right) + \tan^{-1}\sqrt{3}\right]$

$= \quad \cos\left(\frac{2\pi}{3} + \frac{\pi}{3}\right)$

$= \quad \cos\pi$

$= -1.$

Ex.(4) If $|x| \leq 1$, show that

$\sin(\cos^{-1}x) = \cos(\sin^{-1}x).$

Solution :

We have for $|x| \leq 1$.

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\therefore \quad \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x,$

Now using $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

We have $\sin(\cos^{-1}x) = \sin\left(\frac{\pi}{2} - \sin^{-1}x\right)$

$= \cos(\sin^{-1}x)$

$\therefore \quad \sin(\cos^{-1}x) = \cos(\sin^{-1}x)$

Ex.(5) Prove the following

(i) $2\tan^{-1}\left(-\frac{1}{3}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$

(ii) $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

Solution :

(i) $2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{3}\right)$, as $xy = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9} < 1$

$= \quad \tan^{-1}\left(\frac{\frac{1}{3} + \frac{1}{3}}{1 - \frac{1}{3}\frac{1}{3}}\right) = \tan^{-1}\frac{3}{4} = \theta$, (say)

$\therefore \quad \tan\theta = \frac{3}{4} \qquad \therefore \quad 0 < \theta < \frac{\pi}{2}$

$\therefore \quad \sin\theta = \frac{3}{5} \qquad \therefore \ \theta = \sin^{-1}\frac{3}{5}$

$\therefore \quad 2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{3}{4} = \sin^{-1}\frac{3}{5}$

$\therefore \quad 2\tan^{-1}\frac{1}{3} + \cos^{-1}\frac{3}{5} = \sin^{-1}\frac{3}{5} + \cos^{-1}\frac{3}{5} = \frac{\pi}{2}$

(ii) $2\tan^{-1}\frac{1}{3} = \tan^{-1}\frac{3}{4}$ as seen in (i)

$\therefore \quad 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{3}{4} + \tan^{-1}\frac{1}{7}$ and $xy = \frac{3}{4}\times\frac{1}{7} = \frac{3}{28} < 1$

$= \quad \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\frac{1}{7}}\right) = \tan^{-1}1 = \frac{\pi}{4}$

Ex.(6) Prove that $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$

Solution: We use the result:

$\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ if $xy > 1$

Here $xy = 1 \times 2 = 2 > 1$

$\therefore \quad \tan^{-1}1 + \tan^{-1} 2 = \pi + \tan^{-1}\left(\frac{1+2}{1-(1)(2)}\right)$

$= \quad \pi + \tan^{-1}\left(\frac{3}{1-2}\right)$

$= \quad \pi + \tan^{-1}(-3)$

$= \quad \pi - \tan^{-1}3$ (As, $\tan^{-1}(-x) = -\tan^{-1} x$)

$\therefore \quad \tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$

Ex.(7) Prove that $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

Solution: Let $\cos^{-1}\frac{4}{5} = \theta$

Then $0 < \theta < \frac{\pi}{2}$ and $\cos\theta = \frac{4}{5}$

$\therefore \quad \sin\theta = \dfrac{3}{5}$

Let $\cos^{-1}\dfrac{12}{13} = \phi$

Then $0 < \phi < \dfrac{\pi}{2}$ and $\cos\phi = \dfrac{12}{13}$

$\sin\phi = \dfrac{5}{13}$

$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$= \left(\dfrac{4}{2}\right)\left(\dfrac{12}{13}\right) - \left(\dfrac{3}{5}\right)\left(\dfrac{5}{13}\right)$

$= \dfrac{48}{65} - \dfrac{15}{65} = \dfrac{33}{65}$

$\therefore \quad \cos(\theta + \phi) = \dfrac{33}{65}$ .....(1)

Also $0 < \theta < \dfrac{\pi}{2}$ and $0 < \phi < \dfrac{\pi}{2}$

$\therefore \quad 0 < \theta + \phi < \pi$.

$\therefore$ from (1), $\theta + \phi = \cos^{-1}\dfrac{33}{65}$

$\therefore \quad \cos^{-1}\dfrac{4}{5} + \cos^{-1}\dfrac{12}{13} = \cos^{-1}\dfrac{33}{65}$

Exereise 3.3

1) Find the principal values of the following :

(i) $\sin^{-1}\left(\dfrac{1}{2}\right)$ (ii) $\operatorname{cosec}^{-1}(2)$ (iii) $\tan^{-1}(-1)$

(iv) $\tan^{-1}\left(-\sqrt{3}\right)$ (v) $\sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right)$ (vi) $\cos^{-1}\left(-\dfrac{1}{2}\right)$

2) Evaluate the following :

(i) $\tan^{-1}(1) + \cos^{-1}\left(\dfrac{1}{2}\right) + \sin^{-1}\left(\dfrac{1}{2}\right)$

(ii) $\cos^{-1}\left(\dfrac{1}{2}\right) + 2\sin^{-1}\left(\dfrac{1}{2}\right)$

(iii) $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$

(iv) $\operatorname{cosec}^{-1}\left(-\sqrt{2}\right) + \cot^{-1}\left(\sqrt{3}\right)$

3) Prove the following :

(i) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) - 3\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{3\pi}{4}$

(ii) $\sin^{-1}\left(-\frac{1}{2}\right) + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \cos^{-1}\left(-\frac{1}{2}\right)$

(iii) $\sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{12}{13}\right) = \sin^{-1}\left(\frac{56}{65}\right)$

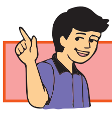
(iv) $\cos^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$

(v) $\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

(vi) $2\tan^{-1}\left(\frac{1}{3}\right) = \tan^{-1}\left(\frac{3}{4}\right)$

(vii) $\tan^{-1}\left[\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right] = \frac{\pi}{4}+\theta$ if $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

(viii) $\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{\theta}{2}$, if $\theta \in (-\pi, \pi)$



Let's remember!

* An equation involving trigonometric function (or functions) is called a trigonometric equation.
* A value of a variable in a trigonometric equation which satisfies the equation is called a solution of the trigonometric equation.
* A solution α of a trigonometric equation is called a principal solution if $0 \le \alpha < 2\pi$.
* The solution of a trigonometric equation which is generalized by using its periodicity is called the general solution.
* The general solution of $\sin\theta = \sin\alpha$ is $\theta = n\pi + (-1)^n \alpha$, where $n \in Z$.
* The general solution of $\cos\theta = \cos\alpha$ is $\theta = 2n\pi + \alpha$, where $n \in Z$.
* The general solution of $\tan\theta = \tan\alpha$ is $\theta = n\pi + \alpha$, where $n \in Z$.
* The general solution of $\sin^2\theta = \sin^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.
* The general solution of $\cos^2\theta = \cos^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.
* The general solution of $\tan^2\theta = \tan^2\alpha$ is $\theta = n\pi \pm \alpha$, where $n \in Z$.

* The Sine Rule : In ΔABC, $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$ = 2R, where R is the circumradius of ΔABC.

Following are the different forms of the Sine rule.

(i) $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$

(ii) a = 2R sinA, b = 2R sinB, c = 2R sinC

(iii) $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}=k$

(iv) b sinA = a sinB , c sinB = b sinC, c sinA = a sinC

(v) $\frac{a}{b}=\frac{\sin A}{\sin B}, \frac{b}{c}=\frac{\sin B}{\sin C}$

* The Cosine Rule : In ΔABC,

$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = c^2 + a^2 - 2ca \cos B$

$c^2 = a^2 + b^2 - 2ab \cos C$

* The Projection Rule : In ΔABC.

a = b cos C + c cos B

b = c cos A + a cos C

c = a cos B + b cos A

* Half angle formulae : In ΔABC , if a + b + c = 2s then

(i) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}, \sin\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{ac}}, \sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

(ii) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}, \cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ac}}, \cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

(iii) $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \tan\frac{B}{2}=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}, \tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

* Heron's Formula : If a,b,c are sides of ΔABC and a + b + c = 2s then

$A(\Delta ABC) = \sqrt{s(s-a)(s-b)(s-c)}$

* Napier's Analogy : In ΔABC , $\tan\left(\frac{B-C}{2}\right)=\frac{(b-c)}{(b+c)}\cot\frac{A}{2}$

* Inverse Trigonometric functions :

(i) $\sin(\sin^{-1}x) = x$, for $x \in [-1,1]$

(ii) $\sin^{-1}(\sin y) = y$, for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(iii) $\cos(\cos^{-1}x) = x$, for $x \in [-1, 1]$

(iv) $\cos^{-1}(\cos y) = y$, for $y \in [0, \pi]$

(v) $\tan(\tan^{-1}x) = x$, for $x \in R$

(vi) $\tan^{-1}(\tan y) = y$, for $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(vii) $\text{cosec}(\text{cosec}^{-1} x) = x$, for $x \in R - (-1, 1)$

(viii) $\text{cosec}^{-1}(\text{cosec } y) = y$, for $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

(ix) $\sec(\sec^{-1} x) = x$, for $x \in R - (-1, 1)$

(x) $\sec^{-1}(\sec y) = y$, for $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

(xi) $\cot(\cot^{-1} x) = x$, for $x \in R$

(xii) $\cot^{-1}(\cot y) = y$, for $y \in (0, \pi)$

*** Properties of inverse trigonometric functions :**

(i) $\sin^{-1} x = \text{cosec}^{-1}\left(\frac{1}{x}\right)$ if $-1 \le x \le 1$ and $x \ne 0$

(ii) $\cos^{-1} x = \sec^{-1}\left(\frac{1}{x}\right)$ if $-1 \le x \le 1$ and $x \ne 0$

(iii) $\tan^{-1} x = \cot^{-1}\left(\frac{1}{x}\right)$ if $x > 0$

(iv) If $-1 \le x \le 1$ then $\sin^{-1}(-x) = -\sin^{-1}(x)$

(v) If $-1 \le x \le 1$ then $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

(vi) For all $x \in R$, $\tan^{-1}(x) = -\tan^{-1}(x)$

(vii) If $|x| \ge 1$ then $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}(x)$

(viii) If $|x| \ge 1$ then $\sec^{-1}(x) = \pi - \sec^{-1}(x)$

(ix) For all $x \in R$, $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

(x) If $-1 \le x \le 1$ then $\sin^{-1}x + \cos^{-1} x = \frac{\pi}{2}$

(xi) For $x \in R$, $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

(xii) For $x \ge 1$, $\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2}$

(xiii) If $x > 0$, $y > 0$ and $xy < 1$ then $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$

(xiv) If $x > 0$, $y > 0$ and $xy > 1$ then $\tan^{-1}x + \tan^{-1} y = \pi + \tan^{-1} \dfrac{x+y}{1-xy}$

(xv) If $x > 0$, $y > 0$ and $xy = 1$ then $\tan^{-1} x + \tan^{-1} y = \dfrac{\pi}{2}$

(xvi) If $x > 0$, $y > 0$ then $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\dfrac{x-y}{1+xy}\right)$

Miscellaneous Exercise 3

I) Select the correct option from the given alternatives.

1) The principal of solutions equation $\sin\theta = \dfrac{-1}{2}$ are ____________.

a) $\dfrac{5\pi}{6}, \dfrac{\pi}{6}$ b) $\dfrac{7\pi}{6}, \dfrac{11\pi}{6}$ c) $\dfrac{\pi}{6}, \dfrac{7\pi}{6}$ d) $\dfrac{7\pi}{6}, \dfrac{\pi}{6}$

2) The principal solution of equation $\cot\theta = \sqrt{3}$ ___________.

a) $\dfrac{\pi}{6}, \dfrac{7\pi}{6}$ b) $\dfrac{\pi}{6}, \dfrac{5\pi}{6}$

c) $\dfrac{\pi}{6}, \dfrac{8\pi}{6}$ d) $\dfrac{7\pi}{6}, \dfrac{\pi}{6}$

3) The general solution of $\sec x = \sqrt{2}$ is __________.

a) $2n\pi \pm \dfrac{\pi}{4}$, $n \in Z$ b) $2n\pi \pm \dfrac{\pi}{2}$, $n \in Z$

c) $n\pi \pm \dfrac{\pi}{2}$, $n \in Z$, d) $2n\pi \pm \dfrac{\pi}{3}$, $n \in Z$

4) If $\cos p\theta = \cos q\theta$, $p \neq q$ rhen _________.

a) $\theta = \dfrac{2n\pi}{p \pm q}$ b) $\theta = 2n\pi$

c) $\theta = 2n\pi \pm p$ d) $n\pi \pm q$

5) If polar co-ordinates of a point are $\left(2, \dfrac{\pi}{4}\right)$ then its cartesian co-ordinates are ______.

a) $(2, \sqrt{2})$ b) $(\sqrt{2}, 2)$

c) $(2, 2)$ d) $(\sqrt{2}, \sqrt{2})$

6) If $\sqrt{3}\cos x - \sin x = 1$, then general value of x is __________.

a) $2n\pi \pm \dfrac{\pi}{3}$ b) $2n\pi \pm \dfrac{\pi}{6}$

c) $2n\pi \pm \dfrac{\pi}{3} - \dfrac{\pi}{6}$ d) $n\pi + (-1)^n \dfrac{\pi}{3}$

7) In ΔABC if $\angle A = 45^0$, $\angle B = 60^0$ then the ratio of its sides are ________.

a) $2 : \sqrt{2} : \sqrt{3}+1$ b) $\sqrt{2} : 2 : \sqrt{3}+1$

c) $2\sqrt{2} : \sqrt{2} : \sqrt{3}$ d) $2 : 2\sqrt{2} : \sqrt{3}+1$

8) In ΔABC, if $c^2 + a^2 - b^2 = ac$, then $\angle B$ = ________.

a) $\frac{\pi}{4}$ b) $\frac{\pi}{3}$ c) $\frac{\pi}{2}$ d) $\frac{\pi}{6}$

9) In ABC, $ac \cos B - bc \cos A$ = ________.

a) $a^2 - b^2$ b) $b^2 - c^2$

c) $c^2 - a^2$ d) $a^2 - b^2 - c^2$

10) If in a triangle, the are in A.P. and $b : c = \sqrt{3} : \sqrt{2}$ then A is equal to ________.

a) 30^0 b) 60^0

c) 75^0 d) 45^0

11) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ = ________.

a) $\frac{7\pi}{6}$ b) $\frac{5\pi}{6}$

c) $\frac{\pi}{6}$ d) $\frac{3\pi}{2}$

11) The value of $\cot(\tan^{-1} 2x + \cot^{-1} 2x)$ is ________.

a) 0 b) 2x

c) $\pi + 2x$ d) $\pi - 2x$

12) The principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is ________.

a) $\left(-\frac{2\pi}{3}\right)$ b) $\frac{4\pi}{3}$

c) $\frac{5\pi}{3}$ d) $-\frac{\pi}{3}$

13) If $\sin^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \sin^{-1}\alpha$, then α = ________.

a) $\frac{63}{65}$ b) $\frac{62}{65}$

c) $\frac{61}{65}$ d) $\frac{60}{65}$

14) If $\tan^{-1}(2x) + \tan^{-1}(3x) = \frac{\pi}{4}$, then x =

a) -1 b) $\frac{1}{6}$ c) $\frac{2}{6}$ d) $\frac{3}{2}$

15) $2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=$ __________.

a) $\tan^{-1}\left(\frac{4}{5}\right)$ b) $\frac{\pi}{2}$

c) 1 d) $\frac{\pi}{4}$

16) $\tan\left(2\tan^{-1}\left(\frac{1}{5}\right)-\frac{\pi}{4}\right)=$ __________.

a) $\frac{17}{7}$ b) $-\frac{17}{7}$

c) $\frac{7}{17}$ d) $-\frac{7}{17}$

17) The principal value branch of $\sec^{-1} x$ is __________.

a) $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$ b) $[0,\pi]-\left[\frac{\pi}{2}\right]$

c) $(0,\pi)$ d) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

18) $\cos\left[\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{2}\right]=$ __________.

a) $\frac{1}{\sqrt{2}}$ b) $\frac{\sqrt{3}}{2}$ c) $\frac{1}{2}$ d) $\frac{\pi}{4}$

19) If $\tan\theta+\tan 2\theta+\tan 3\theta=\tan\theta\tan 2\theta\tan 3\theta$, then the general value of the θ is ________.

a) $n\pi$ b) $\frac{n\pi}{6}$

c) $n\pi\pm\frac{\pi}{4}$ d) $\frac{n\pi}{2}$

20) If any ΔABC, if a cos B = b cos A, then the triangle is _________.

a) Equilateral triangle b) Isosceles triangle

c) Scalene d) Right angled

II: Solve the following

1) Find the principal solutions of the following equations :

(i) $\sin 2\theta=-\frac{1}{2}$ (ii) $\tan 3\theta=-1$ (iii) $\cot\theta=0$

2) Find the principal solutions of the following equations :

(i) $\sin 2\theta=-\frac{1}{\sqrt{2}}$ (ii) $\tan 5\theta=-1$ (iii) $\cot 2\theta=0$

3) Which of the following equations have no solutions ?

(i) $\cos 2\theta = \frac{1}{3}$ (ii) $\cos^2\theta = -1$ (iii) $2\sin\theta = 3$ (iv) $3\sin\theta = 5$

4) Find the general solutions of the following equations :

i) $\tan\theta = -\sqrt{3}$ ii) $\tan^2\theta = 3$ iii) $\sin\theta - \cos\theta = 1$ iv) $\sin^2\theta - \cos^2\theta = 1$

5) In ΔABC prove that $\cos\left(\frac{A-B}{2}\right) = \left(\frac{a+b}{c}\right)\sin\frac{C}{2}$

6) With usual notations prove that $\frac{\sin(A-B)}{\sin(A+B)} = \frac{a^2-b^2}{c^2}$.

7) In ΔABC prove that $(a-b)^2\cos^2\frac{C}{2} + (a+b)^2\sin^2\frac{C}{2} = c^2$

8) In ΔABC if cosA = sin B – cos C then show that it is a right angled triangle.

9) If $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-c)}$ then show that a^2, b^2, c^2, are in A.P.

10) Solve the triangle in which $a = \sqrt{3}+1 \cdot b = \sqrt{3}-1$ and $C = 60^0$

11) In ΔABC prove the following :

(i) a sin A - b sin B = c sin (A – B)

(ii) $\frac{c - b\cos A}{b - c\cos A} = \frac{\cos B}{\cos C}$

(iii) $a^2\sin(B-C) = (b^2-c^2)\sin A$

(iv) $ac\cos B - bc\cos A = (a^2 - b^2)$

(v) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2+b^2+c^2}{2abc}$

(vi) $\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2} = \frac{1}{a^2} - \frac{1}{b^2}$

(vii) $\frac{b-c}{a} = \frac{\tan\frac{B}{2} - \tan\frac{C}{2}}{\tan\frac{B}{2} + \tan\frac{C}{2}}$

12) In ΔABC if a^2, b^2, c^2, are in A.P. then $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ are also in A.P.

13) In ΔABC if C = 90° then prove that $\sin(A - B) = \frac{a^2-b^2}{a^2+b^2}$

14) In ΔABC if $\frac{\cos A}{a} = \frac{\cos B}{b}$ then show that it is an isosceles triangle.

15) In ΔABC if $\sin 2A + \sin 2B = \sin 2C$ then prove that the triangle is a right angled triangle.

16) In ΔABC prove that $a^2(\cos^2 B - \cos^2 C) + b^2(\cos^2 C - \cos^2 A) + c^2(\cos^2 A - \cos^2 B) = 0$

17) With usual notations show that $(c^2 - a^2 + b^2)\tan A = (a^2 - b^2 + c^2)\tan B = (b^2 - c^2 + a^2)\tan C$

18) In ΔABC if $a\cos^2\frac{C}{2} + c\cos^2\frac{A}{2} = \frac{3b}{2}$ then prove that a , b ,c are in A.P.

19) Show that $2\sin^{-1}\frac{3}{5} = \tan^{-1}\frac{24}{7}$

20) Show that $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

21) Prove that $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right) \text{ if } x \in [0,1]$

22) Show that $\frac{9\pi}{5} - \frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

23) Show that $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x, \text{ for } -\frac{1}{\sqrt{2}} \le x \le 1$

24) If $\sin(\sin^{-1}\frac{1}{5} + \cos^{-1}x) = 1$ then find the value of x.

25) If $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$ then find the value of x .

26) If $2\tan^{-1}(\cos x) = \tan^{-1}(\operatorname{cosec} x)$ then find the value of x.

27) Solve: $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x, \text{for } x > 0$

28) If $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$ then find the value of x .

29) If $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{2}$ then find the value of x.

30) Show that $\tan^{-1}\frac{1}{2} - \tan^{-1}\frac{1}{4} = \tan^{-1}\frac{2}{9}$

31) Show that $\cot^{-1}\frac{1}{3} - \tan^{-1}\frac{1}{3} = \cot^{-1}\frac{3}{4}$

32) Show that $\tan^{-1}\frac{1}{2} = \frac{1}{3}\tan^{-1}\frac{11}{2}$

33) Show that $\cos^{-1}\frac{\sqrt{3}}{2}+2\sin^{-1}\frac{\sqrt{3}}{2}=\frac{5\pi}{6}$

34) Show that $2\cot^{-1}\frac{3}{2}+\sec^{-1}\frac{13}{12}=\frac{\pi}{2}$

35) Prove the following :

(i) $\cos^{-1} x = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $x < 0$. (ii) $\cos^{-1} x = \pi + \tan^{-1}\frac{\sqrt{1-x^2}}{x}$ if $x < 0$.

36) If $|x| < 1$, then prove that $2\tan^{-1}x=\tan^{-1}\frac{2x}{1+x^2}=\sin^{-1}\frac{2x}{1+x^2}=\cos^{-1}\frac{1-x^2}{1+x^2}$

37) If x, y, z, are positive then prove that $\tan^{-1}\frac{x-y}{1+xy}+\tan^{-1}\frac{y-z}{1+yz}+\tan^{-1}\frac{z-x}{1+zx}=0$

38) If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$ then, show that $xy + yx + zx = 1$

39) If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$ then show that $x^2 + y^2 + z^2 + 2xyz = 1$.

✦✦✦✦✦

4 Pair of Straight Lines

Let's Study

4.1 INTRODUCTION

We know that equation $ax + by + c = 0$, where $a, b, c, \in, R$, (a and b not zero simultaneously), represents a line in XY plane. We are familiar with different forms of equations of line. Now let's study two lines simultaneously. For this we need the concept of the combined equation of two lines.

Let's learn.

4.1 Combined equation of a pair of lines :

An equation which represents two lines is called the combined equation of those two lines. Let $u \equiv a_1x+b_1 y + c_1$ and $v \equiv a_2x+b_2 y + c_2$. Equation $u = 0$ and $v = 0$ represent lines. We know that equation $u + kv = 0, k \in R$ represents a family of lines. Let us interpret the equation $uv = 0$.

Theorem 4.1:

The equation $uv = 0$ represents,the combined equation of lines $u = 0$ and $v = 0$

Proof : Consider the lines represented by $u = 0$ and $v = 0$

$\therefore \quad a_1x + b_1 y + c_1 = 0$ and $a_2x + b_2 y + c_2 = 0$.

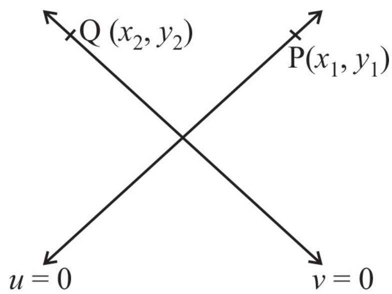


Figure 4.1

Let $P(x_1, y_1)$ be a point on the line $u = 0$.

$\therefore \quad (x_1, y_1)$ satisfy the equation $a_1x + b_1 y + c_1 = 0$

$\therefore \quad a_1x_1 + b_1 y_1 + c_1 = 0$

To show that (x_1, y_1) satisfy the equation $uv = 0$.

$(a_1x_1 + b_1 y_1 + c_1) (a_2x_1 + b_2 y_1 + c_2)$

$= 0 (a_2x_1 + b_2 y_1 + c_2)$

$= 0$

Therefore (x_1, y_1) satisfy the equation $uv = 0$.

This proves that every point on the line $u = 0$ satisfy the equation $uv = 0$.

Similarly we can prove that every point on the line $v = 0$ satisfies the equation $uv = 0$.

Now let $R(x',y')$ be any point which satisfy the equation $uv = 0$.

$\therefore \quad (a_1x' + b_1 y' + c_1) (a_2x' + b_2 y' + c_2) = 0$

$\therefore \quad (a_1x' + b_1y' + c_1) = 0$ or $(a_2x' + b_2y' + c_2) = 0$

Therefore R(x', y') lies on the line $u = 0$ or $v = 0$.

Every points which satisfy the equation $uv = 0$ lies on the line $u = 0$ or the line $v = 0$.

Therefore equation $uv = 0$ represents the combined equation of lines $u = 0$ and $v = 0$.

Remark :

1) The combined equation of a pair of lines is also called as the joint equation of a pair of lines.

2) Equations $u = 0$ *and* $v = 0$ are called separate equations of lines represented by $uv = 0$.

Solved Examples :

Ex. 1) Find the combined equation of lines $x + y - 2 = 0$ and $2x - y + 2 = 0$

Solution : The combined equation of lines $u = 0$ and $v = 0$ is $uv = 0$

$\therefore$ The combined equation of lines $x + y - 2 = 0$ and $2x - y + 2 = 0$ is

$(x + y - 2)(2x - y + 2) = 0$

$\therefore \; x(2x - y + 2) + y(2x - y + 2) - 2(2x - y + 2) = 0$

$\therefore \; 2x^2 - xy + 2x + 2xy - y^2 + 2y - 4x + 2y - 4 = 0$

$\therefore \; 2x^2 + xy - y^2 - 2x + 4y - 4 = 0$

Ex. 2) Find the combined equation of lines $x - 2 = 0$ and $y + 2 = 0$.

Solution : The combined equation of lines $u = 0$ and $v = 0$ is $uv = 0$.

$\therefore$ The combined equation of lines $x - 2 = 0$ and $y + 2 = 0$ is

$(x - 2)(y + 2) = 0$

$\therefore \; xy + 2x - 2y - 4 = 0$

Ex. 3) Find the combined equation of lines $x - 2y = 0$ and $x + y = 0$.

Solution : The combined equation of lines $u = 0$ and $v = 0$ is $uv = 0$.

$\therefore$ The combined equation of lines $x - 2y = 0$ and $x + y = 0$ is

$(x - 2y)(x + y) = 0$

$\therefore \; x^2 - xy - 2y^2 = 0$

Ex. 4) Find separate equation of lines represented by $x^2 - y^2 + x + y = 0$.

Solution : We factorize equation $x^2 - y^2 + x + y = 0$ as

$(x + y)(x - y) + (x + y) = 0$

$\therefore \; (x + y)(x - y + 1) = 0$

Required separate equations are $x + y = 0$ and $x - y + 1 = 0$.

4.2 Homogeneous equation of degree two:

4.2.1 Degree of a term:

Definition: The sum of the indices of all variables in a term is called the degree of the term.

For example, in the expression $x^2 + 3xy - 2y^2 + 5x + 2$ the degree of the term x^2 is two, the degree of the term $3xy$ is two, the degree of the term $-2y^2$ is two, the degree of $5x$ is one. The degree of constant term 2 is zero. Degree of '0' is not defined.

4.2.2 Homogeneous Equation :

Definition: An equation in which the degree of every term is same, is called a homogeneous equation.

For example: $x^2 + 3xy = 0$, $7xy - 2y^2 = 0$, $5x^2 + 3xy - 2y^2 = 0$ are homogeneous equations.

But $3x^2 + 2xy + 2y^2 + 5x = 0$ is not a homogeneous equation.

Homogeneous equation of degree two in x and y has form $ax^2 + 2hxy + by^2 = 0$.

Theorem 4.2 :

The combined equation of a pair of lines passing through the origin is a homogeneous equation of degree two in x and y.

Proof : Let $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$ be any two lines passing through the origin.

Their combined equation is $(a_1x + b_1y)(a_2x + b_2y) = 0$

$a_1a_2x^2 + a_1b_2xy + a_2b_1xy + b_1b_2y^2 = 0$

$(a_1a_2)x^2 + (a_1b_2 + a_2b_1)xy + (b_1b_2)y^2 = 0$

In this if we put $a_1a_2 = a, a_1b_2 + a_2b_1 = 2h$, $b_1b_2 = b$, we get, $ax^2 + 2hxy + by^2 = 0$, which is a homogeneous equation of degree two in x and y.

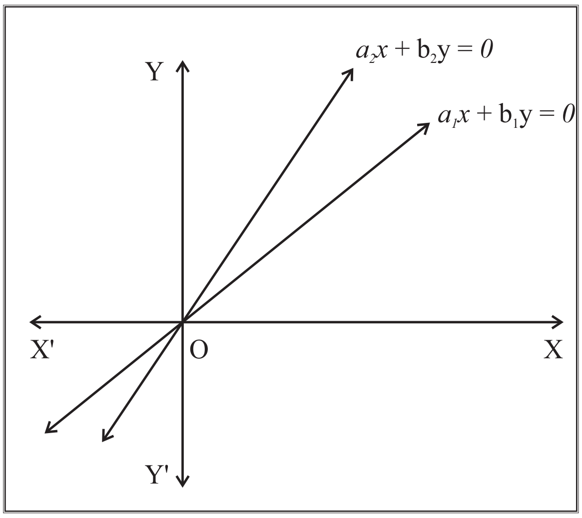


Figure 4.2

Ex.1) Verify that the combined equation of lines $2x + 3y = 0$ and $x - 2y = 0$ is a homogeneous equation of degree two.

Solution :

The combined equation of lines $u = 0$ and $v = 0$ is $uv = 0$.

$\therefore$ The combined equation of lines $2x + 3y = 0$ and $x - 2y = 0$ is

$(2x + 3y)(x - 2y) = 0$

$2x^2 - xy - 6y^2 = 0$, which is a homogeneous equation of degree two.

Remark :

The combined equation of a pair of lines passing through the origin is a homogeneous equation of degree two. But every homogeneous equation of degree two *need not* represent a pair of lines.

Equation $x^2 + y^2 = 0$ is a homogeneous equation of degree two but it does not represent a pair of lines.

How to test whether given homogeneous equation of degree two represents a pair of lines or not?

Let's have a theorem.

Theorem 3 : Homogenous equation of degree two in x and y, $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through the origin if $h^2 - ab \geq 0$.

Proof : Consider the homogeneous equation of degree two in x and y, $ax^2 + 2hxy + by^2 = 0$ ····· (1)

Consider two cases $b = 0$ and $b \neq 0$. These two cases are exhaustive.

Case 1: If $b = 0$ then equation (1) becomes $ax^2 + 2hxy = 0$

$\therefore$ $x(ax + 2hy) = 0$, which is the combined equation of lines

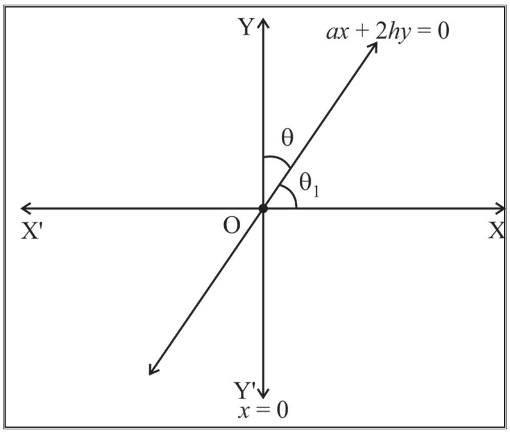


Figure 4.3

$x = 0$ and $ax + 2hy = 0$.

We observe that these lines pass through the origin.

Case 2: If $b \neq 0$ then we multiply equation (1) by b.

$abx^2 + 2hbxy + b^2y^2 = 0$

$b^2y^2 + 2hbxy = -abx^2$

To make L.H.S. complete square we add h^2x^2 to both sides.

$b^2y^2 + 2hbxy + h^2x^2 = h^2x^2 - abx^2$

$(by + hx)^2 = (h^2 - ab)x^2$

$(by + hx)^2 = \left(\sqrt{h^2 - ab}\right)^2 x^2$, as $h^2 - ab \geq 0$

$(by + hx)^2 - \left(\sqrt{h^2 - ab}\right)^2 x^2 = 0$

$(by + hx + \sqrt{h^2 - ab}\,x)(by + hx - \sqrt{h^2 - ab}\,x) = 0$

$[(h + \sqrt{h^2 - ab})x + by] \times [(h - \sqrt{h^2 - ab})x + by] = 0$

Which is the combined equation of lines $(h + \sqrt{h^2 - ab})x + by = 0$ and $(h - \sqrt{h^2 - ab})x + by = 0$.

As $b \neq 0$, we can write these equations in the form $y = m_1 x$ and $y = m_2 x$, where $m_1 = \dfrac{-h - \sqrt{h^2 - ab}}{b}$ and $m_2 = \dfrac{-h + \sqrt{h^2 - ab}}{b}$.

We observe that these lines pass through the origin.

Therefore equation $abx^2 + 2hbxy + by^2 = 0$ represents a pair of lines passing through the origin if $h^2 - ab \geq 0$.

Remarks:

1) If $h^2 - ab > 0$ then line represented by (1) are distinct.
2) If $h^2 - ab = 0$ then lines represented by (1) are coincident.
3) If $h^2 - ab < 0$ then equation (1) does not represent a pair of lines.
4) If $b = 0$ then one of the lines is the Y - axis, whose slope is not defined and the slope of the other line is $-\dfrac{a}{2h}$ (provided that $h \neq 0$).
5) If $h^2 - ab \geq 0$ and $b \neq 0$ then slopes of the lines are $m_1 = \dfrac{-h - \sqrt{h^2 - ab}}{b}$ and $m_2 = \dfrac{-h + \sqrt{h^2 - ab}}{b}$

   Their sum is $m_1 + m_2 = -\dfrac{2h}{b}$ and product is $m_1 m_2 = \dfrac{a}{b}$

   The quadratic equation in m whose roots are m_1 and m_2 is given by

   $m^2 - (m_1 + m_2)\, m + m_2 = 0$

   $\therefore\ m^2 - \left(-\dfrac{2h}{b}\right) m + \dfrac{a}{b} = 0$

$$bm^2 + 2hm + a = 0 \qquad \ldots\ldots (2)$$

Equation (2) is called the **auxiliary** equation of equation (1). Roots of equation (2) are slopes of lines represented by equation (1).

Solved Examples

Ex. 1) Show that lines represented by equation $x^2 - 2xy - 3y^2 = 0$ are distinct.

Solution : Comparing equation $x^2 - 2xy - 3y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get

$a = 1, h = -1$ and $b = -3$.

$$h^2 - ab = (-1)^2 - (1)(-3)$$
$$= 1 + 3$$
$$= 4 > 0$$

As $h^2 - ab > 0$, lines represented by equation $x^2 - 2xy - 3y^2 = 0$ are distinct.

Ex. 2) Show that lines represented by equation $x^2 - 6xy + 9y^2 = 0$ are coincident.

Solution : Comparing equation $x^2 - 6xy + 9y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get

$a = 1$, $h = -3$ and $b = 9$.

$$h^2 - ab = (-3)^2 - (1)(9)$$
$$= 9 - 9 = 0$$

As $h^2 - ab > 0$, lines represented by equation $x^2 - 6xy + 9y^2 = 0$ are coincident.

Ex. 3) Find the sum and the product of slopes of lines represented by $x^2 + 4xy - 7y^2 = 0$.

Solution : Comparing equation $x^2 + 4xy - 7y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get $a = 1$, $h = 2$ and $b = -7$.

If m_1 and m_2 are slopes of lines represented by this equation then

$$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}.$$

$$\therefore \ m_1 + m_2 = \frac{-4}{-7} = \frac{4}{7} \text{ and } m_1 m_2 = \frac{1}{-7} = -\frac{1}{7}$$

Their sum is $\frac{4}{7}$ and products is $-\frac{1}{7}$

Ex. 4) Find the separate equations of lines represented by

i) $x^2 - 4y^2 = 0$

ii) $3x^2 - 7xy + 4y^2 = 0$

iii) $x^2 + 2xy - y^2 = 0$

iv) $5x^2 - 3y^2 = 0$

Solution : i) $x^2 - 4y^2 = 0$

$\therefore \ (x - 2y)(x + 2y) = 0$

Required separate equations are

$x - 2y = 0$ and $x + 2y = 0$

ii) $3x^2 - 7xy + 4y^2 = 0$

$\therefore\ 3x^2 - 3xy - 4xy + 4y^2 = 0$

$\therefore\ 3x(x - y) - 4y(x - y) = 0$

$\therefore\ (x - y)(3x - 4y) = 0$

Required separate equations are

$x - y = 0$ and $3x - 4y = 0$

iii) $x^2 + 2xy - y^2 = 0$

The corresponding auxiliary equation is $bm^2 + 2hm + a = 0$

$\therefore\ -m^2 + 2m + 1 = 0$

$m^2 - 2m - 1 = 0$

$\therefore\ m = \dfrac{2 \pm \sqrt{8}}{2} = \dfrac{2 \pm 2\sqrt{2}}{2}$

$= 1 \pm \sqrt{2}$

Slopes of these lines are

$m_1 = 1 + \sqrt{2}$ and $m_2 = 1 - \sqrt{2}$

$\therefore$ Required separate equations are

$y = m_1 x$ and $y = m_2 x$

$\therefore\ y = (1 + \sqrt{2})x$ and $y = (1 - \sqrt{2})x$

$\therefore\ (1 + \sqrt{2})x - y = 0$ and $(1 - \sqrt{2})x - y = 0$

iv) $5x^2 - 3y^2 = 0$

$\therefore\ (\sqrt{5}x)^2 - (\sqrt{3}y)^2 = 0$

$\therefore\ (\sqrt{5}x - \sqrt{3}y)(\sqrt{5}x + \sqrt{3}y) = 0$

$\therefore$ Required separate equations are

$\sqrt{5}x - \sqrt{3}y = 0$ and $\sqrt{5}x + \sqrt{3}y = 0$

Ex. 5) Find the value of k if $2x + y = 0$ is one of the lines represented by $3x^2 + kxy + 2y^2 = 0$.

Solution : Slope of the line $2x + y = 0$ is -2

As $2x + y = 0$ is one of the lines represented by $3x^2 + kxy + 2y^2 = 0$, -2 is a root of the auxiliary equation $2m^2 + km + 3 = 0$

$\therefore\ 2(-2)^2 + k(-2) + 3 = 0$

$\therefore\ 8 - 2k + 3 = 0$

$\therefore\ -2k + 11 = 0$

$\therefore\ 2k = 11 \qquad \therefore\ k = \dfrac{11}{2}$.

Alternative Method : As $2x + y = 0$ is one of the lines represented by $3x^2 + kxy + 2y^2 = 0$, co-ordinates of every point on the line $2x + y = 0$ satisfy the equation $3x^2 + kxy + 2y^2 = 0$.

As $(1, -2)$ is a point on the line $2x + y = 0$, it must satisfy the combined equation.

$\therefore\ 3(1)^2 + k(1)(-2) + 2(-2)^2 = 0$

$\therefore\ -2k + 11 = 0$

$\therefore\ 2k = 11 \qquad \therefore\ k = \dfrac{11}{2}$.

Ex. 6) Find the condition that the line $3x - 2y = 0$ coincides with one of the lines represented by $ax^2 + 2hxy + by^2 = 0$.

Solution : The corresponding auxiliary equation is $bm^2 + 2hm + a = 0$.

As line $3x - 2y = 0$ coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$, its slope $\frac{3}{2}$ is a root of the auxiliary equation.

$\therefore \frac{3}{2}$ is a root of $bm^2 + 2hm + a = 0$

$$\therefore b\left(\frac{3}{2}\right)^2 + 2h\left(\frac{3}{2}\right) + a = 0$$

$$\therefore \frac{9}{4}b + 3h + a = 0$$

$\therefore 4a + 12h + 9b = 0$ is the required condition.

Ex.7) Find the combined equation of the pair of lines passing through the origin and perpendicular to the lines represented by $3x^2 + 2xy - y^2 = 0$.

Solution : Let m_1 and m_2 are slopes of lines represented by $3x^2 + 2xy - y^2 = 0$.

$$\therefore m_1 + m_2 = -\frac{2h}{b} = \frac{-2}{-1} = 2$$

$$\text{and } m_1 m_2 = \frac{a}{b} = \frac{3}{-1} = -3$$

Now required lines are perpendicular to given lines.

$\therefore$ Their slopes are $-\frac{1}{m_1}$ and $-\frac{1}{m_2}$

And required lines pass through the origin.

$\therefore$ Their equations are $y = -\frac{1}{m_1}x$ and $y = -\frac{1}{m_1}x$

$\therefore m_1 y = -x$ and $m_2 y = -x$

$\therefore x + m_1 y = 0$ and $x + m_2 y = 0$

Their combined equation is $(x + m_1 y)(x + m_2 y) = 0$

$\therefore x^2 + (m_1 + m_2)xy + m_1 m_2 y^2 = 0$

$\therefore x^2 + (2)xy + (-3)y^2 = 0$

$\therefore x^2 + 2xy - 3y^2 = 0$

Ex.8) Find the value of k, if slope of one of the lines represented by $4x^2 + kxy + y^2 = 0$ is four times the slope of the other line.

Solution : Let slopes of the lines represented by $4x^2 + kxy + y^2 = 0$ be m and $4m$,

their sum is $m + 4m = 5m$

But their sum is $\frac{-2h}{b} = \frac{-k}{1} = -k$

$\therefore 5m = -k$

$$\therefore m = \frac{-k}{5} \quad \dots (1)$$

Now their product is $(m)(4m) = 4m^2$

But their product is $\frac{a}{b} = \frac{4}{1} = 4$

$\therefore \; 4m^2 = 4$

$\therefore \; m^2 = 1$..... (2)

From (1) and (2), we get

$$\left(\frac{-k}{5}\right)^2 = 1$$

$\therefore \; k^2 = 25$

$\therefore \; k = \pm 5$

Exercise 4.1

1) Find the combined equation of the following pairs of lines:

i) $2x + y = 0$ and $3x - y = 0$

ii) $x + 2y - 1 = 0$ and $x - 3y + 2 = 0$

iii) Passing through (2,3) and parallel to the co-ordinate axes.

iv) Passing through (2,3) and perpendicular to lines $3x + 2y - 1 = 0$ and $x - 3y + 2 = 0$

v) Passsing through $(-1,2)$,one is parallel to $x + 3y - 1 = 0$ and the other is perpendicular to $2x - 3y - 1 = 0$.

2) Find the separate equations of the lines represented by following equations:

i) $3y^2 + 7xy = 0$

ii) $5x^2 - 9y^2 = 0$

iii) $x^2 - 4xy = 0$

iv) $3x^2 - 10xy - 8y^2 = 0$

v) $3x^2 - 2\sqrt{3}\,xy - 3y^2 = 0$

vi) $x^2 + 2(\operatorname{cosec} \alpha)xy + y^2 = 0$

vii) $x^2 + 2xy \tan \alpha - y^2 = 0$

3) Find the combined equation of a pair of lines passing through the origin and perpendicular to the lines represented by following equations :

i) $5x^2 - 8xy + 3y^2 = 0$

ii) $5x^2 + 2xy - 3y^2 = 0$

iii) $xy + y^2 = 0$

iv) $3x^2 - 4xy = 0$

4) Find k if,

i) the sum of the slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$ is twice their product.

ii) slopes of lines represent by $3x^2 + kxy - y^2 = 0$ differ by 4.

iii) slope of one of the lines given by $kx^2 + 4xy - y^2 = 0$ exceeds the slope of the other by 8.

5) Find the condition that :

i) the line $4x + 5y = 0$ coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$.

ii) the line $3x + y = 0$ may be perpendicular to one of the lines given by $ax^2 + 2hxy + by^2 = 0$.

6) If one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is perpendicular to $px + qy = 0$ then show that $ap^2 + 2hpq + by^2 = 0$.

7) Find the combined equation of the pair of lines passing through the origin and making an equilateral triangle with the line y = 3.

8) If slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is four times the other then show that $16h^2 = 25ab$.

9) If one of the lines given by $ax^2 + 2hxy + by^2 = 0$ bisects an angle between co-ordinate axes then show that $(a + b)^2 = 4h^2$.

4.3 Angle between lines represented by $ax^2 + 2hxy + by^2 = 0$:

If we know slope of a line then we can find the angles made by the line with the co-ordinate axes. In equation $ax^2 + 2hxy + by^2 = 0$ if $b = 0$ then one of the lines is the Y - axis. Using the slope of the other line we can find the angle between them. In the following discussion we assume that $b \neq 0$, so that slopes of both lines will be defined.

If m_1 and m_2 are slopes of these lines then $m_1 m_2 = \frac{a}{b}$

We know that lines having slopes m_1 and m_2 are perpendicular to each other if and only if $m_1 m_2 = -1$.

$$\therefore \frac{a}{b} = -1$$

$$\therefore a = -b$$

$$\therefore a + b = 0$$

Thus lines represented by $ax^2 + 2hxy + by^2 = 0$ are perpendicular to each other if and only if a + b = 0.

If lines are **not perpendicular** to each other then the acute angle between them can be obtained by using the following theorem.

Theorem 4.4 : The acute angle θ between the lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by

$$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

Proof : Let m_1 and m_2 be slopes of lines represented by the equation $ax^2 + 2hxy + by^2 = 0$.

$$\therefore m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b}$$

$$\therefore (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$$

$$= \left(\frac{2h}{b}\right)^2 - 4\left(\frac{a}{b}\right)$$

$$= \frac{4h^2}{b^2} - \frac{4ab}{b^2}$$

$$= \frac{4h^2 - 4ab}{b^2}$$

$$= \frac{4(h^2 - ab)}{b^2}$$

$$\therefore m_1 - m_2 = \pm \frac{2\sqrt{h^2 - ab}}{b}$$

As θ is the acute angle between the lines,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$= \left|\frac{\pm \frac{2\sqrt{h^2 - ab}}{b}}{1 + \frac{a}{b}}\right|$$

$$= \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

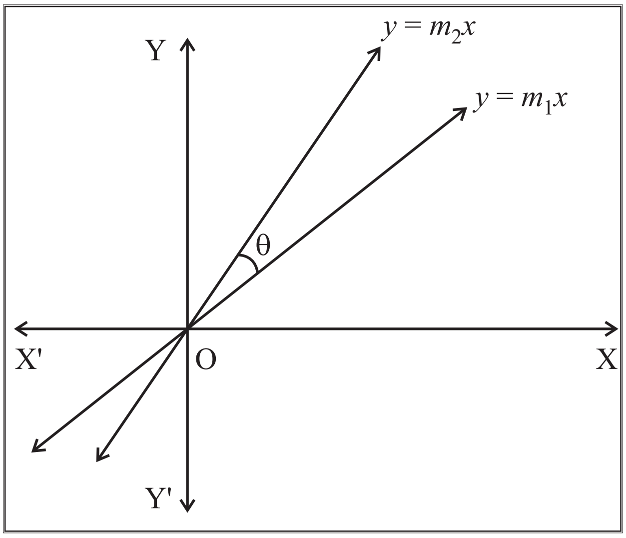


Figure 4.4

Remark : Lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident if and only if $m_1 = m_2$

$\therefore \quad m_1 - m_2 = 0$

$\therefore \quad \frac{2\sqrt{h^2 - ab}}{b} = 0$

$\therefore h^2 - ab = 0$

$\therefore h^2 = ab$

Lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident if and only if $h^2 = ab$.

Solved Examples

Ex.1) Show that lines represented by $3x^2 - 4xy - 3y^2 = 0$ are perpendicular to each other.

Solution : Comparing given equation with $ax^2 + 2hxy + by^2 = 0$ we get $a = 3$, $h = -2$ and $b = -3$.

As $a + b = 3 + (-3) = 0$, lines represented by $3x^2 - 4xy - 3y^2 = 0$ are perpendicular to each other.

Ex. 2) Show that lines represented by $x^2 + 4xy + 4y^2 = 0$ are coincident.

Solution : Comparing given equation with $ax^2 + 2hxy + by^2 = 0$, we get $a = 1$, $h = 2$ and $b = 4$.

As, $h^2 - ab = (2)^2 - (1)(4)$

$= 4 - 4 = 0$

$\therefore$ Lines represented by $x^2 + 4xy + 4y^2 = 0$ are coincident.

Ex.3) Find the acute angle between lines represented by:

i) $x^2 + xy = 0$

ii) $x^2 - 4xy + y^2 = 0$

iii) $3x^2 + 2xy - y^2 = 0$

iv) $2x^2 - 6xy + y^2 = 0$

v) $xy + y^2 = 0$

Solution :

i) Comparing equation $x^2 + xy = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get $a = 1$, $h = \frac{1}{2}$ and $b = 0$.

Let θ be the acute angle between them.

$$\therefore \quad \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right| = \left|\frac{2\sqrt{\frac{1}{4} - 0}}{1}\right| = 1$$

$$\therefore \quad \theta = 45^\circ = \frac{\pi}{4}$$

ii) Comparing equation $x^2 - 4xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get $a = 1$, $h = -2$ and $b = 1$.

Let θ be the acute angle between them.

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$$= \left|\frac{2\sqrt{4-1}}{2}\right| = \sqrt{3}$$

$$\therefore \theta = 60^\circ = \frac{\pi}{3}$$

iii) Comparing equation $3x^2 + 2xy - y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get $a = 3$, $h = 1$ and $b = -1$.

Let θ be the acute angle between them.

$$\therefore \quad \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$$= \left|\frac{2\sqrt{1+3}}{2}\right| = 2.$$

$$\therefore \quad \theta = \tan^{-1}(2)$$

iv) Comparing equation $2x^2 - 6xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get $a = 2$, $h = -3$ and $b = 1$.

Let θ be the acute angle between them.

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$$= \left|\frac{2\sqrt{9-2}}{3}\right| = \frac{2\sqrt{7}}{3} \quad \therefore \theta = \tan^{-1}\left(\frac{2\sqrt{7}}{3}\right).$$

v) Comparing equation $xy + y^2 = 0$ with $ax^2 + 2hxy + by^2 = 0$, we get $a = 0$, $h = \frac{1}{2}$ and $b = 1$.

Let θ be the acute angle between them.

$$\therefore \tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a+b}\right|$$

$$= \left|\frac{2\sqrt{\frac{1}{4} - 0}}{1}\right| = 1$$

$$\therefore \theta = 45° = \frac{\pi}{4}.$$

Ex.4) Find the combined equation of lines passing through the origin and making angle $\frac{\pi}{6}$ with the line $3x + y - 6 = 0$.

Solution : Let m be the slope of one of the lines which make angle $\frac{\pi}{6}$ with the line $3x+y-6 = 0$. Slope of the given line is -3.

$$\therefore \tan\frac{\pi}{6} = \left|\frac{m-(-3)}{1+m(-3)}\right|$$

$$\therefore \frac{1}{\sqrt{3}} = \left|\frac{m+3}{1-3m}\right|$$

$\therefore (1 - 3m)^2 = 3(m + 3)^2$

$\therefore 9m^2 - 6m + 1 = 3(m^2 + 6m + 9)$

$\therefore 9m^2 - 6m + 1 = 3m^2 + 18m + 27$

$\therefore 6m^2 - 24m - 26 = 0$

$\therefore 3m^2 - 12m - 13 = 0$

This is the auxiliary equation of the required combined equation.

The required combined equation is $-13x^2 - 12xy + 3y^2 = 0$

$\therefore 13x^2 + 12xy - 3y^2 = 0$

Ex. 5) Find the combined equation of lines passing through the origin and each of which making angle 60° with the X - axis.

Solution :

Let m be the slope of one of the required lines.

The slope of the X - axis is 0. As required lines make angle 60° with the X - axis,

$$\tan 60° = \left|\frac{m-0}{1+(m)(0)}\right|$$

$\therefore \sqrt{3} = |m|$

$\therefore m^2 = 3$

$\therefore m^2 + 0m - 3 = 0$ is the auxiliary equation.

$\therefore$ The required combined equation is

$-3x^2 + 0xy + y^2 = 0$

$\therefore\ 3x^2 - y^2 = 0$

Alternative Method: As required lines make angle 60° with the X - axis, their inclination are 60° and 120°. Hence their slopes are $\sqrt{3}$ and $-\sqrt{3}$.

Lines pass thorugh the origin. Their equations are $y = \sqrt{3}\,x$ and $y = -\sqrt{3}\,x$

$\therefore\ \sqrt{3}\,x - y = 0$ and $\sqrt{3}\,x + y = 0$

Their combined equation is $(\sqrt{3}\,x - y)(\sqrt{3}\,x + y) = 0$

$\therefore\ 3x^2 - y^2 = 0$

Exercise 4.2

1) Show that lines represented by $3x^2 - 4xy - 3y^2 = 0$ are perpendicular to each other.
2) Show that lines represented by $x^2 + 6xy + gy^2 = 0$ are coincident.
3) Find the value of k if lines represented by $kx^2 + 4xy - 4y^2 = 0$ are perpendicular to each other.
4) Find the measure of the acute angle between the lines represented by:
   i) $3x^2 - 4\sqrt{3}\,xy + 3y^2 = 0$
   ii) $4x^2 + 5xy + 4y^2 = 0$
   iii) $2x^2 + 7xy + 3y^2 = 0$
   iv) $(a^2 - 3b^2)x^2 + 8abxy + (b^2 - 3a^2)y^2 = 0$
5) Find the combined equation of lines passing through the origin each of which making an angle of 30° with the line $3x + 2y - 11 = 0$
6) If the angle between lines represented by $ax^2 + 2hxy + by^2 = 0$ is equal to the angle between lines represented by $2x^2 - 5xy + 3y^2 = 0$ then show that $100(h^2 - ab) = (a + b)^2$.
7) Find the combined equation of lines passing through the origin and each of which making angle 60° with the Y- axis.

4.4 General Second Degree Equation in x and y:

Equation of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, where at least one of a,b,h is not zero, is called a general second degree equation in x and y.

Theorem 4.5 : The combined equation of two lines is a general second degree equation in x and y.

Proof: Let $u \equiv a_1x + b_1y + c_1$ and $v \equiv a_2x + b_2y + c_2$. Equations $u = 0$ and $v = 0$ represent lines. Their combined equation is $uv = 0$.

$\therefore\ (a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$

$a_1a_2x^2 + a_1b_2xy + a_1c_2x + b_1a_2xy + b_1b_2y^2 + b_1c_2y + c_1a_2x + c_1b_2y + c_1c_2 = 0$

Writing $a_1a_2 = a,\ b_1b_2 = b,\ a_1b_2 + a_2b_1 = 2h,\ a_1c_2 + a_2c_1 = 2g,\ b_1c_2 + b_2c_1 = 2f,\ c_1c_2 = c$,

we get, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, which is the general equation of degree two in x and y.

Remark : The converse of the above theorem is **not true**. Every general second degree equation in x and y **need not** represent a pair of lines. For example $x^2 + y^2 = 25$ is a general second degree equation in x and y but it does not represent a pair of lines. It represents a circle.

Equation $x^2+y^2-4x+6y+13=0$ is also a general second degree equation which does not represent a pair of lines. How to identify that whether the given equation represents a pair of lines or not?

4.4.1 The necessary conditions for a general second degree equation.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of lines are:

i) $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ ii) $h^2 - ab \geq 0$

Remarks :

If equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines then

1) These lines are parallel to the line represented by $ax^2 + 2hxy + by^2 = 0$

2) The acute angle between them is given by $\tan\theta = \left|\dfrac{2\sqrt{h^2-ab}}{a+b}\right|$

3) Condition for lines to be perpendicular to each other is $a + b = 0$.

4) Condition for lines to be parallel to each other is $h^2 - ab = 0$.

5) Condition for lines to intersect each other is $h^2 - ab > 0$ and the co-ordinates of their point of intersection are $\left(\dfrac{hf-bg}{ab-h^2}, \dfrac{gh-af}{ab-h^2}\right)$

6) The expression $abc + 2fgh - af^2 - bg^2 - ch^2$ is the expansion of the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

7) The joint equation of the bisector of the angle between the lines represented by $ax^2+2hxy+by^2=0$ is $hx^2-(a-b)xy-hy^2=0$. Here coefficient of x^2 + coefficient of $y^2 = 0$. Hence bisectors are perpendicular to each other

Solved Examples

Ex.1) Show that equation $x^2 - 6xy + 5y^2 + 10x - 14y + 9 = 0$ represents a pair of lines. Find the acute angle between them. Also find the point of their intersection.

Solution: We have $x^2 - 6xy + 5y^2 = (x-5y)(x-y)$

Suppose $x^2 - 6xy + 5y^2 + 10x - 14y + 9 = (x - 5y + c)(x - y + k)$

$\therefore x^2 - 6xy + 5y^2 + 10x - 14y + 9 = x^2 - 6xy + 5y^2 + (c+k)x - (c+5k)y + ck$

$\therefore c + k = 10$, $c + 5k = 14$ and $ck = 9$

We observe that $c = 9$ and $k = 1$ satisfy all three equations.

$\therefore$ Given general equation can be factorized as $(x - 5y + 9)(x - y + 1) = 0$

$\therefore$ Given equation represents a pair of intersecting lines.

The acute angle between them is given by

$$\tan\theta = \left|\frac{2\sqrt{h^2-ab}}{a+b}\right| = \left|\frac{2\sqrt{(-3)^2-(1)(5)}}{1+5}\right| = \frac{2}{3}$$

$$\therefore \theta = \tan^{-1}\left(\frac{2}{3}\right)$$

Their point of intersection is given by

$$\left(\frac{hf-bg}{ab-h^2},\frac{gh-af}{ab-h^2}\right)=\left(\frac{21-25}{5-9},\frac{-15+7}{5-9}\right)=(1,2)$$

Remark :

Note that condition $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ is not sufficient for equation to represent a pair of lines. We can't use this condition to show that given equation represents a pair of lines.

Ex.2) Find the value of k if the equation $2x^2 + 4xy - 2y^2 + 4x + 8y + k = 0$ represents a pair of lines.

Solution: Comparing given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get

$a = 2, b = -2, c = k, f = 4, g = 2, h = 2.$

As given equation represents a pair of lines, it must satisfy the necessary condition.

$\therefore\ abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$\therefore\ (2)(-2)(k) + 2(4)(2)(2) - 2(4)^2 - (-2)(2)^2 - (k)(2)^2 = 0$

$\therefore -4k + 32 - 32 + 8 - 4k = 0$

$\therefore\ 8k = 8$

$\therefore\ k = 1.$

Ex.3) Find p and q if the equatoin $2x^2 + 4xy - py^2 + 4x + qy + 1 = 0$ represents a pair of prependicular lines.

Solution: Comparing given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ we get

$a = 2, b = -p, c = 1, f = \frac{q}{2}, g = 2, h = 2$

As lines are perpendicular to each other, a + b= 0

$\therefore\ 2 + (-p) = 0$

$\therefore\ p = 2$

As given equation represents a pair of lines, it must satisfy the necessary condition.

$\therefore\ abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$\therefore\ (2)(-p)(1) + 2\left(\frac{q}{2}\right)(2)(2) - 2\left(\frac{q}{2}\right)^2 - (-p)(2)^2 - 1(2)^2 = 0$$

$$\therefore\ -2p + 4q - \frac{q^2}{2} + 4p - 4 = 0$$

$$\therefore\ 2p + 4q - \frac{q^2}{2} - 4 = 0 \quad ...(1)$$

substituting $p = 2$ in (1), we get

$$\therefore\ 2(2) + 4q - \frac{q^2}{2} - 4 = 0$$

$$\therefore\ 4q - \frac{q^2}{2} = 0 \ \therefore\ 8q - q^2 = 0$$

$\therefore\ q(8 - q) = 0$

$\therefore\ q = 0$ or $q = 8$

Ex.4) Δ OAB is formed by lines $x^2 - 4xy + y^2 = 0$ and the line $x + y - 2 = 0$. Find the equation of the median of the triangle drawn from O.

Solution : Let the co-ordinates of A and B be (x_1, y_1) and (x_2, y_2) respectively.

The midpoint of segment AB is

$P\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

The co-ordinates of A and B can be obtained by solving equations $x + y - 2 = 0$ and $x^2 - 4xy + y^2 = 0$ simultaneously.

put $y = 2 - x$ in $x^2 - 4xy + y^2 = 0$.

$x^2 - 4x(2 - x) + (2 - x)^2 = 0$

$\therefore 6x^2 - 12x + 4 = 0$

$\therefore 3x^2 - 6x + 2 = 0$

x_1 and x_2 are roots of this equation.

$x_1 + x_2 = -\frac{-6}{3} = 2$

$\therefore \frac{x_1+x_2}{2} = 1$

The x co-ordinate of P is 1.

As P lies on the line $x + y - 2 = 0$

$\therefore 1 + y - 2 = 0 \quad \therefore y = 1$

$\therefore$ Co-ordinates of P are (1,1).

The equation of the median OP is $\frac{y-0}{1-0} = \frac{x-0}{1-0}$

$\therefore y = x \quad \therefore x - y = 0.$

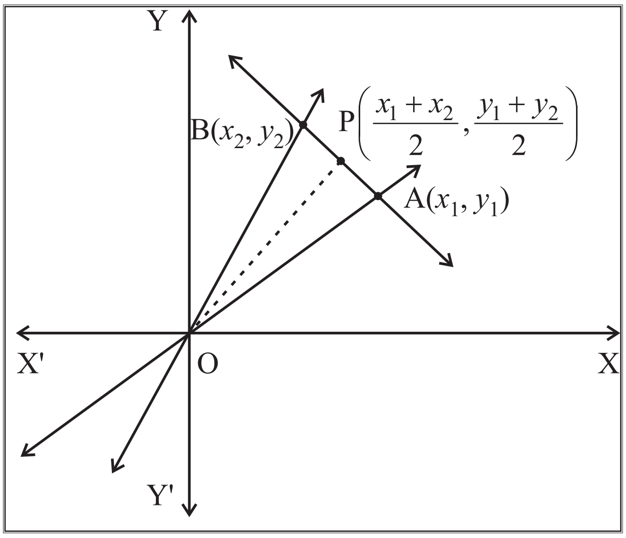


Figure 4.5

Exercise 4.3

1) Find the joint equation of the pair of lines:
   i) Through the point (2, –1) and parallel to lines represented by $2x^2 + 3xy - 9y^2 = 0$
   ii) Through the point (2, –3) and parallel to lines represented by $x^2 + xy - y^2 = 0$

2) Show that equation $x^2 + 2xy + 2y^2 + 2x + 2y + 1 = 0$ does not represent a pair of lines.

3) Show that equation $2x^2 - xy - 3y^2 - 6x + 19y - 20 = 0$ represents a pair of lines.

4) Show the equation $2x^2 + xy - y^2 + x + 4y - 3 = 0$ represents a pair of lines. Also find the acute angle between them.

5) Find the separate equation of the lines represented by the following equations :
   i) $(x - 2)^2 - 3(x - 2)(y + 1) + 2(y + 1)^2 = 0$
   ii) $10(x + 1)^2 + (x + 1)(y - 2) - 3(y - 2)^2 = 0$

6) Find the value of k if the following equations represent a pair of lines :
   i) $3x^2 + 10xy + 3y^2 + 16y + k = 0$
   ii) $kxy + 10x + 6y + 4 = 0$
   iii) $x^2 + 3xy + 2y^2 + x - y + k = 0$

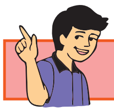
7) Find p and q if the equation $px^2 - 8xy + 3y^2 + 14x + 2y + q = 0$ represents a pair of perpendicular lines.

8) Find p and q if the equation $2x^2 + 8xy + py^2 + qx + 2y - 15 = 0$ represents a pair of parallel lines.

9) Equations of pairs of opposite sides of a parallelogram are $x^2 - 7x + 6 = 0$ and $y^2 - 14y + 40 = 0$. Find the joint equation of its diagonals.

10) ΔOAB is formed by lines $x^2 - 4xy + y^2 = 0$ and the line $2x + 3y - 1 = 0$. Find the equation of the median of the triangle drawn from O.

11) Find the co-ordinates of the points of intersection of the lines represented by $x^2 - y^2 - 2x + 1 = 0$.



Let's remember!

- An equation which represents two lines is called the combined equation of those two lines.
- The equation $uv = 0$ represents the combined equation of lines $u = 0$ and $v = 0$.
- The sum of the indices of all variables in a term is called the degree of the term.
- An equation in which the degree of every term is same, is called a homogeneous equation.
- The combined equation of a pair of lines passing through the origin is a homogeneous equation of degree two in x and y.
- Every homogeneous equation of degree two need not represents a pair of lines.
- A homogeneous equation of degree two in x and y, $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through the origin if $h^2 - ab \geq 0$.
- If $h^2 - ab > 0$ then lines are distinct.
- If $h^2 - ab = 0$ then lines are coincident.
- Slopes of these lines are $m_1 = \dfrac{-h - \sqrt{h^2 - ab}}{b}$ and $m_2 = \dfrac{-h + \sqrt{h^2 - ab}}{b}$
- Their sum is, $m_1 + m_2 = -\dfrac{2h}{b}$ and product is, $m_1 m_2 = \dfrac{a}{b}$
- The quadratic equation in m whose roots are m_1 and m_2 is given by $bm^2 + 2hm + a = 0$, called the **auxiliary** equation.
- Lines represented by $ax^2 + 2hxy + by^2 = 0$ are perpendicular to each other if and only if $a + b = 0$.
- Lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident if and only if $h^2 - ab = 0$.
- The acute angle θ between the lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by

  $$\tan\theta = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$
- Equation of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is called a general second degree equation in x and y.
- The combined equation of two lines is a general second degree equation in x and y.
- The necessary conditions for a general second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of lines are:
  i) $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ ii) $h^2 - ab \geq 0$

- The expression $abc + 2fgh - af^2 - bg^2 - ch^2$ is the expansion of the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

- If equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of lines then
  1) These lines are parallel to the lines represented by $ax^2 + 2hxy + by^2 = 0$.

  2) The acute angle between them is given by $\tan\theta = \left|\dfrac{2\sqrt{h^2 - ab}}{a+b}\right|$

  3) Condition for lines to be perpendicular to each other is $a + b = 0$.
  4) Condition for lines to be parallel to each other is $h^2 - ab = 0$.
  5) Condition for lines to intersect each other is $h^2 - ab \geq 0$ and the co-ordinates of their point of intersection are $\left(\dfrac{hf - bg}{ab - h^2}, \dfrac{gh - af}{ab - h^2}\right)$

MISCELLANEOUS EXERCISE 4

I : Choose correct alternatives.

1) If the equation $4x^2 + hxy + y^2 = 0$ represents two coincident lines, then $h =$ __________.

A) ± 2 B) ± 3

C) ± 4 D) ± 5

2) If the lines represented by $kx^2 - 3xy + 6y^2 = 0$ are perpendicular to each other then __________.

A) $k = 6$ B) $k = -6$

C) $k = 3$ D) $k = -3$

3) Auxiliary equation of $2x^2 + 3xy - 9y^2 = 0$ is __________.

A) $2m^2 + 3m - 9 = 0$ B) $9m^2 - 3m - 2 = 0$

C) $2m^2 - 3m + 9 = 0$ D) $-9m^2 - 3m + 2 = 0$

4) The difference between the slopes of the lines represented by $3x^2 - 4xy + y^2 = 0$ is __________.

A) 2 B) 1

C) 3 D) 4

5) If the two lines $ax^2 + 2hxy + by^2 = 0$ make angles α and β with X-axis, then $\tan(\alpha + \beta) =$ _____.

A) $\dfrac{h}{a+b}$ B) $\dfrac{h}{a-b}$

C) $\dfrac{2h}{a+b}$ D) $\dfrac{2h}{a-b}$

6) If the slope of one of the two lines $\dfrac{x^2}{a} + \dfrac{2xy}{h} + \dfrac{y^2}{b} = 0$ is twice that of the other, then $ab:h^2 =$ ____.

A) 1 : 2 B) 2 : 1

C) 8 : 9 D) 9 : 8

7) The joint equation of the lines through the origin and perpendicular to the pair of lines $3x^2 + 4xy - 5y^2 = 0$ is __________.

A) $5x^2 + 4xy - 3y^2 = 0$ B) $3x^2 + 4xy - 5y^2 = 0$

C) $3x^2 - 4xy + 5y^2 = 0$ D) $5x^2 + 4xy + 3y^2 = 0$

8) If acute angle between lines $ax^2 + 2hxy + by^2 = 0$ is, $\frac{\pi}{4}$ then $4h^2 =$ __________.

A) $a^2 + 4ab + b^2$ B) $a^2 + 6ab + b^2$

C) $(a + 2b)(a + 3b)$ D) $(a - 2b)(2a + b)$

9) If the equation $3x^2 - 8xy + qy^2 + 2x + 14y + p = 1$ represents a pair of perpendicular lines then the values of p and q are respectively __________.

A) -3 and -7 B) -7 and -3

C) 3 and 7 D) -7 and 3

10) The area of triangle formed by the lines $x^2 + 4xy + y^2 = 0$ and $x - y - 4 = 0$ is __________.

A) $\frac{4}{\sqrt{3}}$ Sq. units B) $\frac{8}{\sqrt{3}}$ Sq. units

C) $\frac{16}{\sqrt{3}}$ Sq. units D) $\frac{15}{\sqrt{3}}$ Sq. units

11) The combined equation of the co-ordinate axes is __________.

A) $x + y = 0$ B) $x\,y = k$

C) $xy = 0$ D) $x - y = k$

12) If $h^2 = ab$, then slope of lines $ax^2 + 2hxy + by^2 = 0$ are in the ratio __________.

A) 1 : 2 B) 2 : 1

C) 2 : 3 D) 1 : 1

13) If slope of one of the lines $ax^2 + 2hxy + by^2 = 0$ is 5 times the slope of the other, then $5h^2 =$ __________.

A) ab B) $2\,ab$

C) $7\,ab$ D) $9\,ab$

14) If distance between lines $(x - 2y)^2 + k(x - 2y) = 0$ is 3 units, then $k =$

A) $\pm\ 3$ B) $\pm\ 5\sqrt{5}$

C) 0 D) $\pm\ 3\ \sqrt{5}$

II. Solve the following.

1) Find the joint equation of lines:

i) $x - y = 0$ and $x + y = 0$

ii) $x + y - 3 = 0$ and $2x + y - 1 = 0$

iii) Passing through the origin and having slopes 2 and 3.

iv) Passing through the origin and having inclinations 60° and 120°.

v) Passing through (1,2) amd parallel to the co-ordinate axes.

vi) Passing through (3,2) and parallel to the line $x = 2$ and $y = 3$.

vii)Passing through $(-1,2)$ and perpendicular to the lines $x + 2y + 3 = 0$ and $3x - 4y - 5 = 0$.

viii) Passing through the origin and having slopes $1 + \sqrt{3}$ and $1 - \sqrt{3}$

ix) Which are at a distance of 9 units from the Y - axis.

x) Passing through the point (3,2), one of which is parallel to the line $x - 2y = 2$ and other is perpendicular to the line $y = 3$.

xi) Passing through the origin and perpendicular to the lines $x + 2y = 19$ and $3x + y = 18$.

2) Show that each of the following equation represents a pair of lines.

i) $x^2 + 2xy - y^2 = 0$

ii) $4x^2 + 4xy + y^2 = 0$

iii) $x^2 - y^2 = 0$

iv) $x^2 + 7xy - 2y^2 = 0$

v) $x^2 - 2\sqrt{3}xy - y^2 = 0$

3) Find the separate equations of lines represented by the following equations:

i) $6x^2 - 5xy - 6y^2 = 0$

ii) $x^2 - 4y^2 = 0$

iii) $3x^2 - y^2 = 0$

iv) $2x^2 + 2xy - y^2 = 0$

4) Find the joint equation of the pair of lines through the origin and perpendicular to the lines given by :

i) $x^2 + 4xy - 5y^2 = 0$

ii) $2x^2 - 3xy - 9y^2 = 0$

iii) $x^2 + xy - y^2 = 0$

5) Find k if

i) The sum of the slopes of the lines given by $3x^2 + kxy - y^2 = 0$ is zero.

ii) The sum of slopes of the lines given by $x^2 + kxy - 3y^2 = 0$ is equal to their product.

iii) The slope of one of the lines given by $3x^2 - 4xy + 5y^2 = 0$ is 1.

iv) One of the lines given by $3x^2 - kxy + 5y^2 = 0$ is perpendicular to the $5x + 3y = 0$.

v) The slope of one of the lines given by $3x^2 + 4xy + ky^2 = 0$ is three times the other.

vi) The slopes of lines given by $kx^2 + 5xy + y^2 = 0$ differ by 1.

vii) One of the lines given by $6x^2 + kxy + y^2 = 0$ is $2x + y = 0$.

6) Find the joint equation of the pair of lines which bisect angle between the lines given by $x^2 + 3xy + 2y^2 = 0$

7) Find the joint equation of the pair of lies through the origin and making equilateral triangle with the line $x = 3$.

8) Show that the lines $x^2 - 4xy + y^2 = 0$ and $x + y = 10$ contain the sides of an equilateral triangle. Find the area of the triangle.

9) If the slope of one of the lines represented by $ax^2 + 2hxy + by^2 = 0$ is three times the other then prove that $3h^2 = 4ab$.

10) Find the combined equation of the bisectors of the angles between the lines represented by $5x^2 + 6xy - y^2 = 0$.

11) Find a if the sum of slope of lines represented by $ax^2 + 8xy + 5y^2 = 0$ is twice their product.

12) If line $4x - 5y = 0$ coincides with one of the lines given by $ax^2 + 2hxy + by^2 = 0$ then show that $25a + 40h + 16b = 0$.

13) Show that the following equations represent a pair of lines, find the acute angle between them.

i) $9x^2 - 6xy + y^2 + 18x - 6y + 8 = 0$

ii) $2x^2 + xy - y^2 + x - 4y + 3 = 0$

iii) $(x-3)^2 + (x-3)(y-4) - 2(y-4)^2 = 0$

14) Find the combined equation of pair of lines through the origin each of which makes angle of 60° with the Y-axis.

15) If lines represented by $ax^2 + 2hxy + by^2 = 0$ make angles of equal measures with the co-ordinate axes then show that $a = \pm b$.

16) Show that the combined equation of a pair of lines through the origin and each making an angle of α with the line $x + y = 0$ is $x^2 + 2(\sec 2\alpha)\, xy + y^2 = 0$.

17) Show that the line $3x + 4y + 5 = 0$ and the lines $(3x + 4y)^2 - 3(4x - 3y)^2 = 0$ form an equilateral triangle.

18) Show that lines $x^2 - 4xy + y^2 = 0$ and $x + y = \sqrt{6}$ form an equilateral triangle. Find its area and perimeter.

19) If the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is square of the other then show that $a^2b + ab^2 + 8h^3 = 6abh$.

20) Prove that the product of lengths of perpendiculars drawn from P (x_1, y_1) to the lines repersented by $ax^2 + 2hxy + by^2 = 0$ is $\left|\dfrac{ax_1^2 + 2hx_1y_1 + by_1^2}{\sqrt{(a-b)^2 + 4h^2}}\right|$

21) Show that the difference between the slopes of lines given by $(\tan^2\theta + \cos^2\theta)x^2 - 2xy\tan\theta + (\sin^2\theta)y^2 = 0$ is two.

22) Find the condition that the equation $ay^2 + bxy + ex + dy = 0$ may represent a pair of lines.

23) If the lines given by $ax^2 + 2hxy + by^2 = 0$ form an equilateral triangle with the line $lx + my = 1$ then show that $(3a + b)(a + 3b) = 4h^2$.

24) If line $x + 2 = 0$ coincides with one of the lines represented by the equation $x^2 + 2xy + 4y + k = 0$ then show that $k = -4$.

25) Prove that the combined equation of the pair of lines passing through the origin and perpendicular to the lines represented by $ax^2 + 2hxy + by^2 = 0$ is $bx^2 - 2hxy + ay^2 = 0$

26) If equation $ax^2 - y^2 + 2y + c = 1$ represents a pair of perpendicular lines then find a and c.

✦✦✦✦✦

5 Vectors

Let's Study

1 Vectors and their types.
2. Section formula.
3. Dot Product of Vectors.
4. Cross Product of Vectors.
5. Triple Product of Vectors.

Let's recall.

Scalar quantity : A quantity which can be completely described by magnitude only is called a scalar quantity. e.g. mass, length, temperature, area, volume, time distance, speed, work, money, voltage, density, resistance etc. In this book, scalars are given by real numbers.

Vector quantity : A quantity which needs to be described using both magnitude and direction is called a vector quantity. e.g. displacement, velocity, force, electric field, acceleration, momentum etc.

Let's learn.

5.1 Representation of Vector :

Vector is represented by a directed line segment.

If AB is a segment and its direction is shown with an arrowhead as in figure, then the directed segment AB has magnitude as well as direction. This is an example of vector.

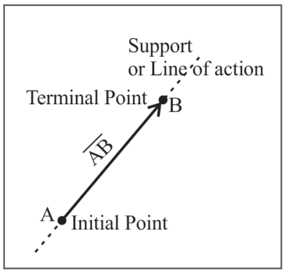


Fig. 5.1

The segment AB with direction from A to B denotes the vector $\overline{AB}$ read as $\overline{AB}$ read as 'AB bar' while direction from B to A denotes the vector $\overline{BA}$.

In vector $\overline{AB}$, the point A is called the initial point and the point B is called the terminal point

The directed line segment is a part of a line of unlimited length which is called the line of support or the line of action of the given vector.

If the initial and terminal point are not specified then the vectors are denoted by $\bar{a}, \bar{b}, \bar{c}$ or **a, b, c** (bold face) etc.

5.1.1 Magnitude of a Vector :

The magnitude (or size or length) of $\overline{AB}$ is denoted by $\left|\overline{AB}\right|$ and is defined as the length of segment AB. *i.e.* $\left|\overline{AB}\right| = l(AB)$

Magnitudes of vectors $\bar{a}, \bar{b}, \bar{c}$ are $|\bar{a}|, |\bar{b}|, |\bar{c}|$ respectively.

The magnitude of a vector does not depend on its direction. Since the length is never negative, $|\bar{a}| \geq 0$

5.1.2 Types of Vectors :

i) **Zero Vector :** A vector whose initial and terminal points coincide, is called a zero vector (or null vector) and denoted as $\bar{0}$. Zero vector cannot be assigned a definite direction and it has zero magnitude or it may be regarded as having any suitable direction. The vectors $\overline{AA}, \overline{BB}$ represent the zero vector and $\left|\overline{AA}\right| = 0$.

ii) **Unit Vector :** A vector whose magnitude is unity (*i.e.* 1 unity) is called a unit vector. The unit vector in the direction of a given vector $\bar{a}$ is denoted by $\hat{a}$, read as 'a-cap' or 'a-hat'.

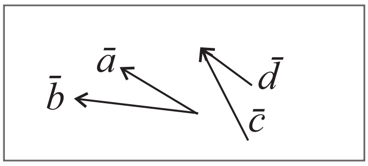


Fig. 5.2

iii) **Co-initial and Co-terminus Vectors :** Vectors having same initial point are called co-initial vectors, whereas vectors having same terminal point are called co-terminus vectors. Here $\bar{a}$ and are $\bar{b}$ co-initial vectors. $\bar{c}$ and $\bar{d}$ are co-terminus vectors.

iv) **Equal Vectors :** Two or more vectors are said to be equal vectors if they have same magnitude and direction.

- As $|\bar{a}| = |\bar{b}|$, and their directions are same regardless of initial point, we write as $\bar{a} = \bar{b}$.
- Here $|\bar{a}| = |\bar{c}|$, but directions are not same, so $\bar{a} \neq \bar{c}$.
- Here directions of $\bar{a}$ and $\bar{d}$ same but $|\bar{a}| \neq |\bar{d}|$, so $\bar{a} \neq \bar{d}$.

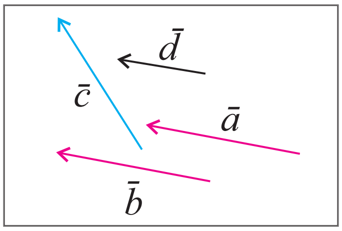


Fig 5.3

v) **Negative of a Vector :** If $\bar{a}$ is a given vector, then the negative of $\bar{a}$ is vector whose magnitude is same as that of $\bar{a}$ but whose direction is opposite to that of $\bar{a}$. It is denoted by $-\bar{a}$.

Thus, if $\overline{PQ} = \bar{a}$, then $\overline{QP} = -\bar{a} = -\overline{PQ}$.

Here $\left|\overline{PQ}\right| = \left|-\overline{QP}\right|$.

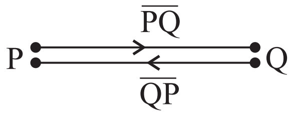


Fig. 5.4

vi) **Collinear Vectors :** Vectors are said to be collinear vectors if they are parallel to same line or they are along the same line.

vii) **Free Vectors :** If a vector can be translated anywhere in the space without changing its magnitude and direction then such a vector is called free vector. In other words, the initial point of free vector can be taken anywhere in the space keeping magnitude and direction same.

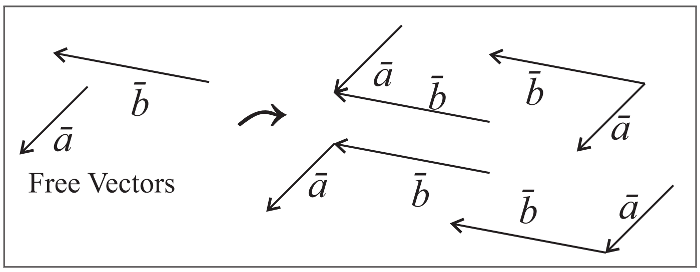


Fig. 5.5

viii) **Localised Vectors :** For a vector of given magnitude and direction, if it's initial point is fixed in space, then such a vector is called localised vector.

For example, if there are two stationary cars A and B on the road and a force is applied to car A, it is a localised vector and only car A moves, while car B is not affected.

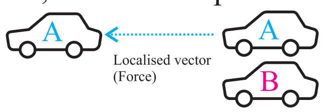


Fig 5.6

Note that in this chapter vectors are treated as free vectors unless otherwise stated.

Activity 1 :

Write the following vectors in terms of vectors $\bar{p}$, $\bar{q}$ and $\bar{r}$.

i) $\overrightarrow{AB} = \square$ ii) $\overrightarrow{BA} = \square$ iii) $\overrightarrow{BC} = \square$

iv) $\overrightarrow{CB} = \square$ v) $\overrightarrow{CA} = \square$ vi) $\overrightarrow{AC} = \square$

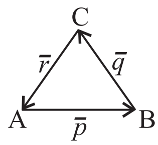


Fig 5.7

Algebra of Vectors :

5.1.3 Scalar Multiplication :

$2\bar{a}$ has the same direction as $\bar{a}$ but is twice as long as $\bar{a}$.

Let $\bar{a}$ be any vector and k be a scalar, then vector $k\bar{a}$, the scalar multiple vector of $\bar{a}$ is defined a vector whose magnitude is $|k\bar{a}| = |k|\,|\bar{a}|$ and vectors $\bar{a}$ and $k\bar{a}$ have the same direction if $k > 0$ and opposite direction if $k < 0$.

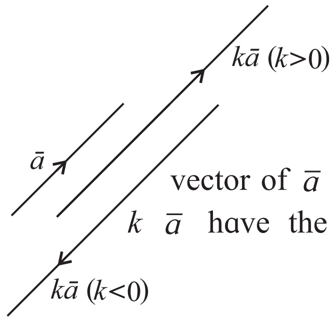


Fig 5.8

Note :

i) If $k = 0$, then $k\bar{a} = \bar{0}$.

ii) $\bar{a}$ and $k\bar{a}$ are collinear or parallel vectors.

iii) Two non zero vectors $\bar{a}$ and $\bar{b}$ are collinear or parallel if $\bar{a} = m\bar{b}$, where $m \neq 0$.

iv) Let $\hat{a}$ be the unit vector along non-zero vector $\bar{a}$ then $\bar{a} = |\bar{a}|\,\hat{a}$ or $\dfrac{\bar{a}}{|\bar{a}|} = \hat{a}$.

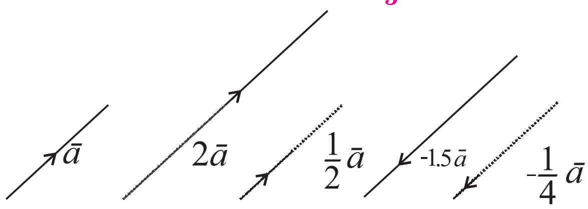


Fig 5.9

v) A vector of length k in the same direction as $\bar{a}$ is $k\hat{a} = k\left(\dfrac{\bar{a}}{|\bar{a}|}\right)$.

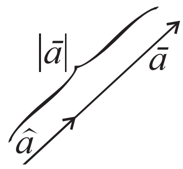


Fig 5.10

Now, consider a boat in a river going from one bank of the river to the other in a direction perpendicular to the flow of the river. Then, it is acted upon by two velocity vectors, one is the velocity imparted to the boat by its engine and other one is the velocity of the flow of river water. Under the simultaneous influence of these two velocities, the boat starts travelling with a different velocity. To have a precise velocity (i.e. resultant velocity) of the boat we use the law of addition of vectors.

5.1.4 Addition of Two Vectors : If $\bar{a}$ and $\bar{b}$ are any two vectors then their addition (or resultant) is denoted by $\bar{a} + \bar{b}$.

There are two laws of addition of two vectors.

Parallelogram Law : Let $\bar{a}$ and $\bar{b}$ be two vectors. Consider $\overline{AB}$ and $\overline{AD}$ along two adjacent sides of a parallelogram, such that $\overline{AB} = \bar{a}$ and $\overline{AD} = \bar{b}$ then $\bar{a} + \bar{b}$ lies along the diagonal of a parallelogram with $\bar{a}$ and $\bar{b}$ as sides.

$\overline{AC} = \overline{AB} + \overline{AD}$ i.e. $\bar{c} = \bar{a} + \bar{b}$.

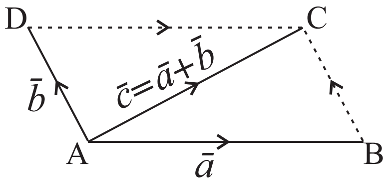


Fig 5.11

Triangle Law of addition of two vectors : Let $\bar{a}$, $\bar{b}$ be any two vectors then consider triangle ABC as shown in figure such that $\overline{AB} = \bar{a}$ and $\overline{BC} = \bar{b}$ then $\bar{a} + \bar{b}$ is given by vector $\overline{AC}$ along the third side of triangle ABC.

Thus, $\bar{a} + \bar{b} = \overline{AB} + \overline{BC} = \overline{AC}$.

This is known as the triangle law of addition of two vectors $\bar{a}$ and $\bar{b}$. The triangle law can also be applied to the ΔADC.

Here, $\overline{AD} = \overline{BC} = \bar{b}$, $\overline{DC} = \overline{AB} = \bar{a}$

Hence, $\overline{AD} + \overline{DC} = \bar{b} + \bar{a} = \overline{AC}$

Thus, $\overline{AC} = \bar{b} + \bar{a} = \bar{a} + \bar{b}$

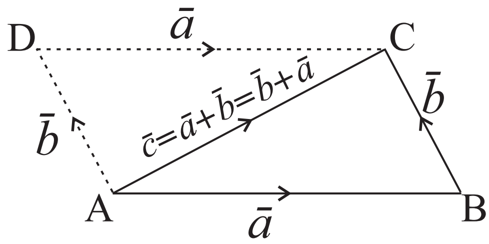


Fig 5.12

5.1.5 Subtraction of two vectors : If $\bar{a}$ and $\bar{b}$ are two vectors, then $\bar{a} - \bar{b} = \bar{a} + (-\bar{b})$, where $-\bar{b}$ is the negative vector of vector $\bar{b}$. Let $\overline{AB} = \bar{a}$, $\overline{BC} = \bar{b}$, now construct a vector $\overline{BD}$ such that its magnitude is same as the vector $\overline{BC}$, but the direction is opposite to that of it.

i.e. $\overline{BD} = -\overline{BC}$.

$\therefore \overline{BD} = -\bar{b}$.

Thus applying triangle law of addition.

We have

$$\therefore \overline{AD} = \overline{AB} + \overline{BD}$$
$$= \overline{AB} - \overline{BC}$$
$$= \bar{a} - \bar{b}$$

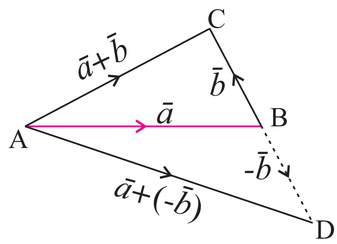


Fig 5.13

Note :

i) If (velocity) vectors $\bar{a}$ and $\bar{b}$ are acting simultaneously then we use parallelogram law of addition.

ii) If (velocity) vectors $\bar{a}$ and $\bar{b}$ are acting one after another then we use triangle law of addition.

iii) Adding vector to its opposite vector gives $\bar{0}$

As $\overline{PQ} + \overline{QP} = \overline{PP} = \bar{0}$ or

As $\overline{PQ} = -\overline{QP}$, then $\overline{PQ} + \overline{QP} = -\overline{QP} + \overline{QP} = \bar{0}$.

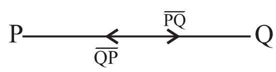


Fig 5.14

iv) In ΔABC, $\overline{AC} = -\overline{CA}$, so $\overline{AB} + \overline{BC} + \overline{CA} = \overline{AA} = \bar{0}$. This means that when the vectors along the sides of a triangle are in order, their resultant is zero as initial and terminal points become same.

v) The addition law of vectors can be extended to a polygon :

Let $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$ be four vectors. Let $\overline{PQ} = \bar{a}$, $\overline{QR} = \bar{b}$, $\overline{RS} = \bar{c}$ and $\overline{ST} = \bar{d}$.

$\therefore\ \bar{a} + \bar{b} + \bar{c} + \bar{d}$

$= \overline{PQ} + \overline{QR} + \overline{RS} + \overline{ST}$

$= \left(\overline{PQ} + \overline{QR}\right) + \overline{RS} + \overline{ST}$

$= \left(\overline{PR} + \overline{RS}\right) + \overline{ST}$

$= \overline{PS} + \overline{ST}$

$= \overline{PT}$

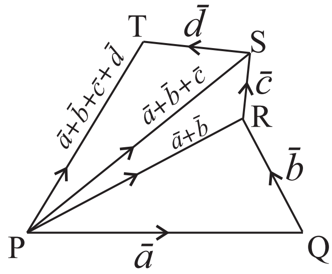


Fig 5.15

Thus, the vector $\overline{PT}$ represents sum of all vectors $\bar{a}, \bar{b}, \bar{c}$ and $\bar{d}$.

This is also called as extended law of addition of vectors or polygonal law of addition of vectors.

vi) $\bar{a} + \bar{b} = \bar{b} + \bar{a}$ (commutative)

vii) $\left(\bar{a} + \bar{b}\right) + \bar{c} = \bar{a} + \left(\bar{b} + \bar{c}\right)$ (associative)

viii) $\bar{a} + \bar{0} = \bar{a}$ ($\bar{0}$ is additive identify)

ix) $\bar{a} + (-\bar{a}) = \bar{0}$ ($-\bar{a}$ is additive inverse)

x) If $\bar{a}$ and $\bar{b}$ are vectors and m and n are scalars, then

i) $(m+n)\,\bar{a} = m\bar{a} + n\bar{a}$ (distributive)

ii) $m(\bar{a} + \bar{b}) = m\bar{a} + m\bar{b}$ (distributive)

iii) $m(n\bar{a}) = (mn)\bar{a} = n(m\bar{a})$

xi) $\left|\bar{a} + \bar{b}\right| \leq \left|\bar{a}\right| + \left|\bar{b}\right|$, this is known as "Triangle Inequality".

This is obtained from the triangle law, as the length of any side of triangle is less than the sum of the other two sides. *i.e.* in triangle ABC, AC < AB + BC,

where, AC = $\left|\bar{a} + \bar{b}\right|$, AB $= \left|\bar{a}\right|$, BC $= \left|\bar{b}\right|$

xii) Any two vectors $\bar{a}$ and $\bar{b}$ determine a plane and vectors $\bar{a} + \bar{b}$ and $\bar{a} - \bar{b}$ lie in the same plane.

Activity 2 :

In quadrilateral PQRS, find a resultant vector

i) $\overline{QR} + \overline{RS} = \square$ ii) $\overline{PQ} + \overline{QR} = \square$

iii) $\overline{PS} + \overline{SR} + \overline{RQ} = \square$ iv) $\overline{PR} + \overline{RQ} + \overline{QS} = \square$

v) $\overline{QR} - \overline{SR} - \overline{PS} = \square$ vi) $\overline{QP} - \overline{RP} + \overline{RS} = \square$

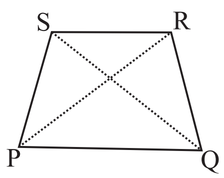


Fig 5.16

Theorem 1:

Two non-zero vectors $\bar{a}$ and $\bar{b}$ are collinear if and only if there exist scalars m and n, at least one of them is non-zero such that $m\bar{a} + n\bar{b} = \bar{0}$.

Proof : Only If - part :

Suppose $\bar{a}$ and $\bar{b}$ are collinear.

$\therefore$ There exists a scalar $t \neq 0$ such that $\bar{a} = t\bar{b}$

$\therefore\ \bar{a} - t\bar{b} = \bar{0}$

i.e. $m\bar{a} + n\bar{b} = \bar{0}$, where $m = 1$ and $n = -$ t.

If - part :

Conversely, suppose $m\bar{a} + n\bar{b} = \bar{0}$ and $m \neq 0$.

$\therefore\ m\bar{a} = -n\bar{b}$

$\therefore\ \bar{a} = \left(-\frac{n}{m}\right)\bar{b}$, where $t = \left(-\frac{n}{m}\right)$ is a scalar.

i.e. $\bar{a} = t\bar{b}$,

$\therefore\ \bar{a}$ is scalar multiple of $\bar{b}$.

$\therefore\ \bar{a}$ and $\bar{b}$ are collinear.

Corollary 1 : If two vectors $\bar{a}$ and $\bar{b}$ are not collinear and $m\bar{a} + n\bar{b} = \bar{0}$, then $m = 0$, $n = 0$. (This can be proved by contradiction assuming $m \neq 0$ or $n \neq 0$).

Corollary 2 : If two vectors $\bar{a}$ and $\bar{b}$ are not collinear and $m\bar{a} + n\bar{b} = p\bar{a} + q\bar{b}$, then $m = p$, $n = q$.

For example, If two vectors $\bar{a}$ and $\bar{b}$ are not collinear and $3\bar{a} + y\bar{b} = x\bar{a} + 5\bar{b}$,

then $3 = x$, $y = 5$.

5.1.6 Coplanar Vectors :

Two or more vectors are coplanar, if they lie in the same plane or in parallel plane.

Vector $\bar{a}$ and $\bar{b}$ are coplanar.

Vector $\bar{a}$ and $\bar{c}$ are coplanar.

Are vectors $\bar{a}$ and $\bar{e}$ coplanar ?

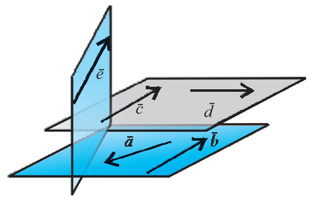


Fig 5.17

Remark : Any two intersecting straight lines OA and OB in space determine a plane. We may choose for convenience the coordinate axes of the plane so that O is origin and axis OX is along one of OA or OB.

Theorem 2 : Let $\bar{a}$ and $\bar{b}$ be non-collinear vectors. A vector $\bar{r}$ is coplanar with $\bar{a}$ and $\bar{b}$ if and only if there exist unique scalars $t_1\ t_2$, such that $\bar{r} = t_1\bar{a} + t_2\bar{b}$.

Proof : Only If-part :

Suppose $\bar{r}$ is coplanar with $\bar{a}$ and $\bar{b}$. To show that their exist unique scalars t_1 and t_2 such that $\bar{r} = t_1\bar{a} + t_2\bar{b}$.

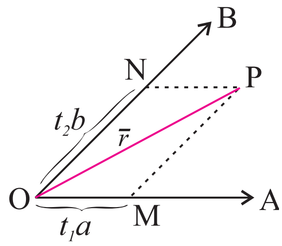


Fig 5.18

Let $\bar{a}$ be along OA and $\bar{b}$ be along OB. Given a vector $\bar{r}$, with initial point O, Let $\overline{OP} = \bar{r}$, draw lines parallel to OB, meeting OA in M and parallel to OA, meeting OB in N.

Then $ON = t_2\bar{b}$ and $OM = t_1\bar{a}$ for some $t_1, t_2 \in$ R. By triangle law or parallelogram law, we have $\bar{r} = t_1\bar{a} + t_2\bar{b}$

If Part : Suppose $\bar{r} = t_1\bar{a} + t_2\bar{b}$, and we have to show that $\bar{r}$, $\bar{a}$ and $\bar{b}$ are co-planar.

As $\bar{a}$, $\bar{b}$ are coplanar, $t_1\bar{a}$, $t_2\bar{b}$ are also coplanar. Therefore $t_1\bar{a} + t_2\bar{b}$, $\bar{a}$, $\bar{b}$ are coplanar.

Therefore $\bar{a}$, $\bar{b}$, $\bar{r}$ are coplanar.

Uniqueness :

Suppose vector $\bar{r} = t_1\bar{a} + t_2\bar{b}$ ... (1)

can also be written as $\bar{r} = s_1\bar{a} + s_2\bar{b}$ ... (2)

Subtracting (2) from (1) we get,

$\bar{0} = (t_1 - s_1)\bar{a} + (t_2 - s_2)\bar{b}$

But, $\bar{a}$ and $\bar{b}$ are non-collinear, vectors

By Corollary 1 of Theorem 1,

$\therefore t_1 - s_1 = 0 = t_2 - s_2$

$\therefore t_1 = s_1$ and $t_2 = s_2$.

Therefore, the uniqueness follows.

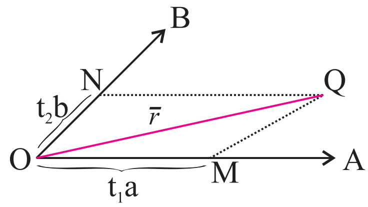


Fig 5.19

Remark :

Linear combination of vectors : If $\bar{a}_1, \bar{a}_2, \bar{a}_3, ..., \bar{a}_n$ are n vectors and $m_1, m_2, m_3, ..., m_n$ are n scalars, then the vector $m_1\bar{a}_1 + m_2\bar{a}_2 + m_3\bar{a}_3 + ... + m_n\bar{a}_n$ is called a linear combination of vectors $\bar{a}_1, \bar{a}_2, \bar{a}_3, ..., \bar{a}_n$. If atleast one m_i is not zero then the linear combination is non zero linear combination

For example : Let $\bar{a}, \bar{b}$ be vectors and *m, n are* scalars then the vector $\bar{c} = m\bar{a} + n\bar{b}$ is called a linear combination of vector $\bar{a}$ and $\bar{b}$. Vectors $\bar{a}, \bar{b}$ and $\bar{c}$ are coplanar vectors.

Theorem 3 : Three vectors $\bar{a}, \bar{b}$ and $\bar{c}$ are coplanar, if and only if there exists a non-zero linear combination $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ with $(x, y, z) \neq (0, 0, 0)$.

Proof : Only If - part :

Assume that $\bar{a}, \bar{b}$ and $\bar{c}$ are coplanar.

Case - 1 : Suppose that any two of $\bar{a}, \bar{b}$ and $\bar{c}$ are collinear vectors, say $\bar{a}$ and $\bar{b}$.

$\therefore$ There exist scalars x, y at least one of which is non-zero such that $x\bar{a} + y\bar{b} = \bar{0}$

i.e. $x\bar{a} + y\bar{b} + 0\bar{c} = \bar{0}$ and $(x, y, 0)$ is the required solution for $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$.

Case - 2 : No two vectors $\bar{a}, \bar{b}$ and $\bar{c}$ are collinear.

As $\bar{c}$ is coplanar with $\bar{a}$ and $\bar{b}$,

$\therefore$ we have scalars x, y such that $\bar{c} = x\bar{a} + y\bar{b}$ (using Theorem 2).

$\therefore x\bar{a} + y\bar{b} - \bar{c} = \bar{0}$ and $(x, y, -1)$ is the required solution for $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$.

If - part : Conversely, suppose $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ where one of x, y, z is non-zero, say $z \neq 0$.

$$\therefore \bar{c} = \frac{-x}{z}\bar{a} - \frac{y}{z}\bar{b}$$

$\therefore \bar{c}$ is coplanar with $\bar{a}$ and $\bar{b}$.

$\therefore \bar{a}, \bar{b}$ and $\bar{c}$ are coplanar vectors.

Corollary 1 : If three vectors $\bar{a}, \bar{b}$ and $\bar{c}$ are not coplanar and $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$, then $x = 0$, $y = 0$ and $z = 0$ because if $(x, y, z) \neq (0, 0, 0)$ then $\bar{a}, \bar{b}, \bar{c}$ are coplanar.

Corollary 2 : The vectors $\bar{a}, \bar{b}$ and $x\bar{a} + y\bar{b}$ are coplanar for all values of x and y.

5.1.7 Vector in Two Dimensions (2-D) :

The plane spanned (covered) by non collinear vectors $\bar{a}$ and $\bar{b}$ is $\{x\bar{a}+y\bar{b} \mid x, y \in \mathbb{R}\}$, where $\bar{a}$ and $\bar{b}$ have same initial point.

This is 2-D space where generators are $\bar{a}$ and $\bar{b}$ or its basis is $\{\bar{a}, \bar{b}\}$

For example, in XY plane, let M = (1, 0) and N = (0, 1) be two points along X and Y axis respectively.

Then, we define unit vectors $\hat{i}$ and $\hat{j}$ as $\overline{OM} = \hat{i}$, $\overline{ON} = \hat{j}$.

Given any other vector say $\overline{OP}$, where P = (3, 4) then

$$\overline{OP} = 3\hat{i} + 4\hat{j}$$

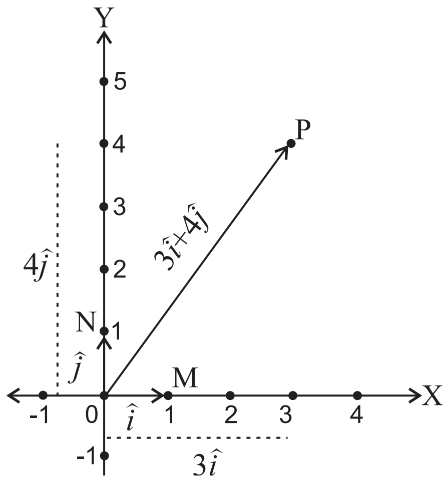


Fig 5.20

5.1.8 Three Dimensional (3-D) Coordinate System :

Any point in the plane is represented as an ordered pair (*a*, *b*) where *a* and *b* are distances (with suitable sign) of point (*a*, *b*) from Y-axis and X-axis respectively.

To locate a point in space, three numbers are required. Here, we need three coordinate axes OX, OY and OZ and to determine a point we need distances of it from three planes formed by these axes.

We represent any point in space by an ordered triple (*a*, *b*, *c*) of real numbers.

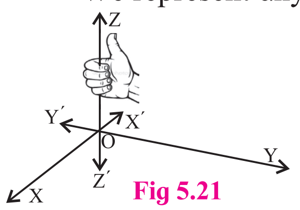


Fig 5.21

O is the origin and three directed lines through O that are perpendicular to each other are the coordinate axes.

Label them as X-axis (XOX'), Y-axis (YOY') and Z-axis (ZOZ'). The direction of Z-axis is determined by right hand rule *i.e.* When you hold your right hand so that the fingers curl from the positive X-axis toward the positive Y-axis, your thumb points along the positive Z-axis, as shown in figure.

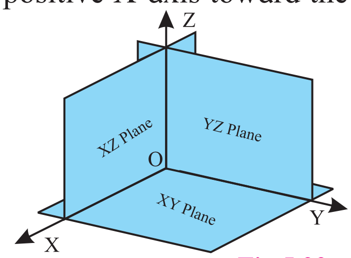


Fig 5.22

Co-ordinates of a point in space :

Let P be a point in the space. Draw perpendiculars PL, PM, PN through P to XY-plane, YZ-plane and XZ-plane respectively, where points L, M and N are feet of perpendicularls in XY, YZ and XZ planes respectively.

For point P(*x*, *y*, *z*), *x*, *y* and *z* are *x*-coordinate, *y*-coordinate and *z*-coordinate respectively.

Point of intersection of all 3 planes is origin O(0, 0, 0).

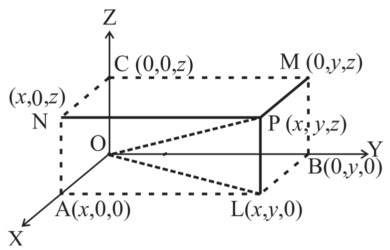


Fig 5.23

Co-ordinates of points on co-ordinate axes :

Points on X-axis, Y-axis and Z-axis have coordinates given by A(*x*, 0, 0), B(0, *y*, 0) and C(0, 0, *z*).

Co-ordinates of points on co-ordinate planes :

Points in XY-plane, YZ-plane and ZX-plane are given by L(*x*, *y*, 0), M(0, *y*, *z*), N(*x*, 0, *z*) respectively.

Distance of P(x, y, z) from co-ordinate planes :

i) Distance of P from XY plane = |PL| = |z|.

ii) Distance of P from YZ plane = |PM| = |x|.

iii) Distance of P from XZ plane = |PN| = |y|.

Distance of any point from origin :

Distance of P (x, y, z) from the origin O(0, 0, 0) from figure 5.23 we have,

$$l(\text{OP}) = \sqrt{\text{OL}^2 + \text{LP}^2} \quad (\Delta\text{OLP right angled triangle})$$

$$= \sqrt{\text{OA}^2 + \text{AL}^2 + \text{LP}^2}$$

$$= \sqrt{\text{OA}^2 + \text{OB}^2 + \text{OC}^2}$$

$$= \sqrt{x^2 + y^2 + z^2}$$

Distance between any two points in space

Distance between two points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) in space is given by distance formula

$$l(\text{AB}) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Distance of a point P(x, y, z) from coordinate axes.

In fig. 5.23, PA is perpendicular to X-axis. Hence distance of P from X-axis is PA.

$$\therefore \text{PA} = \sqrt{(x - x)^2 + (y - 0)^2 + (z - 0)^2}$$

$$= \sqrt{y^2 + z^2}$$

iv) In a right-handed system. Octants II, III and IV are found by rotating anti-clockwise around the positive Z-axis. Octant V is vertically below Octant I. Octants VI, VII and VIII are then found by rotating anti-clockwise around the negative Z-axis.

Signs of coordinates of a point P(x, y, z) in different octants :

Octant (x, y, z)	(I) O-XYZ (+, +, +)	(II) O-X'YZ (–, +, +)	(III) O-XY'Z (+, –, +)	(IV) O-X'Y'Z (–, –, +)
Octant (x, y, z)	(V) O-XYZ' (+, +, –)	(VI) O-X'YZ' (–, +, –)	(VII) O-XY'Z' (+, –, –)	(VIII) O-X'Y'Z' (–, –, –)

Point in Octants

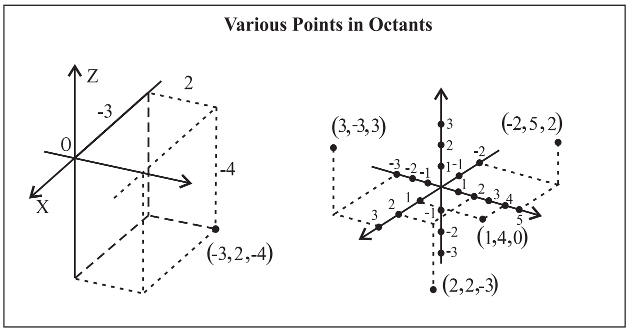


Fig 5.24

Various shapes inb space

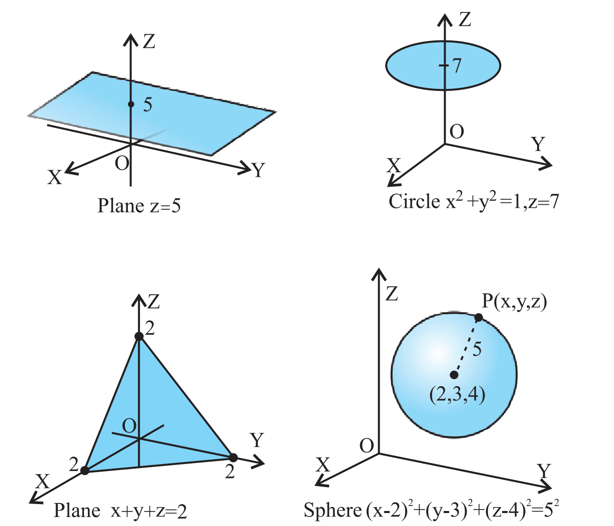


Fig 5.25

5.1.9 Components of Vector :

In order to be more precise about the direction of a vector we can represent a vector as a linear combination of basis vectors.

Take the points A(1, 0, 0), B(0, 1, 0) and C(0, 0, 1) on the X-axis, Y-axis and Z-axis, respectively.

Then $\left|\overline{OA}\right| = \left|\overline{OB}\right| = \left|\overline{OC}\right| = 1$

The vectors $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ each having magnitude 1 are called unit vectors along the axes X, Y, Z respectively. These vectors are denoted by $\hat{i}$, $\hat{j}$, $\hat{k}$ respectively and also called as standard basis vectors or standard unit vectors.

Any vector, along X-axis is a scalar multiple of unit vector $\hat{i}$, along Y-axis is a scalar multiple of $\hat{j}$ and along Z-axis is a scalar multiple of $\hat{k}$. (Collinearity property).

e.g.i) $3\hat{i}$ is a vector along OX with magnitude 3.

ii) $5\hat{j}$ is a vector along OY with magnitude 5.

iii) $4\hat{k}$ is a vector along OZ with magnitude 4.

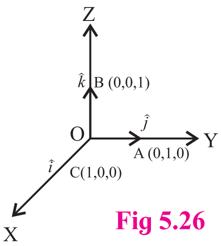


Fig 5.26

Theorem 4 :

If $\bar{a}$, $\bar{b}$, $\bar{c}$ are three non-coplanar vectors, then any vector $\bar{r}$ in the space can be uniquely expressed as a linear combination of $\bar{a}$, $\bar{b}$, $\bar{c}$.

Proof : Let A be any point in the space, take the vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{r}$, so that A becomes their initial point (Fig.5.27).

Let $\overline{AP} = \bar{r}$. As $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar vectors, they determine three distinct planes intersecting at the point A. Through the point P, draw the plane parallel to the plane formed by vectors $\bar{b}$, $\bar{c}$.

This plane intersects line containing $\bar{a}$ at point B. Similarly, draw the other planes and complete the parallelopiped.

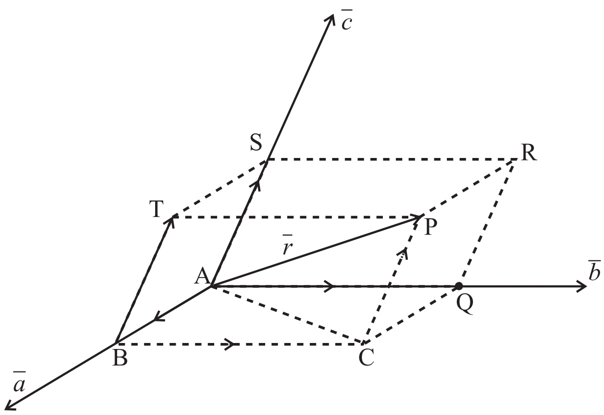


Fig 5.27

Now, $\overline{AB}$ and $\bar{a}$ are collinear.

$\therefore$ $\overline{AB} = x\bar{a}$, where x is a scalar.

Similarly, we have $\overline{AQ} = y\bar{b}$ and $\overline{AS} = z\bar{c}$, where y and z are scalars.

Also, by triangle law of addition of vectors,

$\overline{AP} = \overline{AC} + \overline{CP}$ (In ΔACP)

$= \left(\overline{AB} + \overline{BC}\right) + \overline{CP}$ (In Δ ABC)

$= \overline{AB} + \overline{BC} + \overline{CP}$

$\overline{AP} = \overline{AB} + \overline{BC} + \overline{CP}$

$\therefore$ $\bar{r} = x\bar{a} + y\bar{b} + z\bar{c}$. $\quad$ ($\because \overline{AQ} = \overline{BC}$ and $\overline{AS} = \overline{CP}$)

Therefore, any vector $\bar{r}$ in the space can be expressed as a linear combination of $\bar{a}$, $\bar{b}$ and $\bar{c}$.

Uniqueness :

Suppose $\bar{r} = x_1\bar{a} + x_2\bar{b} + x_3\bar{c}$ and also $\bar{r} = y_1\bar{a} + y_2\bar{b} + y_3\bar{c}$ for some scalars x_1, x_2, x_3 and y_1, y_2, y_3.

We need to prove $x_1 = y_1$, $x_2 = y_2$ and $x_3 = y_3$.

Subtracting one expression from the other we have $(x_1 - y_1)\bar{a} + (x_2 - y_2)\bar{b} + (x_3 - y_3)\bar{c} = \bar{0}$. By Corollary 1 of Theorem 3

As $\bar{a}$, $\bar{b}$, $\bar{c}$ are non-coplanar we must have $x_1 - y_1 = x_2 - y_2 = x_3 - y_3 = 0$ that is $x_1 = y_1$, $x_2 = y_2$, $x_3 = y_3$, as desired.

5.1.10 Position vector of a point P(x, y, z) in space :

Consider a point P in space, having coordinates (x, y, z) with respect to the origin O(0, 0, 0). Then the vector $\overline{OP}$ having O and P as its initial and terminal points, respectively is called the position vector of the point P with respect to O.

Let P (x, y, z) be a point in space.

$\therefore$ OA = PM = x, OB = PN = y, OC = PL = z.

i.e. A $\equiv (x, 0, 0)$, B $\equiv (0, y, 0)$ and C $\equiv (0, 0, z)$.

Let $\hat{i}$, $\hat{j}$, $\hat{k}$ be unit vectors along positive directions of X-axis, Y-axis and Z-axis is respectively.

$\therefore$ $\overline{OA} = x\hat{i}$, $\overline{OB} = y\hat{j}$ and $\overline{OC} = z\hat{k}$

$\overline{OP}$ is the position vector of point P in space with respect to origin O.

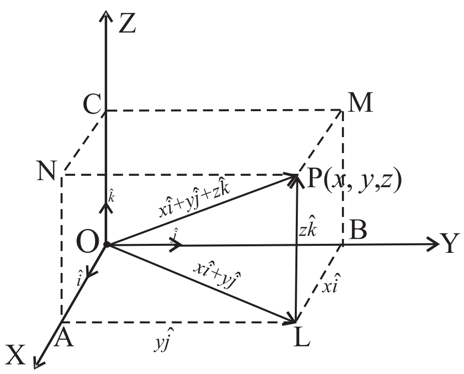


Fig 5.28

Representation of $\overline{OP}$ in terms of unit vector $\hat{i}$, $\hat{j}$, $\hat{k}$.

In ΔOLP we have (see fig. 5.28)

$\overline{OP} = \overline{OL} + \overline{LP}$

$= \overline{OA} + \overline{AL} + \overline{LP}$ $\quad$ (from ΔOAL)

$= \overline{OA} + \overline{OB} + \overline{OC}$ $\quad$ ($\because \overline{AL} = \overline{OB}$)

$= x\hat{i} + y\hat{j} + z\hat{k}$ $\quad$ ... ... (1)

Magnitude of $\overline{\mathrm{OP}}$

Now, $\mathrm{OP}^2 = \mathrm{OL}^2 + \mathrm{LP}^2$ (In right angled ΔOLP)

$= \mathrm{OA}^2 + \mathrm{AL}^2 + \mathrm{LP}^2$ (In right angled ΔOAL)

$= \mathrm{OA}^2 + \mathrm{OB}^2 + \mathrm{OC}^2$

$= x^2 + y^2 + z^2$. ($\because$ AL = OB)

$$\therefore\ l(\mathrm{OP}) = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore\ \left(\overline{\mathrm{OP}}\right) = \sqrt{x^2 + y^2 + z^2}$$

5.1.11 Component form of $\bar{r}$:

If $\bar{r}$ is a position vector (p.v.) of point P w.r.t. O then $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

In this representation x, y, z are called the components of $\bar{r}$ along OX, OY and OZ.

Note that any vector in space is unique linear combination of $\hat{i}$, $\hat{j}$ and $\hat{k}$.

Note : Some authors represent vector $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ using angle brackets as $\bar{r} = \langle x, y, z\rangle$ for the point (x, y, z).

5.1.12 Vector joining two points :

Let O be the origin then $\overline{\mathrm{OA}}$ and $\overline{\mathrm{OB}}$ are the position vectors of points A and B w.r.t. origin 'O'.

In ΔAOB, we have by triangle law.

$$\overline{\mathrm{AB}} = \overline{\mathrm{AO}} + \overline{\mathrm{OB}}$$

$$= -\overline{\mathrm{OA}} + \overline{\mathrm{OB}} \quad (\because \overline{\mathrm{AO}} = -\overline{\mathrm{OA}})$$

$$= \overline{\mathrm{OB}} - \overline{\mathrm{OA}}$$

= position vector of B – position vector of A

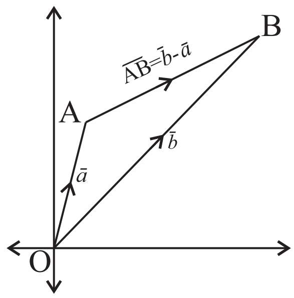


Fig 5.29

That is, $\overline{\mathrm{AB}} = \bar{b} - \bar{a}$

If A $\equiv (x_1, y_1, z_1)$ and B $\equiv (x_2, y_2, z_2)$

Then $\overline{\mathrm{OA}} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$ and $\overline{\mathrm{OB}} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$

$$\therefore \overline{\mathrm{AB}} = \overline{\mathrm{OB}} - \overline{\mathrm{OA}}$$

$$= \left(x_2\hat{i} + y_2\hat{j} + z_2\hat{k}\right) - \left(x_1\hat{i} + y_1\hat{j} + z_1\hat{k}\right)$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$

$$\left|\overline{\mathrm{AB}}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

In general, any non zero vector $\bar{a}$ in space can be expressed uniquely as the linear combination of $\hat{i}$, $\hat{j}$, $\hat{k}$ as $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ where a_1, a_2, a_3 are scalars.

$$\therefore |\bar{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

Note :

(i) If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\bar{a} = \bar{b}$ if $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$.

(ii) If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then

$$\bar{a} + \bar{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) + \left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}.$$

(iii) If k is any scalar then $k\bar{a} = k\left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right) = ka_1\hat{i} + ka_2\hat{j} + ka_3\hat{k}$

Also if $\bar{b}$ and $\bar{a}$ are collinear i.e. $\bar{b} = k\bar{a}$ then $\dfrac{b_1}{a_1} = \dfrac{b_2}{a_2} = \dfrac{b_3}{a_3} = k$.

(iv) Let $\hat{a}$ be a unit vector along a non zero vector $\bar{a}$ in space, Then $\hat{a} = \dfrac{\bar{a}}{|\bar{a}|} = \dfrac{a_1\hat{i} + a_2\hat{j} + a_3\hat{k}}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$.

(v) **(Collinearity of 3-points)** Three distinct points A, B and C with position vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ respectively are collinear if and only if there exist three non-zero scalars x, y and z such that $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$ and $x + y + z = 0$. (Use Theorem 1 and the fact that $\overline{AB}$ and $\overline{BC}$ are collinear).

(vi) **(Coplanarity of 4-points)** Four distinct points A, B, C and D (no three of which are collinear) with position vectors $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ respectively are coplanar if and only if there exist four scalars x, y, z and w, not all zero, such that $x\bar{a} + y\bar{b} + z\bar{c} + w\bar{d} = \bar{0}$,

where $x + y + z + w = 0$. (Use Theorem 2 and the fact that $\overline{AB}$, $\overline{AC}$ and $\overline{AD}$ are coplanar).

(vii) **Linearly dependent vectors :** A set of non-zero vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is said to be linearly dependent if there exist scalars x, y, z not all zero such that $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$.

Such vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar.

(viii) **Linearly independent vectors :** A set of non-zero vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is said to be linearly independent if $x\bar{a} + y\bar{b} + z\bar{c} = \bar{0}$, then $x = y = z = 0$.

Such vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are non-coplanar.

Solved Examples

Ex.1. State the vectors which are :

(i) equal in magnitude

(ii) parallel

(iii) in the same direction

(iv) equal

(v) negatives of one another

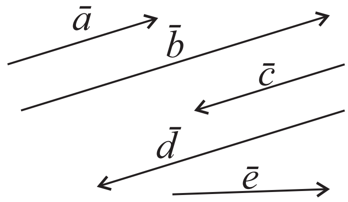


Fig 5.30

Solution :

(i) $\bar{a}$, $\bar{c}$ and $\bar{e}$; $\bar{b}$ and $\bar{d}$

(ii) $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$

(iii) $\bar{a}$ and $\bar{b}$; $\bar{c}$ and $\bar{d}$

(iv) none are equal

(v) $\bar{a}$ and $\bar{c}$, $\bar{b}$ and $\bar{d}$

Ex. 2. In the diagram $\overrightarrow{KL} = \bar{a}$, $\overrightarrow{LN} = \bar{b}$, $\overrightarrow{NM} = \bar{c}$ and $\overrightarrow{KT} = \bar{d}$. Find in terms of $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$. (i) $\overrightarrow{LT}$ (ii) $\overrightarrow{KM}$ (iii) $\overrightarrow{TN}$ (iv) $\overrightarrow{MT}$

Solution :

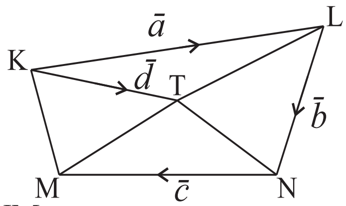


Fig 5.31

(i) In ΔPTQ, using triangle law

$\overline{KL} + \overline{LT} = \overline{KT}$ i.e. $\bar{a} + \overline{LT} = \bar{d}$

$\overline{LT} = \bar{d} - \bar{a}$

(ii) Using polygonal law of addition of vectors for polygon KLNM

$\overline{KM} = \overline{KL} + \overline{LN} + \overline{NM}$.

$= \bar{a} + \bar{b} + \bar{c}$

(iii) Using polygonal law of addition of vectors for polygon TKLN

$\overline{TN} = \overline{TK} + \overline{KL} + \overline{LN}$

$= -\bar{d} + \bar{a} + \bar{b} = \bar{a} + \bar{b} - \bar{d}$

(iv) Using polygonal law of addition of vectors for polygon TKLNM

$\overline{MT} = \overline{MN} + \overline{NL} + \overline{LK} + \overline{KT}$

$= -\bar{c} - \bar{b} - \bar{a} + \bar{d}$

$= \bar{d} - \bar{a} - \bar{b} - \bar{c}$.

Ex.3. Find the magnitude of following vectors :

(i) $\bar{a} = \hat{i} - 2\hat{j} + 4\hat{k}$ (ii) $\bar{b} = 4\hat{i} - 3\hat{j} - 7\hat{k}$

(iii) a vector with initial point : (1, – 3, 4); terminal point : (1, 0, – 1).

Solution :

(i) $|\bar{a}| = \sqrt{1^2 + (-2)^2 + 4^2} = \sqrt{21}$

(ii) $|\bar{b}| = \sqrt{4^2 + (-3)^2 + (-7)^2}$

$|\bar{b}| = \sqrt{16 + 9 + 49} = \sqrt{74}$

(iii) $|\bar{c}| = (\hat{i} - \hat{k}) - (\hat{i} - 3\hat{j} + 4\hat{k}) = 3\hat{j} - 5\hat{k}$

$|\bar{c}| = \sqrt{0 + 3^2 + (-5)^2} = \sqrt{34}$

Ex. 4. A(2, 3), B (– 1, 5), C (– 1, 1) and D (– 7, 5) are four points in the Cartesian plane.

(i) Find $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

(ii) Check if, $\overrightarrow{CD}$ is parallel to $\overrightarrow{AB}$.

(iii) E is the point (*k*, 1) and $\overrightarrow{AC}$ is parallel to $\overrightarrow{BE}$. Find *k*.

Solution : (i) $= 2\hat{i} + 3\hat{j}, \bar{b} = -\hat{i} + 5\hat{j}, \bar{c} = -\hat{i} + \hat{j}, \bar{d} = -7\hat{i} +$

$\overrightarrow{AB} = \bar{b} - \bar{a} = (-\hat{i} + 5\hat{j}) - (2\hat{i} + 3\hat{j}) = -3\hat{i} + 2\hat{j}$

$\overrightarrow{CD} = \bar{d} - \bar{c} = (-7\hat{i} + 5\hat{j}) - (-\hat{i} + \hat{j}) = -6\hat{i} + 4\hat{j}$

(ii) $\overrightarrow{CD} = -6\hat{i} + 4\hat{j} = 2(-3\hat{i} + 2\hat{j} = 2\ \overline{AB}$ therefore $\overrightarrow{CD}$ and $\overrightarrow{AB}$ are parallel.

(iii) $\overrightarrow{BE} = (k\hat{i} + \hat{j}) - (-\hat{i} + 5\hat{j}) = (k+1)\hat{i} - 4\hat{j}$

$\overrightarrow{AC} = (-\hat{i} + \hat{j}) - (2\hat{i} + 3\hat{j}) = -3\hat{i} - 2\hat{j}$

$\overrightarrow{BE} = m\overrightarrow{AC}$

$(k+1)\hat{i} - 4\hat{j} = m(-3\hat{i} - 2\hat{j})$

So – 4 therefore 2 = m

and $k + 1 = -3m$

$k + 1 = -3(2)$

$k = -6 - 1$

$k = -7$

Ex. 5. Determine the values of *c* that satisfy $|c\bar{u}| = 3, \bar{u} = \hat{i} + 2\hat{j} + 3\hat{k}$

Solution : $|c\bar{u}| = \sqrt{c^2 + 4c^2 + 9c^2} = |c|\sqrt{14} = 3$

$\therefore c = \pm\frac{3}{\sqrt{14}}$

Ex. 6. Find a unit vector (i) in the direction of $\bar{u}$ and (ii) in the direction opposite of $\bar{u}$. where $\bar{u} = 8\hat{i} + 3\hat{j} - \hat{k}$.

Solution : (i) $\hat{u} = \frac{\bar{u}}{|\bar{u}|} = \frac{8\hat{i} + 3\hat{j} - 1\hat{k}}{\sqrt{74}}$

$= \frac{1}{\sqrt{74}}(8\hat{i} + 3\hat{j} - \hat{k})$ is the unit vector in direction of $\bar{u}$.

(ii) $-\hat{u} = -\frac{1}{\sqrt{74}}(8\hat{i} + 3\hat{j} - \hat{k})$ is the unit vector in opposite direction of $\bar{u}$.

Ex. 7. Show that the vectors $2\hat{i}-3\hat{j}+4\hat{k}$ and $-4\hat{i}+6\hat{j}-8\hat{k}$ are parallel.

Solution : $\bar{a}=2\hat{i}-3\hat{j}+4\hat{k}$

$\bar{b}=-4\hat{i}+6\hat{j}-8\hat{k}=-2\left(2\hat{i}-3\hat{j}+4\hat{k}\right)=-2\bar{a}$

As $\bar{b}$ is scalar multiple of $\bar{a}$

$\therefore$ $\bar{b}$ and $\bar{a}$ are parallel.

Ex. 8. The non-zero vectors $\bar{a}$ and $\bar{b}$ are not collinear find the value of λ and μ :

(i) $\bar{a}+3\bar{b}=2\lambda\bar{a}-\mu\bar{b}$

(ii) $(1+\lambda)\bar{a}+2\lambda\bar{b}=\mu\bar{a}+4\mu\bar{b}$

(iii) $(3\lambda+5)\bar{a}+\bar{b}=2\mu\bar{a}+(\lambda-3)\bar{b}$

Solution : (i) $2\lambda=1,\ 3=-\mu\ \therefore\ \lambda=½,\ \mu=-3$

(ii) $1+\lambda=\mu,\ 2\lambda=4\mu,\ \lambda=2\mu$

$1+2\mu=\mu,\ 1=-\mu$

$\therefore\mu=-1,\ \lambda=-2$

(iii) $3\lambda+5=2\mu,\ 1=\lambda-3,\ \therefore\ \lambda=1+3=4$

and $3(4)+5=2\mu$

$\therefore 2\mu=17$. So $\mu=\dfrac{17}{2}$

Ex. 9. Are the following set of vectors linearly independent?

(i) $\bar{a}=\hat{i}-2\hat{j}+3\hat{k},\ \bar{b}=3\hat{i}-6\hat{j}+9\hat{k}$

(ii) $\bar{a}=-2\hat{i}-4\hat{k},\quad \bar{b}=\hat{i}-2\hat{j}-\hat{k},\quad \bar{c}=\hat{i}-4\hat{j}+3\hat{k}$. Interpret the results.

Solution : (i) $\bar{a}=\hat{i}-2\hat{j}+3\hat{k},\bar{b}=3\hat{i}-6\hat{j}+9\hat{k}$

$\bar{b}=3\left(\hat{i}-2\hat{j}+3\hat{k}\right)$

$\bar{b}=3\bar{a}$ Here $\bar{a}$ and $\bar{b}$ linearly dependent vectors. Hence $\bar{a}$ and $\bar{b}$ are collinear.

(ii) $\bar{a}=-2\hat{i}-4\hat{k}$, $\bar{b}=\hat{i}-2\hat{j}-\hat{k}$, $\bar{c}=\hat{i}-4\hat{j}+3\hat{k}$

Let $x\bar{a}+y\bar{b}+z\bar{c}=\bar{0}$

$\therefore x\left(-2\hat{i}-4\hat{k}\right)+y\left(\hat{i}-2\hat{j}-\hat{k}\right)+z\left(\hat{i}-4\hat{j}+3\hat{k}\right)=\bar{0}$

$-2x+y+z=0$

$-2y-4z=0$

$-4x-y+3z=0$

$\therefore x=0,y=0,z=0$. Here $\bar{a}$, $\bar{b}$ and $\bar{c}$ are linearly independent vectors.

Hence $\bar{a}$, $\bar{b}$ and $\bar{c}$ are non-coplanar.

Ex. 10. If $\bar{a} = 4\hat{i}+3\hat{k}$ and $\bar{b} = -2\hat{i}+\hat{j}+5\hat{k}$ find (i) $|\bar{a}|$, (ii) $\bar{a}+\bar{b}$, (iii) $\bar{a}-\bar{b}$, (iv) $3b$, (v) $2\bar{a}+5\bar{b}$

Solution : (i) $|\bar{a}| = \sqrt{4^2+0^2+3^2} = \sqrt{25} = 5$

(ii) $\bar{a}+\bar{b} = \left(4\hat{i}+3\hat{k}\right)+\left(-2\hat{i}+\hat{j}+5\hat{k}\right) = 2\hat{i}+\hat{j}+8\hat{k}$

(iii) $\bar{a}-\bar{b} = \left(4\hat{i}+3\hat{k}\right)-\left(-2\hat{i}+\hat{j}+5\hat{k}\right) = 6\hat{i}-\hat{j}-2\hat{k}$

(iv) $3\bar{b} = 3\left(-2\hat{i}+\hat{j}+5\hat{k}\right) = -6\hat{i}+3\hat{j}+15\hat{k}$

(v) $2\bar{a}+5\bar{b} = 2\left(4\hat{i}+3\hat{k}\right)+5\left(-2\hat{i}+\hat{j}+5\hat{k}\right)$

$$= \left(8\hat{i}+6\hat{k}\right)+\left(-10i+5j+25k\right) = -2\hat{i}+5\hat{j}+31\hat{k}$$

Ex. 11. What is the distance from the point (2, 3, 4) to (i) the XY plane? (ii) the X-axis? (iii) origin (iv) point (-2, 7, 3).

Solution :

(a) The distance from (2, 3, 4) to the XY plane is $|z| = 4$ units.

(b) The distance from (2, 3, 4) to the X-axis is $\sqrt{y^2+z^2} = \sqrt{3^2+4^2} = \sqrt{25} = 5$ units.

(c) The distance from $(x, y, z) \equiv (2, 3, 4)$ to origin (0, 0, 0) is

$\sqrt{x^2+y^2+z^2} = \sqrt{2^2+3^2+4^2} = \sqrt{4+9+16} = \sqrt{29}$ units.

(d) The distance from (2, 3, 4) to (-2, 7, 3) is

$\sqrt{(2+2)^2+(3-7)^2+(4-3)^2} = \sqrt{16+16+9} = \sqrt{41}$ units.

Ex. 12. Prove that the line segment joining the midpoints of two sides of a triangle is parallel to and half of the third side.

Solution : Let the triangle be ABC. If M and N are the midpoints of AB and AC respectively, then

$\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AB}$ and $\overrightarrow{AN} = \frac{1}{2}\overrightarrow{AC}$. Thus by triangle law

$\overrightarrow{AN} = \overrightarrow{AM} + \overrightarrow{MN}$

$\therefore \overrightarrow{MN} = \overrightarrow{AN} - \overrightarrow{AM} = \dfrac{\overrightarrow{AC}-\overrightarrow{AB}}{2} = \dfrac{\overrightarrow{BC}}{2}$

Thus, $\overrightarrow{MN}$ is parallel to and half as long as $\overrightarrow{BC}$.

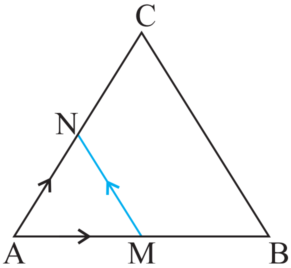


Fig 5.32

Ex. 13. In quadrilateral ABCD, M and N are the mid-points of the diagonals AC and BD respectively. Prove that $\overline{AB} + \overline{AD} + \overline{CB} + \overline{CD} = 4\,\overline{MN}$

Solution : $\overline{AB} = \overline{AM} + \overline{MN} + \overline{NB}$ ... (1)

$\overline{AD} = \overline{AM} + \overline{MN} + \overline{ND}$ ... (2)

$\overline{CB} = \overline{CM} + \overline{MN} + \overline{NB}$ ... (3)

$\overline{CD} = \overline{CM} + \overline{MN} + \overline{ND}$ ... (4)

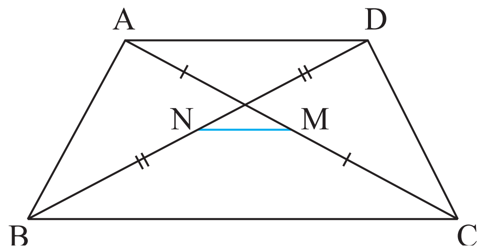


Fig 5.33

Add (1), (2), (3), (4) to get

$$\overline{AB} + \overline{AD} + \overline{CB} + \overline{CD} = 2\,\overline{AM} + 2\,\overline{CM} + 4\,\overline{MN} + 2\,\overline{NB} + 2\,\overline{ND}$$

$$= 2(\overline{AM} + \overline{CM}) + 4\,\overline{MN} + 2(\overline{NB} + \overline{ND})$$

$$= 2(\overline{AM} - \overline{AM}) + 4\,\overline{MN} + 2(\overline{NB} - \overline{NB}) \ (\because\ \overline{MC} = \overline{AM} \text{ and } \overline{DN} = \overline{NB})$$

$$= 2 \times (\overline{0}) + 4\overline{MN} + 2 \times (\overline{0})$$

$$= 4\,\overline{MN}$$

Ex. 14. Express $-\hat{i} - 3\hat{j} + 4\hat{k}$ as the linear combination of the vectors $2\hat{i} + \hat{j} - 4\hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + \hat{j} - 2\hat{k}$.

Solution : Let $\overline{r} = -\hat{i} - 3\hat{j} + 4\hat{k}$ $\overline{a} = 2\hat{i} + \hat{j} - 4\hat{k}$, $\overline{b} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\overline{c} = 3\hat{i} + \hat{j} - 2\hat{k}$

Consider $\overline{r} = x\overline{a} + y\overline{b} + z\overline{c}$

$$-\hat{i} - 3\hat{j} + 4\hat{k} = x\left(2\hat{i} + \hat{j} - 4\hat{k}\right)\hat{i} + \left(2\hat{i} - \hat{j} + 3\hat{k}\right) + z\left(3\hat{i} + \hat{j} - 2\hat{k}\right)$$

$$-\hat{i} - 3\hat{j} + 4\hat{k} = (2x + 2y + 3z)\hat{i} + (x - y + z)\hat{j} + (-4x + 3y - 2z)\hat{k}$$

By equality of vectors, we get $-1 = 2x + 2y + 3z$, $-3 = x - y + z$, $4 = -4x + 3y - 2z$,

Using Cramer's rule we get, $x = 2, y = 2, z = 3$. Therefore $\overline{r} = 2\,\overline{a} + 2\,\overline{b} - 3\,\overline{c}$

Ex. 15. Show that the three points A(1, –2, 3), B(2, 3, –4) and C(0, –7, 10) are collinear.

Solution : If $\overline{a}$, $\overline{b}$ and $\overline{c}$ are the position vectors of the points A, B and C respectively, then

$$\overline{a} = \hat{i} - 2\hat{j} + 3\hat{k}$$

$$\overline{b} = 2\hat{i} + 3\hat{j} - 4\hat{k}$$

$$\overline{c} = 0\hat{i} - 7\hat{j} + 10\hat{k}$$

$$\overline{AB} = \overline{b} - \overline{a} = \hat{i} + 5\hat{j} - 7\hat{k} \qquad \ldots(1) \quad \text{and}$$

$$\overline{AC} = \overline{c} - \overline{a}$$

$$= -\hat{i} - 5\hat{j} + 7\hat{k}$$

$$= (-1)\left[\hat{i} + 5\hat{j} - 7\hat{k}\right]$$

$$\overline{AC} = (-1)\,\overline{AB} \qquad \ldots\text{from}(1)$$

That is, $\overline{AC}$ is a scalar multiple of $\overline{AB}$. Therefore, they are parallel. But point A is in common. Hence, the points A, B and C are collinear.

Ex. 16. Show that the vectors $4\hat{i}+13\hat{j}-18\hat{k}$, $\hat{i}-2\hat{j}+3\hat{k}$ and $2\hat{i}+3\hat{j}-4\hat{k}$ are coplanar.

Solution : Let , $\bar{a}=4\hat{i}+13\hat{j}-18\hat{k},\ \bar{b}=\hat{i}-2\hat{j}+3\hat{k},\ \bar{c}=2\hat{i}+3\hat{j}-4\hat{k}$

Consider $\bar{a}=m\bar{b}+n\bar{c}$

$$\therefore\ 4\hat{i}+13\hat{j}-18\hat{k}=m\left(\hat{i}-2\hat{j}+3\hat{k}\right)+n\left(2\hat{i}+3\hat{j}-4\hat{k}\right)$$

$$\therefore\ 4\hat{i}+13\hat{j}-18\hat{k}=(m+2n)\hat{i}+(-2m+3n)\hat{j}+(3m-4n)\hat{k}$$

By equality of two vectors, we have

$$m+2n=4 \quad \ldots(1)$$

$$-2m+3n=13 \quad \ldots(2)$$

$$3m-4n=-18 \quad \ldots(3)$$

Solving (1) and (2) we get, $\therefore m=-2,\quad n=3$

These values of m and n satisfy equation (3) also.

$$\therefore \bar{a}=-2\bar{b}+3\bar{c}$$

Therefore, $\bar{a}$ is a linear combination of $\bar{b}$ and $\bar{c}$. Hence, $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar.

Exercise 5.1

1. The vector $\bar{a}$ is directed due north and $|\bar{a}|=24$. The vector $\bar{b}$ is directed due west and $|\bar{b}|=7$. Find $|\bar{a}+\bar{b}|$.

2. In the triangle PQR, $\overline{PQ}=2\bar{a}$ and $\overline{QR}=2\bar{b}$. The mid-point of PR is M. Find following vectors in terms of $\bar{a}$ and $\bar{b}$.

   (i) $\overline{PR}$ (ii) $\overline{PM}$ (iii) $\overline{QM}$

3. OABCDE is a regular hexagon. The points A and B have position vectors $\bar{a}$ and $\bar{b}$ respectively, referred to the origin O. Find, in terms of $\bar{a}$ and $\bar{b}$ the position vectors of C, D and E.

4. If ABCDEF is a regular hexagon, show that

   $\overline{AB}+\overline{AC}+\overline{AD}+\overline{AE}+\overline{AF}=6\overline{AO}$, where O is the center of the hexagon.

5. Check whether the vectors $2\hat{i}+2\hat{j}+3\hat{k}$, $-3\hat{i}+3\hat{j}+2\hat{k}$ and $3\hat{i}+4\hat{k}$ form a triangle or not.

6. In the figure 5.34 express $\bar{c}$ and $\bar{d}$ in terms of $\bar{a}$ and $\bar{b}$.

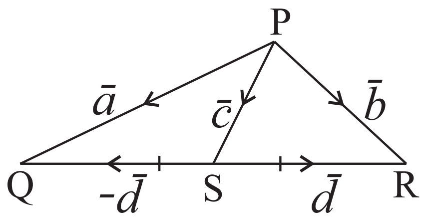


Fig 5.34

   Find a vector in the direction of $\bar{a}=\hat{i}-2\hat{j}$ that has magnitude 7 units.

7. Find the distance from (4, -2, 6) to each of the following :

   (a) The XY-plane (b) The YZ-plane

   (c) The XZ-plane (d) The X-axis

   (e) The Y-axis (f) The Z-axis

8. Find the coordinates of the point which is located :

   (a) Three units behind the YZ-plane, four units to the right of the XZ-plane and five units above the XY-plane.

   (b) In the YZ-plane, one unit to the right of the XZ-plane and six units above the XY-plane.

9. Find the area of the triangle with vertices (1, 1, 0), (1, 0, 1) and (0, 1, 1).

10. If $\overline{\text{AB}} = 2\hat{i} - 4\hat{j} + 7\hat{k}$ and initial point $\text{A} \equiv (1,5,0)$. Find the terminal point B.

11. Show that the following points are collinear :

    (i) A (3, 2, -4), B (9, 8, -10), C (-2, -3, 1).

    (ii) P (4, 5, 2), Q (3, 2, 4), R (5, 8, 0).

12. If the vectors $2\hat{i} - q\hat{j} + 3\hat{k}$ and $4\hat{i} - 5\hat{j} + 6\hat{k}$ are collinear, then find the value of q.

13. Are the four points A(1, –1, 1), B(–1, 1, 1), C(1, 1, 1) and D(2, –3, 4) coplanar? Justify your answer.

14. Express $-\hat{i} - 3\hat{j} + 4\hat{k}$ as linear combination of the vectors $2\hat{i} + \hat{j} - 4\hat{k}$, $2\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + \hat{j} - 2\hat{k}$.

5.2.1 Section Formula :

Theorem 5 : (Section formula for internal division) Let A($\bar{a}$) and B($\bar{b}$) be any two points in the space and R($\bar{r}$) be a point on the line segment AB dividing it internally in the ratio $m : n$.

Then $\bar{r} = \dfrac{m\bar{b} + n\bar{a}}{m+n}$

Proof : As R is a point on the line segment AB (A-R-B) and $\overline{\text{AR}}$ and $\overline{\text{RB}}$ are in same direction.

$$\frac{AR}{\text{RB}} = \frac{m}{n}, \text{ so n (AR) = m (RB)}$$

As $m\left(\overline{\text{RB}}\right)$ and $n\left(\overline{\text{AR}}\right)$ have same direction and magnitude,

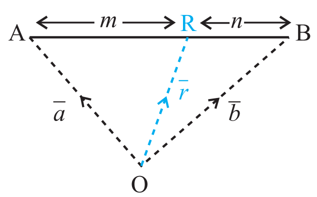


Fig 5.35

$\therefore\ m\left(\overline{\text{RB}}\right) = n\left(\overline{\text{AR}}\right)$

$\therefore\ m\left(\overline{\text{OB}} - \overline{\text{OR}}\right) = n\left(\overline{\text{OR}} - \overline{\text{OA}}\right)$

$\therefore\ m\left(\bar{b} - \bar{r}\right) = n\left(\bar{r} - \bar{a}\right)$

$\therefore\ m\bar{b} - m\bar{r} = n\bar{r} - n\bar{a}$

$\therefore\ m\bar{b} + n\bar{a} = m\bar{r} + n\bar{r} = (m+n)\,\bar{r}$

$\therefore\ \bar{r} = \dfrac{m\bar{b} + n\bar{a}}{m+n}$

Note :

1. If $\text{A} \equiv (a_1, a_2, a_3), \text{B} \equiv (b_1, b_2, b_3)$ and $\text{R} \equiv (r_1, r_2, r_3)$ divides the segment AB in the ratio $m : n$, then $\bar{r}_i = \dfrac{m\bar{b}_i + n\bar{a}_i}{m+n}$, i = 1, 2, 3.

2. **(Midpoint formula)** If R($\bar{r}$) is the mid-point of the line segment AB then m = n so $m : n = 1 : 1$,

$$\therefore\ \bar{r} = \frac{1\bar{b} + 1\bar{a}}{1+1} \text{ that is } \bar{r} = \frac{\bar{a}+\bar{b}}{2}.$$

Theorem 6 : (Section formula for external division) Let A($\bar{a}$) and B($\bar{b}$) be any two points in the space and R($\bar{r}$) be the third point on the line AB dividing the segment AB externally in the ratio $m : n$. Then $\bar{r} = \dfrac{m\bar{b} - n\bar{a}}{m-n}$.

Proof : As the point R divides line segment AB externally, we have either A-B-R or R-A-B.

Assume that A-B-R and $\overline{AR} : \overline{BR} = m : n$

$\therefore \frac{AR}{BR} = \frac{m}{n}$ so n(AR) = m (BR)

As $n\left(\overline{AR}\right)$ and $m\left(\overline{BR}\right)$ have same magnitude and direction,

$\therefore n\left(\overline{AR}\right) = m\left(\overline{BR}\right)$

$\therefore n\left(\bar{r} - \bar{a}\right) = m\left(\bar{r} - \bar{b}\right)$

$\therefore n\bar{r} - n\bar{a} = m\bar{r} - m\bar{b}$

$\therefore m\bar{b} - n\bar{a} = m\bar{r} - n\bar{r} = \left(m-n\right)\bar{r}$

$\therefore \bar{r} = \dfrac{m\bar{b} - n\bar{a}}{m-n}$

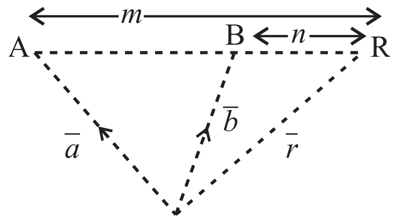


Fig 5.36

Note : 1) Whenever the ratio in which point R divides the join of two points A and B is required, it is convenient to take the ratio as $k : 1$.

Then, $\bar{r} = \dfrac{k\bar{b} + \bar{a}}{k+1}$, if division is internal,

$\bar{r} = \dfrac{k\bar{b} - \bar{a}}{k-1}$, if division is external.

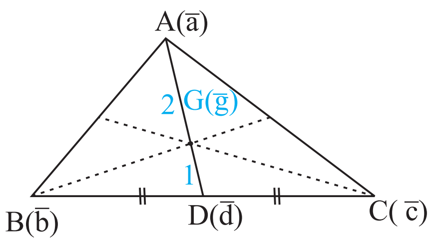


Fig 5.37

2. In ΔABC, centroid G divides the medians internally in ratio 2 : 1 and is given by (see fig. 5.37)

$$\bar{g} = \frac{\bar{a}+\bar{b}+\bar{c}}{3} \text{ (Verify).}$$

3. In tetrahedron ABCD centroid G divides the line joining the vertex of tetrahedron to centroid of opposite triangle in the ratio 3 : 1 and is given by $\bar{g} = \dfrac{\bar{a}+\bar{b}+\bar{c}+\bar{d}}{4}$.

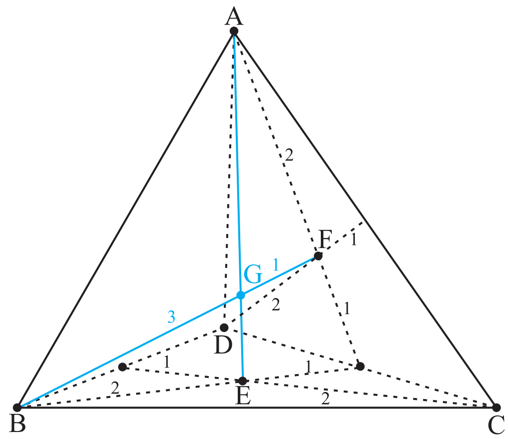


Fig 5.38

Solved Examples

Ex.1. Find the co-ordinates of the point which divides the line segment joining the points A(2, -6, 8)and B(-l, 3, -4). (i) Internally in the ratio 1 : 3. (ii) Externally in the ratio 1: 3.

Solution : If $\bar{a}$ and $\bar{b}$ are position vectors of the points A and B respectively, then

$\bar{a} = 2\hat{i} - 6\hat{j} + 8\hat{k}$ and $\bar{b} = -\hat{i} + 3\hat{j} - 4\hat{k}$ ·

Suppose R($\bar{r}$) is the point which divides the line segment joining the points A($\bar{a}$) and B($\bar{b}$) internally in the ratio 1 : 3 then,

$$\bar{r} = \frac{1(\bar{b}) + 3(\bar{a})}{1+3} = \frac{-1(\hat{i} + 3\hat{j} - 4\hat{k}) + 3(2\hat{i} - 6\hat{j} + 8\hat{k})}{4}$$

$$\therefore \bar{r} = \frac{5\hat{i} - 15\hat{j} + 20\hat{k}}{4}$$

$\therefore$ The coordinates of the point R are $\left(\frac{5}{4}, \frac{-15}{4}, 5\right)$.

Suppose S($\bar{s}$) is the point which divides the line joining the points A($\bar{a}$) and B($\bar{b}$) externally in the ratio 1 : 3 then,

$$\bar{s} = \frac{1\bar{b} - 3\bar{a}}{1-3} = \frac{(-\hat{i} + 3\hat{j} - 4\hat{k}) - 3(2\hat{i} - 6\hat{j} + 8\hat{k})}{-2}$$

$$\therefore \bar{s} = \frac{-7\hat{i} + 21\hat{j} - 28\hat{k}}{-2}$$

$\therefore$ The coordinates of the point S are $\left(\frac{7}{2}, \frac{-21}{2}, 14\right)$.

Ex. 2. If the three points A(3, 2, p), B(q, 8, -10), C(-2, -3, 1) are collinear then find

(i) the ratio in which the point C divides the line segment AB, (ii) the vales of p and q.

Solution : Let $\bar{a} = 3\hat{i} + 2\hat{j} + p\hat{k}$, $\bar{b} = q\hat{i} + 8\hat{j} - 10\hat{k}$ and $\bar{c} = -2\hat{i} - 3\hat{j} + \hat{k}$.

Suppose the point C divides the line segment AB in the ratio t : 1,

then by section formula, $\bar{c} = \frac{t\bar{b} + 1\bar{a}}{t+1}$.

$$\therefore \quad -2\hat{i} - 3\hat{j} + \hat{k} = \frac{t(q\hat{i} + 8\hat{j} - 10\hat{k}) + 1(3\hat{i} + 2\hat{j} + p\hat{k})}{t+1}$$

$$\therefore \quad (-2\hat{i} - 3\hat{j} + \hat{k})(t+1) = (tq+3)\hat{i} + (8t+2)\hat{j} + (-10t+p)\hat{k}$$

$$\therefore \quad -2(t+1)\hat{i} - 3(t+1)\hat{j} + (t+1)\hat{k} = (tq+3)\hat{i} + (8t+2)\hat{j} + (-10t+p)\hat{k}$$

Using equality of two vectors $-2(t+1) = tq + 3$ ... (1)

$-3(t+1) = 8t + 2$ ... (2)

$t + 1 = -10t + p$ ... (3)

Now equation (2) gives $t=-\frac{5}{11}$

Put $t=-\frac{5}{11}$ in equation (1) and equation (3), we get $q = 9$ and $p = -4$.
The negative sign of t suggests that the point C divides the line segment AB externally in the ratio 5:11.

Ex.3. If A(5, 1, p), B(1, q, p) and C(1, –2, 3) are vertices of triangle and $G\left(r,-\frac{4}{3},\frac{1}{3}\right)$ is its centroid, then find the values of p, q and r.

Solution : Let $\bar{a}=5\hat{i}+\hat{j}+p\hat{k}$, $\quad \bar{b}=\hat{i}+q\hat{j}+p\hat{k}$

$\bar{c}=\hat{i}-2\hat{j}+3\hat{k}$ and $\bar{g}=r\hat{i}-\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}$

By centroid formula we have $\bar{g}=\dfrac{\bar{a}+\bar{b}+\bar{c}}{3}$

$$\therefore \quad 3\bar{g}=\bar{a}+\bar{b}+\bar{c}$$

$$\therefore \quad 3\left(r\hat{i}-\frac{4}{3}\hat{j}+\frac{1}{3}\hat{k}\right)=\left(5\hat{i}+\hat{j}+p\hat{k}\right)+\left(\hat{i}+q\hat{j}+p\hat{k}\right)+\left(\hat{i}-2\hat{j}+3\hat{k}\right)$$

$$\therefore \quad (3r)\hat{i}-4\hat{j}+\hat{k}=7\hat{i}+(q-1)\hat{j}+(2p+3)\hat{k}$$

$$\therefore \quad 3r=7, -4=q-1, 1=2p+3$$

$$\therefore \quad r=\frac{7}{3},\quad q=-3,\quad p=-1$$

Ex.4. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are the position vectors of the points A, B, C respectively and $5\bar{a}-3\bar{b}-2\bar{c}=\bar{0}$, then find the ratio in which the point C divides the line segment BA.

Solution : As $5\bar{a}-3\bar{b}-2\bar{c}=\bar{0}$

$$\therefore \quad 2\bar{c}=5\bar{a}-3\bar{b}$$

$$\therefore \quad \bar{c}=\frac{5\bar{a}-3\bar{b}}{2}$$

$$\therefore \quad \bar{c}=\frac{5\bar{a}-3\bar{b}}{5-3}$$

$\therefore$ This shows that the point C divides BA externally in the ratio 5 : 3.

Ex.5. Prove that the medians of a triangle are concurrent.

Solution : Let A, B and C be vertices of a triangle. Let D, E and F be the mid-points of the sides BC, AC and AB respectively. Let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ and $\bar{f}$ be position vectors of points A, B, C, D, E and F respectively.

Therefore, by mid-point formula,

$$\therefore \bar{d}=\frac{\bar{b}+\bar{c}}{2},\quad \bar{e}=\frac{\bar{a}+\bar{c}}{2} \quad \text{and} \quad \bar{f}=\frac{\bar{a}+\bar{b}}{2}$$

$$\therefore 2\bar{d}=\bar{b}+\bar{c},\ 2\bar{e}=\bar{a}+\bar{c} \text{ and } 2\bar{f}=\bar{a}+\bar{b}$$

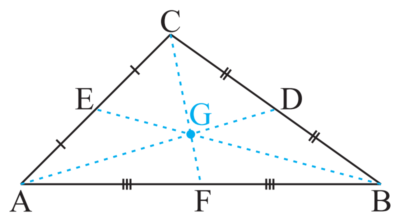


Fig 5.39

$\therefore 2\bar{d}+\bar{a}=\bar{a}+\bar{b}+\bar{c}$, *similarly* $2\bar{e}+\bar{b}=2\bar{f}+\bar{c}=\bar{a}+\bar{b}+\bar{c}$

$$\therefore \frac{2\bar{d}+\bar{a}}{3}=\frac{2\bar{e}+\bar{b}}{3}=\frac{2\bar{f}+\bar{c}}{3}=\frac{\bar{a}+\bar{b}+\bar{c}}{3}=\bar{g} \text{ (say)}$$

Then we have $\bar{g}=\dfrac{\bar{a}+\bar{b}+\bar{c}}{3}=\dfrac{(2)\bar{d}+(1)\bar{a}}{2+1}=\dfrac{(2)\bar{e}+(1)\bar{b}}{2+1}=\dfrac{(2)\bar{f}+(1)\bar{c}}{2+1}$

If G is the point whose position vector is $\bar{g}$, then from the above equation it is clear that the point G lies on the medians AD, BE, CF and it divides each of the medians AD, BE, CF internally in the ratio 2 : 1.

Therefore, three medians are concurrent.

Ex. 6. Prove that the angle bisectors of a triangle are concurrent.

Solution :

Let A, B and C be vertices of a triangle. Let AD, BE and CF be the angle bisectors of the triangle ABD. Let $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{e}$ and $\bar{f}$ be the position vectors of the points A, B, C, D, E and F respectively. Also AB = z BC = x AC = y. Now, the angle bisector AD meets the side BC at the point D. Therefore, the point D divides the line segment BC internally in the ratio AB : AC, that is $z : y$.

Hence, by section formula for internal division, we have $\bar{d}=\dfrac{z\bar{c}+y\bar{b}}{z+y}$

Similarly, we get

$$\bar{e}=\frac{x\bar{a}+z\bar{c}}{x+z} \quad \text{and} \quad \bar{f}=\frac{y\bar{b}+x\bar{a}}{y+x}$$

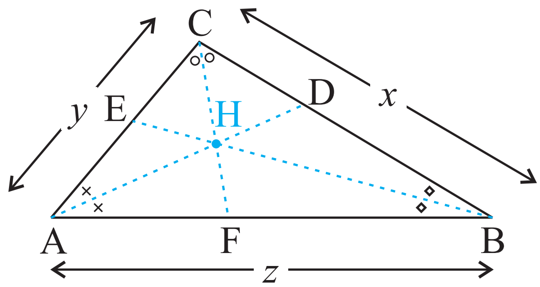


Fig 5.40

As $\quad \bar{d}=\dfrac{z\bar{c}+y\bar{b}}{z+y}$

$\therefore \quad (z+y)\bar{d}=z\bar{c}+y\bar{b}$

i.e. $\quad (z+y)\bar{d}+x\bar{a}=x\bar{a}+y\bar{b}+z\bar{c}$

similarly $\quad (x+z)\bar{e}+y\bar{b}=x\bar{a}+y\bar{b}+z\bar{c}$

and $\quad (x+y)\bar{f}+z\bar{c}=x\bar{a}+y\bar{b}+z\bar{c}$

$$\therefore \quad \frac{(z+y)\bar{d}+x\bar{a}}{x+y+z}=\frac{(x+z)\bar{e}+y\bar{b}}{x+y+z}=\frac{(x+y)\bar{f}+z\bar{c}}{x+y+z}=\frac{x\bar{a}+y\bar{b}+z\bar{c}}{x+y+z}=\bar{h} \text{ (say)}$$

Then we have

$$\bar{h}=\frac{(y+z)\bar{d}+x\bar{a}}{(y+z)+x}=\frac{(x+z)\bar{e}+y\bar{b}}{(x+z)+y}=\frac{(x+y)\bar{f}+z\bar{c}}{(x+y)+z}$$

That is point H($\bar{h}$) divides AD in the ratio $(y+z):x$, BE in the ratio $(x+z):y$ and CF in the ratio $(x+y):z$.

This shows that the point H is the point of concurrence of the angle bisectors AD, BE and CF of the triangle ABC, Thus, the angle bisectors of a triangle are concurrent and H is called incentre of the triangle ABC.

Ex. 7. Using vector method, find the incenter of the triangle whose vertices are A(0, 3, 0), B(0, 0, 4) and C(0, 3, 4).

Solution :

Let $\bar{a} = 3\hat{j}$, $\bar{b} = 4\hat{k}$ and $\bar{c} = 3\hat{j} + 4\hat{k}$

$\therefore \quad \overline{AB} = \bar{b} - \bar{a} = -3\hat{j} + 4\hat{k}$, $\overline{AC} = \bar{c} - \bar{a} = 4\hat{k}$, $\overline{BC} = \bar{c} - \bar{b} = 3\hat{j}$

$\therefore \quad |\overline{AB}| = 5, |\overline{AC}| = 4, |\overline{BC}| = 3$

If H ($\bar{h}$) is the incenter of triangle ABC then,

$$\therefore \quad \bar{h} = \frac{|\overline{BC}|\bar{a} + |\overline{AC}|\bar{b} + |\overline{AB}|\bar{c}}{|\overline{BC}| + |\overline{AC}| + |\overline{AB}|}$$

$$\therefore \quad \bar{h} = \frac{3(3j) + 4(4k) + 5(3j + 4k)}{3 + 4 + 5}$$

$$\therefore \quad = \frac{9j + 16k + 15j + 20k}{12}$$

$$\therefore \quad \bar{h} = \frac{24\hat{j} + 36\hat{k}}{12}$$

$$\therefore \quad \bar{h} = 2\hat{j} + 3\hat{k}$$

And $\quad H \equiv (0, 2, 3)$

Note : In ΔABC,

1) P.V. of Centroid is given by $\dfrac{\bar{a} + \bar{b} + \bar{c}}{3}$

2) P.V. of Incentre is given by $\dfrac{|\overline{AB}|\bar{c} + |\overline{BC}|\bar{a} + |\overline{AC}|\bar{b}}{|\overline{AB}| + |\overline{BC}| + |\overline{AC}|}$

3) P.V. of Orthocentre is given by $\dfrac{\tan A\bar{a} + \tan B\bar{b} + \tan C\bar{c}}{\tan A + \tan B + \tan C}$ (Verify)

Ex. 8. If $4\hat{i} + 7\hat{j} + 8\hat{k}, 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $2\hat{i} + 5\hat{j} + 7\hat{k}$ are the position vectors of the vertices A, B and C respectively of triangle ABC. Find the position vector of the point in which the bisector of $\angle$A meets BC.

Solution : $\bar{a} = 4\hat{i} + 7\hat{j} + 8\hat{k}$, $\bar{b} = 2\hat{i} + 3\hat{j} + 4\hat{j}$, $\bar{c} = 2i + 5j + 7k$

$\overline{AC} = \bar{c} - \bar{a} = -2\hat{i} - 2\hat{j} - \hat{k}$

$\overline{AB} = \bar{b} - \bar{a} = -2\hat{i} - 4\hat{j} - 4\hat{k}$

$\therefore |\overline{AB}| = \sqrt{4 + 16 + 16} = 6$ units

$\therefore |\overline{AC}| = \sqrt{4 + 4 + 1} = 3$ units

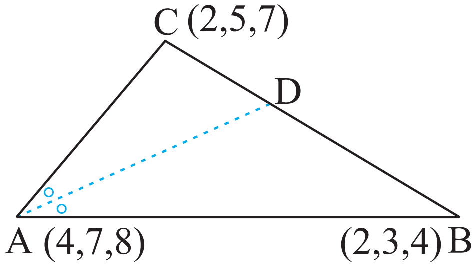


Fig 5.41

Let D be the point where angle bisector of $\angle$A meets BC.

D divides BC in the ratio AB : AC

i.e. $\bar{d} = \dfrac{|\overline{AB}|\bar{c} + |\overline{AC}|\bar{b}}{|\overline{AB}| + |\overline{AC}|}$

$= \dfrac{6\left(2\hat{i} + 5\hat{j} + 7\hat{k}\right) + 3\left(2\hat{i} + 3\hat{j} + 4\hat{k}\right)}{6+3}$

$= \dfrac{\left(12\hat{i} + 30\hat{j} + 42\hat{k}\right) + \left(6\hat{i} + 9\hat{j} + 12\hat{k}\right)}{9}$

$= \dfrac{18\hat{i} + 39\hat{j} + 54\hat{k}}{9}$

$\therefore \bar{d} = 2\hat{i} + \dfrac{13}{3}\hat{j} + 6\hat{k}$

Ex.9. If G(a, 2, -1) is the centroid of the triangle with vertices P(1, 2, 3), Q(3, b, -4) and R(5, 1, c) then find the values of a, b and c.

Solution : As $G(\bar{g})$ is centroid of ΔPQR $\bar{g} = \dfrac{\bar{p} + \bar{q} + \bar{r}}{3}$

$$a\hat{i} + 2\hat{j} - \hat{k} = \frac{\left(\hat{i} + 3\hat{j} + 2\hat{k}\right) + \left(3\hat{i} + b\hat{j} - 4\hat{k}\right) + \left(5\hat{i} + \hat{j} + c\hat{k}\right)}{3} = \frac{(1+3+5)\hat{i} + (3+b+1)\hat{j} + (2-4+c)\hat{k}}{3}$$

by equality of vectors $a = \dfrac{1+3+5}{3} = \dfrac{9}{3} = 3$ $\therefore a = 3$

$\therefore 2 = \dfrac{3+b+1}{3}$ $\therefore 6 = 4 + b$ $\therefore b = 2$

$\therefore -1 = \dfrac{2-4+c}{3}$ $\therefore -3 = -2 + c$ $\therefore c = -1$.

Ex. 10. Find the centroid of tetrahedron with vertices A(3, -5, 7), B(5, 4, 2), C(7, -7, -3), D(1, 0, 2) ?

Solution : Let $\bar{a} = 3\hat{i} - 5\hat{j} + 7\hat{k}, \bar{b} = 5\hat{i} + 4\hat{j} + 2\hat{k}, \bar{c} = 7\hat{i} - 7\hat{j} - 3\hat{k}, \bar{d} = \hat{i} + 2\hat{k}$ be position vectors of vertices A, B, C & D.

By centroid formula, centroid G($\bar{g}$) is given by

$\bar{g} = \dfrac{\bar{a} + \bar{b} + \bar{c} + \bar{d}}{4}$

$= \dfrac{\left(3\hat{i} - 5\hat{j} + 7\hat{k}\right) + \left(5\hat{i} + 4\hat{j} + 2\hat{k}\right) + \left(7\hat{i} - 7\hat{j} - 3\hat{k}\right) + \left(\hat{i} + 2\hat{k}\right)}{4}$

$= \dfrac{(3+5+7+1)\hat{i} + (-5+4-7+0)\hat{j} + (7+2-3+2)\hat{k}}{4} = \dfrac{16\hat{i} - 8\hat{j} + 8\hat{k}}{4}$

$= 4\hat{i} - 2\hat{j} + 2\hat{k}$

Therefore, centroid of tetrahedron is G ≡ (4, –2, 2).

Ex. 11. Find the ratio in which point P divides AB and CD where A(2, –3, 4), B(0, 5, 2), C(–1, 5, 3) and D (2, –1, 3). Also, find its coordinates.

Solution : Let point P divides AB in ratio $m : 1$ and CD in ratio $n : 1$.

By section formula,

$$P \equiv \left(\frac{2}{m+1}, \frac{5m-3}{m+1}, \frac{2m+4}{m+1}\right) \equiv \left(\frac{2n-1}{n+1}, \frac{n+5}{n+1}, \frac{3n+3}{n+1}\right)$$

Equating z-coordinates

$$\therefore \quad \frac{2m+4}{m+1} = \frac{3n+3}{n+1}$$

$$\frac{2m+4}{m+1} = \frac{3(n+1)}{(n+1)}$$

$$2m + 4 = 3(m + 1)$$

$$2m + 4 = 3m + 3$$

$$1 = m$$

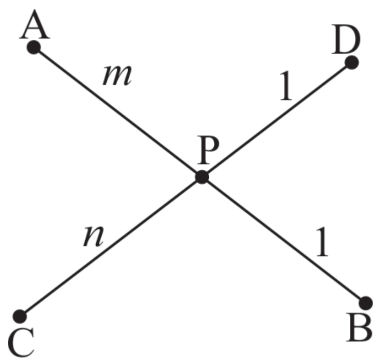


Fig 5.42

Also, by equating x-coordinates

$$\frac{2}{m+1} = \frac{2n-1}{n+1}$$

$$\frac{2}{1+1} = \frac{2n-1}{n+1} (m=1)$$

$$n + 1 = 2n - 1$$

$$2 = n$$

P divides AB in ratio $m : 1$ *i.e.* 1 : 1 and CD in the ratio $n : 1$ *i.e.* 2 : 1.

$$P \equiv \left(\frac{2}{1+1}, \frac{5-3}{1+1}, \frac{2+4}{1+1}\right) \equiv (1,1,3).$$

Ex. 12. In a triangle ABC, D and E are points on BC and AC respectively, such that BD = 2 DC and AE = 3 EC. Let P be the point of intersection of AD and BE. Find BP/PF using vector methods.

Solution : Let $\bar{a}$, $\bar{b}$, $\bar{c}$ be the position vectors of A, B and C respectively with respect to some origin.

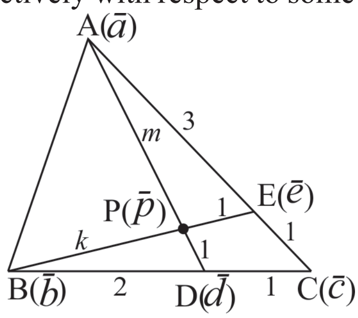


Fig 5.43

D divides BC in the ratio 2 : 1 and E divides AC in the ratio 3 : 1.

$$\therefore \bar{d} = \frac{\vec{b}+2\vec{c}}{3} \quad \bar{e} = \frac{\vec{a}+3\vec{c}}{4}.$$

Let point of intersection P of AD and BE divides BE in the ratio $k : 1$ and AD in the ratio $m : 1$, then position vectors of P in these two cases are $\dfrac{\vec{b}+k\left(\dfrac{\vec{a}+3\vec{c}}{4}\right)}{k+1}$ and $\dfrac{\vec{a}+m\left(\dfrac{\vec{b}+2\vec{c}}{3}\right)}{m+1}$ and respectively.

Equating the position vectors of P we get,

$$\frac{k}{4(k+1)}\bar{a}+\frac{1}{k+1}\bar{b}+\frac{3k}{4(k+1)}\bar{c} = \frac{1}{m+1}\bar{a}+\frac{m}{3(m+1)}\bar{b}+\frac{2m}{3(m+1)}\bar{c}$$

$$\therefore \frac{k}{4(k+1)}=\frac{1}{m+1} \qquad \text{... (1)}$$

$$\frac{1}{k+1}=\frac{m}{3(m+1)} \qquad \text{... (2)}$$

$$\frac{3k}{4(k+1)}=\frac{2m}{3(m+1)} \qquad \text{... (3)}$$

Dividing (3) by (2) we get,

$$\frac{3k}{4}=2 \text{ i.e. } k=\frac{8}{3} \quad \text{therefore } \frac{BP}{PF}=k:1=8:3$$

Exercise 5.2

1. Find the position vector of point R which divides the line joining the points P and Q whose position vectors are $2\hat{i}-\hat{j}+3\hat{k}$ and $-5\hat{i}+2\hat{j}-5\hat{k}$ in the ratio 3 : 2 (i) internally (ii) externally.
2. Find the position vector of mid-point M joining the points L (7, -6,12) and N (5, 4, -2).
3. If the points A(3, 0, p), B (-1, q, 3) and C(-3, 3, 0) are collinear, then find
   (i) The ratio in which the point C divides the line segment AB.
   (ii) The values of p and q.
4. The position vector of points A and B are $6\bar{a}+2\bar{b}$ and $\bar{a}-3\bar{b}$. If the point C divides AB in the ratio 3 : 2 then show that the position vector of C is $3\bar{a}-\bar{b}$.
5. Prove that the line segments joining mid-point of adjacent sides of a quadrilateral form a parallelogram.
6. D and E divide sides BC and CA of a triangle ABC in the ratio 2 : 3 respectively. Find the position vector of the point of intersection of AD and BE and the ratio in which this point divides AD and BE.
7. Prove that a quadrilateral is a parallelogram if and only if its diagonals bisect each other.
8. Prove that the median of a trapezium is parallel to the parallel sides of the trapezium and its length is half the sum of parallel sides.
9. If two of the vertices of the triangle are A(3, 1, 4) and B(-4, 5, -3) and the centroid of a triangle is G(-1, 2, 1), then find the coordinates of the third vertex C of the triangle.
10. In ΔOAB, E is the mid-point of OB and D is the point on AB such that AD : DB = 2 : 1. If OD and AE intersect at P, then determine the ratio OP : PD using vector methods.
11. If the centroid of a tetrahedron OABC is (1, 2, −1) where A = (a, 2, 3), B = (1, b, 2), C = (2, 1, c) respectively, find the distance of P (a, b, c) from the origin.
12. Find the centroid of tetrahedron with vertices K(5, −7, 0), L(1, 5, 3), M(4, – 6, 3), N(6, −4, 2) ?

5.3 Product of vectors :

The product of two vectors is defined in two different ways. One form of product results in a scalar quantity while other form gives a vector quantity. Let us study these products and interpret them geometrically.

Angle between two vectors :

When two non zero vectors $\bar{a}$ and $\bar{b}$ are placed such that their initial points coincide, they form an angle θ of measure $0 \le \theta \le \pi$.

Angle between $\bar{a}$ and $\bar{b}$ is also denoted as $\widehat{\bar{a}\ \bar{b}}$. The angle between the collinear vectors is 0 if they point in the same direction and π if they are in opposite directions.

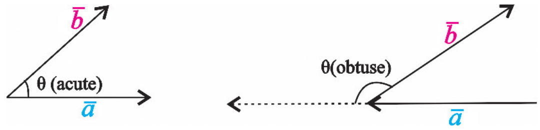


Fig 5.44

5.3.1 Scalar product of two vectors :

The scalar product of two non-zero vectors $\bar{a}$ and $\bar{b}$ is denoted by $\bar{a}\cdot\bar{b}$, and is defined as $\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|$ cos θ, where θ is the angle between $\bar{a}$ and $\bar{b}$.

$\bar{a}\cdot\bar{b}$ is a real number, that is, a scalar. For this reason, the dot product is also called a scalar product.

Note :

1) If either $\bar{a} = 0$ or $\bar{b} = 0$ then θ is not defined and in this case, we define $\vec{a}\cdot\vec{b} = 0$.

2) If θ = 0, then $\bar{a}\cdot\bar{b} = |\bar{a}|\,|\bar{b}|\ \cos 0 = |\bar{a}|\,|\bar{b}|$. In particular, $\bar{a}\cdot\bar{a} = |\bar{a}|^2$, as θ = 0 .

3) If θ = π, then $\bar{a}\cdot\bar{b} = |\bar{a}|\,|\bar{b}|\ \cos\pi = -|\bar{a}|\,|\bar{b}|$.

4) If $\bar{a}$ and $\bar{b}$ are perpendicular or orthogonal then θ = π/2

Conversely if $\bar{a}\cdot\bar{b} = 0$ then either $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$ or θ = π/2.

Also, $\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta = |\bar{b}||\bar{a}|\cos\theta = \bar{b}\cdot\bar{a}$.

5) Dot product is distributive over vector addition. If $\bar{a}$, $\bar{b}$, $\bar{c}$ are any three vectors, then

$\bar{a}\cdot(\bar{b}+\bar{c}) = \bar{a}\cdot\bar{b} + \bar{a}\cdot\bar{c}$.

6) If $\bar{a}$ and $\bar{b}$ are vectors and *m*, *n* are scalars, then

(i) $(m\bar{a})\cdot(n\bar{b}) = mn(\bar{a}\cdot\bar{b})$

(ii) $(m\bar{a})\cdot\bar{b} = m(\bar{a}\cdot\bar{b}) = \bar{a}\cdot(m\bar{b})$

7) $|\bar{a}\cdot\bar{b}| \le |\bar{a}||\bar{b}|$ (This is known as Cauchy Schwartz Inequality).

5.3.2 Finding angle between two vectors :

Angle θ, (0 ≤ θ ≤ π) between two non-zero vectors $\bar{a}$ and $\bar{b}$ is given by $\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$, that is $\theta = \cos^{-1}\left(\dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}\right)$.

Note :

1) If $0 \le \theta < \frac{\pi}{2}$, then $\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} > 0$, that is $\bar{a}\cdot\bar{b} > 0$.

2) If $\theta = \frac{\pi}{2}$, then $\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} = 0$, that is $\bar{a}\cdot\bar{b} = 0$.

3) If $\frac{\pi}{2} < \theta \le \pi$, then $\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|} < 0$, that is $\bar{a}\cdot\bar{b} < 0$.

4) In particular scalar product of $\hat{i}$, $\hat{j}$, $\hat{k}$ vectors are

(i) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$ and (ii) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$.

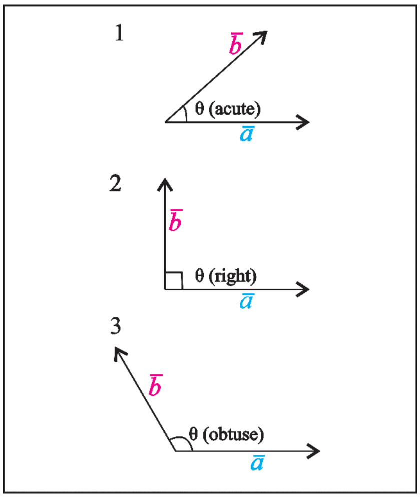


Fig 5.45

The scalar product of vectors $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\bar{a}\cdot\bar{b} = \left(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right)\cdot\left(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}\right) = a_1b_1 + a_2b_2 + a_3b_3$$

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|} = \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\ \sqrt{b_1^2 + b_2^2 b_3^2}}$$

5.3.3 Projections :

$\overline{PQ}$ and $\overline{PR}$ represent the vectors $\bar{a}$ and $\bar{b}$ with same initial point P. If M is the foot of perpendicular from R to the line containing $\overline{PQ}$ then $|\overline{PS}|$ is called the scalar projection of $\bar{b}$ on $\bar{a}$. We can think of it as a shadow of $\bar{b}$ on $\bar{a}$, when sun is overhead.

Scalar Projection of $\bar{b}$ on $\bar{a}$ $= |\overline{PS}| = |\bar{b}|\cos\theta = \frac{|\bar{a}||\bar{b}|\cos\theta}{|\bar{a}|}$

$$\boxed{\text{Scalar Projection of } \bar{b} \text{ on } \bar{a} = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|}}.$$

Vector Projection of $\bar{b}$ on $\bar{a}$ $= \overline{PS}$

$= |\overline{PS}|\hat{a}$, where $\hat{a}$ is unit vector along $\bar{a}$

$= \left(\frac{\bar{a}\cdot\bar{b}}{|\bar{a}|}\right)\hat{a}$

$= \left(\frac{\bar{a}\cdot\bar{b}}{|\bar{a}|}\right)\frac{\bar{a}}{|\bar{a}|}$

$$\boxed{\text{Vector Projection of } \bar{b} \text{ on } \bar{a} = \left(\bar{a}\cdot\bar{b}\right)\frac{\bar{a}}{|\bar{a}|^2}}.$$

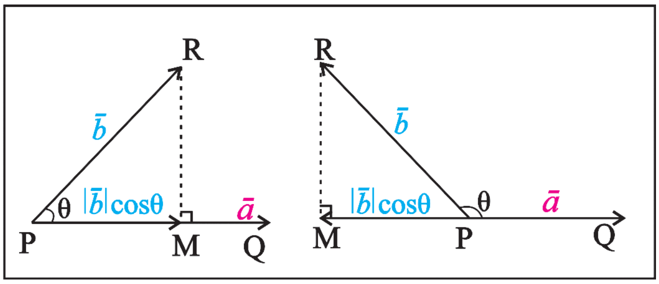


Fig 5.46

5.3.4 Direction Angles and Direction Cosines :

The direction angles of a non-zero vector $\bar{a}$ are the angles α, β,and γ $(\in[0,\pi])$ that $\bar{a}$ makes with the positive X-, Y- and Z-axes respectively. These angles completely determine the direction of the vector $\bar{a}$.

The cosines of these direction angles, that is cos α, cos β, cos γ are called the direction cosines (abbreviated as d.c.s) of vector $\bar{a}$

As α is angle between $\hat{i}$ (unit vector) along X-axis and $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ then.

$$\cos\alpha=\frac{\bar{a}\cdot\hat{i}}{|\bar{a}||\hat{i}|}=\frac{\left(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}\right)\cdot\hat{i}}{|\bar{a}|(1)}=\frac{a_1}{|\bar{a}|},\ \text{Similarly}\ \cos\beta=\frac{a_2}{|\bar{a}|}\ \text{and}\ \cos\gamma=\frac{a_3}{|\bar{a}|},$$

where $|\bar{a}|=\sqrt{a_1^{\,2}+a_2^{\,2}+a_3^{\,2}}$.

By squaring and adding, we get $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=\dfrac{a_1^{\,2}}{|\bar{a}|^2}+\dfrac{a_2^{\,2}}{|\bar{a}|^2}+\dfrac{a_3^{\,2}}{|\bar{a}|^2}$.

As $|\bar{a}|^2=a_1^{\,2}+a_2^{\,2}+a_3^{\,2}$

$\therefore\ \boxed{\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1}$

Also, $\bar{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$$=|\bar{a}|\cos\alpha\hat{i}+|\bar{a}|\cos\beta\hat{j}+|\bar{a}|\cos\gamma\hat{k}$$

$$=|\bar{a}|\left(\cos\alpha\hat{i}+\cos\beta\hat{j}+\cos\gamma\hat{k}\right)$$

That is, $\dfrac{\bar{a}}{|\bar{a}|}=\cos\alpha\hat{i}+\cos\beta\hat{j}+\cos\gamma\hat{k}=\hat{a}$

Which means that the direction cosines of $\bar{a}$, are components of the unit vector in the direction of $\bar{a}$

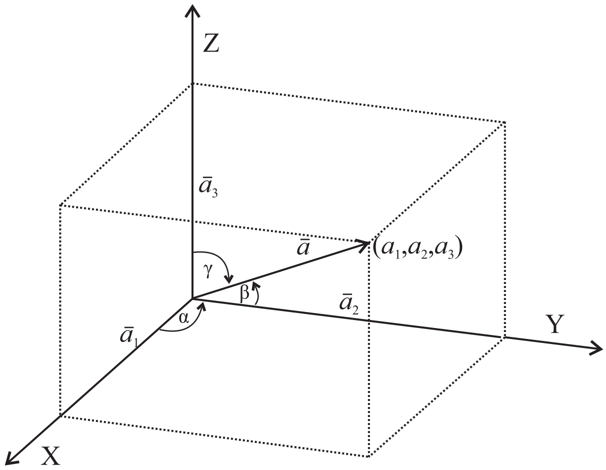


Fig 5.47

Direction cosines (d.c.s) of any line along a vector $\bar{a}$ has same direction cosines as that of $\bar{a}$.

Direction cosines are generally denoted by l, m, n, where $l=$ cos α, $m=$ cos β, $n=$ cos γ.

As the unit vectors along X, Y- and Z- axes are $\hat{i}$, $\hat{j}$, $\hat{k}$. Then $\hat{i}$ makes the direction angles $0,\dfrac{\pi}{2},\dfrac{\pi}{2}$ so its direction cosines are $\cos 0,\cos\dfrac{\pi}{2},\cos\dfrac{\pi}{2}$ that is $1, 0, 0$. Similarly direction cosines of Y-and Z- axes are $0,1,0$ and $0,0,1$ respectively.

Let $\overline{OL}$ and $\overline{OL'}$ be the vectors in the direction of line LL′. If α, β,and γ are direction angles of $\overline{OL}$ then the direction angles of $\overline{OL'}$ are π − α, π − β,and π − γ. Therefore, direction cosines of $\overline{OL}$ are cos α, cos β, cos γ *i.e.* l, m, n whereas direction cosines of $\overline{OL'}$ are cos(π − α), cos(π − β) and cos (π − γ). *i.e.* -cos α, -cos β and -cos γ. *i.e.* $-l,-m,-n$. Therefore direction cosines of line LL' are same as that of vectors $\overline{OL}$ or $\overline{OL'}$ in the direction of line LL'. i.e. either l,m,n or $-l,-m,-n$. As $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ so $l^2 + m^2 + n^2 = 1$.

Direction ratios :

Any 3 numbers which are proportional to direction cosines of the line are called the direction ratios (abbreviated as *d.r.s*) of the line. Generally the direction ratios are denoted by a, b, c.

If l, m, n are the direction cosines and a, b, c are direction ratios then

$a = \lambda l$, $b = \lambda m$, $c = \lambda n$, for some $\lambda \in R$.

For Example.: If direction cosines of the line are $0, \frac{1}{2}, \frac{\sqrt{3}}{2}$ then $0, 1, \sqrt{3}$ or $0, \sqrt{3}, 3$ or $0, 2\sqrt{3}, 6$ are also direction ratios of the same line.

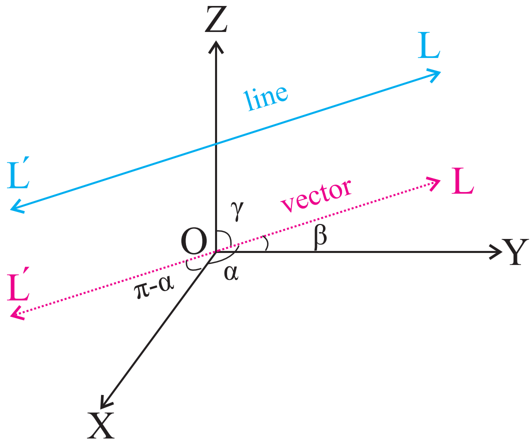


Fig 5.48

Note : A line has infinitely many direction ratios but unique direction cosines.

Relation between direction ratios and direction cosines :

Let a, b, c be direction ratios and l, m, n be direction cosines of a line.

By definition of d.r.s, $\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \lambda$ i.e. $l = \lambda a$, $m = \lambda b$, $n = \lambda c$.

But $l^2 + m^2 + n^2 = 1$

$\therefore \quad (\lambda a)^2 + (\lambda b)^2 + (\lambda c)^2 = 1$

$\therefore \quad \lambda^2(a^2 + b^2 + c^2 = 1$

$$\therefore \quad \lambda = \pm\frac{1}{\sqrt{a^2+b^2+c^2}}$$

$$\therefore \quad l = \pm\frac{a}{\sqrt{a^2+b^2+c^2}}, m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}} \text{ and } n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}.$$

Note that direction cosines are similar to the definition of unit vector, that is if $\overline{x} = a\hat{i} + b\hat{j} + c\hat{k}$ be any vector (d.r.s) then $\hat{x} = \pm\frac{\overline{x}}{|\overline{x}|} = \pm\frac{a\hat{i} + b\hat{j} + c\hat{k}}{\sqrt{a^2+b^2+c^2}}$ is unit vector (d.c.s) along $\overline{x}$.

Solved Examples

Ex.1. Find $\bar{a}\cdot\bar{b}$ if $|\bar{a}| = 3, |\bar{b}| = \sqrt{6}$, the angle between $\bar{a}$ and $\bar{b}$ is 45º.

Solution : $\bar{a}\cdot\bar{b} = |\bar{a}|\,|\bar{b}|\cos\theta = (3)(\sqrt{6})\cos 45° = 3\sqrt{6}\left(\frac{\sqrt{2}}{2}\right) = \frac{3}{2}\cdot 2\sqrt{3} = 3\sqrt{3}$

Ex.2. If $\bar{a} = 3\hat{i} + 4\hat{j} - 5\hat{k}$ and $\bar{b} = 3\hat{i} - 4\hat{j} - 5\hat{k}$

(i) find $\bar{a}\cdot\bar{b}$

(ii) the angle between $\bar{a}$ and $\bar{b}$.

(iii) the scalar projection of $\bar{a}$ in the direction of $\bar{b}$.

(iv) the vector projection of $\bar{b}$ along $\bar{a}$.

Solution : Here $\bar{a} = 3\hat{i}+4\hat{j}-5\hat{k}$ and $\bar{b} = 3\hat{i}-4\hat{j}-5\hat{k}$

i) $\bar{a}\cdot\bar{b} = (3)(3)+(4)(-4)+(-5)(-5) = 9 - 16 + 25 = 18$

ii) $|\bar{a}| = \sqrt{a+16+25} = \sqrt{50},\ |\bar{b}| = \sqrt{9+16+25} = \sqrt{50}$

The angle between $\bar{a}$ and $\bar{b}$ is $\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\,|\bar{b}|} = \frac{18}{50}\ \therefore \theta = \cos^{-1}\left(\frac{18}{50}\right)$

iii) The scalar projectio of $\bar{a}$ in the direction of $\bar{b}$ is $\frac{\bar{a}\cdot\bar{b}}{|\bar{b}|} = \frac{18}{\sqrt{50}} = \frac{18}{5\sqrt{2}}$.

iv) The vector projection of $\bar{b}$ along is $\bar{a}$ is $= (\bar{a}\cdot\bar{b})\frac{\bar{a}}{|\bar{a}|^2} = \frac{18}{50}(3\hat{i}+4\hat{j}-5\hat{k}) = \frac{9}{25}(3\hat{i}+4\hat{j}-5\hat{k})$.

Ex.3. Find the value of a for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} + a\hat{j} + 3\hat{k}$ are (i) perpendicular (ii) parallel

Solution : Let $\bar{p} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\bar{q} = \hat{i} + a\hat{j} + 3\hat{k}$

(i) The two vectors are perpendicular if $\bar{p}\cdot\bar{q} = 0$ *i.e.* $(3\hat{i}+2\hat{j}+9\hat{k})\cdot(\hat{i}+a\hat{j}+3\hat{k}) = 0$

i.e. 3(1) + 2(a) + 9(3) = 0. i.e. 2a + 30 = 0 or a = -15.

(ii) The two vectors are parallel if $\frac{3}{1} = \frac{2}{a} = \frac{9}{3}$ *i.e* $3a = 2$ *i.e.* $a = \frac{2}{3}$.

Ex.4. If $\vec{a} = \hat{i}+2\hat{j}-3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$ find the angle between the vectors $2\vec{a}+\vec{b}$ and $\vec{a} + 2\vec{b}$.

Solution : $2\bar{a} + \bar{b} = 2(\hat{i}+2\hat{j}-3\hat{k}) + (3\hat{i} - \hat{j} + 2\hat{k}) = 5\hat{i} + 3\hat{j} - 4\hat{k} = '\bar{m}'\ (say)$

$\bar{a} + 2\bar{b} = (\hat{i}+2\hat{j}-3\hat{k})+2(3\hat{i}-\hat{j}+2\hat{k}) = 7\hat{i} + \hat{k} = '\bar{n}'(say)$

$\therefore \bar{m}\cdot\bar{n} = (2\bar{a} + \bar{b})\cdot(\bar{a} + 2\bar{b}) = (5\hat{j} + 3\hat{j} - 4\hat{k})\cdot(7\hat{i} + \hat{k})$

= (5) (7) + (3) (0) + (-4) (1) = 31

$$|\overline{m}| = \sqrt{(5)^2+(3)^2+(-4)^2} = \sqrt{50}$$

$$|\overline{n}| = \sqrt{(7)^2+(0)^2+(1)^2} = \sqrt{50}$$

If θ is the angle between $\overline{m}$ and $\overline{n}$ then

$$\cos\theta = \frac{\overline{m}\cdot\overline{n}}{|\overline{m}|\,|\overline{n}|} = \frac{31}{\sqrt{50}\times\sqrt{50}} = \frac{31}{50}$$

$$\therefore \theta = \cos^{-1}\left(\frac{31}{50}\right)$$

Ex 5. : If a line makes angle 90º, 60º and 30º with the positive direction of X, Yand Z axes respectively, find its direction cosines.

Solution : Let the d.c.s. of the lines be l, m, n then $l = \cos 90º = 0$, $m = \cos 60º = \frac{1}{2}$

$n = \cos 30º = \frac{\sqrt{3}}{2}$. Therefore, l, m, n are $0, \frac{1}{2}, \frac{\sqrt{3}}{2}$

Ex. 6 : Find the vector projection of $\vec{PQ}$ on $\vec{AB}$ where P, Q, A, B are the points (-2, 1, 3), (3, 2, 5) (4, -3, 5) and (7, -5, -1) respectively.

Solution : Let the position vectors of P, Q, A, B are $\overline{p}, \overline{q}, \overline{a}, \overline{b}$ respectively

$$\overline{p} = -2\hat{i}+\hat{j}+3\hat{k},\ \overline{q} = 3\hat{i}+2\hat{j}+5\hat{k}$$
$$\overline{a} = 4\hat{i}-3\hat{j}+5\hat{k},\ \overline{b} = 7\hat{i}-5\hat{j}-\hat{k}$$

$$\therefore \quad \vec{PQ} = \overline{q}-\overline{p} = (3\hat{i}+2\hat{j}+5\hat{k})-(-2\hat{i}+\hat{j}+3\hat{k}) = 5\hat{i}+\hat{j}+2\hat{k}$$

$$\text{and } \vec{AB} = \overline{b}-\overline{a} = (7\hat{i}-5\hat{j}-\hat{k})-(4\hat{i}-3\hat{j}+5\hat{k}) = 3\hat{i}-2\hat{j}-6\hat{k}$$

∴ Vector Projection of $\overline{PQ}$ on $\overline{AB}$

$$= \frac{\overline{PQ}\cdot\overline{AB}}{|\overline{AB}|^2}\overline{AB} = \frac{(5)(3)+(1)(-2)+(2)(-6)}{(3)^2+(-2)^2+(-6)^2}\overline{AB}$$

$$= \frac{1}{49}(3\hat{i}-2\hat{j}-6\hat{k}) = \frac{3}{49}\hat{i}-\frac{2}{49}\hat{j}-\frac{6}{49}\hat{k}$$

Ex. 7 : Find the values of λ for which the angle between the vectors

$$\vec{a} = 2\lambda^2\hat{i}+4\lambda\hat{j}+\hat{k} \text{ and } \vec{b} = 7\hat{i}-2\hat{j}+\lambda\hat{k}$$

Solution : If θ is the angle between $\vec{a}$ and $\vec{b}$, then $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$

If θ is obtuse then $\cos\theta < 0$

$\therefore \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} < 0$ i.e. $\vec{a}\cdot\vec{b} < 0$ $\quad\left[\because |\vec{a}||\vec{b}| > 0\right]$

$\therefore \left(2\lambda^2\hat{i} + 4\lambda\hat{j} + \hat{k}\right)\cdot\left(7\hat{i} - 2\hat{j} + \lambda\hat{k}\right) < 0$

$\therefore\ 14\lambda^2 - 8\lambda + \lambda < 0$ i.e. $14\lambda^2 - 7\lambda < 0$

$\therefore\ 7\lambda(2\lambda - 1) < 0$ i.e. $\lambda\left(\lambda - \dfrac{1}{2}\right) < 0$ i.e. $0 < \lambda < \dfrac{1}{2}$

Thus the angle between $\bar{a}$ and $\bar{b}$ is abtuse if $0 < \lambda < \dfrac{1}{2}$

Ex. 8 Find the direction cosines of the vector $2\hat{i} + 2\hat{j} - \hat{k}$

Solutions : Let $|\bar{a}| = 2\hat{i} + 2\hat{j} - \hat{k}$

$$\therefore |\bar{a}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3$$

$$\therefore\ \hat{a} = \frac{\bar{a}}{|\bar{a}|} = \frac{2\hat{i} + 2\hat{j} + \hat{k}}{3} = \frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$$

$\therefore$ The direction cosines of $\bar{a}$ are $\dfrac{2}{3}, \dfrac{2}{3}, -\dfrac{1}{3}$

Ex. 9 : Find the position vector of a point P such that AB is inclined to X axis at 45^0 and to Y axis at 60^0 and OP = 12 units.

Solution : We have $l = \cos 45^\circ = \dfrac{1}{\sqrt{2}},\ m = \cos 60^\circ = \dfrac{1}{2},\ n = \cos\gamma$

Now $l^2 + m^2 + n^2 = 1$

$$\therefore\ \frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1, \text{ i.e. } \cos^2\gamma = \frac{1}{4}, \text{ i.e. } n = \cos\gamma = \pm\frac{1}{2}$$

$$\text{Now } \vec{r} = |\vec{r}|\hat{r} = |\vec{r}|\left(l\hat{i} + m\hat{j} + n\hat{k}\right) = 12\left(\frac{1}{\sqrt{12}}\hat{i} + \frac{1}{2}\hat{j} \pm \frac{1}{2}\hat{k}\right)$$

Hence $\vec{r} = 6\sqrt{2}\hat{i} + 6\hat{j} \pm 6\hat{k}$

Ex. 10 A line makes angles of measure 45º and 60º with the positive direction of the Y and Z axes respectively. Find the angle made by the line with the positive directions of the X-axis.

Let α, β, γ be the angles made by the line with positive direction of X, Y and Z axes respectively. Given $\beta = 45^\circ$ and $\gamma = 60^\circ$.

Now $\cos\beta = \cos 45^\circ$ and $\cos\gamma = \cos 60^\circ = \dfrac{1}{2}$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$\therefore \quad \cos^2\alpha + \frac{1}{2} + \frac{1}{4} = 1$

$\therefore \quad \cos^2\alpha = \frac{1}{4}$

$\therefore \quad \cos\alpha = \pm\frac{1}{2}$

$\therefore \quad \alpha = 60^\circ$ or 120° There are two lines satisfying given conditions. Their direction angles are 45°, 60°, 60° and 45°, 60° , 120°

Ex. 11. A line passes through the points (6, -7, –1) and (2, -3, 1). Find the direction ratios and the direction cosines of the line so that the angle α is acute.

Solution : Let A(6, -7, -1) and B(2, -3, 1) be the given points. So $\bar{a} = 6\hat{i} - 7\hat{j} - \hat{k}, \bar{b} = 2\hat{i} - 3\hat{j} + \hat{k}$

$\overrightarrow{AB} = \bar{b} - \bar{a} = (2-6)\hat{i} + (-3+7)\hat{j} + (1+1)\hat{k} = -4\hat{i} + 4\hat{j} + 2\hat{k}$

the direction ratios of $\overrightarrow{AB}$ are –4, 4, 2 .

Let the direction cosines of $\overrightarrow{AB}$ be -4k, 4k, 2k. Then

$$(-4k)^2 + (4k)^2 + (2k)^2 = 1$$

i.e. $16k^2 + 16k^2 + 4k^2 = 1$ *i.e.* $36k^2 = 1$ *i.e.* $k = \pm\frac{1}{6}$

Since the line AB is so directed that the angle α which it makes with the x-Axis is acute,

$\therefore \quad \cos\alpha = -4k > 0$

$\therefore \quad$ As $k < 0 \quad \therefore k = -\frac{1}{6}$

$\therefore \quad$ the direction cosines of $\overrightarrow{AB}$ are $-4\left(-\frac{1}{6}\right), 4\left(-\frac{1}{6}\right), 2\left(-\frac{1}{6}\right)$ *i.e.* $\frac{2}{3}, \frac{-2}{3}, \frac{-1}{3}$

Ex. 12 Prove that the altitudes of a triangle are concurrent.

Solution : Let A, B and C be the vertices of a triangle

Let AD, BE and CF be the altitudes of the triangle ABC, therefore AD $\perp$ BC, BE $\perp$ AC, CF $\perp$ AB.

Let $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}$ be the position vectors of *A, B, C, D, E, F* respectively. Let P be the point of intersection of the altitudes *AD* and *BE* with $\bar{p}$ as the position vector.

Therefore, $\overline{AP} \perp \overline{BC}, \overline{BP} \perp \overline{AC}$ ....(1)

To show that the altitudes *AD, BE* and *CF* are concurrent, it is sufficient to show that the altitude CF passes through the point *P*. We will have to prove that $\overline{CF}$ and $\overline{CP}$ are collinear vectors. This can be achieved by showing $\overline{CP} \perp \overline{AB}$

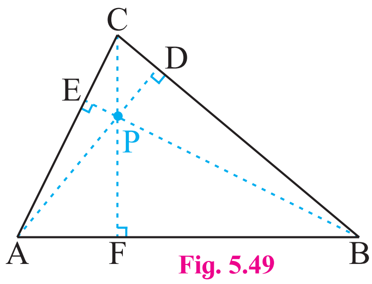


Fig. 5.49

Now from (1) we have

$$\overline{AP} \perp \overline{BC} \quad \text{and } \overline{BP} \perp \overline{AC}$$

$$\overline{AP} \perp \overline{BC} = 0 \quad \text{and } \overline{BP} \perp \overline{AC} = 0$$

$$\therefore (\overline{p} - \overline{a}).(\overline{c} - \overline{b}) = 0 \text{ and } (\overline{p} - \overline{b}).(\overline{c} - \overline{a}) = 0$$

$$\therefore \overline{p}.\overline{c} - \overline{p}.\overline{b} - \overline{a}.\overline{c} + \overline{a}.\overline{b} = 0 \quad \text{..... (2)}$$

$$\overline{p}.\overline{c} - \overline{p}.\overline{a} - \overline{b}.\overline{c} + \overline{b}.\overline{a} = 0 \quad \text{...... (3)}$$

Therefore, subtracting equation (2) from equation (3), we get

$$-\overline{p}.\overline{a} + \overline{p}.\overline{b} - \overline{b}.\overline{c} + \overline{a}.\overline{c} = 0 \quad (\text{Since } \overline{a}.\overline{b} = \overline{b}.\overline{a})$$

$\therefore \quad \overline{p}(\overline{b} - \overline{a}) - \overline{c}(\overline{b} - \overline{a}) = 0$

$\therefore \quad (\overline{p} - \overline{c}) \cdot (\overline{b} - \overline{a}) = 0$

$\therefore \quad \overline{CP} \cdot \overline{AB} = 0$

$\therefore \quad \overline{CP} \perp \overline{AB}$

Hence the proof.

Exercise 5.3

1. Find two unit vectors each of which is perpendicular to both $\overline{u}$ and $\overline{v}$, where $\overline{u} = 2\hat{i} + \hat{j} - 2\hat{k}, \overline{v} = \hat{i} + 2\hat{j} - 2\hat{k}$

2. If $\overline{a}$ and $\overline{b}$ are two vectors perpendicular to each other, prove that $(\overline{a} + \overline{b})^2 = (\overline{a} - \overline{b})^2$

3. Find the values of c so that for all real x the vectors $xc\hat{i} - 6\hat{j} + 3\hat{k}$ *and* $x\hat{i} + 2\hat{j} + 2cx\hat{k}$ make an obtuse angle.

4. Show that the sum of the length of projections of $p\hat{i} + q\hat{j} + r\hat{k}$ on the coordinate axes, where $p = 2$, $q = 3$ and $r = 4$, is 9.

5. Suppose that all sides of a quadrilateral are equal in length and opposite sides are parallel. Use vector methods to show that the diagonals are perpendicular.

6. Determine whether $\overline{a}$ and $\overline{b}$ are orthogonal, parallel or neither.

   $i)\ \overline{a} = -9\hat{i} + 6\hat{j} + 15\hat{k}, \overline{b} = 6\hat{i} - 4\hat{j} - 10\hat{k}$ $\qquad ii)\ \overline{a} = 2\hat{i} + 3\hat{j} - \hat{k}, \overline{b} = 5\hat{i} - 2\hat{j} + 4\hat{k}$

   $iii)\ \overline{a} = -\frac{3}{5}\hat{i} + \frac{1}{2}\hat{j} + \frac{1}{3}\hat{k}, \overline{b} = 5\hat{i} + 4\hat{j} + 3\hat{k}$ $\qquad iv)\ \overline{a} = 4\hat{i} - \hat{j} + 6\hat{k}, \overline{b} = 5\hat{i} - 2\hat{j} + 4\hat{k}$

7. Find the angle P of the triangle whose vertices are P(0, -1, -2), Q(3, 1, 4) and R(5, 7, 1).

8. If $\hat{p}, \hat{q}$ and $\hat{r}$ are unit vectors, find $i)\ \hat{p} \cdot \hat{q}$ $\quad ii)\ \hat{p} \cdot \hat{r}$ (see fig.5.50)

9. Prove by vector method that the angle subtended on semicircle is a right angle.

10. If a vector has direction angles 45º and 60º find the third direction angle.

11. If a line makes angles 90º, 135º, 45º with the X, Y and Z axes respectively, then find its direction cosines.

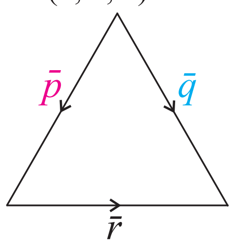


Fig.5.50

12. If a line has the direction ratios, 4, -12, 18 then find its direction cosines.

13. The direction ratios of $\overline{AB}$ are -2, 2, 1. If A = (4, 1, 5) and l (AB) = 6 units, find B.

14. Find the angle between the lines whose direction cosines l, m, n satisfy the equations

$5l+m+3n = 0$ and $5mn - 2nl + 6lm = 0$.

5.4.1 Vector Product of two vectors

In a plane, to describe how a line is tilting we used the notions of slope and angle of inclination. In space, we need to know how plane is tilting. We get this by multiplying two vectors in the plane together to get the third vector perpendicular to the plane. Third vector tell us inclination of the plane. The product we use for finding the third vector is called vector product.

Let $\bar{a}$ and $\bar{b}$ be two nonzero vectors in space. If $\bar{a}$ and $\bar{b}$ are not collinear, they determine a plane. We choose a unit vector $\hat{n}$ perpendicular to the plane by the right-hand rule. Which means $\hat{n}$ points in the way, right thumb points when our fingers curl through the angle from (See fig 5.51). Then we define a new vector $\bar{a}\times\bar{b}$ as $\bar{a}\times\bar{b} = |\bar{a}|\,|\bar{b}|\sin\theta\,\hat{n}$

The vector product is also called as the cross product of two vectors.

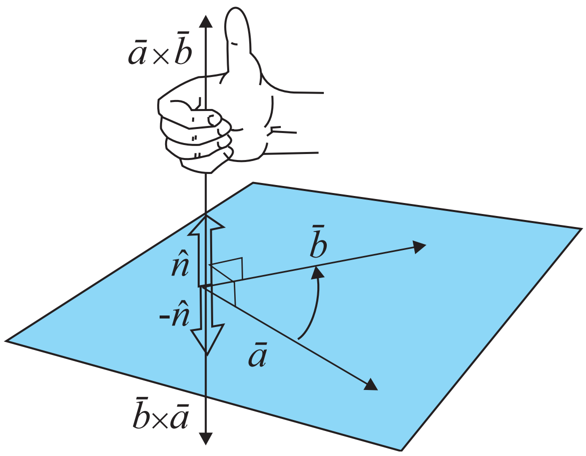


Fig.5.51

Remarks :

(i) $|\bar{a}\times\bar{b}| = |\bar{a}||\bar{b}|\sin\theta$ as $|\hat{n}| = 1$

(ii) $\bar{a}\times\bar{b}$ is perpendicular vector to the plane of $\bar{a}$ and $\bar{b}$.

(iii) The unit vector

$\hat{n}$ along $\bar{a}\times\bar{b}$ is given by $\hat{n} = \dfrac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}$

iv) If $\bar{a}$ and $\bar{b}$ are any two coplanar (but non collinear) vectors then any vector $\bar{c}$ in the space can be given by $\bar{c} = x\bar{a} + y\bar{b} + z\left(\bar{a}\times\bar{b}\right)$ This is because $\bar{a}\times\bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{b}$ and thus $\bar{a}$, $\bar{b}$ and $\bar{a}\times\bar{b}$ span the whole space.

v) If $\bar{a}, \bar{b}, \hat{n}$ form a right handed triplet, then $\bar{b}, \bar{a}, -\hat{n}$ also form a right handed triplet and

$\bar{b}\times\bar{a} = |\bar{a}||\bar{b}|\sin\theta\left(-\hat{n}\right) = -|\bar{a}||\bar{b}|\sin\theta\left(\hat{n}\right) = -\bar{a}\times\bar{b}$ .Thus vector product is anticommutative.

vi) If $\bar{a}$ and $\bar{b}$ are non zero vectors such that $\bar{a}$ is parallel to $\bar{b}$.

$\therefore \theta = 0$ i.e $\sin\theta = 0$ i.e $\bar{a}\times\bar{b} = \bar{0}$

Conversely if $\bar{a}\times\bar{b} = \bar{0}$, then either $\bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$ or $\sin\theta = 0$ that is $\theta = 0$.

Thus the cross product of two non zero vectors is zero only when $\bar{a}$ and $\bar{b}$ are collinear. In particular if $\bar{a} = k\bar{b}$ then $\bar{a} \times \bar{b} = k\bar{b} \times \bar{b} = k(\bar{0}) = \bar{0}$

(4) If $\bar{a}, \bar{b}, \bar{c}$ are any three vectors, then

(i) $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$ Left disributive law

(ii) $(\bar{b} + \bar{c}) \times \bar{a} = \bar{b} \times \bar{a} + \bar{c} \times \bar{a}$ Right disributive law

viii) If $\bar{a}$ and $\bar{b}$ are any two vectors and m, n are two scalars then

(i) $m(\bar{a} \times \bar{b}) = (m\bar{a}) \times \bar{b} = \bar{a} \times (m\bar{b})$

(ii) $m\bar{a} \times n\bar{b} = mn(\bar{a} \times \bar{b})$

ix) If $\hat{i}, \hat{j}, \hat{k}$ are the unit vectors along the co-ordinate axes then

(i) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \bar{0}$

$\hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{k} = \hat{i}, \hat{k} \times \hat{i} = \hat{j}$

(ii) $\hat{j} \times \hat{i} = -\hat{k}, \hat{k} \times \hat{j} = -\hat{i}, \hat{i} \times \hat{k} = -\hat{j}$

(Since $\hat{i}, \hat{j}, \hat{k}$ from a right handed triplet)

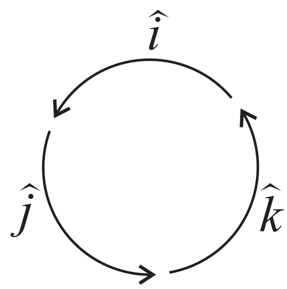


Fig.5.52

x) If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are two vectors in space then

$$\bar{a} \times \bar{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

This is given using determinant by $\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Angle between two vectors: Let θ be the angle between $\bar{a}$ and $\bar{b}$ (so $0 \leq \theta < \pi$),

then $|\bar{a} \times \bar{b}| = |\bar{a}|\,|\bar{b}| \sin\theta$, so $\sin\theta = \dfrac{|\bar{a} \times \bar{b}|}{|\bar{a}||\bar{b}|}$

Geometrical meaning of vector product of $\bar{a}$ and $\bar{b}$:

If $\bar{a}$ and $\bar{b}$ are represented by directed line segments with the same initial point, then they determine a parallelogram with base $|\bar{a}|$, height $|\bar{b}| \sin\theta$ and area of parallelogram

$A = \text{(Base) (Height)} = |\bar{a}|\,(|\bar{b}|\sin\theta) = |\bar{a} \times \bar{b}|$

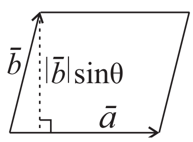


Fig.5.53

Ex. 1 Find the cross product $\bar{a} \times \bar{b}$ and verify that it is orthagonal (perpendicular) to both $\bar{a}$ and $\bar{b}$

Solved Examples

(i) $\bar{a} = \hat{i} + \hat{j} - \hat{k},\ \bar{b} = 2\hat{i} + 4\hat{j} + 6\hat{k}$ (ii) $\bar{a} = \hat{i} + 3\hat{j} - 2\hat{k}\ \bar{b} = -\hat{i} + 5\hat{k}$

Solution : (i) $\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 2 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 4 & 6 \end{vmatrix}\hat{i} - \begin{vmatrix} 1 & -1 \\ 2 & 6 \end{vmatrix}\hat{j} + \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix}\hat{k}$

$= [6-(-4)]\hat{i} - [6-(-2)]\hat{j} + (4-2)\hat{k} = 10\hat{i} - 8\hat{j} + 2\hat{k}$

Now $\bar{a}\times\bar{b}\cdot\bar{a} = (10\hat{i} - 8\hat{j} + 2\hat{k})\cdot(\hat{i} + \hat{j} - \hat{k}) = 10 - 8 - 2 = 0$ and

$\bar{a}\times\bar{b}\cdot\bar{b} = (10\hat{i} - 8\hat{j} + 2\hat{k})\cdot(2\hat{i} + 4\hat{j} - 6\hat{k}) = 20 - 32 + 12 = 0$

so $\bar{a}\times\bar{b}$ is orthogonal to both $\bar{a}$ and $\bar{b}$.

(ii) $\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ -1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 0 & 5 \end{vmatrix}\hat{i} \begin{vmatrix} 1 & -2 \\ -1 & 5 \end{vmatrix}\hat{j} + \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix}\hat{k}$

$= (15-0)\hat{i} - (5-2)\hat{j} + [0-(-3)]\hat{k} = 15\hat{i} - 3\hat{j} + 3\hat{k}$

Now $(\bar{a}\times\bar{b})\cdot\bar{a} = (15\hat{i} - 3\hat{j} + 3\hat{k})\cdot(\hat{i} + 3\hat{j} - 2\hat{k}) = 15 - 9 - 6 = 0,$

and $(\bar{a}\times\bar{b})\cdot\bar{b} = (15\hat{i} - 3\hat{j} + 3\hat{k})\cdot(-\hat{i} + 5\hat{k}) = 15 + 0 + 15 = 0,$ $\bar{a}\times\bar{b}$ is orthogonal to both $\bar{a}$ and $\bar{b}$

Ex.2: Find all vectors of magnitude $10\sqrt{3}$ that are perpendicular to the plane of $\hat{i} + 2\hat{j} + \hat{k}$ and $-\hat{i} + 3\hat{j} + 4\hat{k}$

Solution : Let $\bar{a} = \hat{i} + 2\hat{j} + \hat{k}$ and $\bar{b} = -\hat{i} + 3\hat{j} + 4\hat{k}$. Then

$$\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ -1 & 3 & 4 \end{vmatrix} = \hat{i}(8-3) - \hat{j}(4+1) + \hat{k}(3+2) = 5\hat{i} - 5\hat{j} + 5\hat{k} = \bar{m}\,(\text{say})$$

$$\therefore |\bar{a}\times\bar{b}| = \sqrt{(5)^2 + (-5)^2 + (5)^2} = \sqrt{3(5)^2} = 5\sqrt{3} = |\bar{m}|$$

Therefore, unit vector perpendicular to the plane of $\bar{a}$ and $\bar{b}$ is given by

$$\hat{m} = \frac{\bar{m}}{|\bar{m}|} = \frac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|} = \frac{5\hat{i}-5\hat{j}+5\hat{k}}{5\sqrt{3}} = \frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}$$

Hence, vectors of magnitude of $10\sqrt{3}$ that are perpendicular to plane of $\bar{a}$ and $\bar{b}$ are

$$\pm 10\sqrt{3}\,\hat{m} = \pm 10\sqrt{3}\left(\frac{\hat{i}-\hat{j}+\hat{k}}{\sqrt{3}}\right), \text{ i.e. } \pm 10\left(\hat{i}-\hat{j}+\hat{k}\right).$$

Ex.3: If $\bar{u}+\bar{v}+\bar{w}=\bar{0}$, show that $\bar{u}\times\bar{v}=\bar{v}\times\bar{w}=\bar{w}\times\bar{u}.$

Solution : Suppose that $\bar{u}+\bar{v}+\bar{w}=0.$ Then

$$\left(\bar{u}+\bar{v}+\bar{w}\right)\times\bar{v}=\bar{0}\times\bar{v}$$

$$\bar{u}\times\bar{v}+\bar{v}\times\bar{v}+\bar{w}\times\bar{v}=\bar{0}.$$

But $\bar{v}\times\bar{v}=\bar{0}$

Thus $\bar{u}\times\bar{v}+\bar{w}\times\bar{v}=\bar{0}.$

Thus $\bar{u}\times\bar{v}=-\bar{w}\times\bar{v}=\bar{v}\times\bar{w}.$

Similarly, we have $\bar{v}\times\bar{w}=\bar{w}\times\bar{u}.$

Ex 5. If $\vec{a}=3\hat{i}-\hat{j}+2\hat{k}, \vec{b}=2\hat{i}+\hat{j}-\hat{k}, \vec{c}=\hat{i}-2\hat{j}+2\hat{k}$, find $\left(\vec{a}\times\vec{b}\right)\times\vec{c}$ and $\vec{a}\times\left(\vec{b}\times\vec{c}\right)$ and hence show that $\left(\vec{a}\times\vec{b}\right)\times\vec{c} \neq \vec{a}\times\left(\vec{b}\times\vec{c}\right)$.

Solution :

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -1 & 2\\ 2 & 1 & -1\end{vmatrix} = -\hat{i}+7\hat{j}+5\hat{k}$$

$$\left(\bar{a}\times\bar{b}\right)\times c=\left(-\hat{i}+7\hat{j}+5\hat{k}\right)\times\left(\hat{i}-2\hat{j}+2\hat{k}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & 7 & 5\\ 1 & -2 & 2\end{vmatrix}$$

$$= 24\hat{i}+7\hat{j}-5\hat{k} \qquad ....(1)$$

Now, $\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & -1\\ 1 & -2 & 2\end{vmatrix}=-5\hat{j}-5\hat{k}$

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=(3\hat{i}-\hat{j}+2\hat{k})\times\left(-5\hat{j}-5\hat{k}\right)$$

$$\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -1 & 2\\ 0 & -5 & -5\end{vmatrix}=15(\hat{i}+\hat{j}-\hat{k}) \quad ....(2)$$

from (1) and (2), we conclude that

$$\left(\vec{a}\times\vec{b}\right)\times\vec{c}\neq\vec{a}\times\left(\vec{b}\times\vec{c}\right)$$

Ex. 6 Find the area of the triangle with vertices (1,2,0), (1,0,2), and (0,3,1).

Solution : If A = (1, 2, 0), B = (1, 0, 2) and C = (0, 3, 1), then $\overline{AB}=-2j+2k,\ \overline{AC}=-i+j+k$ and

the area of triangle ABC is $\frac{1}{2}\left|\overline{AB}\times\overline{AC}\right|$ *and* $\overline{AB}\times\overline{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & -2 & 2\\ -1 & 1 & 1\end{vmatrix}=-4\hat{i}-2\hat{j}-2\hat{k}$

$$\frac{\left|\overline{AB}\times\overline{AC}\right|}{2}=\frac{\left|-4i-2j-2k\right|}{2}=\frac{\sqrt{16+4+4}}{2}=\frac{\sqrt{24}}{2}=\frac{2\sqrt{6}}{2}=\sqrt{6}sq.units$$

Ex. 7 Find the area of the parallelogram with vertices K(1, 2, 3), L(1, 3, 6), M(3, 8, 6) and N(3, 7,3)

Solution : The parallelogram is determined by the vectors $\overline{KL}=\hat{j}+3\hat{k}$ and $\overline{KN}=2\hat{i}+5\hat{j}$, so the area of parallelogram KLMN is

$$\left|\overline{KL}\times\overline{KN}\right|=\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & 3\\ 2 & 5 & 0\end{matrix}\right\|=\left|(-15)\hat{i}-(-6)\hat{j}+(-2)\hat{k}\right|=\left|-15\hat{i}+6\hat{j}-2\hat{k}\right|=\sqrt{265}\text{ square units}$$

Ex. 8 Find $\left|\bar{u}\times\bar{v}\right|$ if

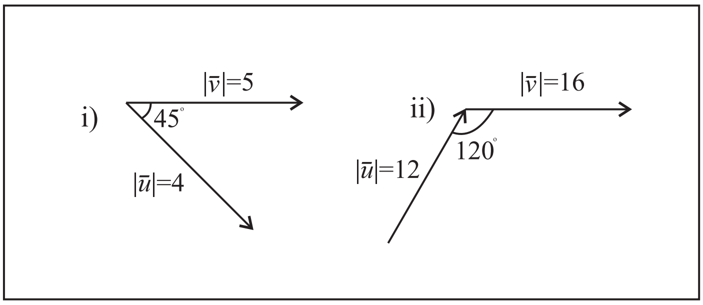


Fig 5.54

Solution : i) We have $\left|\bar{u}\times\bar{v}\right|=\left|\bar{u}\right|\left|\bar{v}\right|\sin\theta=(4)(5)\sin 45^\circ=20\,\frac{1}{\sqrt{2}}=10\sqrt{2}$

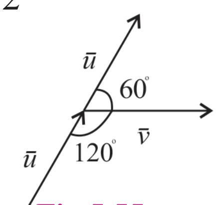


Fig 5.55

ii) If we sketch $\bar{u}$ and $\bar{v}$ starting from the same initial point, we see that the angle between them is 60º. we have $\left|\bar{u}\times\bar{v}\right| = \left|\bar{u}\right|\left|\bar{v}\right|\sin\theta = (12)(16)\sin 60^\circ = 192\,\frac{\sqrt{3}}{2} = 96\sqrt{3}$.

Ex. 9 Show that $\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)=2\left(\vec{a}\times\vec{b}\right)$

Solution : Using distributive property $\left(a-b\right)\times\left(a+b\right)=a\times a + a\times b-b\times a-b\times b\ \left(\because a\times b=-b\times a\right)$

$$=\bar{0}+\bar{a}\times\bar{b}+\bar{a}\times\bar{b}-\bar{0}\left(\because \bar{a}\times\bar{a}=\bar{0}\right)$$

$$=2\left(\bar{a}\times\bar{b}\right)$$

Ex. 10 Show that the three points with position vectors $3\hat{j}-2\hat{j}+4\hat{k},\ \hat{i}+\hat{j}+\hat{k}$ and $-\hat{i}+4\hat{j}-2\hat{k}$ respectively are collinear.

Solution : Let A, B, C be the given three points.

$\bar{a}=3\hat{j}-2\hat{j}+4\hat{k},\ \bar{b}=\hat{i}+\hat{j}+\hat{k}$ and $\bar{c}=-\hat{i}+4\hat{j}-$ To show that points A, B, C are collinear.

Now $\overrightarrow{AB}=\bar{b}-\bar{a}=-2\hat{i}+3\hat{j}-3\hat{k},\ \overrightarrow{AC}=\bar{c}-\bar{a}=-4\hat{i}+6\hat{j}-6\hat{k}$.

$$\overline{AB}\times\overline{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 3 & -3\\ -4 & 6 & -6\end{vmatrix}$$

$$=\left(-18+18\right)\hat{i}-\left(12-12\right)\hat{j}+\left(-12+12\right)\hat{k}$$

$$=0\hat{i}-0\hat{j}+0\hat{k}$$

$$=\bar{0}$$

Vectors $\overline{AB}$ and $\overline{AC}$ are collinear, but the point A is common, therefore points A, B, C are collinear.

Ex. 11 Find a unit vector perpendicular to $\overline{PQ}$ and $\overline{PR}$ where P ≡ (2,2,0), Q ≡ (0,3,5) and R ≡(5,0,3). Also find the sine of angle between $\overline{PQ}$ and $\overline{PR}$

Solution : $\overline{PQ}=\bar{q}-\bar{p}=-2\hat{i}+\hat{j}+5\hat{k}$

and $\overline{PR}=\bar{r}-\bar{p}=3\hat{i}-2\hat{j}+3\hat{k}$

Now $\left|\overline{PQ}\right|=\sqrt{4+1+25}=\sqrt{30}$

and $\left|\overline{PR}\right|=\sqrt{9+4+9}=\sqrt{22}$

$$\therefore\ \overline{PQ}\times\overline{PR}=\begin{vmatrix}i & j & k\\ -2 & 1 & 5\\ 3 & -2 & 3\end{vmatrix}$$

$$= (3+10)\hat{i} - (-6-15)\hat{j} + (4-3)\hat{k}$$

$$= 13\hat{i} + 21\hat{j} + \hat{k}$$

$$\therefore \left|\overline{PQ} \times \overline{PR}\right| = \sqrt{169+441+1} = \sqrt{611}$$

If $\hat{n}$ is a unit vector perpendicular to $\overline{PQ}$ and $\overline{PR}$, then

$$\hat{n} = \frac{\overline{PQ} \times \overline{PR}}{\left|\overline{PQ} \times \overline{PR}\right|} = \frac{13\hat{i} + 21\hat{j} + \hat{k}}{\sqrt{611}}$$

If θ is the angle between $\overline{PQ}$ and $\overline{PR}$ then $\sin\theta = \left|\frac{\overline{PQ} \times \overline{PR}}{\left|\overline{PQ}\right| \left|\overline{PR}\right|}\right| = \frac{\sqrt{611}}{\sqrt{30}\sqrt{22}}$

Ex. 12 If $|\bar{a}| = 5$, $|\bar{b}| = 13$ and $|\bar{a} \times \bar{b}| = 25$, find $\bar{a} \cdot \bar{b}$.

Solution : Given $|\bar{a} \times \bar{b}| = 25$

$\therefore$ $|\bar{a}| \cdot |\bar{b}| \sin\theta = 25$ (θ is the angle between $\bar{a}$ and $\bar{b}$)

$5 \times 13 \sin\theta = 25$

$$\sin\theta = \frac{25}{5\times 13} = \frac{5}{13}$$

$$\therefore \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\frac{25}{169}} = \pm\frac{12}{13}$$

$$\therefore \bar{a} \cdot \bar{b} = |\bar{a}| \cdot |\bar{b}| \cos\theta$$

$$= 5\times 13\times\left(\pm\frac{12}{13}\right)$$

$$= \underline{+}\,60$$

Thus $\bar{a} \cdot \bar{b} = 60$ if $0 < \theta^{o} < \pi/2$ and

$\bar{a} \cdot \bar{b} = -60$ if $\theta/2 < \theta < \pi$.

Ex. 13.: Direction ratios of two lines satisfy the relation 2a-b+2c = 0 and ab+bc+ca =0. Show that the lines are perpendicular.

Solution : Given equations are 2a-b+2c = 0 i.e b = 2a+2c .... (I) and ab+bc+ca = 0....(II)

Put b = 2a+2c in equation (II), we get

a (2a+2c) + (2a+2c) c + ca = 0

$2a^2+2ac+2ac+2c^2+ac = 0$

$2a^2+5ac+2c^2 = 0$

$\therefore$ (2a+c) (a+2c) = 0

Case I : i.e. 2a+c = 0 $\therefore$ 2a = –c...(III)

Using this equation (I) becomes b = –c + 2c = c ie. b = c..... (IV) from (III) and (IV) we get,

$\frac{a}{-\frac{1}{2}} = \frac{b}{1} = \frac{c}{1}$ Direction ratios of 1st line are i.e. $-\frac{1}{2},1,1$ *i.e.* $-1,2,2 = \bar{p}$(say)

Case II: i.e. a + 2c=0, $\therefore$ a = –2c ..... (V)

Using this equation (I) becomes

b = 2(–2c) + 2c = –2c i.e. b = –2c ....(VI)

From (V) and (VI), we get

$$\frac{a}{-2} = \frac{b}{-2} = \frac{c}{1}$$

$\therefore$ Direction ratios of second line are –2, –2, 1 i.e. 2,2 –1 = $\bar{q}$ (Say)

Now $\bar{p}\cdot\bar{q} = \left(-1\hat{i}+2\hat{j}+2\hat{k}\right)\cdot\left(2\hat{i}+2\hat{j}-\hat{k}\right) = -2+4-2 = 0$

$\therefore$ The lines are perpendicular.

Ex. 13 Find the direction cosines of the line which is perpendicular to the lines with direction ratios -1, 2, 2 and 0, 2, 1.

Solution : Given –1, 2,2 and 0, 2, 1 be direction ratios of lines L_1 and L_2.

Let *l, m, n* be direction cosines of line L. As line L is perpendicular to lines L_1 and L_2.

Then $-l+2m+2n = 0$ and $2m + n = 0$

$2m+n = 0$

$\therefore$ $2m = -n$

$\therefore$ $\frac{m}{-1} = \frac{n}{2}$ ....(I) and

$-l + 2m + 2n = 0$ becomes

$-l - n + 2n = 0$

$-l + n = 0$

$l = n$

$\frac{l}{1} = \frac{n}{1}$ i.e. $\frac{1}{2} = \frac{n}{2}$ .....(II)

from (I) and (II) $\frac{l}{2} = \frac{m}{-1} = \frac{n}{2}$

The direction ratios of line L are 2, –1, 2 and the direction cosines of line L are $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$.

Ex. 15 If M is the foot of the perpendicular drawn from A(4, 3, 2) on the line joining the points B(2, 4, 1) and C (4, 5, 3), find the coordinates of M.

Let the point M divides BC internally in the ratio k:1

$$\therefore\ M \equiv \left(\frac{4k+2}{k+1}, \frac{5k+4}{k+1}, \frac{3k+1}{k+1}\right) \quad .....(I)$$

$\therefore$ Direction ratios of AM are

$$\frac{4k+2}{k+1}-4, \frac{5k+4}{k+1}-3, \frac{3k+1}{k+1}-2 = \bar{p} \text{ (say)}$$

i.e. $\frac{-2}{K+1}, \frac{2k+1}{k+1}, \frac{k-1}{k+1}$ and direction ratios of BC are 4–2, 5–4, 3-1 i.e. 2, 1, 2 $= \bar{q}$ (say)

since AM is perpendicular to BC, $\bar{p}\cdot\bar{q}=0$

$$\text{i.e. } 2\frac{(-2)}{k+1}+1\frac{(2k+1)}{k+1}+2\frac{(k-1)}{k+1}=0$$

i.e. -4+2k + 1+2k – 2 = 0

$\therefore$ 4k–5=0

$\therefore$ $k = \frac{5}{4}$

from (I) $M \equiv \left(\frac{28}{9}, \frac{41}{9}, \frac{19}{9}\right)$

Exercise 5.4

1. If $\bar{a} = 2\hat{i}+3\hat{j}-\hat{k}$, $\bar{b}=\hat{i}-4\hat{j}+2\hat{k}$ find $(\bar{a}+\bar{b})\times(\bar{a}-\bar{b})$

2. Find a unit vector perpendicular to the vectors $\hat{j}+2\hat{k}$ and $\hat{i}+\hat{j}$.

3. If $\bar{a}\cdot\bar{b} = \sqrt{3}$ and $\bar{a}\times\bar{b} = 2\hat{i}+\hat{j}+2\hat{k}$, find the angle between $\bar{a}$ and $\bar{b}$.

4. If $\bar{a} = 2\hat{i}+\hat{j}-3\hat{k}$ and $\bar{b}=\hat{i}-2\hat{j}+\hat{k}$, find a vector of magnitude 5 perpendicular to both $\bar{a}$ and $\bar{b}$.

5. Find i) $\bar{u}\cdot\bar{v}$ if $|\bar{u}|=2, |\bar{v}| = 5, |\bar{u}\times\bar{v}| =8$ ii) $|\bar{u}\times\bar{v}|$ if $|\bar{u}| = 10, |\bar{v}|=2, \bar{u}\cdot\bar{v} =12$

6. Prove that $2(\bar{a}-\bar{b}) \times 2(\bar{a} + \bar{b}) = 4(\bar{a} \times \bar{b})$

7. If $\bar{a}=\hat{i}-2\hat{j}+3\hat{k}$, $\bar{b}=4\hat{i}-3\hat{j}+\hat{k}$ and $\bar{c}=\hat{i}-\hat{j}+2\hat{k}$ verify that $\bar{a}\times(\bar{b}+\bar{c}) = \bar{a}\times\bar{b}+\bar{a}\times\bar{c}$

8. Find the area of the parallelogram whose adjacent sides are the vectors $\bar{a}=2\hat{i}-2\hat{j}+\hat{k}$ and $\bar{b}=\hat{i}-3\hat{j}-3\hat{k}$.

9. Show that vector area of a quadrilateral $ABCD$ is $\frac{1}{2}\left(\overline{AC}\times\overline{BD}\right)$, where AC and BD are its diagonals.

10. Find the area of parallelogram whose diagonals are determined by the vectors

$\bar{a} = 3i - \hat{j} - 2\hat{k}$, and $\bar{b} = -\hat{i} + 3\hat{j} - 3\hat{k}$

11. If $\bar{a}, \bar{b}, \bar{c}$ *and* $\bar{d}$ are four distinct vectors such that $\bar{a} \times \bar{b} = \bar{c} \times \bar{d}$ *and* $\bar{a} \times \bar{c} = \bar{b} \times \bar{d}$, prove that $\bar{a} - \bar{d}$ is parallel to $\bar{b} - \bar{c}$

12. If $\bar{a} = \hat{i} + \hat{j} + \hat{k}$ and $\bar{c} = \hat{j} - \hat{k}$, find a vector $\bar{b}$ satisfying $\bar{a} \times \bar{b} = \bar{c}$ and $\bar{a} \cdot \bar{b} = 3$

13. Find $\bar{a}$, if $\bar{a} \times \hat{i} + 2\bar{a} - 5\hat{j} = 0$.

14. If $|\bar{a} \cdot \bar{b}| = |\bar{a} \times \bar{b}|$ and $\bar{a} \cdot \bar{b} < 0$, then find the angle between $\bar{a}$ and $\bar{b}$

15. Prove by vector method that $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$.

16. Find the direction ratios of a vector perpendicular to the two lines whose direction ratios are
    (i) –2,1,–1 and –3,–4,1
    (ii) 1, 3, 2 and –1, 1,2

17. Prove that two vectors whose direction cosines are given by relations $al+bm+cn=0$ and $fmn+gnl+hlm=0$ are perpendicular if $\frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$

18. If A(1, 2, 3) and B(4, 5, 6) are two points, then find the foot of the perpendicular from the point B to the line joining the origin and point A.

5.5.1 Scalar Triple Product :

We define the scalar triple product of three vectors $\bar{a}, \bar{b}, \bar{c}$ (Order is important) which is denoted by $[\bar{a}\ \bar{b}\ \bar{c}]$ and is defined as $\bar{a} \cdot (\bar{b} \times \bar{c})$

For $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\bar{b} = b_1\hat{i}, b_2\hat{j} + b_3\hat{k}$ and $\bar{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Scalar triple product is also called as box product.

Properties of scalar triple product:

Using the properties of determinant, we get following properties of scalar triple product.

(1) A cyclic change of vectors $\bar{a}, \bar{b}, \bar{c}$ in a scalar triple product does not change its value

i.e. $[\bar{a}\ \bar{b}\ \bar{c}] = [\bar{c}\ \bar{a}\ \bar{b}] = [\bar{b}\ \bar{c}\ \bar{a}]$

This follows as a cyclic change is equivalent to interchanging a pair of rows in the determinant two times.

(2) A single interchange of vectors in a scalar triple product changes the sign of its value.

i.e. $[\bar{a}\ \bar{b}\ \bar{c}] = -[\bar{b}\ \bar{a}\ \bar{c}] = -[\bar{c}\ \bar{b}\ \bar{a}] = -[\bar{a}\ \bar{c}\ \bar{b}]$

This follows as interchange of any 2 rows changes the value of determinant by sign only.

(3) If a row of determinant can be expressed as a linear combination of other rows then the determinant is zero. Using this fact we get following properties.

The scalar triple product of vectors is zero if any one of the following is true.

(i) One of the vectors is a zero vector.

(ii) Any two vectors are collinear.

(iii) The three vectors are coplanar.

(4) An interchange of 'dot' and 'cross' in a scalar triple product does not change its value

i.e. $\bar{a}\cdot(\bar{b}\times\bar{c}) = (\bar{a}\times\bar{b})\cdot\bar{c}$

This is followed by property (1) and the commutativity of dot product.

Theorem 9 : The volume of parallelopiped with coterminus edges as $\bar{a}, \bar{b}$ and $\bar{c}$ is $[\bar{a}\ \ \bar{b}\ \ \bar{c}]$

Proof : Let $\overline{OA} = \bar{a}$, $\overline{OB} = \bar{b}$ and $\overline{OC} = \bar{c}$ be coterminus edges of parallelopiped.

Let AP be the height of the parallelopiped.

Volume of Parallelopiped = (Area of base parallelogram OBDC) (Height AP)

But AP = Scalar Projection of $\bar{a}$ on $(\bar{b} \times \bar{c}) = \dfrac{(\bar{b}\times\bar{c})\cdot\bar{a}}{|\bar{b}\times\bar{c}|}$ $\left(\because \text{ scalar projection of } \vec{p} \text{ on } \vec{q} \text{ is } \dfrac{\vec{p}\cdot\vec{q}}{\bar{q}}\right)$

and area of parallelogram OBDC = $|\bar{b}\times\bar{c}|$

volume of parallelopiped $= \dfrac{(\bar{b}\times\bar{c})\cdot\bar{a}}{|\bar{b}\times\bar{c}|}\ |\bar{b}\times\bar{c}|$

$= \bar{a}\cdot(\bar{b}\times\bar{c}) = [\bar{a}\ \ \bar{b}\ \ \bar{c}]$

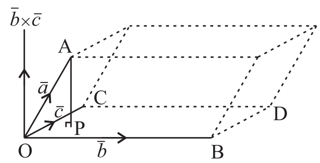


Fig . 5.56

Theorem 10. The volume of a tetrahedron with coterminus edges $\bar{a}, \bar{b}$ and $\bar{c}$ is $\frac{1}{6}[\bar{a}\ \ \bar{b}\ \ \bar{c}]$.

Proof : Let $\overline{OA} = \bar{a}$, $\overline{OB} = \bar{b}$ and $\overline{OC} = \bar{c}$ be coterminus edges of tetrahedron OABC.

Let AP be the height of tetrahedron

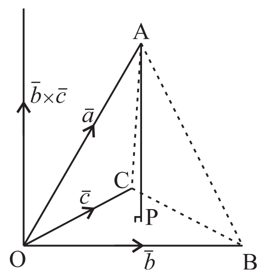


Fig. 5.57

Volume of tetrahedron $= \frac{1}{3}$(Area of base ΔOCB) (Height AP)

But AP = Scalar Projection of $\bar{a}$ on $(\bar{b} \times \bar{c}) = \dfrac{(\bar{b}\times\bar{c})\cdot\bar{a}}{|\bar{b}\times\bar{c}|}$ $\left(\because \text{ scalar projection of } \vec{p} \text{ on } \vec{q} \text{ is } \dfrac{\vec{p}\cdot\vec{q}}{\bar{q}}\right)$

Area of $\Delta OBC = \frac{1}{2}\left|\bar{b}\times\bar{c}\right|$

Volume of tetrahedron $= \frac{1}{3}\times\frac{1}{2}\left|\bar{b}\times\bar{c}\right| \frac{\left(\bar{b}\times\bar{c}\right).\bar{a}}{\left|\bar{b}\times\bar{c}\right|}$

$$= \frac{1}{6}\left[\left(\bar{b}\times\bar{c}\right)\cdot\bar{a}\right] = \frac{1}{6}\left[\bar{a}\ \bar{b}\ \bar{c}\right]$$

5.5.2 : Vector triple product :

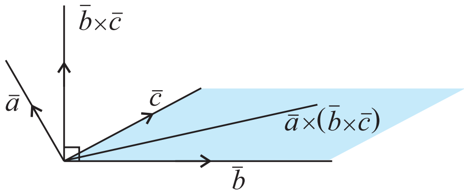


Fig. 5.58

For vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ in the space,

we define the vector triple product without proof as

$$\bar{a}\times\left(\bar{b}\times\bar{c}\right) = \left(\bar{a}\cdot\bar{c}\right)\bar{b} - \left(\bar{a}\cdot\bar{b}\right)\bar{c}.$$

Properties of vector triple product

1) $\bar{a}\times\left(\bar{b}\times\bar{c}\right) = -\left(\bar{b}\times\bar{c}\right)\times\bar{a}\ \left(\because\ \bar{p}\times\bar{q} = -\bar{q}\times\bar{p}\right)$

2) $\left(\vec{a}\times\vec{b}\right)\times\vec{c} = \left(\vec{a}\cdot\vec{c}\right)\ \vec{b} - \left(\vec{b}\cdot\vec{c}\right)\vec{a}$

3) $\vec{a}\times\left(\vec{b}\times\vec{c}\right) \neq \left(\vec{a}\times\vec{b}\right)\times\vec{c}$

4) $\hat{i}\times\left(\hat{j}\times\hat{k}\right) = \vec{0}$

5) $\bar{a}\times\left(\bar{b}\times\bar{c}\right)$ is linear combination of $\bar{b}$ and $\bar{c}$, hence it is coplanar with $\bar{b}$ and $\bar{c}$.

Solved Examples

Ex. 1 Find the volume of the parallelepiped determined by the vectors $\bar{a}, \bar{b}$ and $\bar{c}$

(i) $\bar{a} = \hat{i}+2\hat{j}+3\hat{k},\ \bar{b} = -\hat{i}+\hat{j}+2\hat{k},\ \bar{c} = 2\hat{i}+\hat{j}+4\hat{k}$

(ii) $\bar{a} = \hat{i}+\hat{j},\ \bar{b} = \hat{j}+\hat{k},\ \bar{c} = \hat{i}+\hat{j}+\hat{k}$

Solution (i) The volume of the parallelepiped determined by $\bar{a}, \bar{b}$ and $\bar{c}$ is the magnitude of their scalar triple product $\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)$

and $\bar{a}\cdot\left(\bar{b}\times\bar{c}\right) = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 2 & 1 & 4 \end{vmatrix} = 1\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 2 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix}$ = 1(4-2) -2(-4-4)+3(-1-2) = 9.

Thus the volume of the parallelepiped is 9 cubic units.

(ii) $\bar{a}\cdot\left(\bar{b}\times\bar{c}\right) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 0+1+0=1.$

So the volume of the parallelepiped is 1 cubic unit.

Ex. 2. Find the scalar triple product $\left[\bar{u}\ \ \bar{v}\ \ \bar{w}\right]$ and verify that the vectors $\bar{u} = \hat{i}+5\hat{j}-2\hat{k}, \bar{v} = 3\hat{i}-\hat{j}$ and $\bar{w} = 5\hat{i}+9\hat{j}-4\hat{k}$ are coplanar.

Solution :

$$\bar{u}\cdot\left(\bar{v}\times\bar{w}\right) = \begin{vmatrix} 1 & 5 & -2 \\ 3 & -1 & 0 \\ 5 & 9 & -4 \end{vmatrix} = 1\begin{vmatrix} -1 & 0 \\ 9 & -4 \end{vmatrix} - 5\begin{vmatrix} 3 & 0 \\ 5 & -4 \end{vmatrix} + (-2)\begin{vmatrix} 3 & -1 \\ 5 & 9 \end{vmatrix} = 4+60-64 = 0$$

i.e. volume of the parallelepiped is 0 and thus these three vectors are coplanar.

Ex. 3. Find the vector which is orthogonal to the vector $3\hat{i}+2\hat{j}+6\hat{k}$ and is co-planar with the vectors $2\hat{i}+\hat{j}+\hat{k}$ and $\hat{i}-\hat{j}+\hat{k}$

Solution : Let, $\bar{a}=3\hat{i}+2\hat{j}+6\hat{k}$, $\bar{b}=2\hat{i}+\hat{j}+\hat{k}$, $\bar{c}=\hat{i}-\hat{j}+\hat{k}$

Then by definition, a vector orthogonal to $\bar{a}$ and co-planar to $\bar{b}$ and $\bar{c}$ is given by

$$\bar{a}\times\left(\bar{b}\times\bar{c}\right)=\left(\bar{a}\cdot\bar{c}\right)\bar{b}-\left(\bar{a}\cdot\bar{b}\right)\bar{c}$$

$$=7\left(2\hat{i}+\hat{j}+\hat{k}\right)-14\left(\hat{i}+\hat{j}+\hat{k}\right)=21\hat{j}-7\hat{k}$$

Ex. 4. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{a}\cdot\vec{b}=1$ and $\vec{a}\times\vec{b}=\hat{j}-\hat{k}$ then prove that $\bar{b}=\hat{i}$.

Solution : As $\left(\vec{a}\times\vec{b}\right)\times\vec{a}=\left(\vec{a}\cdot\vec{a}\right)\vec{b}-\left(\vec{a}\cdot\vec{b}\right)\vec{a}$

$$\vec{a}\cdot\vec{a}=\left|\bar{a}\right|^2=1+1+1=3 \text{ and } \vec{a}\cdot\vec{b}=1$$

$$\therefore\left(\hat{j}-\hat{k}\right)\times\left(\hat{i}+\hat{j}+\hat{k}\right)=3\left(\vec{b}\right)-\left(\hat{i}+\hat{j}+\hat{k}\right)$$

$$i.e.\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & -1\\ 1 & 1 & 1\end{vmatrix}=3\left(\vec{b}\right)-\left(\hat{i}+\hat{j}+\hat{k}\right)$$

$$\left(2\hat{i}-\hat{j}-\hat{k}\right)+\left(\hat{i}+\hat{j}+\hat{k}\right)=3\bar{b}$$

$$i.e.\ \hat{i}=\bar{b}$$

Ex. 5. Prove that : $\bar{a}\times\left(\bar{b}\times\bar{c}\right)+\bar{b}\times\left(\bar{c}\times\bar{a}\right)+\bar{c}\times\left(\bar{a}\times\bar{b}\right)=\bar{0}$

Solution : $\bar{a}\times\left(\bar{b}\times\bar{c}\right)+\bar{b}\times\left(\bar{c}\times\bar{a}\right)+\bar{c}\times\left(\bar{a}\times\bar{b}\right)$

$$\left[\left(\bar{a}\cdot\bar{c}\right)\bar{b}\left(\bar{a}\cdot\bar{b}\right)\bar{c}\right]+\left[\left(\bar{b}\cdot\bar{a}\right)\bar{c}\left(\bar{b}\cdot\bar{c}\right)\bar{a}\right]+\left[\left(\bar{c}\cdot\bar{b}\right)\bar{a}\left(\bar{c}\cdot\bar{a}\right)\bar{b}\right]$$

$$\left(\bar{a}\cdot\bar{c}\right)\bar{b}\left(\bar{a}\cdot\bar{b}\right)\bar{c}+\left(\bar{a}\cdot\bar{b}\right)\bar{c}\left(\bar{b}\cdot\bar{c}\right)\bar{a}+\left(\bar{b}\cdot\bar{c}\right)\bar{a}\left(\bar{a}\cdot\bar{c}\right)\bar{b}=\bar{0}$$

Ex. 6. Show that the points $A(2,-1,0)$ $B(-3,0,4)$, $C(-1,-1,4)$ and $D(0,-5,2)$ are non coplanar.

Solution: Let $\bar{a}=2\hat{i}-\hat{j}$, $\bar{b}=-3\hat{i}+4\hat{k}$, $\bar{c}=-\hat{i}-\hat{j}+4\hat{k}$, $\bar{c}=\hat{i}+\hat{j}+4\hat{k}$, $\bar{d}=5\hat{i}+2\hat{k}$

$$\overline{AB}=\bar{b}-\bar{c}=-5\hat{i}+\hat{i}+4\hat{k}$$

$$\overline{AC}=\bar{c}-\bar{a}=-3\hat{i}+4\hat{k}$$

$$\overline{AD}=\bar{d}-\bar{a}=-2\hat{i}-4\hat{j}+2\hat{k}$$

Consider : $\left(\overline{AB}\right)\cdot\left(\overline{AC}\right)\times\left(\overline{AD}\right)=\begin{bmatrix}-5 & 1 & 4\\ -3 & 0 & 4\\ -2 & -4 & 2\end{bmatrix}$

$= -5[0 + 16) -1\ [-6 + 8] + 4\ [12]$

$= -\ 80 - 2 + 48$

$= 34 \neq 0$

Therefore, the points *A, B, C, D* are non-coplanar.

Ex. 7:. If $\overline{a}$, $\overline{b}$, $\overline{c}$ are non coplanar vectors, then show that the four points $2\overline{a}+\overline{b}, \overline{a}+2\overline{b}+\overline{c}$, $4\overline{a}-2\overline{b}-\overline{c}$ and $3\overline{a}+4\overline{b}-5\overline{c}$ are coplanar.

Solution : Let 4 prints be, P ($\overline{p}$), Q ($\overline{q}$), R($\overline{r}$) and S($\overline{s}$)

$$\overline{P} = 2\overline{a}+\overline{b}, q = \overline{a}+2\overline{b}+\overline{c},$$

$$\overline{r} = 4\overline{a}-2\overline{b}-\overline{c}, \overline{s}=3\overline{a}+4\overline{b}-5\overline{c}$$

Let us form 3 coinitial vectors

$$\overline{PQ} = \overline{q}-\overline{p}=\left(\overline{a}+2\overline{b}+\overline{c}\right) \ -\left(2\overline{a}+\overline{b}\right) = -\overline{a}+\overline{b}+\overline{c}$$

$$\overline{PR} = \overline{r}-\overline{p} = \left(4\overline{a}-2\overline{b}-\overline{c}\right)-\left(2\overline{a}+\overline{b}\right) = 2\overline{a}-3\overline{b}-\overline{c}$$

$$\overline{PS} = \overline{s}-\overline{p} = \left(3\overline{a}+4\overline{b}-5\overline{c}\right)-\left(2\overline{a}+\overline{b}\right) = \overline{a}+3\overline{b}-5\overline{c}$$

If P, Q, R, S, are coplanar then these three vectors $\overline{PQ}$, $\overline{PR}$ and $\overline{PS}$ are also coplanar this is possible only if $\overline{PQ}\cdot\left(\overline{PR}\times\overline{PS}\right) = 0$

$$\overline{PQ}\cdot\left(\overline{PR}\times\overline{PS}\right) = \begin{vmatrix}-1 & 1 & 1\\ 2 & -3 & -1\\ 1 & 3 & -5\end{vmatrix} \qquad = -1(15+3)\ -1\ (-10 + 1)\ +1\ (6 + 3) = -\ 18 + 9 + 9 = 0$$

Exercise 5.5

1. Find $\overline{a}\cdot\left(\overline{b}\times\overline{c}\right)$, if $\overline{a} = 3\hat{i}-\hat{j}+4\hat{k}, \overline{b} = 2\hat{i}+3\hat{j}-\hat{k}$ and $c = -5\hat{i}+2\hat{j}+3\hat{k}$

2. If the vectors $3\hat{i}+5\hat{k}$, $4\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+\hat{j}+4\hat{k}$ are to co-terminus edges of the parallelo piped, then find the volume of the parallelopiped.

3. If the vectors $-3\hat{i}+4\hat{j}-2\hat{k}, \hat{i}+2\hat{k}$ *and* $\hat{i}-p\hat{j}$ are coplanar, then find the value of *p*.

4. Prove that :

(i) $\left[\bar{a}\,\bar{b}+\bar{c}\;\;\bar{a}+\bar{b}+\bar{c}\right]=0$ (ii) $\left(\bar{a}+2\bar{b}-\bar{c}\right)\left[\left(\bar{a}-\bar{b}\right)\times\bar{a}-\bar{b}-\bar{c}\right]=3\left[\bar{a}-\bar{b}-\bar{c}\right]$

5. If , $\bar{c}=3\bar{a}-2\bar{b}$. then prove that $\left[\bar{a}\,\bar{b}\,\bar{c}\right]=0$

6. If $u=\hat{i}-2\hat{j}+\hat{k},\ \bar{r}=3\hat{i}+\hat{k}\ and\ w=\hat{j},\ \hat{k}$ are given vectors, then find

(i) $\left[\bar{u}+\bar{w}\right]\cdot\left[\left(\bar{w}\times\bar{r}\right)\times\left(\bar{r}\times\bar{w}\right)\right]$

7. Find the volume of a tetrahedron whose vertices are A(-1, 2, 3) B(3, -2, 1), C(2, 1, 3) and D(–1, –2, 4).

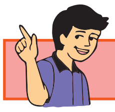
8. If $\bar{a}=\hat{i}+2\hat{j}+3,\ \bar{b}=3\hat{i}+2\hat{j}+\text{and}\ c=2\hat{i}+\hat{j}+3$ then verify that

$\bar{a}\times\left(\bar{b}\times\bar{c}\right)=\left(\bar{a}\cdot\bar{c}\right)\bar{b}-\left(\bar{a}\cdot\bar{b}\right)\bar{c}$

9. If, $\bar{a}=\hat{i}-2\hat{j},\ \bar{b}=\hat{i}+2\hat{j}\ and\ \bar{c}=2\hat{i}+\hat{j}-2$ then find

(i) $\bar{a}\times\left(\bar{b}\times\bar{c}\right)$ (ii) $\left(\bar{a}\times\bar{b}\right)\times\bar{c}$ Are the results same? Justify.

10. Show that $\bar{a}\times\left(\bar{b}\times\bar{c}\right)+\bar{b}\times\left(\bar{c}\times\bar{a}\right)+\bar{c}\times\left(\bar{a}\times\bar{b}\right)=0$



Let's remember!

- If two vectors $\bar{a}$ and $\bar{b}$ are represented by the two adjacent sides of a parallelogram then the diagonal of the parallelogram represents $\bar{a} + \bar{b}$.
- If two vectors $\bar{a}$ and $\bar{b}$ are represented by the two adjacent sides of a triangle so that the initial point of $\bar{b}$ coincides with the terminal point of $\bar{a}$, then the vector $\bar{a} + \bar{b}$ is represented by the third side.
- Unit vector along $\bar{a}$ is denoted by $\hat{a}$ and is given by $\hat{a} = \dfrac{\bar{a}}{|\bar{a}|}$
- The vector $\overline{OP}$ where the origin O as initial point and P terminal point, is the position vector (P, V) of the point P with respect to O. $\overline{OP} = \bar{p}$.
- $\overline{AB} = \bar{b} - \bar{a}$
- Unit vector along positive X-axis, Y-axis and Z-axis denoted by $\hat{i}$, $\hat{j}$, $\hat{k}$ respectively.
- If P ≡ (x, y, z) is any point in space and O is the origin then.

  $OP = x\hat{i} + y\hat{j} + z\hat{k}$ and $|\overline{OP}| = \sqrt{x^2 + y^2 + z^2}$

- Two non zero vectors $\bar{a}$ and $\bar{b}$ are said to be collinear if $\bar{a} = k\,\bar{b}$ $(k \neq 0)$
- Three non zero vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are said to be coplanar if $\bar{a} = m\bar{b} + n\bar{c}$ $(m, n \neq 0)$
- Section formula for the internal division : If R($\bar{r}$) divides the line segment joining the points A($\bar{a}$) and B($\bar{b}$) internally in the ratio m: n then $\bar{r} = \dfrac{m\bar{b} + n\bar{a}}{m+n}$
- Section formula for the external division.

  If R($\bar{r}$) is any point on the line AB such that points $A(\bar{a}), B(\bar{b})$, R$(\bar{r})$ are collinear (i.e. *A-B-R* or *R-A-B*) and $\dfrac{AR}{BR} = \dfrac{m}{n}$, and where m, n are scalars then $\bar{r} = \dfrac{m\bar{b} - n\bar{a}}{m-n}$.
- Mid Point Formula : If M($\bar{m}$) is the mid-point of the ine segment joining the points A($\bar{a}$) and B($\bar{b}$)then $\bar{m} = \dfrac{(\bar{a}+\bar{b})}{2}$.
- Centroid Formula : If G ($\bar{g}$) is the centroid of the triangle whose vertices are the point A($\bar{a}$), B($\bar{b}$),C($\bar{c}$) then $(\bar{g}) = \dfrac{\bar{a}+\bar{b}+\bar{c}}{3}$.
- If H($\bar{h}$) is incenter of Δ ABC then $\bar{h} = \dfrac{|BC|\bar{a} + |AC|\bar{b} + |AB|\bar{c}}{|BC| + |AC| + |AB|}$
- If G ($\bar{g}$) is centroiod of tetrahedron whose vertices are A($\bar{a}$), B($\bar{b}$),C($\bar{c}$) and D($\bar{d}$) then

$$\bar{g} = \frac{\bar{a}+\bar{b}+\bar{c}+\bar{d}}{4}$$

- The scalar product of two non-zero vectors and b denoted by $\bar{a}\cdot\bar{b}$, is given by $\bar{a}\,.\,\bar{b} = |\bar{a}|\;|\bar{b}|\cos\theta$

  Where θ is the angle between $\bar{a}$ and $\bar{b}$, $0 \le \theta \le \pi$
- If $\bar{a}$ is perpendicular to $\bar{b}$ then $\bar{a}\cdot\bar{b} = 0$

  $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1 \quad \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
- The angle θ between two non-zero vectors $\bar{a}$ and $\bar{b}$ is given by $\cos\theta = \dfrac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}$
- If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\bar{a}\cdot\bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$
- Scalar projection of a vector $\bar{a}$ on vector $\bar{b}$ is given $\dfrac{\bar{a}\cdot\bar{b}}{|\bar{b}|}$
- Vector projection of $\bar{a}$ on $\bar{b}$ is given by $\left(\dfrac{\bar{a}\cdot\bar{b}}{|\bar{b}|}\right)(\hat{b})$
- The vector product or cross product of two non-zero vectors $\bar{a}$ and $\bar{b}$, denoted by $\bar{a}\times\bar{b}$ is given by $\bar{a}\times\bar{b} = |\bar{a}|\cdot|\bar{b}|\sin\theta\,\hat{n}$

  where θ is the angle between $\bar{a}$ and $\bar{b}$, $0 \le \theta \le \pi$ and $\hat{n}$ is a unit vector perpendicular to both $\bar{a}$ and $\bar{b}$.
- $\bar{a}\times\bar{b} = \bar{0}$ if and only if $\bar{a}$ and $\bar{b}$ are collinear.
- $\bar{a}\times\bar{b} = -\bar{b}\times\bar{a}$
- The angle between two vectors $\bar{a}$ and $\bar{b}$ may be given as $\dfrac{|\bar{a}\times\bar{b}|}{|\bar{a}||\bar{b}|}$
- The $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$ also

  $\hat{i}\times\hat{i} = \bar{0}$, $\hat{j}\times\hat{j} = \bar{0}$, $\hat{k}\times\hat{k} = \bar{0}$
- For a plane containing two vectors $\bar{a}$ and $\bar{b}$ the two perpendicular directions are given by $\pm(\bar{a}\times\bar{b})$.
- If $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- For $\bar{a}$ and $\bar{b}$ represent the adjacent sides of a parallelogram then its area is given by $|\bar{a}\times\bar{b}|$.

- For $\bar{a}$ and $\bar{b}$ represent the adjacent sides of a triangle then its area is given by $\frac{1}{2}\left|\bar{a}\times\bar{b}\right|$.

- If $\overline{OP}$ makes angles α, β, γ with coordinate axes, then α, β, γ are known as direction angles cos α, cos β, cos γ are known as direction cosines.

  Thus we have d.c.s of $\overline{OP}$ are $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$. i.e. l, m, n

∴ Direction cosines of $\overline{PO}$ are $-l, -m, -n$

- If l, m, n are direction cosines of a vector $\bar{r}$,

  Then (i) $l^2+m^2+n^2=1$

  (ii) $\bar{r} = |\bar{r}|\left(l\hat{i}+m\hat{j}+n\hat{k}\right)$

  (iii) $\hat{r} = l\hat{i}+m\hat{j}+n\hat{k}$

- If l, m, n are direction cosines of a vector $\bar{r}$ and a, b, c are three real numbers such that $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$ then a, b, c are called as direction ratios of vector $\bar{r}$ and

  $$l=\frac{a}{\sqrt{a^2+b^2+c^2}}, m=\frac{b}{\sqrt{a^2+b^2+c^2}}, \text{n}=\frac{c}{\sqrt{a^2+b^2+c^2}}$$

- Scalar Triple Product (or Box Product): The dot product of $\bar{a}$ and $\bar{b}\times\bar{c}$ is called the scalar triple product of $\bar{a}$, $\bar{b}$ and $\bar{c}$. It is denoted by $\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)$ or $\left[\bar{a}\ \bar{b}\ \bar{c}\right]$.

- $\left[\bar{a}\,\bar{b}\,\bar{c}\right]=\left[\bar{b}\,\bar{c}\,\bar{a}\right]=\left[\bar{c}\,\bar{a}\,\bar{b}\right]$.

- $\left[\bar{a}\,\bar{b}\,\bar{c}\right]=-\left[\bar{b}\,\bar{a}\,\bar{c}\right]=-\left[\bar{a}\,\bar{c}\,\bar{b}\right]=-\left[\bar{c}\,\bar{b}\bar{a}\right]$

- Scalar Triple product is zero, if at least one of the vectors is a zero vector or any two vectors are collinear or all vectors are coplanar.

- Four points A($\bar{a}$), B($\bar{b}$),C($\bar{c}$)and D($\bar{d}$)are coplanar if and only if $\overline{AB}\cdot\left(\overline{AC}\times\overline{AD}\right)=0$

- The volume of the parallelopiped whose coterminus edges are represented by he vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ is given by $\left|\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)\right|$.

- The volume of the tetrahedron whose coterminus edges are given by $\bar{a}$, $\bar{b}$ and $\bar{c}$ is $\frac{1}{6}\left[\bar{a}\,\bar{b}\,\bar{c}\right]$

Miscellaneous Exercise 5

I) Select the correct option from the given alternatives :

1) If $|\bar{a}| = 2$, $|\bar{b}| = 3$ $|\bar{c}| = 4$ then [$\bar{a}$ + $\bar{b}$ $\bar{b}$ + $\bar{c}$ $\bar{c}$ − $\bar{a}$] is equal to

   A) 24   B) –24   C) 0   D) 48

2) If $|\bar{a}| = 3$, $|\bar{b}| = 4$, then the value of λ for which $\bar{a} + \lambda \bar{b}$ is perpendicular to $\bar{a} - \lambda \bar{b}$,is

A) $\frac{9}{16}$ B) $\frac{3}{4}$ C) $\frac{3}{2}$ D) $\frac{4}{3}$

3) If sum of two unit vectors is itself a unit vector, then the magnitude of their difference is

A) $\sqrt{2}$ B) $\sqrt{3}$ C) 1 D) 2

4) If $|\bar{a}| = 3$, $|\bar{b}| = 5$, $|\bar{c}| = 7$ and $\bar{a} + \bar{b} + \bar{c} = 0$, then the angle between $\bar{a}$ and $\bar{b}$ is

A) $\frac{\pi}{2}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{4}$ D) $\frac{\pi}{6}$

5) The volume of tetrahedron whose vertices are (1, –6, 10), (–1, –3, 7), (5, – 1, λ) and (7, –4, 7) is 11 cu. units then the value of λ is

A) 7 B) $\frac{\pi}{3}$ C) 1 D) 5

6) If α, β, γ are direction angles of a line and $\alpha = 60^\circ$, $\beta = 45^\circ$, the γ =

A) 30º or 90º B) 45º or 60º C) 90º or 30º D) 60º or 120º

7) The distance of the point (3, 4, 5) from Y- axis is

A) 3 B) 5 C) $\sqrt{34}$ D) $\sqrt{41}$

8) The line joining the points (–2, 1, –8) and (a, b, c) is parallel to the line whose direction ratios are 6, 2, 3. The value of a, b, c are

A) 4, 3, –5 B) $1, 2, \frac{-13}{2}$ C)10, 5, –2 D) 3, 5, 11

9) If cos α, cos β, cos γ are the direction cosines of a line then the value of $\sin^2 \alpha + \sin^2\beta + \sin^2\gamma$ is

A) 1 B) 2 C) 3 D) 4

10) If l, m, n are direction cosines of a line then $l\hat{i} + m\hat{j} + n\hat{k}$ is

A) null vector B) the unit vector along the line

C) any vector along the line D) a vector perpendicular to the line

11) If $|\bar{a}| = 3$ and $-1 \leq k \leq 2$, then $|k\vec{a}|$ lies in the interval

A) [0, 6] B) [–3, 6] C) [3, 6] D) [1,2]

12) Let α, β, γ be distinct real numbers. The points with position vectors $\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$,

$\beta\hat{i} + \gamma\hat{j} + \alpha\hat{k}$, $\gamma\hat{i} + \alpha\hat{j} + \beta\hat{k}$

A) are collinear B) form an equilateral triangle

C) form a scalene triangle D) form a right angled triangle

13) Let $\bar{p}$ and $\bar{q}$ be the position vectors of P and Q respectively, with respect to O and $|\bar{p}| = p$, $|\bar{q}| = q$. The points R and S divide PQ internally and externally in the ratio 2 : 3 respectively. If OR and OS are perpendicular then.

A) $9p^2 = 4q^2$ B) $4p^2 = 9q^2$ C) $9p = 4q$ D) $4p = 9q$

14) The 2 vectors $\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represents the two sides AB and AC, respectively of a ΔABC. The length of the median through A is

A) $\frac{\sqrt{34}}{2}$ B) $\frac{\sqrt{48}}{2}$ C) $\sqrt{18}$ D) None of these

15) If $\vec{a}$ and $\vec{b}$ are unit vectors, then what is the angle between $\vec{a}$ and $\vec{b}$ for $\sqrt{3}\bar{a} - \bar{b}$ to be a unit vector ?

A) 30° B) 45° C) 60° D) 90°

16) If θ be the angle between any two vectors $\bar{a}$ and $\bar{b}$, then $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$, when θ is equal to

A) 0 B) $\frac{\pi}{4}$ C) $\frac{\pi}{2}$ D) π

17) The value of $\hat{i} \cdot (\hat{j} \times \hat{k}) + \hat{j} \cdot (\hat{i} \times \hat{k}) + \hat{k} \cdot (\hat{i} \times \hat{j})$

A) 0 B) –1 C) 1 D) 3

18) Let a, b, c be distinct non-negative numbers. If the vectors $a\hat{i} + a\hat{j} + c\hat{k}$, $\hat{i} + \hat{k}$ and $c\hat{i} + c\hat{j} + b\hat{k}$ lie in a plane, then c is

A) The arithmetic mean of a and b B) The geometric mean of a and b

C) The harmonic man of a and b D) 0

19) Let $\bar{a} = \hat{i} \;\; \hat{j}$, $\bar{b} = \hat{j} \;\; \hat{k}$, $\bar{c} = \hat{k} \;\; \hat{i}$. If $\bar{d}$ is a unit vector such that $\bar{a} \cdot \bar{d} = 0 = [\,\bar{b} \;\; \bar{c} \;\; \bar{d}\,]$, then $\bar{d}$ equals.

A) $\pm\frac{\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{6}}$ B) $\pm\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$ C) $\pm\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$ D) $\pm\hat{k}$

20) If $\vec{a}\ \vec{b}\ \vec{c}$ are non coplanar unit vectors such that $\bar{a} \times (\bar{b} \times \bar{c}) = \frac{(\bar{b} + \bar{c})}{\sqrt{2}}$ then the angle between $\vec{a}$ and $\vec{b}$ is

A) $\frac{3\pi}{4}$ B) $\frac{\pi}{4}$ C) $\frac{\pi}{2}$ D) π

II Answer the following :

1) ABCD is a trapezium with AB parallel to DC and DC = 3AB. M is the mid-point of DC, $\overrightarrow{AB} = \bar{p}$ and $\overrightarrow{BC} = \bar{q}$. Find in terms of $\bar{p}$ and $\bar{q}$.

i) $\overrightarrow{AM}$ ii) $\overrightarrow{BD}$ iii) $\overrightarrow{MB}$ iv) $\overrightarrow{DA}$

2) The points A, B and C have position vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ respectively. The point P is midpoint of AB. Find in terms of $\bar{a}$, $\bar{b}$ and $\bar{c}$ the vector $\overrightarrow{PC}$

3) In a pentagon ABCDE

Show that $\overline{AB} + \overline{AE} + \overline{BC} + \overline{DC} + \overline{ED} = 2\overline{AC}$

4) If in parallelogram ABCD, diagonal vectors are $\overline{AC} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\overrightarrow{BD} = -6\hat{i} + 7\hat{j} - 2\hat{k}$, then find the adjacent side vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$

5) If two sides of a triangle are $\hat{i} + 2\hat{j}$ and $\hat{i} + \hat{k}$, then find the length of the third side.

6) If $|\bar{a}| = |\bar{b}| = 1$ $\bar{a} \cdot \bar{b} = 0$ and $\bar{a} + \bar{b} + \bar{c} = 0$ then find $|\bar{c}|$

7) Find the lengths of the sides of the triangle and also determine the type of a triangle.

i) A(2, –1, 0), B(4, 1, 1,), C(4, –5, 4) ii) L(3, –2, –3), M(7, 0,1), N (1, 2, 1)

8) Find the component form of if $\bar{a}$ if

i) It lies in YZ plane and makes 60º with positive Y-axis and $|\bar{a}| = 4$

ii) It lies in XZ plane and makes 45º with positive Z-axis and $|\bar{a}| = 10$

9) Two sides of a parallelogram are $3\hat{i} + 4\hat{j} - 5\hat{k}$ and $-2\hat{j} + 7\hat{k}$. Find the unit vectors parallel to the diagonals.

10) If *D, E, F* are the mid-points of the sides *BC, CA, AB* of a triangle *ABC*, prove that

$\overline{AD} + \overline{BE} + \overline{CF} = \bar{0}$

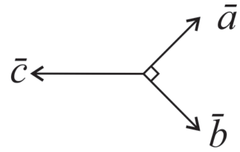


Fig.5.59

11) Find the unit vectors that are parallel to the tangent line to the parabola y = x^2 at the point (2, 4)

12) Express the vector $\hat{i} + 4\hat{j} - 4\hat{k}$ as a linear combination of the vectors $2\hat{i} - \hat{j} + 3\hat{k}$, $\hat{i} - 2\hat{j} + 4\hat{k}$ and $-\hat{i} + 3\hat{j} - 5\hat{k}$

13) If $\overline{OA} = \bar{a}$ and $\overline{OB} = \bar{b}$ then show that the vector along the angle bisector of angle AOB is given by $\bar{d} = \lambda\left(\frac{\bar{a}}{|\bar{b}|} + \frac{\bar{b}}{|\bar{b}|}\right)$

14) The position vectors f three consecutive vertices of a parallelogram are $\hat{i} + \hat{j} + \hat{k}$,

$\hat{i} + 3\hat{j} + 5\hat{k}$ and $7\hat{i} + 9\hat{j} + 11\hat{k}$ Find the position vector of the fourth vertex.

15) A point P with $p.v. \dfrac{-14\hat{i}+39\hat{j}+28\hat{k}}{5}$ divides the line joining A(–1, 6, 5) and B in the ratio 3:2 then find the point B.

16) Prove that the sum of the three vectors determined by the medians of a triangle directed from the vertices is zero.

17) ABCD is a parallelogram E, F are the mid points of BC and CD respectively. AE, AF meet the diagonal BD at Q and P respectively. Show that P and Q trisect DB.

18) If aBC is a triangle whose orthocenter is P and the circumcenter is Q, then prove that

$\overline{PA} + \overline{PC} + \overline{PB} = 2\,\overline{PQ}$

19) If P is orthocenter, Q is circumcenter and G is centroid of a triangle ABC, then prove that

$\overline{QP} = 3\,\overline{QG}$

20) In a triangle OAB, E is the midpoint of BO and D is a point on AB such that AD: DB = 2:1. If OD and AE intersect at P, determine the ratio OP:PD using vector methods.

21) Dot-product of a vector with vectors $3\hat{i} - 5\hat{k}$, $2\hat{i} + 7\hat{j}$ and $\hat{i} + \hat{j} + \hat{k}$ are respectively –1, 6 and 5. Find the vector.

22) If $\vec{a}$, $\vec{b}$, $\vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = 0$, then find the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$

23) If a parallelogram is constructed on the vectors $\vec{a} = 3\overline{p} - \overline{q}$, $\overline{b} = \overline{p} + 3\overline{q}$ and $|\overline{p}| = |\overline{q}| = 2$ and angle between $\overline{p}$ and $\overline{q}$ is π/3 show that the ratio of the lengths of the sides is $\sqrt{7} : \sqrt{13}$

24) Express the vector $\vec{a} = 5\hat{i} - 2\hat{j} + 5\hat{k}$ as a sum of two vectors such that one is parallel to the vector $\overline{b} = 3\hat{i} + \hat{k}$ and other is perpendicular to $\overline{b}$.

25) Find two unit vectors each of which makes equal angles with $\overline{u}$, $\overline{v}$ and $\overline{w}$.

$\overline{u} = 2\hat{i} + \hat{j} - 2\hat{k}$, $\overline{v} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\overline{w} = 2\hat{i} - 2\hat{j} + \hat{k}$

26) Find the acute angles between the curves at their points of intersection. $y = x^2$, $y = x^3$

27) Find the direction cosines and direction angles of the vector.

i) $2\hat{i} + \hat{j} + 2\hat{k}$ (ii) $(1/2)\hat{i} + \hat{j} + \hat{k}$

28) Let b = $4\hat{i} + 3\hat{j}$ and $\vec{c}$ be two vectors perpendicular to each other in the XY-plane. Find vectors in the same plane having projection 1 and 2 along $\overline{b}$ and $\overline{c}$, respectively, are given y.

29) Show that no line in space can make angle $\pi/6$ and $\pi/4$ with X- axis and Y-axis.

30) Find the angle between the lines whose direction cosines are given by the equation $6mn-2nl+5lm=0$, $3l+m+5n=0$

31) If Q is the foot of the perpendicular from P(2,4,3) on the line joining the points A(1,2,4) and B(3,4,5), find coordinates of Q.

32) Show that the area of a triangle ABC, the position vectors of whose vertices are a, b and c is $\frac{1}{2}\left[\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}\right]$

33) Find a unit vector perpendicular to the plane containing the point (a, 0, 0), (0, b, 0), and (0, 0, c). What is the area of the triangle with these vertices?

34) State whether each expression is meaningful. If not, explain why ? If so, state whether it is a vector or a scalar.

(a) $\bar{a}\cdot(\bar{b}\times\bar{c})$	(b) $\bar{a}\times(\bar{b}\cdot\bar{c})$
(c) $\bar{a}\times(\bar{b}\times\bar{c})$	(d) $\bar{a}\cdot(\bar{b}\cdot\bar{c})$
(e) $(\bar{a}\cdot\bar{b})\times(\bar{c}\cdot\bar{d})$	(f) $(\bar{a}\times\bar{b})\cdot(\bar{c}\times\bar{d})$
(g) $(\bar{a}\cdot\bar{b})\cdot\bar{c}$	(h) $(\bar{a}\cdot\bar{b})\bar{c}$
(i) $(\lvert\bar{a}\rvert)(\bar{b}\cdot\bar{c})$	(j) $\bar{a}\cdot(\bar{b}+\bar{c})$
(k) $\bar{a}\cdot\bar{b}+\bar{c}$	(l) $\lvert\bar{a}\rvert\cdot(\bar{b}+\bar{c})$

35. Show that, for any vectors $\bar{a},\bar{b},\bar{c}$

$(\bar{a}+\bar{b}+\bar{c})\times\bar{c}+(\bar{a}+\bar{b}+\bar{c})\times\bar{b}+(\bar{b}+\bar{c})\times\bar{a}\ \ 2\bar{a}\times\bar{c}$

36. Suppose that $\bar{a}=0$.

(a) If $\bar{a}\cdot\bar{b}=\bar{a}\cdot\bar{c}$ then is $\bar{b}=\bar{c}$?

(b) If $\bar{a}\times\bar{b}=\bar{a}\times\bar{c}$ then is $\bar{b}=\bar{c}$?

(c) If $\bar{a}\cdot\bar{b}=\bar{a}\cdot\bar{c}$ and $\bar{a}\times\bar{b}=\bar{a}\times\bar{c}$ then is $\bar{b}=\bar{c}$?

37. If A(3, 2, -1), B(-2, 2, -3), C(3, 5, -2), D(-2, 5, -4) then (i) verify that the points are the vertices of a parallelogram and (ii) find its area.

38. Let A, B, C, D be any four points in space. Prove that

$$\left|\overline{AB}\times\overline{CD}+\overline{BC}\times\overline{AD}+\overline{CA}\times\overline{BD}\right| = 4 \text{ (area of } \Delta ABC)$$

39. Let $\hat{a},\hat{b},\hat{c}$ be unit vectors such that $\hat{a}\cdot\hat{b}=\hat{a}\cdot\hat{c}=0$ and the angle between $\hat{b}$ and $\hat{c}$ be $\pi/6$.

Prove that $\hat{a}=\pm\, 2\left(\hat{b}\times\hat{c}\right)$

40. Find the value of 'a' so that the volume of parallelopiped a formed by $\hat{i}+\hat{j}+\hat{k}+a\hat{k}$ and $aj+\hat{k}$ becomes minimum.

41. Find the volume of the parallelepiped spanned by the diagonals of the three faces of a cube of side a that meet at one vertex of the cube.

42. If $\bar{a},\bar{b},\bar{c}$ are three non-coplanar vectors, then show that

$$\frac{\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)}{\left(\bar{c}\times\bar{a}\right)\cdot\bar{b}}+\frac{\bar{b}\cdot\left(\bar{a}\times\bar{c}\right)}{\left(\bar{c}\times\bar{a}\right)\cdot\bar{b}}=0$$

43. Prove that $\left(\bar{a}\times\bar{b}\right)\cdot\left(\bar{c}\times\bar{d}\right)\begin{vmatrix}\bar{a}\cdot\bar{c} & \bar{b}\cdot\bar{c}\\ \bar{a}\cdot\bar{d} & \bar{b}\cdot\bar{d}\end{vmatrix}$

44. Find the volume of a parallelopiped whose coterminus edges are represented by the vector $\hat{j}+\hat{k}$. $\hat{i}+\hat{k}$ and $\hat{i}+\hat{j}$. Also find volume of tetrahedron having these coterminous edges.

45. Using properties of scalar triple product, prove that $[\bar{a}+\bar{b} \quad \bar{b}+\bar{c} \quad \bar{c}+\bar{a}]=2[\bar{a} \quad \bar{b} \quad \bar{c}]$.

46) If four points A($\bar{a}$), B($\bar{b}$), C($\bar{c}$) and D($\bar{d}$) are coplanar then show that

$$[\bar{a}\ \bar{b}\ \bar{d}]+[\bar{b}\ \bar{c}\ \bar{d}]+[\bar{c}\ \bar{a}\ \bar{d}]=[\bar{a}\ \bar{b}\ \bar{c}]$$

47) If $\bar{a}$ $\bar{b}$ and $\bar{c}$ are three non coplanar vectors, then $\left(\bar{a}+\bar{b}+\bar{c}\right)\cdot\left[\left(\bar{a}+\bar{b}\right)\times\left(\bar{a}+\bar{c}\right)\right]=-\left[\bar{a}\quad\bar{b}\quad\bar{c}\right]$

48) If in a tetrahedron, edges in each of the two pairs of opposite edges are perpendicular, then show that the edges in the third pair are also perpendicular.

♦ ♦ ♦ ♦ ♦

6 Line and Plane

Let's Study

6.1 Vector and Cartesian equations of a line.

6.1.1 Passing through a point and parallel to a vector.

6.1.2 Passing through two points.

6:2 Distance of a point from a line.

6.3 Skew lines

6.3.1 Distance between skew lines

6.3.2 Distance between parallel lines.

6.4 Equations of Plane:

6.4.1 Passing through a point and perpendicular to a vector.

6.4.2 Passing through a point and parallel to two vectors.

6.4.3 Passing through three non-collinear points.

6.4.4 In normal form.

6.4.5 Passing through the intersection of two planes.

6.5 Angle between planes:

6.5.1 Angle between two planes.

6.5.2 Angle between a line and a plane.

6.6 Coplanarity of two lines.

6.7 Distance of a point from a plane.

Let's recall.

A line in space is completely determined by a point on it and its direction. Two points on a line determine the direction of the line. Let us derive equations of lines in different forms and discuss parallel lines.

6.1 Vector and Cartesian equations of a line:

Line in space is a locus. Points on line have position vectors. Position vector of a point determines the position of the point in space. In this topic position vector of a variable point on line will be denoted by $\bar{r}$.

6.1.1 Equation of a line passing through a given point and parallel to given vector:

Theorem 6.1 :

The vector equation of the line passing through A($\overline{a}$) and parallel to vector $\overline{b}$ is $\overline{r} = \overline{a} + \lambda\overline{b}$.

Proof:

Let L be the line which passes through A($\overline{a}$) and parallel to vector $\overline{b}$.

Let $P(\overline{r})$ be a variable point on the line L.

$\therefore$ $\overline{AP}$ is parallel to $\overline{b}$.

$\therefore$ $\overline{AP} = \lambda\overline{b}$, where λ is a scalar .

$\therefore$ $\overline{OP} - \overline{OA} = \lambda\overline{b}$

$\therefore$ $\overline{r} - \overline{a} = \lambda\overline{b}$

$\therefore$ $\overline{r} = \overline{a} + \lambda\overline{b}$

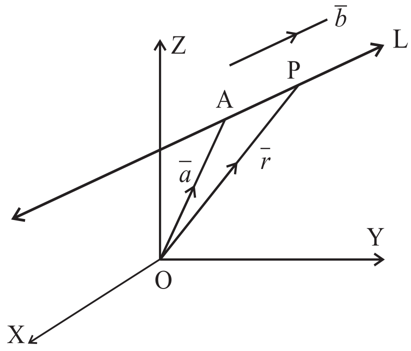


Fig. 6.1

is the required vector equation of the line.

Remark: Each real value of λ corresponds to a point on line L and conversely each point on L determines unique value of λ. There is one to one correspondence between points on L and values of λ. Here λ is called a parameter and equation $\overline{r} = \overline{a} + \lambda\overline{b}$ is called the **parametric form of vector equation** of line.

Activity: Write position vectors of any three points on the line $\overline{r} = \overline{a} + \lambda\overline{b}$.

Remark: The equation of line passing through A($\overline{a}$) and parallel to vector $\overline{b}$ can also be expressed as $(\overline{r} - \overline{a}) \times \overline{b} = \overline{0}$. This equation is called the non-parametric form of vector equation of line.

Theorem 6.2 :

The Cartesian equations of the line passing through A(x_1, y_1, z_1) and having direction ratios a, b, c are $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$.

Proof: Let L be the line which passes through A(x_1, y_1, z_1) and has direction ratios a, b, c.

Let P(x, y, z) be a variable point on the line L other than A.

$\therefore$ Direction ratios of L are $x - x_1, y - y_1, z - z_1$.

But direction ratios of line L are a, b, c.

$\therefore$ $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ are the required Cartesian equations of the line.

In Cartesian form line cannot be represented by a single equation.

Remark :

- If $\overline{b} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ then a_1, b_1, c_1 are direction ratios of the line and conversely if a_1, b_1, c_1 are direction ratios of a line then $\overline{b} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ is parallel to the line.
- The equations $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \lambda$ are called the symmetric form of Cartesian equations of line.
- The equations $x = x_1 + \lambda a, y = y_1 + \lambda b, z = z_1 + \lambda c$ are called parametric form of the Cartesian equations of line.

- The co-ordinates of variable point P on the line are $(x_1 + \lambda a, y_1 + \lambda b, z_1 + \lambda c)$.
- Corresponding to each real value of λ there is one point on the line and conversely corresponding to each point on the line there is unique real value of λ.
- Whenever we write the equations of line in symmetric form, it is assumed that, none of a, b, c is zero. If atleast one of them is zero then we write the equations of line in parametric form and not in symmetric form.
- In place of direction ratios a, b, c if we take direction cosines l, m, n then the co-ordinates of the variable point P will be $(x_1 + \lambda l, y_1 + \lambda m, z_1 + \lambda n)$, A (x_1, y_1, z_1)

  $\therefore \quad AP^2 = (x_1 + \lambda l - x_1)^2 + (y_1 + \lambda m - y_1)^2 + (z_1 + \lambda n - z_1)^2$

  $= (\lambda l)^2 + (\lambda m)^2 + (\lambda n)^2 = \lambda^2\{(l)^2 + (m)^2 + (n)^2\} = \lambda^2$

  $AP^2 = \lambda^2 \quad \therefore AP = |\lambda|$ and $\lambda = \pm AP$

  Thus parameter of point on a line gives its distance from the base point of the 1ine.

6.1.2 Equation of a line passing through given two points.

Theorem 6.3 : The equation of the line passing through A $(\bar{a})$ and B$(\bar{b})$ is $\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$.

Proof: Let L be the line which passes through A$(\bar{a})$ and B $(\bar{b})$.

Let P$(\bar{r})$ be a variable point on the line L other than A.

$\therefore \quad \overline{AP}$ and $\lambda \overline{AB}$ are collinear.

$\therefore \quad \overline{AP} = \lambda \overline{AB}$, where λ is a scalar .

$\therefore \quad \bar{r} - \bar{a} = \lambda(\bar{b} - \bar{a}) \qquad \bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$

$\therefore \quad \bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$ is the required equation of the line.

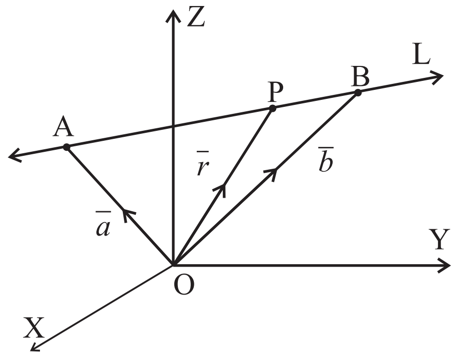


Fig. 6.2

Remark: The equation of the line passing through A$(\bar{a})$ and B$(\bar{b})$ can also be expressed as $(\bar{r} - \bar{a}) \times (\bar{b} - \bar{a}) = \bar{0}$.

Theorem 6.4 : The Cartesian equations of the line passing through $A(x_1, y_1 z_1)$ and B(x_2, y_2, z_2) is $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$.

Proof: Let L be the line which passes through A$(x_1, y_1 z_1)$ and B(x_2, y_2, z_2)

Let P(x, y, z) be a variable point on the line L other than A.

$\therefore$ Direction ratios of L are $x - x_1, y - y_1, z - z_1$

But direction ratios of line L are $x_2 - x_1, y_2 - y_1, z_2 - z_1$

$\therefore \quad \dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$ are the required Cartesian equations of the line.

Solved Examples

Ex.(l) Verify that point having position vector $4\hat{i} - 11\hat{j} + 2\hat{k}$ lies on the line

$\bar{r} = (6\hat{i} - 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 7\hat{j} + 3\hat{k})$.

Solution: Replacing $\bar{r}$ by $4\hat{i} - 11\hat{j} + 2\hat{k}$ we get,

$4\hat{i} - 11\hat{j} + 2\hat{k} = (6\hat{i} - 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 7\hat{j} + 3\hat{k})$

$\therefore$ $6 + 2\lambda = 4, -4 + 7\lambda = -11, 5 + 3\lambda = 2$

From each of these equations we get the same value of λ.

$\therefore$ Given point lies on the given line.

Alternative Method: Equation $\bar{r} = \bar{a} + \lambda\bar{b}$ can be written as $\bar{r} - \bar{a} = \lambda\bar{b}$

Thus point P($\bar{r}$) lies on the line if and only if $\bar{r} - \bar{a}$ is a scalar multiple of $\bar{b}$.

Equation of line is $\bar{r} = (6\hat{i} - 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 7\hat{j} + 3\hat{k})$.

$\therefore$ $\bar{a} = 6\hat{i} - 4\hat{j} + 5\hat{k}$ and $\bar{b} = 2\hat{i} + 7\hat{j} + 3\hat{k}$

The position vector of given point is $\bar{r} = 4\hat{i} - 11\hat{j} + 2\hat{k}$

$\bar{r} - \bar{a} = (4\hat{i} - 11\hat{j} + 2\hat{k}) - (6\hat{i} - 4\hat{j} + 5\hat{k}) = -2\hat{i} - 7\hat{j} - 3\hat{k}$

$= -(2\hat{i} + 7\hat{j} + 3\hat{k})$

$= -1\bar{b}$, a scalar multiple of $\bar{b}$.

$\therefore$ Given point lies on the given line.

Ex.(2) Find the vector equation of the line passing through the point having position vector $4\hat{i} - \hat{j} + 2\hat{k}$ and parallel to vector $-2\hat{i} - \hat{j} + \hat{k}$

Solution: The equation of the line passing through A($\bar{a}$) and parallel to vector $\bar{b}$ is $\bar{r} = \bar{a} + \lambda\bar{b}$.
The equation of the line passing through $4\hat{i} - \hat{j} + 2\hat{k}$ and parallel to vector $-2\hat{i} - \hat{j} + \hat{k}$ is
$\bar{r} = (4\hat{i} - \hat{j} + 2\hat{k}) + \lambda(-2\hat{i} - \hat{j} + \hat{k})$.

Ex.(3) Find the vector equation of the line passing through the point having position vector $2\hat{i} + \hat{j} - 3\hat{k}$ and perpendicular to vectors $\hat{i} + \hat{j} + \hat{k}$. and $\hat{i} + 2\hat{j} - \hat{k}$

Solution:

Let $a = 2\hat{i} + \hat{j} - 3\hat{k}$, $\bar{b} = \hat{i} + \hat{j} + \hat{k}$ and $c = \hat{i} + 2\hat{j} - \hat{k}$

We know that $\bar{b} \times \bar{c}$ is perpendicular to both $\bar{b}$ and $\bar{c}$.

$\therefore$ $\bar{b} \times \bar{c}$ is parallel to the required line.

$$\bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -3\hat{i} + 2\hat{j} + \hat{k}$$

Thus required line passes through $a = 2\hat{i} + \hat{j} - 3\hat{k}$ and parallel to $-3\hat{i} + 2\hat{j} + \hat{k}$.

$\therefore$ Its equation is $\bar{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \lambda(-3\hat{i} + 2\hat{j} + \hat{k})$

Ex.(4) Find the vector equation of the line passing through $2\hat{i} + \hat{j} - \hat{k}$ and parallel to the line joining points $-\hat{i} + \hat{j} + 4\hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$.

Solution :

Let A, B, C be points with position vectors $\bar{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\bar{b} = -\hat{i} + \hat{j} + 4\hat{k}$ and $\bar{c} = \hat{i} + 2\hat{j} + 2\hat{k}$ respectively.

$\overline{BC} = \bar{c} - \bar{b} = (\hat{i} + 2\hat{j} + 2\hat{k}) - (-\hat{i} + \hat{j} + 4\hat{k}) = 2\hat{i} + \hat{j} - 2\hat{k}$

The required line passes through $2\hat{i} + \hat{j} - \hat{k}$ and is parallel to $2\hat{i} + \hat{j} - 2\hat{k}$

Its equation is $\bar{r} = (2\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j} - 2\hat{k})$

Ex.(5) Find the vector equation of the line passing through A(1, 2, 3) and B(2, 3, 4).

Solution: Let position vectors of points A and B be $\bar{a}$ and $\bar{b}$.

$\therefore$ $\bar{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\bar{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore$ $\bar{b} - \bar{a} = (2\hat{i} + 3\hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (\hat{i} + \hat{j} + \hat{k})$

The equation of the line passing through A $(\bar{a})$ and B $(\bar{b})$ is $\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$.

The equation of the required line is $\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} + \hat{j} + \hat{k})$

Activity: Verify that position vector of B satisfies the above equation.

Ex.(6) Find the Cartesian equations of the line passing through A(1, 2, 3) and having direction ratios 2, 3, 7.

Solution:

The Cartesian equations of the line passing through A(x_1, y_1, z_1) and having direction ratios a, b, c are $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.

Here $(x_1, y_1, z_1) = (1, 2, 3)$ and direction ratios are 2, 3, 7.

Required equation $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{7}$.

Ex.(7) Find the Cartesian equations of the line passing through A(l, 2, 3) and B(2, 3. 4).

Solution: The Cartesian equations of the line passing through A(x_1, y_1, z_1) and B(x_2, y_2, z_2) are $\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$

Here $(x_1, y_1, z_1) = (1, 2, 3)$ and $(x_2, y_2, z_2) = (2, 3, 4)$.

$\therefore$ Required Cartesian equations are $\frac{x-1}{2-1} = \frac{y-2}{3-2} = \frac{z-3}{4-3}$.

$\therefore$ $\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-3}{1}$.

$\therefore$ $x - 1 = y - 2 = z - 3$.

Activity: Verify that co-ordinates of B satisfy the above equation.

Ex.(8) Find the Cartesian equations of the line passing through the point A(2, 1, –3) and perpendicular to vectors $\bar{b} = \hat{i} + \hat{j} + \hat{k}$ and $\bar{c} = \hat{i} + 2\hat{j} - \hat{k}$

Solution: We know that $\bar{b} \times \bar{c}$ is perpendicular to both b and c.

$$\therefore \bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -3\hat{i} + 2\hat{j} + \hat{k} \text{ is parallel to the required line.}$$

The direction ratios of the required line are –3, 2, 1 and it passes through A(2, 1, –3).

$\therefore$ Its Cartesian equations are $\dfrac{x-2}{-3} = \dfrac{y-1}{2} = \dfrac{z+3}{1}$.

Ex.(9) Find the angle between lines $\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and $\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + \hat{k})$

Solution: Let $\bar{b}$ and $\bar{c}$ be vectors along given lines .

$\therefore \quad \bar{b} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\bar{c} = \hat{i} + 2\hat{j} + 2\hat{k}$

Angle between lines is same as the angle between $\bar{b}$ and $\bar{c}$.

The angle between $\bar{b}$ and $\bar{c}$ is given by,

$$\cos\theta = \frac{\bar{b}\cdot\bar{c}}{|\bar{b}|\cdot|\bar{c}|} = \frac{(2\hat{i}-2\hat{j}+\hat{k})\cdot(\hat{i}+2\hat{j}+2\hat{k})}{3\times 3} = \frac{0}{9} = 0$$

$\therefore \cos\theta = 0 \qquad \therefore \theta = 90^{\circ}$

Lines are perpendicular to each other.

Ex.(10) Show that lines $\bar{r} = (-\hat{i} - 3\hat{j} + 4\hat{k}) + \lambda\ (-10\hat{i} - \hat{j} + \hat{k})$ and $\bar{r} = (-10\hat{i} - \hat{j} + \hat{k})\ \mu(-\hat{i} - 3\hat{j} + 4\hat{k})$ intersect each other.

Find the position vector of their point of intersection.

Solution: The position vector of a variable point on the line $\bar{r} = (-\hat{i} - 3\hat{j} + 4\hat{k}) + \lambda(-10\hat{i} - \hat{j} + \hat{k})$ is

$(-1 - 10\lambda)\ \hat{i} + (-3 - \lambda)\hat{j} + (4 + \lambda)\ \hat{k}$

The position vector of a variable point on the line

$\bar{r} = (-10\hat{i} - \hat{j} + \hat{k}) + \mu(-\hat{i} - 3\hat{j} + 4\hat{k})$ is $(-10 - 1\mu)\hat{i} + (-1 - 3\mu)\hat{j} + (1 + 4\mu)\hat{k}$

Given lines intersect each other if there exist some values of λ and μ for which

$(-1 - 10\lambda)\hat{i} + (-3 - \lambda)\hat{j} + (4 + \lambda)\hat{k} = (-10 - 1\mu)\hat{i} + (-1 - 3\mu)\hat{j} + (1 + 4\mu)\ \hat{k}$

$\therefore \quad -1 - 10\lambda = -10 - 1\mu,\ -3 - \lambda = -1 - 3\mu$ and $4 + \lambda = 1 + 4\mu$

$\therefore \quad 10\lambda - \mu, = 9\ ,\ \lambda - 3\mu = -2$ and $\lambda - 4\mu = -3$ ... ... (1)

Given lines intersect each other if this system is consistent

$$\text{As} \begin{vmatrix} 10 & -1 & 9 \\ 1 & -3 & -2 \\ 1 & -4 & -3 \end{vmatrix} = 10(9-8) + 1\ (-3+2) + 9\ (-4+3) = 10 - 1 - 9 = 0$$

$\therefore$ The system (1) is consistent and lines intersect each other.

Solving any two equations in system (1), we get. $\lambda = 1, \mu = 1$

Substituting this value of λ in $(-1 - 10\lambda)\hat{i} + (-3 - \lambda)\hat{j} + (4 + \lambda)\hat{k}$ we get, $-11\hat{i} - 4\hat{j} + 5\hat{k}$

$\therefore$ The position vector of their point of intersection is $-11\hat{i} - 4\hat{j} + 5\hat{k}$.

Ex.(11) Find the co-ordinates of points on the line $\frac{x+1}{2} = \frac{y-2}{3} = \frac{z+3}{6}$, which are at 3 unit distance from the base point A(–1, 2, –3).

Solution: Let Q $(2\lambda - 1, 3\lambda + 2, 6\lambda - 3)$ be a point on the line which is at 3 unit distance from the point A(–1,2,–3) $\therefore$ AQ = 3

$\therefore \sqrt{(2\lambda)^2 + (3\lambda)^2 + (6\lambda)^2} = 3 \quad \therefore \left((2\lambda)^2 + (3\lambda)^2 + (6\lambda)^2\right) = 9 \quad \therefore 49\lambda^2 = 9$

$\therefore \lambda = -\frac{3}{7} or \frac{3}{7}$

$\therefore$ There are two points on the line which are at a distance of 3 units from P.

Their co-ordinates are Q $(2\lambda - 1, 3\lambda + 2, 6\lambda - 3)$

Hence, the required points are

$\left(-\frac{1}{7}, 3\frac{2}{7}, -\frac{3}{7}\right)$ *and* $\left(-1\frac{6}{7}, \frac{5}{7}, -5\frac{4}{7}\right)$.

Exercise 6.1

(1) Find the vector equation of the line passing through the point having position vector $-2\hat{i} + \hat{j} + \hat{k}$ and parallel to vector $4\hat{i} - \hat{j} + 2\hat{k}$.

(2) Find the vector equation of the line passing through points having position vectors $3\hat{i} + 4\hat{j} - 7\hat{k}$ and $6\hat{i} - \hat{j} + \hat{k}$.

(3) Find the vector equation of line passing through the point having position vector $5\hat{i} + 4\hat{j} + 3\hat{k}$ and having direction ratios –3, 4, 2.

(4) Find the vector equation of the line passing through the point having position vector $\hat{i} + 2\hat{j} + 3\hat{k}$ and perpendicular to vectors $\hat{i} + \hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} + \hat{k}$.

(5) Find the vector equation of the line passing through the point having position vector $-\hat{i} - \hat{j} + 2\hat{k}$ and parallel to the line $\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} + \hat{k})$.

(6) Find the Cartesian equations of the line passing through A(–1, 2, 1) and having direction ratios 2, 3, 1.

(7) Find the Cartesian equations of the line passing through A(2, 2,1) and B(l, 3, 0).

(8) A(–2, 3, 4), B(l, 1, 2) and C(4, –1, 0) are three points. Find the Cartesian equations of the line AB and show that points A, B, C are collinear.

(9) Show that lines $\frac{x+1}{-10} = \frac{y+3}{-1} = \frac{z-4}{1}$ and $\frac{x+10}{-1} = \frac{y+1}{-3} = \frac{z-1}{4}$ intersect each other. Find the co-ordinates of their point of intersection.

(10) A line passes through (3, –1, 2) and is perpendicular to lines $\bar{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and $\bar{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \mu(\hat{i} - 2\hat{j} + 2\hat{k})$. Find its equation.

(11) Show that the line $\frac{x-2}{1}=\frac{y-4}{2}=\frac{z+4}{-2}$ passes through the origin.

6.2 Distance of a point from a line:

Theorem 6.5:

The distance of point P($\bar{\alpha}$) from the line $\bar{r}$ = $\bar{a}+\lambda\bar{b}$ is

$$\sqrt{|\bar{\alpha}-\bar{a}|^2-\left[\frac{(\bar{\alpha}-\bar{a})\cdot b}{|\bar{b}|}\right]^2}$$

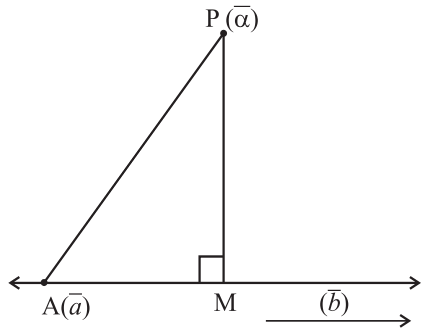


Fig. 6.3

Proof: The line $\bar{r}$ = $\bar{a}+\lambda\bar{b}$ passes through A($\bar{a}$).

Let M be the foot of the perpendicular drawn from P to the line $\bar{r}=\bar{a}+\lambda\bar{b}$

$\therefore$ $AM=|\overline{AM}|$ = the projection of $\overline{AP}$ on the line.

= the projection of $\overline{AP}$ on $\bar{b}$. (As line is parallel to $\bar{b}$)

$=\frac{\overline{AP}\cdot\bar{b}}{|\bar{b}|}$

Now ΔAM P is a right angled triangle. $\therefore$ $PM^2 = AP^2 - AM^2$

$$PM^2=|\overline{AP}|^2-|\overline{AM}|^2=|\bar{\alpha}-\bar{a}|^2-\left[\frac{(\bar{\alpha}-\bar{a})\cdot\bar{b}}{|\bar{b}|}\right]^2$$

$$\therefore\quad PM=\sqrt{|\bar{\alpha}-\bar{a}|^2-\left[\frac{(\bar{\alpha}-\bar{a})\cdot\bar{b}}{|\bar{b}|}\right]^2}$$

The distance of point P($\bar{a}$) from the line $\bar{r}$ = $(\bar{a}+\lambda\bar{b})$ is $PM=\sqrt{|\bar{\alpha}-\bar{a}|^2-\left[\frac{(\bar{\alpha}-\bar{a})\cdot\bar{b}}{|\bar{b}|}\right]^2}$

Ex.(12) : Find the length of the perpendicular drawn from the point P(3, 2,1) to the line

$$\bar{r}=\left(7\hat{i}+7\hat{j}+6\hat{k}\right)+\lambda\left(-2\hat{i}+2\hat{j}+3\hat{k}\right)$$

Solution:

The length of the perpendicular is same as the distance of P from the given line.

The distance of point P($\bar{\alpha}$) from the line $\bar{r}=\bar{a}+\lambda\bar{b}$ is $\sqrt{|\bar{\alpha}-\bar{a}|^2-\left[\frac{(\bar{\alpha}-\bar{a})\cdot\bar{b}}{|\bar{b}|}\right]^2}$

Here $\overline{\alpha} = 3\hat{i} + 2\hat{j} + \hat{k}$, $\overline{a_2} = 7\hat{i} + 7\hat{j} + 6\hat{k}$, $\overline{b} = -2\hat{i} + 2\hat{j} + 3\hat{k}$

$\therefore \overline{\alpha} - \overline{a} = (3\hat{i} + 2\hat{j} + \hat{k}) - (7\hat{i} + 7\hat{j} + 6\hat{k}) = -4\hat{i} - 5\hat{j} - 5\hat{k}$

$$|\overline{\alpha} - \overline{a}| = \sqrt{(-4)^2 + (-5)^2 + (-5)^2} = \sqrt{16 + 25 + 25} = \sqrt{66}$$

$$(\overline{\alpha} - \overline{a}).\overline{b} = (-4\hat{i} - 5\hat{j} - 5\hat{k}).(-2\hat{i} + 2\hat{j} + 3\hat{k}) = 8 - 10 - 15 = -17$$

$$|\overline{b}| = \sqrt{(-2)^2 + (2)^2 + (3)^2} = \sqrt{17}$$

The require length =

$$\sqrt{|\overline{a} - \overline{a}|^2 - \left[\frac{(\overline{a} - \overline{a}).\overline{b}}{|\overline{b}|}\right]^2} = \sqrt{66 - \left|\frac{-17}{\sqrt{17}}\right|^2} = \sqrt{66 - 17} = \sqrt{49} = 7 \text{ unit}$$

Ex.(13) : Find the distance of the point P(0, 2, 3)from the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$.

Solution:

Let M be the foot of the perpendicular drawn from the point P(0, 2, 3) to the given line.

M lies on the line. Let co-ordinates of M be $(5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$.

The direction ratios of PM are $(5\lambda - 3) - 0, (2\lambda + 1) - 2, (3\lambda - 4) - 3$

i.e. $5\lambda - 3, 2\lambda - 1, 3\lambda - 7$

The direction ratios of given line are 5, 2, 3 and

PM is perpendicular to the given line .

$\therefore$ $5(5\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 7) = 0$

$\therefore$ $\lambda = 1$

$\therefore$ The co-ordinates of M are (2,3, –1).

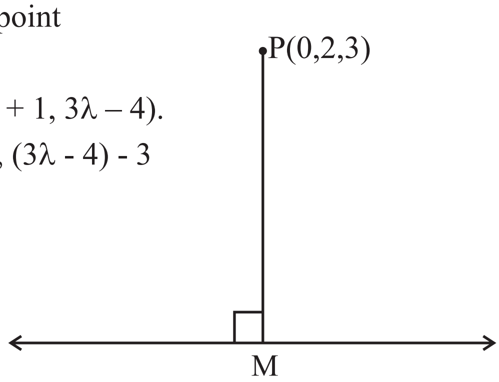


Fig. 6.4

The distance of P from the line is PM = $\sqrt{(2-0)^2 + (3-2)^2 + (-1-3)^2} = \sqrt{21}$ unit

6.3 Skew lines:

If two lines in space intersect at a point then the shortest distance between them is zero. If two lines in space are parallel to each other then the shortest distance between them is the perpendicular distance between them.

A pair of lines in space which neither intersect each other nor are parallel to each other are called skew lines. Skew lines are non-coplanar. Lines in the same plane either intersect or are parallel to each other.

In the figure 6.5, line CP that goes diagonally across the plane CSPR and line SQ passes across the plane SAQP are skew lines The shortest distance between skew lines is the length of the segment which is perpendicular to both the lines.

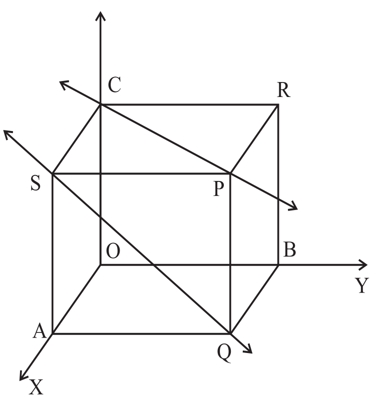


Fig. 6.5

6.3.1 Distance between skew lines :

Theorem 6.6: The distance between lines $\bar{r} = \bar{a}_1 + \lambda_1 \bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2 \bar{b}_2$ is $\left| \dfrac{(\bar{a}_2 - \bar{a}_1).\bar{b}_1 \times \bar{b}_2}{|\bar{b}_1 \times \bar{b}_2|} \right|$

Proof: Let L_1 and L_2 be the lines whose equations are $\bar{r} = \bar{a}_1 + \lambda_1 \bar{b}_1$, $\bar{r} = \bar{a}_2 + \lambda_2 \bar{b}_2$ respectively. Let PQ be the segment which is perpendicular to both L_1 and L_2 .

To find the length of segment PQ.

Lines L_1 and L_2 pass through points A($\bar{a}_1$) and B($\bar{a}_2$) respectively.

Lines L_1 and L_2 are parallel to $\bar{b}_1$ and $\bar{b}_2$ respectively.

As PQ is perpendicular to both L_1 and L_2, it is parallel to $\bar{b}_1 \times \bar{b}_2$

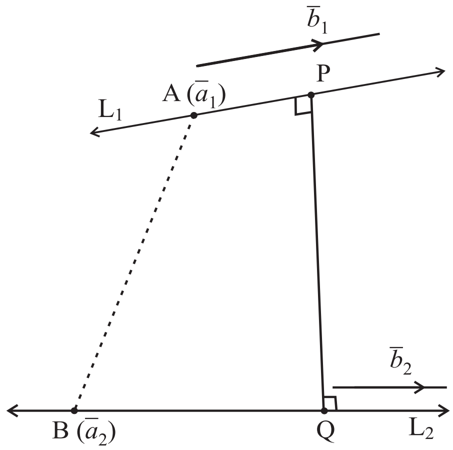


Fig. 6.6

The unit vector along $\overline{PQ}$ = unit vector along $\bar{b}_1 \times \bar{b}_2 = \hat{n}$ (say)

PQ = The projection of $\overline{AB}$ on $\overline{PQ}$ = $\overline{AB} \cdot \hat{n}$

$$PQ = \left| \frac{(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|} \right|$$

The shortest distance between lines $\bar{r} = \bar{a}_1 + \lambda_1 \bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2 \bar{b}_2$ is $\left| \dfrac{(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|} \right|$

Remark

- Two lines intersect each other if and only if the shortest distance between them is zero.
- Lines $\bar{r} = \bar{a}_1 + \lambda_1 \bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2 \bar{b}_2$ intersect each other if and only if $(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times \bar{b}_2) = 0$

  Lines $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$ intersect each other if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Ex.(14) Find the shortest distance between lines $\bar{r} = (2\hat{i} - \hat{j}) + \lambda(2\hat{i} + \hat{j} - 3\hat{k})$ and $\bar{r} = \hat{i} - \hat{j} + 2\hat{k} + \mu(2\hat{i} + \hat{j} - 5\hat{k})$

Solution: The shortest distance between lines $\bar{r} = \bar{a}_1 + \lambda_1 \bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2 \bar{b}_2$ is $\left| \dfrac{(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|} \right|$

Here $\bar{a}_1 = 2\hat{i} - \hat{j}$, $\bar{a}_2 = \hat{i} - \hat{j} + 2\hat{k}$, $\bar{b}_1 = 2\hat{i} + \hat{j} + 3\hat{k}$, $\bar{b}_2 = 2\hat{i} + \hat{j} - 5\hat{k}$

$$\bar{a}_2 - \bar{a}_1 = (\hat{i} - \hat{j} + 2\hat{k}) - (2\hat{i} - \hat{j}) = -\hat{i} + 2\hat{k}$$

And $\bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 2 & 1 & -5 \end{vmatrix} = -2\hat{i} + 4\hat{j}$

$\therefore$ $\left|\bar{b}_1 \times \bar{b}_2\right| = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

$(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2) = (-\hat{i} + 2\hat{k}) \cdot (-2\hat{i} + 4\hat{j}) = 2$

The required shortest distance $\left|\dfrac{(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times \bar{b}_2)}{\left|\bar{b}_1 \times \bar{b}_2\right|}\right| = \left|\dfrac{2}{2\sqrt{5}}\right| = \dfrac{1}{\sqrt{5}}$ unit.

Ex.(15) Find the shortest distance between lines $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ and $\dfrac{x-2}{3} = \dfrac{y-4}{4} = \dfrac{z-5}{5}$

Solution:

The vector equations of given lines are $\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$ and $\bar{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 4\hat{j} + 5\hat{k})$

The shortest distance between lines $\bar{r} = \bar{a}_1 + \lambda\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda\bar{b}_2$ is $\left|\dfrac{(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times \bar{b}_2)}{\left|\bar{b}_1 \times \bar{b}_2\right|}\right|$ and

Here $\bar{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$, $\bar{a}_2 = 2\hat{i} + 4\hat{j} + 5\hat{k}$, $\bar{b}_1 = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\bar{b}_2 = 3\hat{i} + 4\hat{j} + 5\hat{k})$

$\therefore$ $\bar{a}_2 - \bar{a}_1 = (2\hat{i} + 4\hat{j} + 5k) - (\hat{i} + 2\hat{j} + 3\hat{k}) = \hat{i} + 2\hat{j} + 2\hat{k}$

And $\bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -\hat{i} + 2\hat{j} - \hat{k}$

$\left|\bar{b}_1 \times \bar{b}_2\right| = \sqrt{1+4+1} = \sqrt{6}$

$(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2) = (\hat{i} + 2\hat{j} + 2\hat{k}) . (-\hat{i} + 2\hat{j} - \hat{k}) = 1$

$\therefore$ The required shortest distance $\left|\dfrac{(a_2 - a_1).(\bar{b}_1 \times \bar{b}_2)}{\left|\bar{b}_1 \times \bar{b}_2\right|}\right| = \dfrac{1}{\sqrt{6}}$ unit.

Ex.(16) Show that lines :

$\bar{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and $\bar{r} = (4\hat{i} - 3\hat{j} + 2\hat{k}) + \mu(\hat{i} - 2\hat{j} + 2\hat{k})$ intersect each other.

Solution: Lines $\bar{r} = \bar{a}_1 + \lambda_1 \bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2 \bar{b}_2$ intersect each other if and only if

$$(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2) = 0$$

Here $\bar{a}_1 = \hat{i} + \hat{j} - \hat{k}$, $\bar{a}_2 = 4\hat{i} - 3\hat{j} + 2\hat{k}$, $\bar{b}_1 = 2\hat{i} - 2\hat{j} + \hat{k}$, $\bar{b}_2 = \hat{i} - 2\hat{j} + 2\hat{k}$,

$\bar{a}_2 - \bar{a}_1 = 3\hat{i} - 4\hat{j} + 3\hat{k}$

$$(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times \bar{b}_2) = \begin{vmatrix} 3 & -4 & 3 \\ 2 & -2 & 1 \\ 1 & -2 & 2 \end{vmatrix} = 3(-2) + 4(3) + 3(-2) = -6 + 12 - 6 = 0$$

Given lines intersect each other.

6.3.2 Distance between parallel lines:

Theorem 6.7: The distance between parallel lines $\bar{r} = \bar{a}_1 + \lambda\bar{b}$ and $\bar{r} = \bar{a}_2 + \lambda\bar{b}$ is $|(\bar{a}_2 - \bar{a}_1) \times \hat{b}|$

Proof: Let lines represented by $\bar{r} = \bar{a}_1 + \lambda\bar{b}$ and $\bar{r} = \bar{a}_2 + \lambda\bar{b}$ be l_1 and l_2

Lines L_1 and L_2 pass through A($\bar{a}_1$) and B($\bar{a}_2$) respectively.

Let BM be perpendicular to L_1. To find BM

ΔAMB is a right angle triangle. Let m $\angle BAM = \theta$

$$\sin\theta = \frac{BM}{AB}$$

$\therefore$ BM = AB sin θ = AB.1 sin θ = AB.$|\hat{b}|$. sin θ

$$|\overline{AB} \times \hat{b}| = |(\bar{a}_2 - \bar{a}_1) \times \hat{b}|$$

$\therefore$ The distance between parallel lines

$\bar{r} = \bar{a}_1 + \lambda\bar{b}$ and $\bar{r} = \bar{a}_2 + \lambda_1\bar{b}$ is given by

$$d = BM = |(\bar{a}_2 - \bar{a}_1) \times \hat{b}|$$

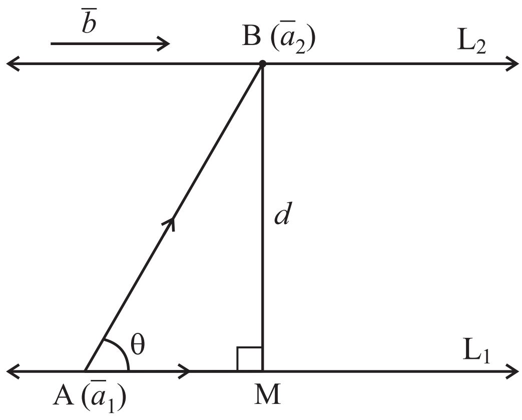


Fig. 6.7

Ex.(17) Find the distance between parallel lines $\bar{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} + \hat{j} - 2\hat{k})$ and $\bar{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + \hat{j} - 2\hat{k})$

Solution: The distance between parallel lines $\bar{r} = \bar{a}_1 + \lambda\bar{b}$ and $\bar{r} = \bar{a}_2 + \lambda\bar{b}$ given by

$$d = |(\bar{a}_2 - \bar{a}_1) \times \hat{b}|$$

Here $\bar{a}_1 = 2\hat{i} - \hat{j} + \hat{k}$, $\bar{a}_2 = \hat{i} - \hat{j} + 2\hat{k}$, $\bar{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$\therefore \bar{a}_2 - \bar{a}_1 = \left(\hat{i} - \hat{j} + 2\hat{k}\right) - \left(2\hat{i} - \hat{j} + \hat{k}\right) = -\hat{i} + \hat{k}$ and $\hat{b} = \dfrac{2\hat{i} + \hat{j} - 2\hat{k}}{3}$

$$\therefore (\bar{a}_2 - \bar{a}_1) \times \hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{vmatrix} = \frac{1}{3}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \frac{1}{3}\left\{-\hat{i} - \hat{k}\right\}$$

$$d = \left|(\bar{a}_2 - \bar{a}_1) \times \hat{b}\right| = \frac{\sqrt{2}}{3} \text{ unit}$$

Alternative Method:

The distance between parallel lines $\bar{r} = \bar{a}_1 + \lambda \bar{b}$ and $\bar{r} = \bar{a}_2 + \lambda \bar{b}$ is same as the distance of point $\mathrm{A}(\bar{a}_1)$ from the line $\bar{r} = \bar{a}_2 + \lambda \bar{b}$. This distance is given by

$$d = \sqrt{|\bar{a}_2 - \bar{a}_1|^2 - \left[\frac{(\bar{a}_2 - \bar{a}_1) \cdot \bar{b}}{|\bar{b}|}\right]^2}$$

Now $\bar{a}_2 - \bar{a}_1 = \left(\hat{i} - \hat{j} + 2\hat{k}\right) - \left(2\hat{i} - \hat{j} + \hat{k}\right) = -\hat{i} + \hat{k}$

$\therefore |\bar{a}_2 - \bar{a}_1| = \sqrt{2}$

As $(\bar{a}_2 - \bar{a}_1) \cdot \bar{b} = \left(-\hat{i} + \hat{k}\right) \cdot \left(2\hat{i} + \hat{j} - 2\hat{k}\right) = -2 + 0 - 2 = -4$

$\bar{b} = 2\hat{i} + \hat{j} - 2\hat{k}, \qquad |\bar{b}| = 3$

$$d = \sqrt{|\bar{a}_1 - \bar{a}_2|^2 - \left[\frac{(\bar{a}_1 - \bar{a}_2) \cdot \bar{b}}{|\bar{b}|}\right]^2} = \sqrt{2 - \left(-\frac{4}{3}\right)^2} = \sqrt{2 - \frac{16}{9}} = \frac{\sqrt{2}}{3} \text{ unit}$$

Ex.(18) Find the distance between parallel lines $\dfrac{x}{2} = \dfrac{y}{-1} = \dfrac{z}{2}$ and $\dfrac{x-1}{2} = \dfrac{y-1}{-1} = \dfrac{z-1}{2}$

Solution:

The vector equations of given lines $\bar{r} = \lambda\left(2\hat{i} - \hat{j} + 2\hat{k}\right)$

and $\bar{r} = \left(\hat{i} + \hat{j} + \hat{k}\right) + \mu\left(2\hat{i} - \hat{j} + 2\hat{k}\right)$

The distance between parallel lines $\bar{r} = \bar{a}_1 + \lambda \bar{b}$ and $\bar{r} = \bar{a}_2 + \lambda \bar{b}$ is given by $d = \left|(\bar{a}_2 - \bar{a}_1) \times \hat{b}\right|$

Here $\bar{a}_1 = \bar{0}$, $\bar{a}_2 = \hat{i} + \hat{j} + \hat{k}$, $\bar{b} = 2\hat{i} - \hat{j} + 2\hat{k}$

$\therefore \bar{a}_2 - \bar{a}_1 = \hat{i} + \hat{j} + \hat{k}$

And $\hat{b} = \dfrac{2\hat{i} - \hat{j} + 2\hat{k}}{3}$

$$\therefore (\bar{a}_2 - \bar{a}_1) \times \hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{vmatrix} = \frac{1}{3}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & 2 \end{vmatrix} = \frac{1}{3}\left\{3\hat{i} - 3\hat{k}\right\} = \hat{i} - \hat{k}$$

$$d = \left|(\bar{a}_2 - \bar{a}_1) \times \hat{b}\right| = \sqrt{2} \text{ unit}$$

Exercise 6.2

(1) Find the length of the perpendicular from (2, –3, 1) to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+1}{-1}$

(2) Find the co-ordinates of the foot of the perpendicular drawn from the point $2\hat{i} - \hat{j} + 5\hat{k}$ to the line $\bar{r} = \left(11\hat{i} - 2\hat{j} - 8\hat{k}\right) + \lambda\left(10\hat{i} - 4\hat{j} - 11\hat{k}\right)$. Also find the length of the perpendicular.

(3) Find the shortest distance between the lines $\bar{r} = \left(4\hat{i} - \hat{j}\right) + \lambda\left(\hat{i} + 2\hat{j} - 3\hat{k}\right)$ and $\bar{r} = \left(\hat{i} - \hat{j} + 2\hat{k}\right) + \mu\left(\hat{i} + 4\hat{j} - 5\hat{k}\right)$

(4) Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

(5) Find the perpendicular distance of the point (1, 0, 0) from the line $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$

Also find the co-ordinates of the foot of the perpendicular.

(6) A(1, 0, 4), B(0, -11, 13), C(2, -3, 1) are three points and D is the foot of the perpendicular from A to BC. Find the co-ordinates of D.

(7) By computing the shortest distance, determine whether following lines intersect each other.

(i) $\bar{r} = \left(\hat{i} - \hat{j}\right) + \lambda\left(2\hat{i} + \hat{k}\right)$ and $\bar{r} = \left(2\hat{i} - \hat{j}\right) + \mu\left(\hat{i} + \hat{j} - \hat{k}\right)$

(ii) $\frac{x-5}{4} = \frac{y-7}{-5} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{y-7}{1} = \frac{z-5}{3}$

(8) If lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect each other then find k.

Miscellaneous Exercise 6 A

(1) Find the vector equation of the line passing through the point having position vector $3\hat{i} + 4\hat{j} - 7\hat{k}$ and parallel to $6\hat{i} - \hat{j} + \hat{k}$.

(2) Find the vector equation of the line which passes through the point (3, 2, 1) and is parallel to the vector $2\hat{i} + 2\hat{j} - 3\hat{k}$.

(3) Find the Cartesian equations of the line which passes through the point (–2, 4, –5) and parallel to the line $\frac{x+2}{3}=\frac{y-3}{5}=\frac{z+5}{6}$.

(4) Obtain the vector equation of the line $\frac{x+5}{3}=\frac{y+4}{5}=\frac{z+5}{6}$.

(5) Find the vector equation of the line which passes through the origin and the point (5, –2, 3).

(6) Find the Cartesian equations of the line which passes through points (3, –2, –5) and (3, –2, 6).

(7) Find the Cartesian equations of the line passing through A(3, 2, 1) and B(1, 3, 1).

(8) Find the Cartesian equations of the line passing through the point A(1, 1, 2) and perpendicular to vectors $\bar{b}=\hat{i}+2\hat{j}+\hat{k}$ and $\bar{c}=3\hat{i}+2\hat{j}-\hat{k}$.

(9) Find the Cartesian equations of the line which passes through the point (2, 1, 3) and perpendicular to lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ and $\frac{x}{-3}=\frac{y}{2}=\frac{z}{5}$.

(10) Find the vector equation of the line which passes through the origin and intersect the line $x-1=y-2=z-3$ at right angle.

(11) Find the value of λ so that lines $\frac{1-x}{3}=\frac{7y-14}{2\lambda}=\frac{z-3}{2}$ and $\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$ are at right angle.

(12) Find the acute angle between lines $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-3}{2}$ and $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{1}$.

(13) Find the acute angle between lines $x=y, z=0$ and $x=0, z=0$.

(14) Find the acute angle between lines $x=-y, z=0$ and $x=0, z=0$.

(15) Find the co-ordinates of the foot of the perpendicular drawn from the point (0, 2, 3) to the line $\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}$.

(16) By computing the shortest distance determine whether following lines intersect each other.

(i) $\bar{r}=\left(\hat{i}+\hat{j}-\hat{k}\right)+\lambda\left(2\hat{i}-\hat{j}+\hat{k}\right)$ and $\bar{r}=\left(2\hat{i}+2\hat{j}-3\hat{k}\right)+\mu\left(\hat{i}+\hat{j}-2\hat{k}\right)$

(ii) $\frac{x-5}{4}=\frac{y-7}{5}=\frac{z+3}{5}$ and $x-6=y-8=z+2$.

(17) If lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-2}{1}=\frac{y+m}{2}=\frac{z-2}{1}$ intersect each other then find m.

(18) Find the vector and Cartesian equations of the line passing through the point (–1, –1, 2) and parallel to the line $2x-2=3y+1=6z-2$.

(19) Find the direction cosines of the line $\bar{r}=\left(-2\hat{i}+\frac{5}{2}\hat{j}-\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}\right)$.

(20) Find the Cartesian equation of the line passing through the origin which is perpendicular to $x-1=y-2=z-1$ and intersects the $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$.

(21) Write the vector equation of the line whose Cartesian equations are $y=2$ and $4x-3z+5=0$.

(22) Find the co-ordinates of points on the line $\frac{x-1}{1}=\frac{y-2}{-2}=\frac{z-3}{2}$ which are at the distance 3 unit from the base point A(l, 2, 3).

6.4 Equations of Plane :

Introduction : A plane is a surface such that the line joining any two points on it lies entirely on it.

Plane can be determined by

(i) two intersecting lines

(ii) two parallel lines

(iii) a line and a point outside it

(iv) three non collinear points.

Definition: A line perpendicular to a plane is called a normal to the plane. A plane has several normals. They all have the proportional direction ratios. We require only direction ratios of normal therefore we refer normal as **the normal** to a plane.

Direction ratios of the normal to the XY plane are 0, 0, 1.

6.4.1 Equation of plane passing through a point and perpendicular to a vector.

Theorem 6.8: The equation of the plane passing through the point $A(\bar{a})$ and perpendicular to vector $\bar{n}$ is $\bar{r}\cdot\bar{n}=\bar{a}\cdot\bar{n}$.

Proof: Let $P(\bar{r})$ be any point on the plane.

$\therefore \overline{AP}$ is perpendicular to $\bar{n}$.

$\therefore \overline{AP}\cdot\bar{n}=0$

$\therefore (\bar{r}-\bar{a})\cdot\bar{n}=0$

$\therefore \bar{r}\cdot\bar{n}-\bar{a}\cdot\bar{n}=0$

$\bar{r}\cdot\bar{n}=\bar{a}\cdot\bar{n}$

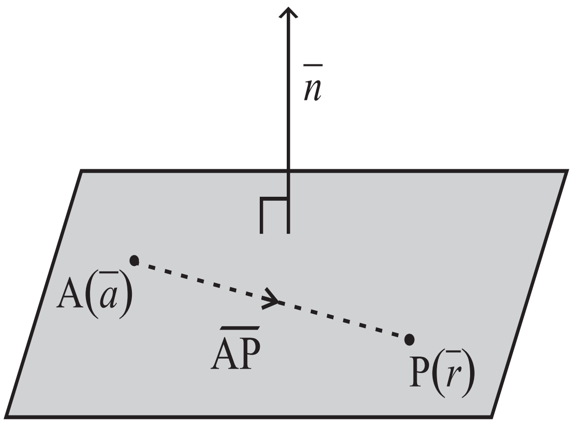


Fig. 6.8

This is the equation of the plane passing through the point $A(\bar{a})$ and perpendicular to the vector $\bar{n}$.

Remark :

- Equation $\bar{r}\cdot\bar{n}=\bar{a}\cdot\bar{n}$ is called the **vector equation of plane in scalar product form .**
- If $\bar{a}\cdot\bar{n}=d$ then equation $\bar{r}\cdot\bar{n}=\bar{a}\cdot\bar{n}$ takes the form $\bar{r}\cdot\bar{n}=d$.

Cartesian form :

Theorem 6.9 : The equation of the plane passing through the point A(x_1, y_1, z_1) and direction ratios of whose normal are a, b, c is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$.

Proof: Let P(x, y, z) be any point on the plane.

The direction ratios of AP are $x-x_1, y-y_1, z-z_1$.

The direction ratios of the normal are a, b, c. And AP is perpendicular to the normal.

$\therefore a(x-x_1)+b(y-y_1)+c(z-z_1)=0$.. This is the required equation of plane.

Remark: Above equation may be written as $ax+by+cz+d=0$

Ex.(1) Find the vector equation of the plane passing through the point having position vector $2\hat{i}+3\hat{j}+4\hat{k}$ and perpendicular to the vector $2\hat{i}+\hat{j}-2\hat{k}$.

Solution: We know that the vector equation of the plane passing through $A(\bar{a})$ and normal to vector $\bar{n}$ is given by $\bar{r}\cdot\bar{n}=\bar{a}\cdot\bar{n}$.

Here $\bar{a}=2\hat{i}+3\hat{j}+4\hat{k}$, $\bar{n}=2\hat{i}+\hat{j}-2\hat{k}$

$$\bar{a}\cdot\bar{n}=\left(2\hat{i}+3\hat{j}+4\hat{k}\right)\cdot\left(2\hat{i}+\hat{j}-2\hat{k}\right)=4+3-8=-1$$

The vector equation of the plane is $\bar{r}\cdot\left(2\hat{i}+\hat{j}-2\hat{k}\right)=-1$.

Ex.(2) Find the Cartesian equation of the plane passing through A(1, 2, 3) and the direction ratios of whose normal are 3, 2, 5.

Solution: The equation of the plane passing through A(x_1, y_1, z_1) and normal to the line having direction ratios a, b, c is $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$.

Here $(x_1, y_1, z_1) \equiv (1, 2, 3)$ and direction ratios of the normal are 3, 2. 5.

The Cartesian equation of the plane is $3(x-1)+2(y-2)+5(z-3)=0$.

$\therefore 3x+2y+5z-22=0$.

Ex.(3) The foot of the perpendicular drawn from the origin to a plane is M(2, 1, -2). Find vector equation of the plane.

Solution: OM is normal to the plane.

$\therefore$ The direction ratios of the normal are 2, 1, -2.

The plane passes through the point having position vector $2\hat{i}+\hat{j}-2\hat{k}$ and vector $\overline{OM}=2\hat{i}+\hat{j}-2\hat{k}$ is normal to it.

Its vector equation is $\bar{r}\cdot\bar{n}=\bar{a}\cdot\bar{n}$.

$\bar{r}\cdot\left(2\hat{i}+\hat{j}-2\hat{k}\right)=\left(2\hat{i}+\hat{j}-2\hat{k}\right)\cdot\left(2\hat{i}+\hat{j}-2\hat{k}\right)$

$\bar{r}\cdot\left(2\hat{i}+\hat{j}-2\hat{k}\right)=9$.

6.4.2 The vector equation of the plane passing through point $A(\bar{a})$ and parallel to $\bar{b}$ and $\bar{c}$:

Theorem 6.10 : The vector equation of the plane passing through the point $A(\bar{a})$ and parallel to non-zero and non-parallel vectors $\bar{b}$ and $\bar{c}$ is $(\bar{r}-\bar{a})\cdot(\bar{b}\times\bar{c})=0$.

Proof: As vectors $\bar{b}$ and $\bar{c}$ are parallel to the plane, vector $\bar{b}\times\bar{c}$ is normal to the plane. Plane passes through $A(\bar{a})$.

Let $P(\bar{r})$ be any point on the plane.

$\therefore \overline{AP}$ is perpendicular to $\bar{b}\times\bar{c}$

$\therefore \overline{AP}\cdot(\bar{b}\times\bar{c})=0$

$\therefore (\bar{r}-\bar{a})\cdot(\bar{b}\times\bar{c})=0$

$\therefore \bar{r}\cdot(\bar{b}\times\bar{c})=\bar{a}\cdot(\bar{b}\times\bar{c})$ is the required equation.

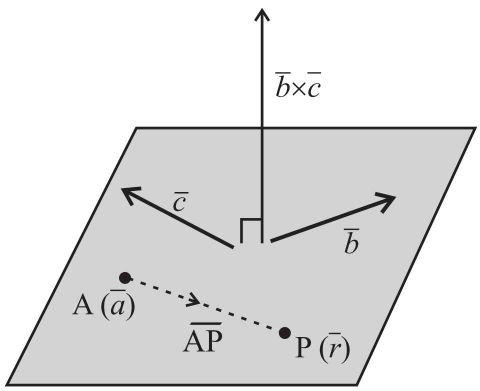


Fig. 6.9

Remark: As $\overline{AP}$, $\bar{b}$ and $\bar{c}$ are parallel to the same plane, they are coplanar vectors. Therefore $\overline{AP}$ can be expressed as the linear combination of $\bar{b}$ and $\bar{c}$. Hence $\overline{AP}=\lambda\bar{b}+\mu\bar{c}$ for some scalars λ and μ.

$\therefore \bar{r}-\bar{a}=\lambda\bar{b}+\mu\bar{c}$

$\therefore \bar{r}=\bar{a}+\lambda\bar{b}+\mu\bar{c}$ This equation is called the **vector equation of plane in parametric form.**

Ex(4) Find the vector equation of the plane passing through the point A(–1, 2, –5) an parallel to vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $\hat{i}+\hat{j}-\hat{k}$.

Solution: The vector equation of the plane passing through point $A(\bar{a})$ and parallel to $\bar{b}$ and $\bar{c}$ is $\bar{r}\cdot(\bar{b}\times\bar{c})=\bar{a}\cdot(\bar{b}\times\bar{c})$.

Here $\bar{a}=-\hat{i}+2\hat{j}-5\hat{k}$, $\bar{b}=4\hat{i}-\hat{j}+3\hat{k}$, $\bar{c}=\hat{i}+\hat{j}-\hat{k}$

$$\bar{b}\times\bar{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 3\\ 1 & 1 & -1\end{vmatrix}=-2\hat{i}+7\hat{j}+5\hat{k}$$

$$\bar{a}\cdot(\bar{b}\times\bar{c})=\left(-\hat{i}+2\hat{j}-5\hat{k}\right)\cdot\left(-2\hat{i}+7\hat{j}+5\hat{k}\right)=-9$$

The required equation is $\bar{r}\cdot\left(-2\hat{i}+7\hat{j}+5\hat{k}\right)=-9$

Ex.(5) Find the Cartesian equation of the plane $\bar{r}=\left(\hat{i}-\hat{j}\right)+\lambda\left(\hat{i}+\hat{j}+\hat{k}\right)+\mu\left(\hat{i}-2\hat{j}+3\hat{k}\right)$

Solution: Given plane is perpendicular to vector $\bar{n}$, where

$$\bar{n}=\bar{b}\times\bar{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 1 & -2 & 3\end{vmatrix}=5\hat{i}-2\hat{j}-3\hat{k}$$

$\therefore$ The direction ratios of the normal are 5, –2, –3.

And plane passes through A(1, -1, 0).

$\therefore$ Its Cartesian equation is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\therefore 5(x - 1) - 2(y + 1) - 3(z - 0) = 0$

$\therefore 5x - 2y - 3z - 7 = 0$

6.4.3 The vector equation of plane passing through three non-collinear points :

Theorem 6.11 : The vector equation of the plane passing through non-collinear points $A(\bar{a})$, $B(\bar{b})$ and $C(\bar{c})$ is $(\bar{r}-\bar{a})\cdot(\bar{b}-\bar{a})\times(\bar{c}-\bar{a})=0$

Proof: Let $P(\bar{r})$ be any point on the plane passing through non-collinear points $A(\bar{a})$, $B(\bar{b})$ and $C(\bar{c})$.

$\therefore \overline{AP}$, $\overline{AB}$ and $\overline{AC}$ are coplanar .

$\therefore \overline{AP}\cdot\overline{AB}\times\overline{AC}=0$

$\therefore (\bar{r}-\bar{a})\cdot\overline{AB}\times\overline{AC}=0$

$\therefore (\bar{r}-\bar{a})\cdot(\bar{b}-\bar{a})\times(\bar{c}-\bar{a})=0$

$\therefore \left[\bar{r}-\bar{a}\quad \bar{b}-\bar{a}\quad \bar{c}-\bar{a}\right]=0$

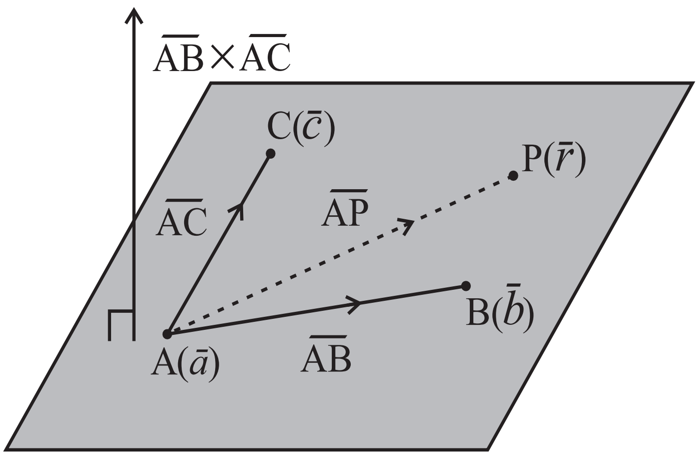


Fig. 6.10

This is the required equation of plane.

Cartesian form of the above equation is

$$\begin{vmatrix}x-x_1 & y-y_1 & z-z_1\\ x_2-x_1 & y_2-y_1 & z_2-z_1\\ x_3-x_1 & y_3-y_1 & z_3-z_1\end{vmatrix}=0$$

Ex.(6) Find the vector equation of the plane passing through points A (l, 1, 2), B (0, 2, 3) and C (4, 5, 6).

Solution; Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be position vectors of points A, B and C respectively .

$\therefore \bar{a} = \hat{i} + \hat{j} + 2\hat{k},\ \bar{b} = 2\hat{j} + 3\hat{k},\ \bar{c} = 4\hat{i} + 5\hat{j} + 6\hat{k}$

$\therefore \bar{b} - \bar{a} = -\hat{i} + \hat{j} + \hat{k}$ and $\bar{c} - \bar{a} = 3\hat{i} + 4\hat{j} + 4\hat{k}$

$$\therefore (\bar{b} - \bar{a}) \times (\bar{c} - \bar{a}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 3 & 4 & 4 \end{vmatrix} = 7\hat{j} - 7\hat{k}$$

The plane passes through $\bar{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $7\hat{j} - 7\hat{k}$ is normal to the plane.

$\therefore$ Its equation is $(\bar{r} - \bar{a}) \cdot (7\hat{j} - 7\hat{k}) = 0$

$\therefore \left(\bar{r} - (\hat{i} + \hat{j} + 2\hat{k})\right) \cdot (7\hat{j} - 7\hat{k}) = 0$

$\therefore \bar{r} \cdot (7\hat{j} - 7\hat{k}) = (\hat{i} + \hat{j} + 2\hat{k}) \cdot (7\hat{j} - 7\hat{k})$

$\therefore \bar{r} \cdot (7\hat{j} - 7\hat{k}) = -7$ is the required equation.

6.4.4 The normal form of equation of plane.

Theorem 6.12: The equation of the plane at distance p unit from the origin and to which unit vector $\hat{n}$ is normal is $\bar{r} \cdot \hat{n} = p$

Proof: Let ON be the perpendicular to the plane $\therefore$ ON $= p$

As $\hat{n}$ is the unit vector along ON, $\overline{ON} = p\hat{n}$

Let $P(\bar{r})$ be any point on the plane.

$\therefore \overline{NP} \perp \hat{n}$ $\qquad \therefore \overline{NP} \cdot \hat{n} = 0$

$\therefore (\bar{r} - p\hat{n}) \cdot \hat{n} = 0$ $\qquad \therefore \bar{r} \cdot \hat{n} - p\hat{n} \cdot \hat{n} = 0$

$\therefore \bar{r} \cdot \hat{n} - p = 0$ $\qquad \therefore \bar{r} \cdot \hat{n} = p$

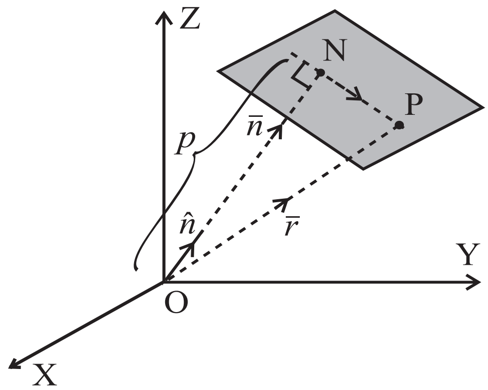


Fig. 6.11

This is called the **normal form of vector equation** of plane.

Remark:

- If l, m, n are direction cosines of the normal to a plane then $\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$.
- If N is the foot of the perpendicular drawn from the origin to the plane and ON = p then the co-ordinates of N are (pl, pm, pn).
- The equation of the plane is $lx + my + nz = p$. This is the **normal form of the Cartesian equation** of the plane.
- There are two planes at distance p units from origin and having $\hat{n}$ as unit vector along normal, namely $\bar{r} \cdot \hat{n} = \pm p$

Ex.(7) Find the vector equation of the plane which is at a distance of 6 unit from the origin and to which the vector $2\hat{i}-\hat{j}+2\hat{k}$ is normal.

Solution: Here $p = 6$ and $\bar{n} = 2\hat{i}-\hat{j}+2\hat{k}$ $\quad\therefore |\bar{n}| = 3$

$$\therefore \hat{n} = \frac{\bar{n}}{|\bar{n}|} = \frac{2\hat{i}-\hat{j}+2\hat{k}}{3}$$

The required equation is $\therefore \bar{r}\cdot\hat{n} = p$

$$\therefore \bar{r}\cdot\left(\frac{2\hat{i}-\hat{j}+2\hat{k}}{3}\right) = 6$$

$$\therefore \bar{r}\cdot\left(2\hat{i}-\hat{j}+2\hat{k}\right) = 18$$

Ex.(8) Find the perpendicular distance of the origin from the plane $x - 3y + 4z - 6 = 0$

Solution : First we write the given Cartesian equation in normal form.

i.e. in the form $lx + my + nz = p$

Direction ratios of the normal are 1, –3, 4.

$\therefore$ Direction cosines are $\frac{1}{\sqrt{26}}, \frac{-3}{\sqrt{26}}, \frac{4}{\sqrt{26}}$

Given equation can be written as $\frac{1}{\sqrt{26}}x - \frac{-3}{\sqrt{26}}y + \frac{4}{\sqrt{26}}z = \frac{6}{\sqrt{26}}$

$\therefore$ The distance of the origin from the plane is $\frac{6}{\sqrt{26}}$

Ex.(9) Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $2x + y - 2z = 18$

Solution:

The Direction ratios of the normal to the plane $2x + y - 2z = 18$ are 2, 1, –2.

$\therefore$ Direction cosines are $\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}$

The normal form of the given Cartesian equation is $\frac{2}{3}x + \frac{1}{3}y - \frac{2}{3}z = 6$

$\therefore \quad p = 6$

The coordinates of the foot of the perpendicular are $(lp, mp, np) = \left(6\left(\frac{2}{3}\right), 6\left(\frac{1}{3}\right), 6\left(-\frac{2}{3}\right)\right) \equiv (4, 2, -4)$

Ex.(10) Reduce the equation $\bar{r}\cdot\left(3\hat{i}-4\hat{j}+12\hat{k}\right)=8$ to the normal form and hence find

(i) the length of the perpendicular from the origin to the plane

(ii) direction cosines of the normal.

Solution: Here $\bar{n}=3\hat{i}-4\hat{j}+12\hat{k}$ $\quad\therefore |\bar{n}|=13$

The required normal form is $\bar{r}\cdot\dfrac{\left(3\hat{i}-4\hat{j}+12\hat{k}\right)}{13}=\dfrac{8}{13}$

(i) the length of the perpendicular from the origin to the plane is $\dfrac{8}{13}$

(ii) direction cosines of the normal are $\dfrac{3}{13},\dfrac{-4}{13},\dfrac{12}{13}$.

6.4.5 Equation of plane passing through the intersection of two planes :

If planes $\left(\bar{r}\cdot\bar{n}_1-d_1\right)=0$ and $\bar{r}\cdot\bar{n}_2-d_2=0$ intersect each other, then for every real value of λ equation $\bar{r}\cdot\left(\bar{n}_1+\lambda\bar{n}_2\right)=\left(d_1+\lambda d_2\right)$ represents a plane passing through the line of their intersection

If planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ intersect each other, then for every real value of λ, equation $\left(a_1x+b_1y+c_1z+d_1\right)+\lambda\left(a_2x+b_2y+c_2z+d_2\right)=0$ represents a plane passing through the line of their intersection.

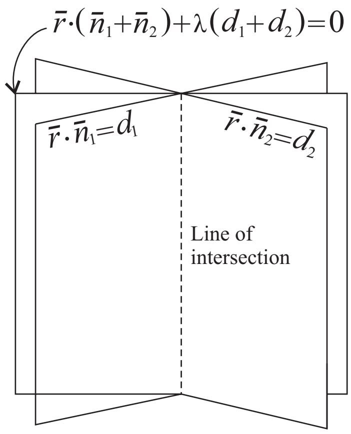


Fig. 6.12

Ex.(11) Find the vector equation of the plane passing through the point (1, 0. 2) and the line of intersection of planes $\bar{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=8$ and $\bar{r}\cdot\left(2\hat{i}+3\hat{j}+4\hat{k}\right)=3$

Solution: The equation of the required plane is of the form equation $\therefore \bar{r}\cdot\left(\bar{n}_1+\lambda\bar{n}_2\right)-\left(d_1+\lambda d_2\right)=0$

$$\therefore \bar{r}\cdot\left[\left(\hat{i}+\hat{j}+\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+4\hat{k}\right)\right]=8+3\lambda \quad \text{... (1)}$$

$$\therefore \bar{r}\cdot\left(\left(1+2\lambda\right)\hat{i}+\left(1+3\lambda\right)\hat{j}+\left(1+4\lambda\right)\hat{k}\right)=8+3\lambda$$

The plane passes through the point (1, 0, 2).

$$\therefore \left(\hat{i}+2\hat{k}\right)\cdot\left(\left(1+2\lambda\right)\hat{i}+\left(1+3\lambda\right)\hat{j}+\left(1+4\lambda\right)\hat{k}\right)=8+3\lambda$$

$$\therefore \left(1+2\lambda\right)+2\left(1+4\lambda\right)=8+3\lambda$$

$$\therefore 1+2\lambda+2+8\lambda=8+3\lambda$$

$\therefore 7\lambda = 5$

$\therefore \lambda = \dfrac{5}{7}$ ... (2)

From (1) and (2) we get

$$\therefore \bar{r} \cdot \left(\left(\hat{i} + \hat{j} + \hat{k} \right) + \frac{5}{7}\left(2\hat{i} + 3\hat{j} + 4\hat{k} \right) \right) = 8 + 3\left(\frac{5}{7} \right)$$

$$\therefore \bar{r} \cdot \left(17\hat{i} + 22\hat{j} + 27\hat{k} \right) = 71$$

Exercise 6.3

(1) Find the vector equation of a plane which is at 42 unit distance from the origin and which is normal to the vector $2\hat{i} + \hat{j} - 2\hat{k}$.

(2) Find the perpendicular distance of the origin from the plane $6x - 2y + 3z - 7 = 0$.

(3) Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $2x + 6y - 3z = 63$.

(4) Reduce the equation $\bar{r} \cdot \left(3\hat{i} + 4\hat{j} + 12\hat{k} \right) = 78$ to normal form and hence find

(i) the length of the perpendicular from the origin to the plane (ii) direction cosines of the normal.

(5) Find the vector equation of the plane passing through the point having position vector $\hat{i} + \hat{j} + \hat{k}$ and perpendicular to the vector $4\hat{i} + 5\hat{j} + 6\hat{k}$.

(6) Find the Cartesian equation of the plane passing through A(-1, 2, 3), the direction ratios of whose normal are 0, 2, 5.

(7) Find the Cartesian equation of the plane passing through A(7, 8, 6) and parallel to the XY plane.

(8) The foot of the perpendicular drawn from the origin to a plane is M(1,0,0). Find the vector equation of the plane.

(9) Find the vector equation of the plane passing through the point A(–2, 7, 5) and parallel to vectors $4\hat{i} - \hat{j} + 3\hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$.

(10) Find the Cartesian equation of the plane $\bar{r} = \left(5\hat{i} - 2\hat{j} - 3\hat{k} \right) + \lambda\left(\hat{i} + \hat{j} + \hat{k} \right) + \mu\left(\hat{i} - 2\hat{j} + 3\hat{k} \right)$.

(11) Find the vector equation of the plane which makes intercepts 1, 1, 1 on the co-ordinates axes.

6.5 Angle between planes: In this article we will discuss angles between two planes, angle between a line and a plane.

6.5.1 Angle between planes: Angle between planes can be determined from the angle between their normals. Planes $\bar{r}\cdot\bar{n}_1 = d_1$ and $\bar{r}\cdot\bar{n}_2 = d_2$ are perpendicular to each other if and only if $\bar{n}_1\cdot\bar{n}_2 = 0$
Planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular to each other if and only if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Definition : If two planes are not perpendicular to each other then **the angle between them** is defined as the **acute angle** between their normals.

The angle between the planes $\bar{r}\cdot\bar{n}_1 = d_1$ and $\bar{r}\cdot\bar{n}_2 = d_2$ is given by $\cos\theta = \left|\dfrac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1|\cdot|\bar{n}_2|}\right|$

Ex.(12) Find the angle between planes $\bar{r}\cdot\left(\hat{i}+\hat{j}-2\hat{k}\right) = 8$ and $\bar{r}\cdot\left(-2\hat{i}+\hat{j}+\hat{k}\right) = 3$

Solution: Normal to the given planes are $\bar{n}_1 = \hat{i}+\hat{j}-2\hat{k}$ and $\bar{n}_2 = -2\hat{i}+\hat{j}+\hat{k}$

The acute angle θ between normal is given by

$$\cos\theta = \left|\frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1|\cdot|\bar{n}_2|}\right|$$

$$\therefore \cos\theta = \left|\frac{\left(\hat{i}+\hat{j}-2\hat{k}\right)\cdot\left(-2\hat{i}+\hat{j}+\hat{k}\right)}{\sqrt{6}\cdot\sqrt{6}}\right| = \left|\frac{-3}{6}\right| = \frac{1}{2}$$

$$\therefore \cos\theta = \frac{1}{2} \qquad \therefore \theta = 60^\circ = \frac{\pi}{3}$$

The acute angle between normals $\bar{n}_1$ and $\bar{n}_2$ is 60°.

∴ The angle between given planes is 60°.

6.5.2 Angle between a line and a plane: Line $\bar{r} = \bar{a} + \lambda\bar{b}$ is perpendicular to the plane $\bar{r}\cdot\bar{n} = d$ if and only if $\bar{b}$ and $\bar{n}$ are collinear. *i.e.* if $\bar{b} = t\bar{n}$ for some $t \in$ R Line $\bar{r} = \bar{a} + \lambda\bar{b}$ is parallel to the plane of $\bar{r}\cdot\bar{n} = d$ and only if $\bar{b}$ and $\bar{n}$ are perpendicular to each other. *i.e.* if $\bar{b}\cdot\bar{n} = 0$.

Definition : The angle between a line and a plane is defined as the complementary angle of the acute angle between the normal to the plane and the line.

Because of the definition, the angle between a line and a plane can't be obtuse.

If θ is the angle between the line $\bar{r} = \bar{a} + \lambda\bar{b}$ and the plane $\bar{r}\cdot\bar{n} = d$ then the acute angle between the line and the normal to the plane is $\dfrac{\pi}{2} - \theta$.

$$\therefore \cos\left(\frac{\pi}{2} - \theta\right) = \left|\frac{\bar{b}\cdot\bar{n}}{|\bar{b}|\cdot|\bar{n}|}\right|$$

$$\sin\theta = \left|\frac{\bar{b}\cdot\bar{n}}{|\bar{b}|\cdot|\bar{n}|}\right|$$

Ex.(13) Find the angle between the line $\bar{r} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) + \lambda\left(\hat{i} + \hat{j} + \hat{k}\right)$ and the plane $\bar{r} \cdot \left(2\hat{i} - \hat{j} + \hat{k}\right) = 8$

Solution: The angle between the line $\bar{r} = \bar{a} + \lambda\bar{b}$ and the plane $\bar{r} \cdot \bar{n} = d$ is given by $\sin\theta = \left|\dfrac{\bar{b} \cdot \bar{n}}{|\bar{b}| \cdot |\bar{n}|}\right|$

Here $\bar{b} = \hat{i} + \hat{j} + \hat{k}$ and $\bar{n} = 2\hat{i} - \hat{j} + \hat{k}$

$\therefore \bar{b} \cdot \bar{n} = \left(\hat{i} + \hat{j} + \hat{k}\right) \cdot \left(2\hat{i} - \hat{j} + \hat{k}\right) = 2 - 1 + 1 = 2$

$|\bar{b}| = \sqrt{1+1+1} = \sqrt{3}$ and $|\bar{n}| = \sqrt{4+1+1} = \sqrt{6}$

$$\therefore \sin\theta = \left|\frac{\bar{b} \cdot \bar{n}}{|\bar{b}| \cdot |\bar{n}|}\right| = \frac{2}{\sqrt{3} \cdot \sqrt{6}} = \frac{\sqrt{2}}{3}$$

$$\therefore \theta = \sin^{-1}\left(\frac{\sqrt{2}}{3}\right)$$

6.6 Coplanarity of two lines:

We know that two parallel lines are always coplanar. If two non-parallel lines are coplanar then the shortest distance between them is zero. Conversely if the distance between two non-parallel lines is zero then they are coplanar.

Thus lines $\bar{r} = \bar{a}_1 + \lambda_1\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2\bar{b}_2$ are coplanar if and only if $\left(\bar{a}_2 - \bar{a}_1\right) \cdot \left(\bar{b}_1 \times \bar{b}_2\right) = 0$

The plane determined by them passes through $A\left(\bar{a}_1\right)$ and $\bar{b}_1 \times \bar{b}_2$. is normal to the plane .

$\therefore$ Its equation is $\left(\bar{r} - \bar{a}_1\right) \cdot \left(\bar{b}_1 \times \bar{b}_2\right) = 0$.

Lines $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$ are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

The equation of the plane determined by them is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$.

Ex.(14) Show that lines $\bar{r} = \left(\hat{i} + \hat{j} - \hat{k}\right) + \lambda\left(2\hat{i} - 2\hat{j} + \hat{k}\right)$ and $\bar{r} = \left(4\hat{i} - 3\hat{j} + 2\hat{k}\right) + \mu\left(\hat{i} - 2\hat{j} + 2\hat{k}\right)$ are coplanar. Find the equation of the plane determined by them.

Solution: Lines $\bar{r} = \bar{a}_1 + \lambda_1\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2\bar{b}_2$ are coplanar if and only if $(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2) = 0$

Here $\bar{a}_1 = \hat{i} + \hat{j} - \hat{k}$, $\bar{a}_2 = 4\hat{i} - 3\hat{j} + 2\hat{k}$

$\bar{b}_1 = 2\hat{i} - 2\hat{j} + \hat{k}$, $\bar{b}_2 = \hat{i} - 2\hat{j} + 2\hat{k}$

$\therefore \bar{a}_2 - \bar{a}_1 = 3\hat{i} - 4\hat{j} + 3\hat{k}$

$$(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2) = \begin{vmatrix} 3 & -4 & 3 \\ 2 & -2 & 1 \\ 1 & -2 & 2 \end{vmatrix} = 3(-2) + 4(3) + 3(-2) = -6 + 12 - 6 = 0$$

$\therefore$ Given lines are coplanar.

$$\text{Now } \bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & -2 & 2 \end{vmatrix} = -2\hat{i} - 3\hat{j} - 2\hat{k}$$

The equation of the plane determined by them is $(\bar{r} - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2) = 0$

$\therefore \bar{r}\cdot(\bar{b}_1 \times \bar{b}_2) = \bar{a}_1\cdot(\bar{b}_1 \times \bar{b}_2)$

$\therefore \bar{r}\cdot\left(-2\hat{i} - 3\hat{j} - 2\hat{k}\right) = \left(\hat{i} + \hat{j} - \hat{k}\right)\cdot\left(-2\hat{i} - 3\hat{j} - 2\hat{k}\right)$

$\therefore \bar{r}\cdot\left(-2\hat{i} - 3\hat{j} - 2\hat{k}\right) = -3$

$\therefore \bar{r}\cdot\left(2\hat{i} + 3\hat{j} + 2\hat{k}\right) = 3$

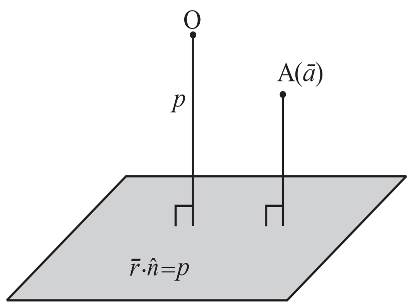


Fig. 6.13

6.7 Distance of a point from a plane.

To find the distance of the point $A(\bar{a})$ from the plane $\bar{r}\cdot\hat{n} = p$

The distance of the origin from the plane $\bar{r}\cdot\hat{n} = p$ is p.

The equation of the plane passing through $A(\bar{a})$ and parallel to the plane

$\bar{r}\cdot\hat{n} = p$ is $\bar{r}\cdot\hat{n} = \bar{a}\cdot\hat{n}$.

The distance of the origin from the plane $\bar{r}\cdot\hat{n} = \bar{a}\cdot\hat{n}$ is $\left|\bar{a}\cdot\hat{n}\right|$

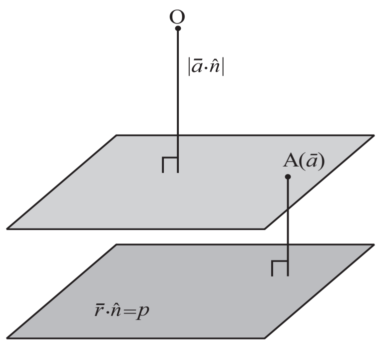


Fig. 6.14

The distance of the point $A(\bar{a})$ from the plane $\bar{r}\cdot\hat{n}=p$ is given by $\left|p-\left|\bar{a}\cdot\hat{n}\right|\right|$.

Remark : For finding distance of a point from a plane, the equation of the plane must be in the normal form.

Ex.(15) Find the distance of the point $4\hat{i}-3\hat{j}+2\hat{k}$ from the plane $\bar{r}\,.\left(-2\hat{i}+\hat{j}-2\hat{k}\right)=6$

Solution: Here $\bar{a}=4\hat{i}-3\hat{j}+2\hat{k},\bar{n}=-2\hat{i}+\hat{j}-2\hat{k}$

$$\therefore |\bar{n}|=\sqrt{(-2)^2+(1)^2+(-2)^2}=3$$

$$\therefore \hat{n}=\frac{\left(-2\hat{i}+\hat{j}-2\hat{k}\right)}{3}$$

The normal form of the equation of the given plane is

$$\bar{r}\cdot\frac{\left(-2\hat{i}+\hat{j}-2\hat{k}\right)}{3}=2 \quad \therefore p=2$$

Now, $\bar{a}\cdot\hat{n}=\left(4\hat{i}-3\hat{j}+2\hat{k}\right)\cdot\dfrac{\left(-2\hat{i}+\hat{j}-2\hat{k}\right)}{3}$

$$=\frac{\left(4\hat{i}-3\hat{j}+2\hat{k}\right)\cdot\left(-2\hat{i}+\hat{j}-2\hat{k}\right)}{3}=\frac{15}{3}=-5$$

$$\therefore \left|\bar{a}\cdot\hat{n}\right|=5$$

The required distance is given by $\left|p-\left|\bar{a}\cdot\hat{n}\right|\right|=|2-5|=3$

Therefore the distance of the point $4\hat{i}-3\hat{j}+2\hat{k}$ from the plane $\left(-2\hat{i}+\hat{j}-2\hat{k}\right)=6$ is 3 unit.

Exercise 6.4

(1) Find the angle between planes $\bar{r}\cdot\left(\hat{i}+\hat{j}+2\hat{k}\right)=13$ and $\bar{r}\cdot\left(2\hat{i}-\hat{j}+\hat{k}\right)=31$.

(2) Find the acute angle between the line $\bar{r}\cdot\left(\hat{i}+2\hat{j}+2\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}-6\hat{k}\right)$ and the plane $\bar{r}\cdot\left(2\hat{i}-\hat{j}+\hat{k}\right)=0$.

(3) Show that lines $\bar{r}=\left(2\hat{j}-3\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}+3\hat{k}\right)$ and $\bar{r}=\left(2\hat{i}+6\hat{j}+3\hat{k}\right)+\mu\left(2\hat{i}+3\hat{j}+4\hat{k}\right)$ are coplanar. Find the equation of the plane determined by them.

(4) Find the distance of the point $4\hat{i}-3\hat{j}+\hat{k}$ from the plane $\bar{r}\cdot\left(2\hat{i}+3\hat{j}-6\hat{k}\right)=21$.

(5) Find the distance of the point (1, 1, -l) from the plane $3x+4y-12z+20=0$.

Remember This: Line

- The vector equation of the line passing through $A(\bar{a})$ and parallel to vector $\bar{b}$ is $\bar{r} = \bar{a} + \lambda\bar{b}$
- The vector equation of the line passing through $A(\bar{a})$ and $B(\bar{b})$ is $\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$.
- The Cartesian equations of the line passing through $A(x_1, y_1, z_1)$ and having direction ratios *a, b, c* are $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$
- The distance of point $P(\bar{\alpha})$ from the line $\bar{r} = \bar{a} + \lambda\bar{b}$ is given by $\sqrt{|\bar{\alpha} - \bar{a}|^2 - \left[\frac{(\bar{\alpha} - \bar{a}) \cdot \bar{b}}{|\bar{b}|}\right]^2}$
- The shortest distance between lines $\bar{r} = \bar{a}_1 + \lambda_1\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2\bar{b}_2$ is given by $d = \left|\frac{(\bar{a}_2 - \bar{a}_1) \cdot (\bar{b}_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|}\right|$
- Lines $\bar{r} = \bar{a}_1 + \lambda_1\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2\bar{b}_2$ intersect each other if and only if $(\bar{a}_2 - \bar{a}_1) \cdot (\bar{b}_1 \times \bar{b}_2) = 0$
- Lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ intersect each other if and only if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$
- The distance between parallel lines $\bar{r} = \bar{a}_1 + \lambda\bar{b}$ and $\bar{r} = \bar{a}_2 + \lambda\bar{b}$ is given by $d = |(\bar{a}_2 - \bar{a}_1) \times \hat{b}|$

Plane

- The vector equation of the plane passing through $A(\bar{a})$ and normal to vector $\bar{n}$ *is* $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$
- Equation $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$ is called the **veetor equation of plane in scalar product form.**
- If $\bar{a} \cdot \bar{n} = d$ then equation $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$ takes the form $\bar{r} \cdot \bar{n} = d$
- The equation of the plane passing through $A(x_1, y_1, z_1)$ and normal to the line having direction ratios *a,b,c* is $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$.

- The vector equation of the plane passing through point $A(\bar{a})$ and parallel to $\bar{b}$ and $\bar{c}$ is $\bar{r}\cdot(\bar{b}\times\bar{c}) = a\cdot(\bar{b}\times\bar{c})$
- Equation $\bar{r} = \bar{a} + \lambda\bar{b} + \mu\bar{c}$ is called the vector equation of plane in parametric form.
- The vector equation of the plane passing through non-collinear points $A(\bar{a})$, $B(\bar{b})$ and $C(\bar{c})$ is $(\bar{r}-\bar{a})\cdot(\bar{b}-\bar{a})\times(\bar{c}-\bar{a}) = 0$
- Cartesian from of the above equation is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$
- The equation of the plane at distance p unit from the origin and to which unit vector $\hat{n}$ is normal is $\bar{r}\cdot\hat{n} = p$
- If l, m, n are direction cosines of the normal to a plane which is at distance p unit from the origin then its equation is $lx + my + nz = p$.
- If N is the foot of the perpendicular drawn from the origin to a plane and $ON = p$ then the co-ordinates of N are (pl, pm, pn).
- If planes $(\bar{r}\cdot\bar{n}_1 - d_1) = 0$ and $\bar{r}\cdot\bar{n}_2 - d_2 = 0$ intersect each other, then for every real value of λ,equation $\bar{r}\cdot(\bar{n}_1 + \lambda\bar{n}_2) - (d_1 + \lambda d_2) = 0$ represents a plane passing through the line of their intersection.
- If planes and $a_1x+b_1y+c_1z+d=0$ and $a_2x+b_2y + c_2z + d_2 = 0$ intersect each other, then for every real value of λ, equation $(a_1x+b_1y + c_1z + d_1) + \lambda\ (a_2x+b_2y\ c_2z + d_2) = 0$ represents a plane passing through the line of their intersection.
- The angle between the planes $\bar{r}\cdot\bar{n}_1 = d_1$ and $\bar{r}\cdot\bar{n}_2 = d_2$ is given by $\cos\theta = \left|\frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1||\bar{n}_2|}\right|$
- The acute angle between the line $\bar{r} = \bar{a} + \lambda\bar{b}$ and the plane $\bar{r}\cdot\bar{n} = d$ is given by $\sin\theta = \left|\frac{\bar{b}\cdot\bar{n}}{|\bar{b}_1||\bar{n}|}\right|$
- Lines $\bar{r} = a_1 + \lambda_1 b_1$ and $\bar{r} = \bar{a}_2 + \lambda_2\bar{b}_2$ are coplanar if and only if $(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1\times\bar{b}_2) = 0$ and the quation of the plane determined by them is $(\bar{r} - \bar{a}_1)\cdot(\bar{b}_1\times\bar{b}_2) = 0$
- Lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$.

are coplanar if and only if $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$, and the equation of the plane determined by them is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

- The distance of the point $A(\bar{a})$ from the plane $\bar{r}\cdot\hat{n} = p$ is given by $\left|p - \left|\bar{a}\cdot\hat{n}\right|\right|$

Miscellenous Exercise 6 (B)

I **Choose correct alternatives.**

1) If the line $\frac{x}{3} = \frac{y}{4} = z$ is perpendicular to the line $\frac{x-1}{k} = \frac{y+2}{3} = \frac{z-3}{k-1}$ then the value of k is:

A) $\frac{11}{4}$ B) $-\frac{11}{4}$ C) $\frac{11}{2}$ D) $\frac{4}{11}$

2) The vector equation of line $2x-1 = 3y+2 = z-2$ is

A) $\bar{r} = \left(\frac{1}{2}\bar{i} - \frac{2}{3}\bar{j} + 2\bar{k}\right) + \lambda\left(3\bar{i} + 2\bar{j} + 6\bar{k}\right)$

B) $\bar{r} = \bar{i} - \bar{j} + \left(2\bar{i} + \bar{j} + \bar{k}\right)$

C) $\bar{r} = \left(\frac{1}{2}\bar{i} - \bar{j}\right) + \lambda\left(\bar{i} - 2\bar{j} + 6\bar{k}\right)$

D) $\bar{r} = \left(\bar{i} + \bar{j}\right) + \lambda\left(\bar{i} - 2\bar{j} + 6\bar{k}\right)$

3) The direction ratios of the line which is perpendicular to the two lines $\frac{x-7}{2} = \frac{y+17}{-3} = \frac{z-6}{1}$ and $\frac{x+5}{1} = \frac{y+3}{2} = \frac{z-6}{-2}$ are

A) 4,5,7 B) 4, -5, 7 C) 4, -5,-7 D) -4, 5, 8

4) The length of the perpendicular from (1, 6,3) to the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

A) 3 B) $\sqrt{11}$ C) $\sqrt{13}$ D) 5

5) The shortest distance between the lines $\bar{r} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) + \lambda\left(\hat{i} - \hat{j} - \hat{k}\right)$ and $\bar{r} = \left(2\hat{i} - \hat{j} - \hat{k}\right) + \mu\left(2\hat{i} + \hat{j} + 2\hat{k}\right)$ is

A) $\frac{1}{\sqrt{3}}$ B) $\frac{1}{\sqrt{2}}$ C) $\frac{3}{\sqrt{2}}$ D) $\frac{\sqrt{3}}{2}$

6) The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$. and coplanar if

A) k = 1 or -1 B) k = 0 or -3 C) k = $\pm$ 3 D) k = 0 or -1

7) The lines $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ *and* $\frac{x-1}{-2} = \frac{y-2}{-4} = \frac{z-3}{6}$ and are

A) perpendicular B) inrersecting C) skew D) coincident

8) Equation of X-axis is

A) $x = y = z$ B) $y = z$

C) $y = 0, z = 0$ D) $x = 0, y = 0$

9) The angle between the lines 2x = 3y = -z and 6x = –y = -4z is

A) 45° B) 30° C) 0° D) 90°

10) The direction ratios of the line 3x + 1 = 6y–2 = 1–z are

A) 2,1,6 B) 2,1,-6 C) 2,-1,6 D) -2,1,6

11) The perpendicular distance of the plane 2x + 3y - z = k from the origin is $\sqrt{14}$ units, the value of k is

A) 14 B) 196 C) $2\sqrt{14}$ D) $\frac{\sqrt{14}}{2}$

12) The angle between the planes and $\bar{r}.\left(\bar{i} - 2\bar{j} + 3\bar{k}\right) + 4 = 0$ and $\bar{r}\cdot\left(2\bar{i} + \bar{j} - 3\bar{k}\right) + 7 = 0$ is

A) $\frac{\pi}{2}$ B) $\frac{\pi}{3}$ C) $\cos^{-1}\left(\frac{3}{4}\right)$ D) $\cos^{1}\left(\frac{9}{14}\right)$

13) If the planes $\bar{r}\cdot\left(2\bar{i} - \lambda\bar{j} + \bar{k}\right) = 3$ and $\bar{r}\cdot\left(4\bar{i} - \bar{j} + \mu\bar{k}\right) = 5$ are parallel, then the values of λ and μ are respectively.

A) $\frac{1}{2}, -2$ B) $-\frac{1}{2}, 2$ C) $-\frac{1}{2}, -2$ D) $\frac{1}{2}, 2$

14) The equation of the plane passing through (2, -1, 3) and making equal intercepts on the coordinate axes is

A) $x + y + z = 1$ B) $x + y + z = 2$ C) $x + y + z = 3$ D) $x + y + z = 4$

15) Measure of angle between the planes $5x-2y+3z-7=0$ and $15x-6y+9z+5=0$ is

A) 0^o B) 30^o

C) 45^o D) 90^o

16) The direction cosines of the normal to the plane 2x-y+2z=3 are

A) $\frac{2}{3},\frac{-1}{3},\frac{2}{3}$ B) $\frac{-2}{3},\frac{1}{3},\frac{-2}{3}$

C) $\frac{2}{3},\frac{1}{3},\frac{2}{3}$ D) $\frac{2}{3},\frac{-1}{3},\frac{-2}{3}$

17) The equation of the plane passing through the points (1, -1, 1), (3, 2, 4) and parallel to Y-axis is :

A) 3x+2z –1= 0 B) 3x –2z=1 C) 3x+2z+1=0 D) 3x+2z=2

18) The equation of the plane in which the line $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{y-4}{1}=\frac{z+5}{3}$ lie, is

A) 17x - 47y - 24z + 172 = 0

B) 17x + 47y - 24z + 172 = 0

C) 17x + 47y + 24z +172 = 0

D) 17x - 47y + 24z + 172 = 0

19) If the line $\frac{x+1}{2}=\frac{y-m}{3}=\frac{z-4}{6}$ lies in the plane 3x-14y+6z+49=0, then the value of *m* is:

A) 5 B) 3 C) 2 D) -5

20) The foot of perpendicular drawn from the point (0,0,0) to the plane is (4, -2, -5) then the equation of the plane is

A) 4x + y + 5z = 14 B) 4x - 2y - 5z = 45

C) x - 2y - 5z = 10 D) 4x + y + 6z = 11

II. Solve the following :

(1) Find the vector equation of the plane which is at a distance of 5 unit from the origin and which is normal to the vector $2\hat{i}+\hat{j}+2\hat{k}$

(2) Find the perpendicular distance of the origin from the plane 6x+2y+3z-7=0

(3) Find the coordinates of the foot of the perpendicular drawn from the origin to the plane 2x+3y+6z=49.

(4) Reduce the equation $\bar{r}.\left(6\hat{i}+8\hat{j}+24\hat{k}\right)=13$ to normal form and hence find

(i) the length of the perpendicular from the origin to the plane

(ii) direction cosines of the normal.

(5) Find the vector equation of the plane passing through the points A(l, -2,l), B(2, -1, -3) and C(0,l,5).

(6) Find the Cartesian equation of the plane passing through A(1, - 2, 3) and the direction ratios of whose normal are 0. 2, 0.

(7) Find the Cartesian equation of the plane passing through A(7, 8, 6) and parallel to the plane $\bar{r}.\left(6\hat{i}+8\hat{j}+7\hat{k}\right)=0$

(8) The foot of the perpendicular drawn from the origin to a plane is M(l, 2,0). Find the vector equation of the plane.

(9) A plane makes non zero intercepts a, b, c on the co-ordinates axes. Show that the vector equation of the plane is $\bar{r}.\left(bc\hat{i}+ca\hat{j}+ab\hat{k}\right)=abc$

(10) Find the vector equation of the plane passing through the point A(-2, 3,5) and parallel to vectors $4\hat{i}+3\hat{k}$ *and* $\hat{i}+\hat{j}$

(11) Find the Cartesian equation of the plane $\bar{r}=\lambda\left(\hat{i}+\hat{j}-\hat{k}\right)+\mu\left(\hat{i}+2\hat{j}+3\hat{k}\right)$

(12) Find the vector equations of planes which pass through A(l, 2,3), B(3, 2, 1) and make equal intercepts on the co-ordinates axes.

(13) Find the vector equation of the plane which makes equal non-zero intercepts on the co-ordinates axes and passes through (1,1,1).

(14) Find the angle between planes $\bar{r}\cdot\left(-2\hat{i}+\hat{j}+2\hat{k}\right)=17$ and $\bar{r}\cdot\left(2\hat{i}+2\hat{j}+\hat{k}\right)=71$.

(15) Find the acute angle between the line $\bar{r}=\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$ and the plane $\bar{r}\cdot\left(2\hat{i}-\hat{j}+\hat{k}\right)=23$

(16) Show that lines $\bar{r}=\left(\hat{i}+4\hat{j}\right)+\lambda\left(\hat{i}+2\hat{j}+3\hat{k}\right)$ and .$\bar{r}=\left(3\hat{j}-\hat{k}\right)+\mu\left(2\hat{i}+3\hat{j}+4\hat{k}\right)$

are coplanar. Find the equation of the plane determined by them.

(17) Find the distance of the point $3\hat{i}+3\hat{j}+\hat{k}$ from the plane $\bar{r}\cdot\left(2\hat{i}+3\hat{j}+6\hat{k}\right)=21$

(18) Find the distance of the point (13, 13,-13) from the plane 3x + 4y - 12z = 0.

(19) Find the vector equation of the plane passing through the origln and containing the line $\bar{r}=\left(\hat{i}+4\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}+\hat{k}\right)$

(20) Find the vector equation of the plane which bisects the segment joining A(2,3, 6) and B(4,3, -2) at right angle.

(21) Show that lines $x=y, z=0$ and $x+y=0, z=0$ intersect each other. Find the vector equation of the plane determined by them.

✦✦✦✦✦

7 Linear Programming

Let's Study

Let's Recall

A linear equation in two variables $ax + by + c = 0$, where $a, b, c \in R$ a and b are not zero simultaneously represents a straight line. A straight line makes disjoint parts of the plane. The points lying on the straight line and two half planes on either side, which are represented by $ax + by + c < 0$ and $ax + by + c > 0$. We will now study the two half planes made by a line.

The sets of points $\{(x, y) \mid ax + by + c < 0\}$ and $\{(x, y) \mid ax + by + c > 0\}$ are two open half planes. The sets of points $\{(x, y) \mid ax + by + c \le 0\}$ and $\{(x, y) \mid ax + by + c \ge 0\}$ are two half, planes with common points. The sets $\{(x, y) \mid ax + by + c \le 0\}$ and $\{(x, y) \mid ax + by + c \ge 0\}$ have the common boundary $\{(x, y) \mid ax + by + c = 0\}$.

Linear inequation in two variables.

Definition : A linear inequation in two variables x, y is a mathematical expression of the form $ax + by < c$ or $ax + by > c$ where $a \ne 0$, $b \ne 0$ simultaneously and $a, b \in R$.

Activity : Check which of the following ordered pairs is a solution of $2x + 3y - 6 \le 0$.

1. $(1, -1)$ 2. $(2, 1)$ 3. $(-2, 1)$ 4. $(-1, -2)$ 5. $(-3, 4)$

Solution :

Sr. No.	(x, y)	Inequation	Conclusion
1.	(1, -1)	$2(1) + 3(-1) - 6 = -1 < 0$	is a solution
2.	(2, 1)	$2(2) + 3(1) - 6 = 1 > 0$	
3.	(-2, 1)		
4.	(1, -2)		
5.	(-3, 4)		

Graphical representation of linear inequation in two variables $ax + by$ ($\le$ or $\ge$) c is a region on any one side of the straight line ax + by = c in the coordinate system, depending on the sign of inequality.

7.1.1 : Convex set :

Definition : A set of points in a plane is said to be a convex set if line segment joining any two points of the set entirely lies within the set.

The following sets are convex sets :

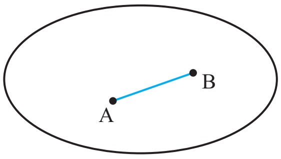


fig 7.1(a)

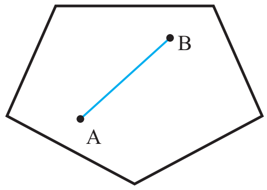


fig 7.1(b)

The following sets are not convex sets :

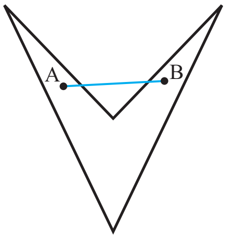


fig 7.1(c)

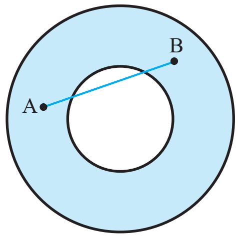


fig 7.1(d)

Note :

(i) The convex sets may be bounded.

Following are bounded convex sets.

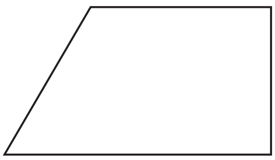


fig 7.2(a)

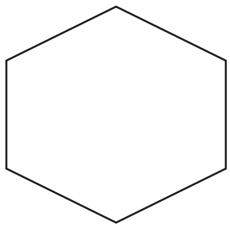


fig 7.2(b)

(ii) Convex sets may be unbounded.

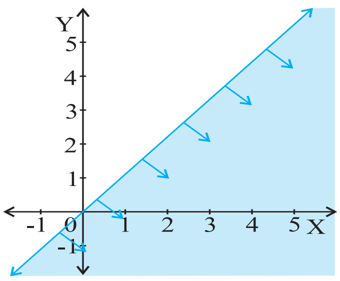


fig 7.3(a)

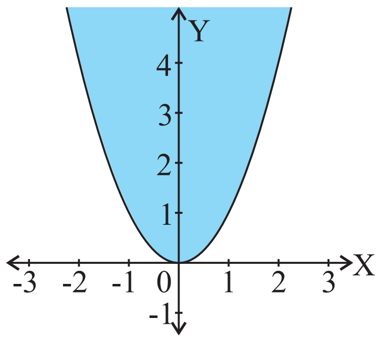


fig 7.3(b)

Note :

1) Graphical representations of $x \leq h$ and $x \geq h$ on the Cartesian coordinate system. Draw the line $x = h$ in XOY plane.

The solution set is the set of points lying on the Left side or Right side of the line $x = h$.

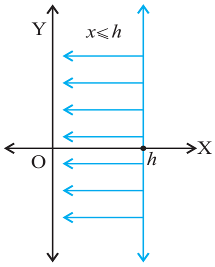


fig 7.4

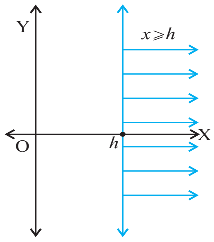


fig 7.5

2) Graphical representation of $y \leq k$ and $y \geq k$ on the Cartesian coordinate system.
Draw the line $y = k$ in XOY plane.
The solution set is the set of points lying below or above the line $y = k$.

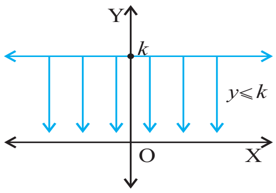


fig 7.6

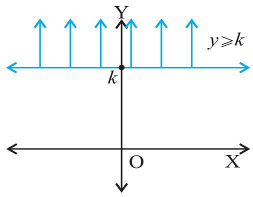


fig 7.7

3) Graphical representations of $ax + by \leq 0$ and $ax+by \geq 0$ on the Cartesian coordinate system.
The line $ax + by = 0$ passes through the origin, see the following graphs.

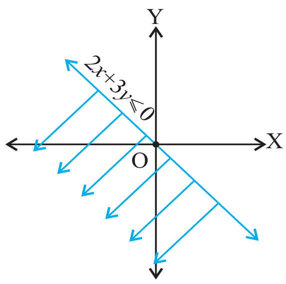


fig 7.8

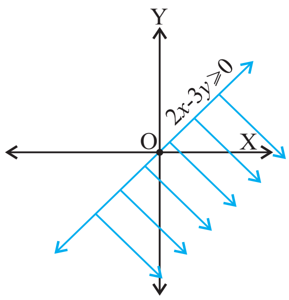


fig 7.9

4) To find the solutions of $ax + by \leq c$ and $ax + by \geq c$ graphically,
Draw the line $ax + by = c$ in XOY system. It divides the plane into two parts, each part is called half plane :

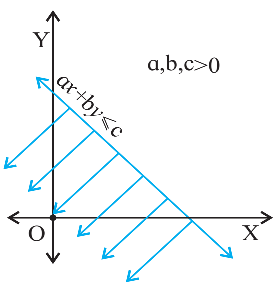


fig 7.10

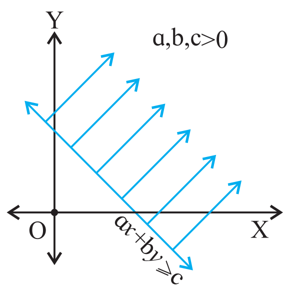


fig 7.11

The half plane H_1, containing origin is called origin side of the line and the other half plane H_2 is non-origin side of the line $ax + by = c$.

For convenience, consider O (0, 0) as test point. If O (0, 0) satisfies the given inequation

$ax + by \leq c$, then the required region is on origin side. Hence shade the region the H_1 otherwise shade the other half plane H_2.

The shaded portion represents the solution set of the given inequation.

Note that the points $\{(x, y)\} \mid ax + by = c\}$ form the common boundary of the two half planes.

Solved Examples

Ex.1 Show the solution sets for the following inequations graphically.

a) $x \leq 3$ b) $y \geq -2$ c) $x + 2y \leq 0$

d) $2x + 3y \geq 6$ e) $2x - 3y \geq -6$ f) $4x - 5y \leq 20$.

Solution :

a) To draw : $x \leq 3$

Draw line: $x = 3$

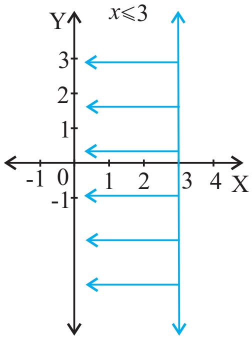


fig 7.12

b) To draw : $y \geq -2$

Draw line: $y = -2$

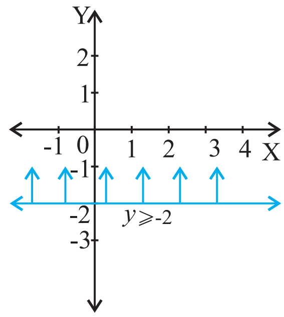


fig 7.13

c) To draw : $x + 2y \leq 0$

Draw line: $x = -2y$

x	0	-2	-4	2
$y = -\frac{x}{2}$	0	1	2	-1

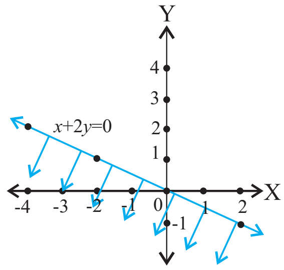


fig 7.14

As the origin lies on the line $x = -2y$, we have to choose another point as a test point. Let us choose $(-2,-2)$ =-2–4 = $-6 < 0$

$x + 2y = -2 - 4 = -6 < 0$

d) To draw : $2x + 3y \geq 6$

Draw line : $2x + 3y = 6$

x	y	(x, y)
3	0	(3, 0)
0	2	(0, 2)

$2x + 3y\big|_{(0,0)} = 0 < 6$. Therefore, the required region is the non-origin side of the line.

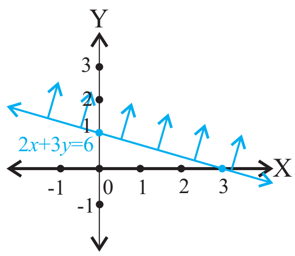


fig 7.15

e) To draw : $2x - 3y \geq -6$

Draw line : $2x - 3y = -6$

x	y	(x, y)
-3	0	$(-3, 0)$
0	2	(0, 2)

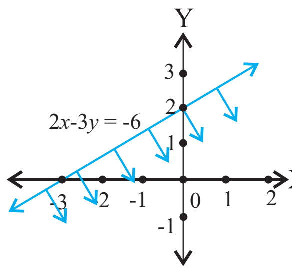


fig 7.16

$2x + 3y\big|_{(0,0)} = 0 > -6$. Therefore, the required region is the origin side of the line.

f) To draw : $4x - 5y \leq 20$

Draw line : $4x - 5y = 20$

x	y	(x, y)
5	0	(5, 0)
0	–4	(0, –4)

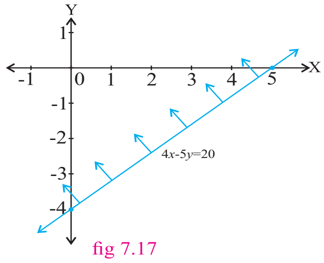


fig 7.17

$4x - 5y|_{(0,0)} = 0 < 20$. Therefore, the required region is the origin side of the line.

Ex. 2 : Represent the solution set of inequation $3x + 2y \leq 6$ graphically.

Solution :

To draw : $3x + 2y \leq 6$

Draw the line : $3x + 2y = 6$

x	y	(x, y)
2	0	(2, 0)
0	3	(0, 3)

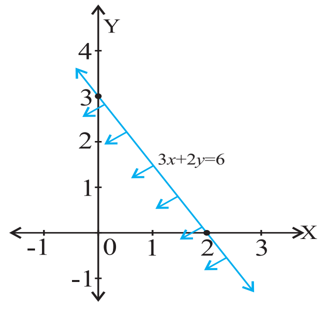


fig 7.18

$3x + 2y|_{(0,0)} = 0 < 6$. Therefore, the required region is the origin side of the line.

Ex. 3 : Find the common region of the solutions of the inequations $x + 2y \geq 4$, $2x - y \leq 6$.

Solution :

To find the common region of : $x + 2y \geq 4$ and $2x - y \leq 6$

Draw the lines : $x + 2y = 4$ and $2x - y = 6$

Equation of line	x	y	Line passes through (x, y)	Sign	Region
$x + 2y = 4$	4	0	(4, 0)	$\geq$	Non-origin side
	0	2	(0, 2)		
$2x - y = 6$	3	0	(3, 0)	$\leq$	Origin side
	0	–6	(0, –6)		

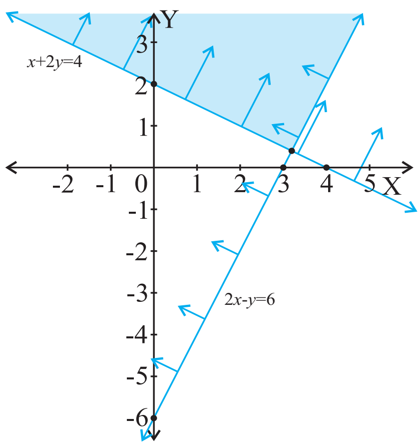


fig 7.19

Notes :

There are two methods of plotting the lines :

1) Find the intersection points of the line with X axis and Y axis.

2) Write the equation of the straight line in the double intercept form.

For example : consider the line $3x + 2y = 6$.

1) The intersection with X axis is given when y = 0, So A (2, 0) is the point of intersection with the x axis, The intersection with Y axis is given when x = 0.

So B (0, 3) is the point of intersection with the y axis. We draw the line through A and B.

2) The equation of line is $3x + 2y = 6$ Divide both sides by 6.

We get the double intercept form $\frac{x}{2} + \frac{y}{3} = 1$

$\therefore$ Intercepts on X axis and Y axis are 2 and 3 respectively.

The points (2, 0), (0, 3) lie on the line.

Ex. 4 : Find the graphical solution of $3x + 4y \le 12$, and $x - 4y \le 4$

Solution : To find the graphical solution of : $3x + 4y \le 12$ and $x - 4y \le 4$

Draw the lines : $L_1 : 3x + 4y = 12$ and $L_2 : x - 4y = 4$.

Equation of line	x	y	Line passes through (x, y)	Sign	Region
$3x + 4y = 12$	4	0	(4, 0)	$\le$	Origin side
	0	3	(0, 3)		
$x - 4y = 4$	4	0	(4, 0)	$\le$	Origin side
	0	-1	(0, -1)		

The common shaded region is graphical solution.

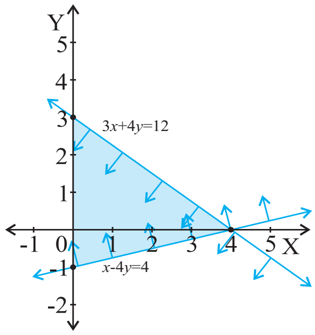


fig 7.20

Exercise 7.1

1) Solve graphically :

i) $x \ge 0$ ii) $x \le 0$ iii) $y \ge 0$ iv) $y \le 0$

2) Solve graphically :

i) $x \ge 0$ and $y \ge 0$ ii) $x \le 0$ and $y \ge 0$ iii) $x \le 0$ and $y \le 0$ iv) $x \ge 0$ and $y \le 0$

3) Solve graphically :

i) $2x - 3 \ge 0$ ii) $2y - 5 \ge 0$ iii) $3x + 4 \le 0$ iv) $5y + 3 \le 0$

4) Solve graphically :

i) $x + 2y \le 6$ ii) $2x - 5y \ge 10$ iii) $3x + 2y \ge 0$ iv) $5x - 3y \le 0$

5) Solve graphically :

i) $2x + y \geq 2$ and $x - y \leq 1$

ii) $x - y \leq 2$ and $x + 2y \leq 8$

iii) $x + y \geq 6$ and $x + 2y \leq 10$

iv) $2x + 3y \leq 6$ and $x + 4y \geq 4$

v) $2x + y \geq 5$ and $x - y \leq 1$

Solved Examples

Ex. 1 : Find the graphical solution of the system of inequations.

$$2x + y = 10,\ 2x - y = 2,\ x \geq = 0,\ y \geq = 0$$

Solution : To find the solution of the system of given inequations -

Draw lines	x	y	Line passes through (x, y)	Sign	Region
L_1 $2x + y = 10$	0	10	(0, 10)	$\leq$	Origin side of L_1
	5	0	(5, 0)		
L_2 $2x - y = 2$	0	-2	(0, -2)	$\geq$	Origin side of L_2
	1	0	(1, 0)		

The common shaded region OABCO is the graphical solution. This graphical solution is known as feasible solution.

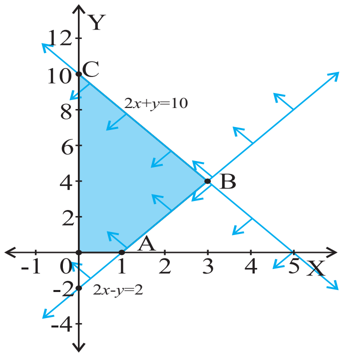


fig 7.21

Remark : The restrictions $x \geq 0$, $y \geq 0$, are called non-negativity constraints.

Definition : A solution which satisfies all the constraints is called a feasible solution.

Ex. 2 : Find the feasible solution of the system of inequations $3x + 4y \geq 12$, $2x + 5y \geq 10$, $x \geq 0$, $y \geq 0$.

To draw (Inequations)	Draw line	x	y	Line passes through (x, y)	Sign	Region
$3x + 4y \geq 12$	L_1 $3x + 4y = 12$	0	3	(0, 3)	$\geq$	Non-origin side of Line L_1
		4	0	(4, 0)		
$2x + 5y \geq 10$	L_2 $2x + 5y = 10$	0	2	(0, 2)	$\geq$	Non-origin side of Line L_2
		5	0	(5, 0)		

Solution :

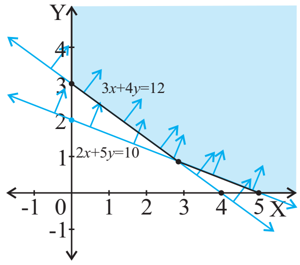


fig 7.22

Common shaded region is the feasible solution.

Ex. 3 :

A manufacturer produces two items A and B. Both are processed on two machines I and II. A needs 2 hours on machine I and 2 hours on machine II. B needs 3 hours on machine I and 1 hour on machine II. If machine I can run maximum 12 hours per day and II for 8 hours per day, construct a problem in the form of inequations and find its feasible solution graphically.

Solution :

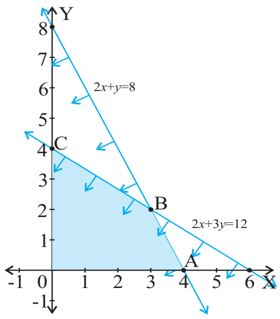


fig 7.23

Let x units of product A and y units of product B be produced.

$x \geq 0, y \geq 0$.

Tabular form is:

Machine	Product A (x)	Product B (y)	Availability
I	2	3	12
II	2	1	08

Inequations are $2x + 3y \leq 12$, $2x + y \leq 8$, $x \geq 0$, $y \geq 0$.

To draw graphs of the above inequations :

To draw	Draw line	x	y	Line passes through (x, y)	Sign	Region lie on
$2x + 3y \leq 12$	L_1 $2x + 3y = 12$	0	4	(0, 4)	$\leq$	Origin side of Line L_1
		6	0	(6, 0)		
$2x + 4 \leq 8$	L_2 $2x + 4 = 8$	0	8	(0, 8)	$\leq$	Origin side of Line L_2
		4	0	(4, 0)		

The common shaded region OABCO the feasible region.

Exercise 7.2

I) Find the feasible solution of the following inequations graphically.

1) $3x + 2y \leq 18$, $2x + y \leq 10$, $x \geq 0$, $y \geq 0$
2) $2x + 3y \leq 6$, $x + y \geq 2$, $x \geq 0$, $y \geq 0$
3) $3x + 4y \geq 12$, $4x + 7y \leq 28$, $y \geq 1$, $x \geq 0$
4) $x + 4y \leq 24$, $3x + y \leq 21$, $x + y \leq 9$, $x \geq 0$, $y \geq 0$
5) $0 \leq x \leq 3$, $0 \leq y \leq 3$, $x + y \leq 5$, $2x + y \geq 4$
6) $x - 2y \leq 2$, $x + y \geq 3$, $-2x + y \leq 4$, $x \geq 0$, $y \geq 0$
7) A company produces two types of articles A and B which requires silver and gold. Each unit of A requires 3 gm of silver and 1 gm of gold, while each unit of B requires 2 gm of silver and 2 gm of gold. The company has 6 gm of silver and 4 gm of gold. Construct the inequations and find the feasible solution graphically.
8) A furniture dealer deals in tables and chairs. He has Rs.1,50,000 to invest and a space to store at most 60 pieces. A table costs him Rs.1500 and a chair Rs.750. Construct the inequations and find the feasible solution.

7.2 Linear Programming Problems (L.P.P.) :

L.P.P. is an optimization technique used in different fields such as management, planning, production, transportation etc. It is developed during the second world war to optimize the utilization of limited resources to get maximum returns. Linear Programming is used to minimize the cost of production and maximizing the profit. These problems are related to efficient use of limited resources like raw materials, man-power, availability of machine time and cost of the material and so on.

Linear Programming is mathematical technique designed to help managers in the planning and decision making. Programming problems are also known as optimization problems. The mathematical programming involves optimization of a certain function, called objective function, subject to given conditions or restrictions known as constraints.

7.2.1 Meaning of L.P.P. :

Linear implies all the mathematical functions contain variables of index at most one. A L.P.P. may be defined as the problem of maximizing or minimizing a linear function subject to linear constraints. These constraints may be equations or inequations.

Now, we formally define the terms related to L.P.P. as follows :

1) **Decision variables :** The variables involved in L.P.P. are called decision variables.
2) **Objective function :** A linear function of variables which is to be optimized, i.e. either maximized or minimized, is called an objective function.
3) **Constraints :** Conditions under which the objective function is to be optimized are called constraints. These constraints are in the form of equations or inequations.
4) **Non-negativity constraints :** In some situations, the values of the variables under considerations may be positive or zero due to the imposed constraints, Such constraints are referred as non-negativity constraints.

7.2.2 Mathematical formulations of L.P.P. :

Step 1) : Identify the decision variables (x, y) or (x_1, x_2).

Step 2) : Identify the objective function and write it as mathematical expression in terms of decision variables.

Step 3) : Identify the different constraints and express them as mathematical equations / inequations.

Note : i) We shall study L.P.P. with at most two variables.

ii) We shall restrict ourselves to L.P.P. involving non-negativity constraints.

Solved examples

Ex. 1 : A Toy manufacturer produces bicycles and tricycles, each of which must be processed through two machine A and B. Machine A has maximum of 120 hours available and machine B has a maximum of 180 hours available. Manufacturing a bicycle requires 4 hours on machine A and 10 hours on machine B. Manufacturing a tricycle required 6 hours on machine A and 3 hours on machine B. If profits are Rs.65 for a bicycle and Rs.45 for a tricycle, formulate L.P.P. to have maximum profit.

Solution :

Let z be the profit, which can be made by manufacturing and selling x tricycles and y bicycles.

$x \geq 0, y \geq 0$

$\therefore$ Total Profit $z = 45x + 65y$

Maximize $z = 45x + 65y$

It is given that

Machine	Tricycles (x)	Bicycles (y)	Availability
A	6	4	120
B	3	10	180

From the above table, remaining conditions are $6x + 4y \leq 120$, $3x + 10y \leq 180$.

$\therefore$ The required formulated L.P.P. is as follows :

Maximize $z = 45x + 65y$

Subject to the constraints $x \geq 0, y \geq 0$

$6x + 4y \leq 120$

$3x + 10y \leq 18\ 0$

Ex. 2 : A company manufactures two types of toys A and B. Each toy of type A requires 2 minutes for cutting and 1 minute for assembling. Each toy of type B requires 3 minutes for cutting and 4 minutes for assembling. There are 3 hours available for cutting and 2 hours are available for assembling. On selling a toy of type A the company gets a profit of Rs.10 and that on toy of type B is Rs. 20. Formulate the L.P.P. to maximize profit.

Solution :

Suppose, the company manufacters x toys of type A and y toys of type B.

$x \geq 0, y \geq 0$

Let P be the total profit

On selling a toy of type A, company gets Rs.10 and that on a toy of type B is Rs.20.

$\therefore$ total profit on selling x toy s of type A and y toys of type B is $p = 10x + 20y$.

$\therefore$ maximize $p = 10x + 20y$.

The conditions are

$2x + 3y \leq 180$, $x + 4y \leq 120$, $x \geq 0$, $y \geq 0$.

Ex. 3 : A horticulturist wishes to mix two brands of fertilizers that will provide a minimum of 15 units of potash, 20 units of nitrate and 24 units of phosphate. One unit of brand I provides 3 units of potash, 1 unit of nitrate, 3 units of phosphate. One unit of brand II provides 1 unit of potash, 5 units of nitrate and 2 units of phosphates. One unit of brand I costs Rs. 120 and one unit of brand II costs Rs.60 per unit. Formulate this problems as L.P.P. to minimize the cost.

Solution :

Let z be the cost of mixture prepared by mixing x units of brand I and y units of brand II.

Then $x \geq 0, y \geq 0$.

Since, 1 unit of brand I costs Rs.120.

1 unit of brand II costs Rs.60.

$\therefore$ total cost $z = 120x + 60y$.

$\therefore$ Minimize $z = 120x + 60y$.

Brand / Content	I per unit	II per unit	Minimum requirement
Potash	3	1	15
Nitrate	1	5	20
Phosphate	3	2	24

The conditions are $3x + y \geq 15$,

$x + 5y \geq 20$,

$3x + 2y \geq 24$.

The L.P.P. is

Maximize $z = 120x + 60y$ subject to the above constraints.

Exercise 7.3

1) A manufacturing firm produces two types of gadgets A and B, which are first processed in the foundry and then sent to machine shop for finishing. The number of man hours of labour required in each shop for production of A and B and the number of man hours available for the firm are as follows :

Gadgets	Foundry	Machine Shop
A	10	5
B	6	4
Time available (hour)	60	35

Profit on the sale of A is Rs. 30 and B is Rs. 20 per units. Formulate the L.P.P. to have maximum profit.

2) In a cattle breading firm, it is prescribed that the food ration for one animal must contain 14, 22 and 1 units of nutrients A, B and C respectively. Two different kinds of fodder are available. Each unit of these two contains the following amounts of these three nutrients :

Fodder / Nutrient	Fodder 1	Fodder 2
Nutrients A	2	1
Nutrients B	2	3
Nutrients C	1	1

The cost of fodder 1 is Rs.3 per unit and that of fodder Rs. 2, Formulate the L.P.P. to minimize the cost.

3) A company manufactures two types of chemicals A and B. Each chemical requires two types of raw material P and Q. The table below shows number of units of P and Q required to manufacture one unit of A and one unit of B and the total availability of P and Q.

Chemical / Raw Material	A	B	Availability
P	3	2	120
Q	2	5	160

The company gets profits of Rs.350 and Rs.400 by selling one unit of A and one unit of B respectively. (Assume that the entire production of A and B can be sold). How many units of the chemicals A and B should be manufactured so that the company get maximum profit? Formulate the problem as L.P.P. to maximize the profit.

4) A printing company prints two types of magazines A and B. The company earns Rs. 10 and Rs. 15 on magazines A and B per copy. These are processed on three machines I, II, III. Magazine A requires 2 hours on Machine I, 5 hours on Machine II and 2 hours on Machine III. Magazine B requires 3 hours on Machine I, 2 hours on Machine II and 6 hours on Machine III. Machines I, II, III are available for 36, 50, 60 hours per week respectively. Formulate the L.P.P. to determine weekly production of A and B, so that the total profit is maximum.

5) A manufacture produces bulbs and tubes. Each of these must be processed through two machines M_1 and M_2. A package of bulbs require 1 hour of work on Machine M_1 and 3 hours of work on M_2. A package of tubes require 2 hours on Machine M_1 and 4 hours on Machine M_2. He earns a profit of Rs. 13.5 per package of bulbs and Rs. 55 per package of tubes. Formulate the LLP to maximize the profit, if he operates the machine M_1, for atmost 10 hours a day and machine M_2 for atmost 12 hours a day.

6) A company manufactures two types of fertilizers F_1 and F_2. Each type of fertilizer requires two raw materials A and B. The number of units of A and B required to manufacture one unit of fertilizer F_1 and F_2 and availability of the raw materials A and B per day are given in the table below :

Fertilizers / Raw Material	F1	F2	Availability
A	2	3	40
B	1	4	70

By selling one unit of F_1 and one unit of F_2, company gets a profit of Rs. 500 and Rs. 750 respectively. Formulate the problem as L.P.P. to maximize the profit.

7) A doctor has prescribed two different units of foods A and B to form a weekly diet for a sick person. The minimum requirements of fats, carbohydrates and proteins are 18, 28, 14 units respectively. One unit of food A has 4 units of fats. 14 units of carbohydrates and 8 units of protein. One unit of food B has 6 units of fat, 12 units of carbohydrates and 8 units of protein. The price of food A is 4.5 per unit and that of food B is 3.5 per unit. Form the L.P.P. so that the sick person's diet meets the requirements at a minimum cost.

8) If John drives a car at a speed of 60 kms/hour he has to spend Rs. 5 per km on petrol. If he drives at a faster speed of 90 kms/hour, the cost of petrol increases to 8 per km. He has Rs. 600 to spend on petrol and wishes to travel the maximum distance within an hour. Formulate the above problem as L.P.P.

9) The company makes concrete bricks made up of cement and sand. The weight of a concrete brick has to be least 5 kg. Cement costs Rs.20 per kg. and sand costs of Rs.6 per kg. strength consideration dictate that a concrete brick should contain minimum 4 kg. of cement and not more than 2 kg. of sand. Form the L.P.P. for the cost to be minimum.

7.2.3 Formal definitions related to L.P.P. :

1) **Solution of L.P.P. :** A set of values of the decision variables x_1, x_2, ..., x_n which satisfy the conditions of given linear programming problem is called a solution to that problem.

2) **Feasible solution :** A solution which satisfies the given constraints is called a feasible solution.

3) **Optimal feasible solution :** A feasible solution which maximizes or minimizes the objective function as per the requirements is called an optimal feasible solution.

4) **Feasible region :** The common region determined by all the constraints of the L.P.P. is called the feasible region.

Solution of L.P.P. :

There are two methods to find the solution of L.P.P. :

1) Graphical method, 2) Simplex method.

Note : We shall restrict ourselves to graphical method.

Some definitions :

Solution :

A set of values of the variables which satisfies all the constraints of the L.P.P. is called the solution of the L.P.P.

Optimum feasible solution :

A feasible solution which optimizes (either maximizes or minimizes) the objective function of L.P.P. is called optimum feasible solution.

Theorems (without proof) :

Theorem 1 : The set of all feasible solutions of L.P.P. is a convex set.

Theorem 2 : The objective function of L.P.P. attains its optimum value (either maximum or minimum) at least at one of the vertices of convex polygon. This is known as convex polygon theorem.

Corner - Point Method :

1) Convert all inequations of the constraints into equations..
2) Draw the lines in X - Y plane.
3) Locate common region indicated by the constraints. This common region is feasible region.
4) Find the vertices of feasible region.
5) Find the value of the objective function z at all vertices of feasible region.

Suppose, we are expected to maximize or minimize a given objective function $z = ax + by$ in the feasible region. The feasible region is a convex region bounded by straight lines. If any linear function $z = ax + by$, is maximized in the feasible region at some point, then the point is vertex of the polygon. This can be verified by drawing a line $ax + by = c$ which passes through the feasible region and moves with different values of c.

Solve graphically the following Linear Programming Problems :

Example 1 : Maximize : $z = 9x + 13y$ subject to $2x + 3y \leq 18$, $2x + y \leq 10$, $x \geq 0$, $y \geq 0$.

Solution : To draw $2x + 3y \leq 18$ and $2x + y \leq 10$

Draw line $2x + 3y = 18$ and $2x + y = 10$

To draw	x	y	Line passes through (x, y)	Sign	Region lies on
L_1 $2x + 3y = 18$	0	6	(0, 6)	$\leq$	Origin side of Line L_1
	9	0	(9, 0)		
L_2 $2x + y = 10$	0	10	(0, 10)	$\leq$	Origin side of Line L_2
	5	0	(5, 0)		

The common shaded region is O A B C O is a feasible region with vertices O (0, 0), A (5, 0), B (3, 4), C (0, 6).

(x, y) Vertex of S	Value of z = 9x + 13y at (x, y)
O (0, 0)	0
A (5, 0)	45
B (3, 4)	79
C (0, 6)	78

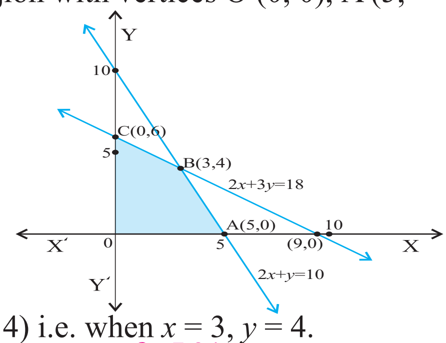


fig 7.24

From the table, maximum value of $z = 79$, occurs at B (3, 4) i.e. when $x = 3$, $y = 4$.

Solve graphically the following Linear Programming Problems :

Example 2 : Maximize : $z = 5x + 2y$ subject to $5x + y \geq 10$, $x + y \geq 6$, $x \geq 0$, $y \geq 0$.

Solution : To draw $5x + y \geq 10$ and $x + y \geq 6$

Draw line $5x + y = 10$ and $x + y = 6$.

To draw	x	y	Line passes through (x, y)	Sign	Region lies on
L_1 $5x + y = 10$	0	10	(0, 10)	$\geq$	Non-origin side of Line L_1
	2	0	(2, 0)		
L_2 $x + y = 6$	0	6	(0, 6)	$\geq$	Non-origin side of Line L_2
	6	0	(6, 0)		

The common shaded region is feasible region with vertices A (6, 0), B (1, 5), C (0, 10).

(x, y) Vertex of S	Value of z = 5x + 2y at (x, y)
A (6, 0)	30
B (1, 5)	15
C (0, 10).	20

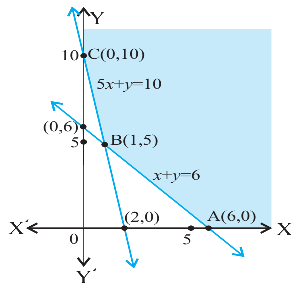


fig 7.25

From the table, maximum value of $z = 15$, occurs at B (1, 5) i.e. when $x = 1, y = 5$.

Example 3: Maximize: $z = 3x+4y$ subject to $x-y \geq 0$, $-x + 3y \leq 3$, $x \geq 0$, $y \geq 0$.

Solution : To draw $x - y \geq 0$ and $-x + 3y \leq 3$

Draw line $x - y \geq 0$ and $-x + 3y = 3$.

To draw	Draw line	x	y	Line passes through (x, y)	Sign	Region lies on
$x - y \geq 0$	L_1 $x = y$	0	0	(0, 0)	$\geq$	A side
		1	1	(1, 1)		
$-x + 3y \leq 3$	L_2 $-x+3y = 3$	0	1	(0, 1)	$\leq$	Origin side of Line L_2
		–3	0	(–3, 0)		

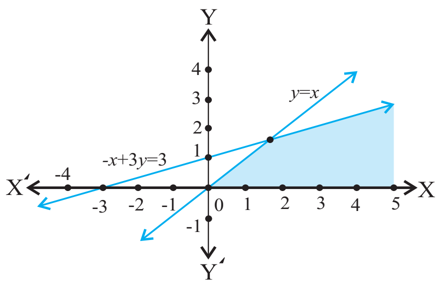


fig 7.26

From graph, we can see that the common shaded area is the feasible region which is unbounded (not a polygon). In such cases, the iso-profit lines can be moved away form the origin indefinitely. ∴ There is no finite maximum value of z within the feasible region.

Example 4 : Maximize : $z = 5x + 2y$ subject to $3x + 5y \leq 15$, $5x + 2y \leq 10$, $x \geq 0$, $y \geq 0$.

Solution : To draw $3x + 5y \leq 15$ and $5x + 2y \leq 10$

Draw line $3x + 5y = 15$ and $5x + 2y = 10$.

To draw	x	y	Line passes through (x, y)	Sign	Region lies on
L_1 $3x + 5y = 15$	5	0	(5, 0)	$\le$	Origin side of Line L_1
	0	3	(0, 3)		
L_2 $5x + 2y = 10$	2	0	(2, 0)	$\le$	Origin side of Line L_2
	0	5	(0, 5)		

The shaded region O A B C is the feasible region with the vertices O (0, 0), A (2, 0), $B\left(\frac{20}{19}, \frac{45}{19}\right)$, C (0, 3)

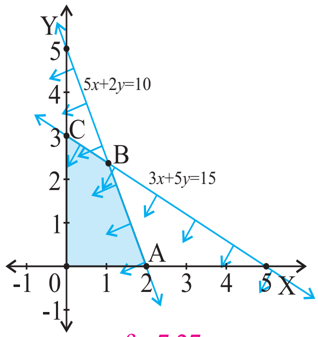


fig 7.27

$Z_0 = 0$, $Z_A = 10$, $Z_B = 10$, $Z_C = 6$.

Maximum value of z occurs at A and B and is $z = 10$.

Maximum value of z occurs at every point lying on the segment AB.

Hence there are infinite number of optimal solutions.

Note : If the two distinct points produce the same minimum value then the minimum value of objective function occurs at every point on the segment joining them.

Exercise 7.4

Solve the following L.P.P. by graphical method :

1) Maximize : $z = 11x + 8y$ subject to $x \le 4$, $y \le 6$, $x + y \le 6$, $x \ge 0$, $y \ge 0$.

2) Maximize : $z = 4x + 6y$ subject to $3x + 2y \le 12$, $x + y \ge 4$, $x, y \ge 0$.

3) Maximize : $z = 7x + 11y$ subject to $3x + 5y \le 26$ $5x + 3y \le 30$, $x \ge 0$, $y \ge 0$.

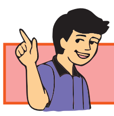
4) Maximize : $z = 10x + 25y$ subject to $0 \le x \le 3$, $0 \le y \le 3$, $x + y \le 5$ also find maximum value of z.

5) Maximize : $z = 3x + 5y$ subject to $x + 4y \le 24$, $3x + y \le 21$, $x + y \le 9$, $x \ge 0$, $y \ge 0$ also find maximum value of z.

6) Minimize : $z = 7x + y$ subject to $5x + y \ge 5$, $x + y \ge 3$, $x \ge 0$, $y \ge 0$.

7) Minimize : $z = 8x + 10y$ subject to $2x + y \ge 7$, $2x + 3y \ge 15$, $y \ge 2$, $x \ge 0$, $y \ge 0$.

8) Minimize : $z = 6x + 21y$ subject to $x + 2y \ge 3$, $x + 4y \ge 4$, $3x + y \ge 3$, $x \ge 0$, $y \ge 0$.



Let's remember!

* Working rule to formulate the L.P.P. :

Step 1 : Identify the decision variables and assign the symbols x, y or x_1, x_2 to them. Introduce non-negativity constraints.

Step 2 : Identify the set of constraints and express them as linear inequation in terms of the decision variables.

Step 3 : Identify the objective function to be optimized (i.e. maximized or minimized) and express it as a linear function of decision variables.

* Let R be the feasible region (convex polygon) for a L.P.P. and Let $z = ax + by$ be the objective functions then the optimal value (maximum or minimum) of z occurs at least one of the corner points (vertex) of the feasible region.

* Corner point method for solving L.P.P. graphically :

Step 1 : Find the feasible region of the L.P.P.

Step 2 : Determine the vertices of the feasible region either by inspection or by solving the two equations of the lines intersecting at that points.

Step 3 : Find the value of the objective function z, at all vertices of feasible region.

Step 4 : Determine the feasible solution which optimizes the value of the objective function.

Miscelleneous Exercise

I) Select the appropriate alternatives for each of the following :

1) The value of objective function is maximum under linear constraints _______.

A) at the centre of feasible region
B) at (0, 0)
C) at a vertex of feasible region
D) the vertex which is of maximum distance from (0, 0)

2) Which of the following is correct _______.

A) every L.P.P. has an optimal solution
B) a L.P.P. has unique optimal solution
C) if L.P.P. has two optimal solutions then it has infinite number of optimal solutions
D) the set of all feasible solution of L.P.P. may not be convex set

3) Objective function of L.P.P. is _______.

A) a constraint
B) a function to be maximized or minimized
C) a relation between the decision variables
D) equation of a straight line

4) The maximum value of $z = 5x + 3y$ subjected to the constraints $3x + 5y \leq 15$, $5x + 2y \leq 10$, $x, y \geq 0$ is _______.

A) 235 B) $\frac{235}{9}$
C) $\frac{235}{19}$ D) $\frac{235}{3}$

5) The maximum value of $z = 10x + 6y$ subjected to the constraints $3x + y \leq 12$, $2x + 5y \leq 34$, $x \geq 0$, $y \geq 0$. _______.

A) 56 B) 65
C) 55 D) 66

6) The point at which the maximum value of $x + y$ subject to the constraints $x + 2y \leq 70$, $2x + y \leq 95$, $x \geq 0$, $y \geq 0$ is obtained at _______.

A) (30, 25) B) (20, 35)
C) (35, 20) D) (40, 15)

7) Of all the points of the feasible region, the optimal value of z obtained at the point lies _______.

A) inside the feasible region B) at the boundary of the feasible region
C) at vertex of feasible region D) outside the feasible region

8) Feasible region is the set of points which satisfy _______.

A) the objective function B) all of the given constraints

C) some of the given constraints D) only one constraint

9) Solution of L.P.P. to minimize $z = 2x + 3y$ s.t. $x \geq 0, y \geq 0, 1 \leq x + 2y \leq 10$ is _______.

A) $x = 0,\ y = \frac{1}{2}$ B) $x = \frac{1}{2},\ y = 0$

C)x = 1, y = 2 D) $x = \frac{1}{2},\ y = \frac{1}{2}$

10) The corner points of the feasible solution given by the inequation $x + y \leq 4,\ 2x + y \leq 7,$ $x \geq 0, y \geq 0$ are _______.

A) (0, 0), (4, 0), (7, 1), (0, 4) B) (0, 0), $(\frac{7}{2}, 0)$, (3, 1), (0, 4)

C) (0, 0), $(\frac{7}{2}, 0)$, (3, 1), (0, 7) D) (0, 0), (4, 0), (3, 1), (0, 7)

11) The corner points of the feasible solution are (0, 0), (2, 0), $\left(\frac{12}{7}, \frac{3}{7}\right)$, (0, 1). Then Z = $7x + y$ is maximum at _______.

A)(0, 0) B)(2, 0)

C)$\left(\frac{12}{7}, \frac{3}{7}\right)$ D)(0, 1)

12) If the corner points of the feasible solution are (0, 0), (3, 0), (2, 1) and $\left(0, \frac{7}{3}\right)$ the maximum value of $z = 4x + 5y$ is _______.

A) 12 B) 13

C) $\frac{35}{3}$ D) 0

13) If the corner points of the feasible solution are (0, 10), (2, 2) and (4, 0) then the point of minimum $z = 3x + 2y$ is _______.

A) (2, 2) B) (0, 10)

C) (4, 0) D) (3 ,4)

14) The half plane represented by $3x + 2y < 8$ constraints the point _______.

A) $\left(1, \frac{5}{2}\right)$ B) (2, 1)

C) (0, 0) D) (5, 1)

15) The half plane represented by $4x + 3y > 14$ contains the point _______.

A) (0, 0) B) (2, 2)

C) (3, 4) D) (1, 1)

II) Solve the following :

1) Solve each of the following inequations graphically using X Y plane.

i) $4x - 18 \geq 0$ ii) $-11x - 55 \leq 0$

iii) $5y - 12 \geq 0$ iv) $y \leq -3.5$

2) Sketch the graph of each of following inequations in XOY co-ordinate system.

i) $x \geq 5y$ ii) $x + y \leq 0$

iii) $2y - 5x \geq 0$ iv) $1x + 51 \leq y$

3) Find graphical solution for each of the following system of linear inequation.

i) $2x + y \geq 2,\ x - y \leq 1$ ii) $x + 2y \geq 4,\ 2x - y \leq 6$

iii) $3x + 4y \leq 12,\ x - 2y \geq 2,\ y \geq -1$

4) Find feasible solution for each of the following system of linear inequations graphically.

i) $2x + 3y \leq 12,\ 2x + y \leq 8,\ x \geq 0,\ y \geq 0$

ii) $3x + 4y \geq 12,\ 4x + 7y \leq 28,\ x \geq 0,\ y \geq 0$

5) Solve each of the following L.P.P.

i) Maximize $z = 5x_1 + 6x_2$ subject to $2x_1 + 3x_2 \le 18$, $2x_1 + x_2 \le 12$, $x_1 \ge 0$, $x_2 \ge 0$

ii) Maximize $z = 4x + 2y$ subject to $3x + y \ge 27$, $x + y \ge 21$

iii) Maximize $z = 6x + 10y$ subject to $3x + 5y \le 10$, $5x + 3y \le 15$, $x \ge 0$, $y \ge 0$

iv) Maximize $z = 2x + 3y$ subject to $x - y \ge 3$, $x \ge 0$, $y \ge 0$

6) Solve each of the following L.P.P.

i) Maximize $z = 4x_1 + 3x_2$ subject to $3x_1 + x_2 \le 15$, $3x_1 + 4x_2 \le 24$, $x_1 \ge 0$, $x_2 \ge 0$

ii) Maximize $z = 60x + 50y$ subject to $x + 2y \le 40$, $3x + 2y \le 60$, $x \ge 0$, $y \ge 0$

iii) Maximize $z = 4x + 2y$ subject to $3x + y \ge 27$, $x + y \ge 21$, $x + 2y \ge 30$; $x \ge 0$, $y \ge 0$

7) A carpenter makes chairs and tables. Profits are Rs.140/- per chair and < 210/- per table. Both products are processed on three machines : Assembling, Finishing and Polishing. The time required for each product in hours and availability of each machine is given by following table:

Product Machine	Chair (x)	Table (y)	Available time (hours)
Assembling	3	3	36
Finishing	5	2	50
Polishing	2	6	60

Formulate the above problem as L.P.P. Solve it graphically.

Formulate and solve the following Linear Programming Problems using graphical method :

8) A company manufactures bicycles and tricycles, each of which must be processed through two machines A and B. Maximum availability of Machine A and B is respectively 120 and 180 hours. Manufacturing a bicycle requires 6 hours on Machine A and 3 hours on Machine B. Manufacturing a tricycles requires 4 hours on Machine A and 10 hours on Machine B. If profits are Rs.180/- for a bicycle and Rs.220/- for a tricycle. Determine the number of bicycles and tricycles that should be manufactured in order to maximize the profit.

9) A factory produced two types of chemicals A and B. The following table gives the units of ingredients P and Q (per kg) of chemicals A and B as well as minimum requirements of P and Q and also cost per kg. chemicals A and B :

Chemicals in units Ingredients per kg.	A (x)	B (y)	Minimum requirements in units
P	1	2	80
Q	3	1	75
Cost (in Rs.)	4	6	--

Find the number of units of chemicals A and B should be produced sp as to minimize the cost.

10) A company produces mixers and food processors. Profit on selling one mixer and one food processor is Rs. 2,000/- and Rs. 3,000/- respectively. Both the products are processed through three Machines A, B, C. The time required in hours by each product and total time available in hours per week on each machine are as follows :

Product / Machine	Mixer (per unit)	Food Processor (per unit)	Available time
$A \rightarrow A$	3	3	36
$B \rightarrow B$	5	2	50
$C \rightarrow C$	2	6	60

How many mixers and food processors should be produced to maximize the profit?

11) A chemical company produces a chemical containing three basic elements A, B, C so that it has at least 16 liters of A, 24 liters of B and 18 liters of C. This chemical is made by mixing two compounds I and II. Each unit of compound I has 4 liters of A, 12 liters of B, 2 liters of C. Each unit of compound II has 2 liters of A, 2 liters of B and 6 liters of C. The cost per unit of compound I is Rs.800/- and that of compound II is Rs.640/-. Formulate the problem as L.P.P. and solve it to minimize the cost.

12) A person makes two types of gift items A and B requires the services of a cutter and a finisher. Gift item A requires 4 hours of cutter's time and 2 hours of finisher's time. B requires 2 hours of cutter's time and 4 hours of finisher's time. The cutter and finisher have 208 hours and 152 hours available times respectively every month. The profit of one gift item of type A is Rs.75/- and on gift item B is Rs.125/-. Assuming that the person can sell all the gift items produced, determine how many gift items of each type should he make every month to obtain the best returns?

13) A firm manufactures two products A and B on which profit earned per unit Rs.3/- and Rs.4/- respectively. Each product is processed on two machines M_1 and M_2. The product A requires one minute of processing time on M_1 and two minute of processing time on M_2, B requires one minute of processing time on M_1 and one minute of processing time on M_2. Machine M_1 is available for use for 450 minutes while M_2 is available for 600 minutes during any working day. Find the number of units of product A and B to be manufactured to get the maximum profit.

14) A firm manufacturing two types of electrical items A and B, can make a profit of Rs.20/- per unit of A and Rs.30/- per unit of B. Both A and B make use of two essential components a motor and a transformer. Each unit of A requires 3 motors and 2 transformers and each units of B requires 2 motors and 4 transformers. The total supply of components per month is restricted to 210 motors and 300 transformers. How many units of A and B should the manufacture per month to maximize profit? How much is the maximum profit?

✦✦✦✦✦

ANSWERS

1. Mathematical Logic

Exercise 1.1

1) (i) Statement, F
(ii) Not statement
(iii) Not statement
(iv) Statement, T
(v) Not statement
(vi) Statement, T
(vii) Not statement
(viii) Statement, T
(ix) Not statement
(x) Statement, T
(xi) Statement, F
(xii) Not statement
(xiii) Statement, T
(xiv) Statement, T
(xv) Not statement

2) (i) $p \wedge q$
(ii) $p \vee q$
(iii) $p \leftrightarrow q$
(iv) $\sim p \wedge \sim q$
(v) $p \rightarrow q$
(vi) $p \leftrightarrow q$
(vii) $\sim p \wedge q$

3) (i) F
(ii) F
(iii) F
(iv) T
(v) T
(vi) T
(vii) T

4) (i) T
(ii) T
(iii) F
(iv) T
(v) F
(vi) F
(vii) T
(viii) T

5) (i) Tirupati is not in Andhra Pradesh.
(ii) 3 is a root of the equation $x^2 + 3x - 18 = 0$.
(iii) $\sqrt{2}$ is not a rational number.
(iv) Polygon ABCDE is not a pentagon.
(v) $7 + 3 \not> 5$

Exercise 1.2

1) (i) TTFT
(ii) FFFF
(iii) TTFT FFFT
(iv) FTTTTTTT
(v) FFFF
(vi) TFFT
(vii) TTTT
(viii) TTTTTTTT
(ix) FTTTTTTT
(x) TFTFTTFF

3) (i) Tautology
(ii) Tautology
(iii) Contingency
(iv) Contingency
(v) Tautology
(vi) Contingency
(vii) Contingency
(viii) Contingency
(ix) Contingency
(x) Contradiction

Exercise 1.3

1) (i) T
(ii) T
(iii) F
(iv) F
(v) T
(vi) T

2) (i) $p \wedge (q \vee r)$ (ii) $p \vee (q \vee r)$ (iii) $(p \wedge q) \vee (r \wedge s)$

(iv) $p \vee \sim q$ (v) $(\sim p \wedge q) \vee (\sim r \vee s)$

(vi) $\sim p \vee (\sim q \vee (p \wedge q) \vee \sim r)$

(vii) $[\sim(p \wedge q) \vee [p \wedge \sim (q \vee \sim s)]$

(viii) $t \vee \{ p \vee (q \wedge r)\}$

(ix) $\sim p \wedge (q \vee r) \vee c$

(x) $(p \wedge q) \wedge t$

3) (i) $x + 8 \leq 11$ and $y - 3 \neq 6$

(ii) $11 \geq 15$ and $25 \leq 20$

(iii) Quadrilateral is a square but not rhombus or quadrilateral is a rhombus but not a square.

(iv) It is not cold or not raining.

(v) It is raining and we will not go or not play football.

(vi) $\sqrt{2}$ is not a rational number.

(vii) Some natural numbers are not whole numbers.

(viii) $\exists\, n \in$ N, $n^2 + n + 2$ is not divisible by 4.

(ix) $\forall\, x \in$ N, $x - 17 \geq 20$.

4) (i) Converse :If $x^2 < y^2$ then $x < y$

Inverse :If $x \geq y$ then $x^2 \geq y^2$.

Contrapositive :If $x^2 \geq y^2$ then $x \geq y$.

(ii) Converse :If a family becomes literate then the woman in it is literate.

Inverse :If the woman in the family is not literate then the family does not become literate.

Contrapositive :If a family does not become literate then the woman in the family is not literate.

(iii) Converse :If pressure increases then surface area decreases.

Inverse :If surface area does not decrease then pressure does not increase.

Contrapositive :If pressure does not increase then surface area does not decrease.

(iv) Converse :If current decreases then voltage increases.

Inverse :If voltage does not increase then current does not decrease.

Contrapositive :If current does not decrease then voltage does not increase.

Exercise 1.4

1) (i) $\sim q \wedge \sim p$ (ii) $\sim p \vee q$ (iii) $\sim p \wedge q$

(iv) $(\sim p \wedge q) \vee \sim r$ (v) $p \wedge (\sim p \wedge q)$ (vi) $(p \wedge q) \wedge (\sim p \wedge q)$

(vii) $(p \vee \sim q) \wedge (\sim p \vee q)$ (viii) $(p \wedge q) \wedge (\sim p \vee q)$

2) (i) A man is not a judge or he is honest.

(ii) 2 is not rational number or is $\sqrt{2}$ irrational number.

(iii) $f(2) \neq 0$ or $f(x)$ is divisible by $(x - 2)$.

Exercise 1.5

1) i) $p \vee (q \wedge r)$ 11111000

ii) $(\sim p \wedge q) \vee (p \wedge \sim q)$ 0110

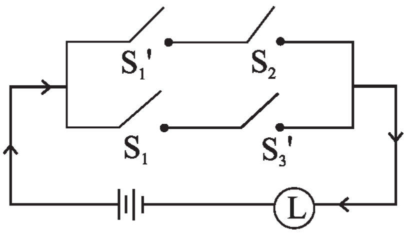
iii) $[(p \wedge (\sim q \vee r)] \vee [\sim q \wedge \sim r]$ 10110001

iv) $(p \vee q) \wedge \sim r \wedge (\sim p \vee r)$ 01000100

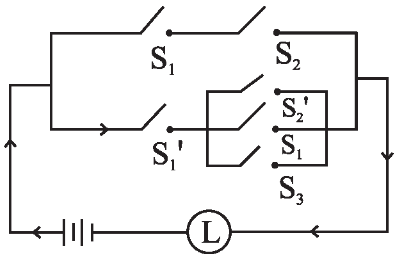
v) $[p \vee (\sim p \wedge \sim q) \vee (p \wedge q)$ 1101

vi) $(p \vee q) \wedge (q \vee r) \wedge (r \vee p)$ 11101000

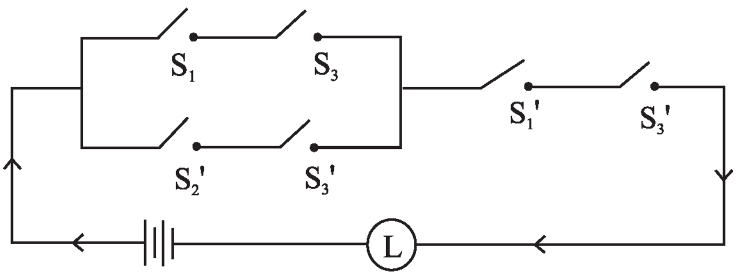
2) (i)



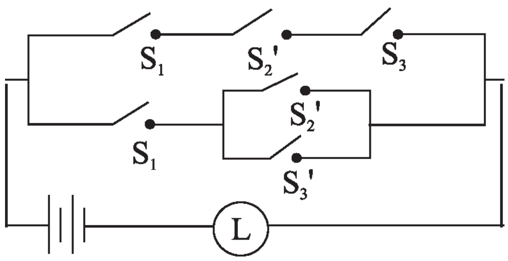
(ii)



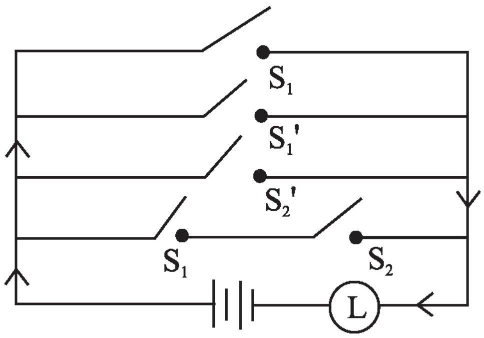
(iii)



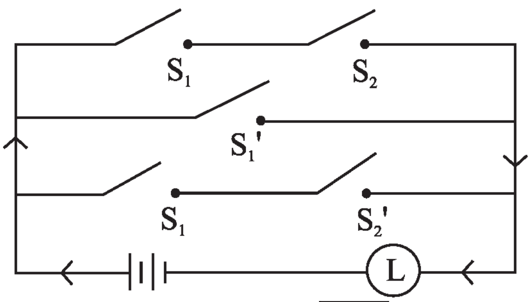
(iv)



(v)



(vi)



4) (i) $(p \vee \sim q) \vee (\sim p \wedge q)$ 1111

The lamp will glow irrespective of the status of the switches.

(ii) $[p \vee (\sim p \wedge \sim q)] \vee (p \wedge q)$ 1101

The lamp will not glow when switch S_1 is OFF and S_2 is ON otherwise it will glow.

(iii) $[p \vee \sim q \vee \sim r)] \wedge [(p \vee (q \wedge \sim r)]$ 11110000

The lamp will glow if S_1 is ON and any status of S_2.

5) (i) P

(ii) $\sim p \vee \sim q$

(iii) P

(iv) $(q \wedge r) \vee$

Miscelleanous Exercise - 1

1)

i	ii	iii	iv	v	vi	vii
B	A	C	B	A	D	C

2) (i) Statement , T (ii) Statement, T (iii) Statement, F (iv) Not a statement
(v) Statement, T (vi) Statement, T

3) (i) T(ii) F (iii) T (iv) T(v) T (vi) F

4) (i) T(ii) F (iii) T (iv) F

5) (i) $\exists\, n \in N$ such that $n+7 \not> 6$. $\exists\, n \in N$ such that $n + t \leq 6$

(ii) $\forall\, x \in A, x + 9 \not> 15$ on x A, $\forall\, x + 9 > 15$.

(iii) All triangles are not equilateral triangles.

6) (i)

p	q	$q \to p$	$p \to (q \to p)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

(ii)

(I)	(II)	(III)	(IV)	(V)	(VI)	(VII)	(VIII)
p	q	~p	~q	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$	(vi) $\leftrightarrow$ (vii)
T	T	F	F	T	F	F	T
T	T	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	T	T	T	F	T	T	T

(iii)

(I)	(II)	(III)	(IV)	(V)	(VI)	(VII)
p	q	~p	~q	$\sim p \wedge \sim q$	$\sim (\sim p \wedge \sim q)$	$\sim (\sim p \wedge \sim q) \wedge q$
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

(iv)

(I)	(II)	(III)	(IV)	(V)	(VI)	(VII)	(IX)
p	q	r	$p \wedge q$	$(p \wedge q) \vee q$	$\sim r$	$\sim r \vee$ (iv)	(v) $\wedge$ (vii)
T	T	T	T	T	F	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	F	F	F
T	F	F	F	F	T	T	F
F	T	T	F	T	F	F	F
F	T	F	F	F	T	T	F
F	F	T	F	T	F	F	F
F	F	F	F	F	T	T	F

(v)

(I)	(II)	(III)	(IV)	(V)	(VI)	(VII)	(VIII)	(IX)
p	q	r	$\sim p$	$\sim p \wedge q$	$q \rightarrow r$	$p \rightarrow r)$	(i) $\wedge$ (vi)	(viii $\rightarrow$ vii)
T	T	T	F	T	T	T	T	T
T	T	F	F	T	F	F	F	T
T	F	T	F	F	T	T	F	T
T	F	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	F	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

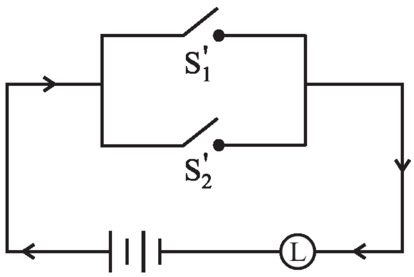
7) (i) Tautology (ii) Contradiction (iii) Contradiction (iv) Tautology
(v) Tautology (vi) Tautology (vii) Contingency (viii) Tautology

8) (i) T, T (ii) T, F (iii) T, F or F, T or F, F

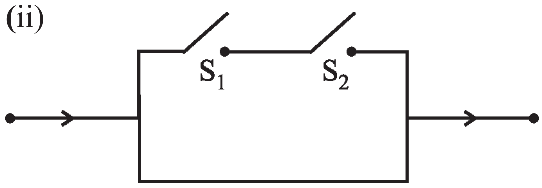
11) (i) $\sim q \wedge (\sim p \vee r)$ (ii) $\sim p \vee (\sim q \wedge \sim r)$ (iii) $(p \wedge \sim q) \vee r$ (iv) $(p \vee \sim q) \wedge (\sim p \vee q)$

12) (i) $(p \wedge q) \vee \sim p \vee (p \wedge \sim q)$ 1111 (ii) $(p \vee q) \wedge (p \vee r)$ 11111000

13) (i)

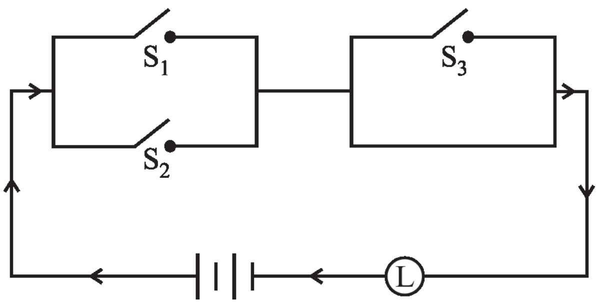


(ii)



14) (i) Logically equivalent (ii) Logically equivalent

15)

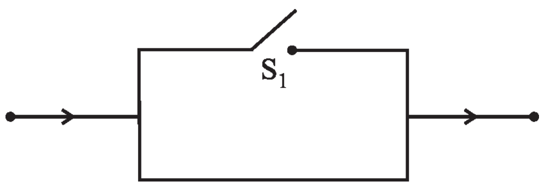


16) Current always flows

17) $(p \vee \sim q \vee \sim r) \wedge [p \vee (q \wedge r)]$

1 1 1 1 0 0 0 0 which is same as p.

Hence we can conclude that the given switching circuit is equivalent to a simple circuit with only one switch S_1.



♦ ♦ ♦ ♦ ♦

2. Matrics

Exercise 2.1

1) $\begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}$

2) $\begin{bmatrix} -1 & -6 & -1 \\ 2 & 5 & 4 \end{bmatrix}$

3) $A \sim \begin{bmatrix} 4 & 5 \\ 3 & 1 \end{bmatrix}$ $B \sim \begin{bmatrix} 4 & 5 \\ 3 & 1 \end{bmatrix}$ The new matrices are equal.

4) $\begin{bmatrix} -2 & 4 & -7 \\ 2 & 6 & 8 \end{bmatrix}$

5) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 9 & 9 & 21 \end{bmatrix}$

6) $\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 2 \\ 9 & 9 & 21 \end{bmatrix}$ ∴ The transformations are commutative.

7) $\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$

9) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & -1 \\ 0 & 0 & \frac{17}{3} \end{bmatrix}$

Exercise 2.2

1) (i) 4, 3, –2, –1.

(ii) –3, –12, 6, –1, 3, 2, –11, –9, 1.

2) (i) $\begin{bmatrix} -1 & -4 \\ -3 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -11 & -10 & -6 \\ 6 & -5 & 3 \\ -2 & -7 & 1 \end{bmatrix}$

3) (i) $\begin{bmatrix} 5 & 3 \\ -3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} -3 & -1 & -11 \\ -12 & 3 & -9 \\ 6 & 2 & 1 \end{bmatrix}$

5) (i) $\frac{1}{13}\begin{bmatrix} 2 & -5 \\ 3 & -1 \end{bmatrix}$ (ii) $\frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$

(iii) $-\frac{1}{3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$ (iv) $-\frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

6) (i) $-\frac{1}{5}\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & 3 & 1 \\ \frac{5}{3} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$ (iv) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

Miscelleanous Exercise - 2(A)

1) Using $C_1 - 2C_2$, $C_1 + 3C_3$ and $C_2 - 3C_3$, We get the required result.

2) Using $R_1 - R_2$, $R_3 - R_2$, $-R_2$, $R_1 - R_2$, $R_3 - R_2$, $-R_3$, $R_1 - R_3$, $R_2 - R_3$, we get the required result.

(There can be another sequence of the transformations.)

3) The invertible matrices are (i), (iii), (v), (vi),(vii) and not invertible matrices are (ii), (iv)(viii) and (ix).

4) AB = $\begin{bmatrix} 6 & -3 \\ -4 & 1 \end{bmatrix}$ and it is invertible.

5) $A^{-1} = \begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix}$ $\qquad$ $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

6) (i) $X = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{1}{2} \end{bmatrix}$

7) (i) $\frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$ $\qquad$ (ii) $-\frac{1}{3}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}$

(iii) $\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$ $\qquad$ (iv) $\frac{1}{29}\begin{bmatrix} 7 & 3 \\ -5 & 2 \end{bmatrix}$

(v) $\begin{bmatrix} 4 & -1 \\ -7 & 2 \end{bmatrix}$ $\qquad$ (vi) $\begin{bmatrix} 7 & -10 \\ 2 & -3 \end{bmatrix}$

(vii) $-\frac{1}{25}\begin{bmatrix} 10 & 0 & -15 \\ -5 & -5 & 0 \\ -10 & 5 & 10 \end{bmatrix}$ $\qquad$ (viii) $\frac{1}{25}\begin{bmatrix} 25 & -10 & -15 \\ -10 & 4 & 11 \\ -15 & 1 & 9 \end{bmatrix}$

(ix) $\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ $\qquad$ (x) $\begin{bmatrix} 3 & 6 & 2 \\ 1 & 2 & 1 \\ 2 & 5 & 2 \end{bmatrix}$

8) $A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9) $AB = \begin{bmatrix} 11 & 3 \\ 7 & 2 \end{bmatrix}, (AB)^{-1} = \begin{bmatrix} 2 & -3 \\ -7 & 11 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ $B^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

11) $X = \begin{bmatrix} \frac{4}{5} & 1 \\ \frac{2}{5} & 1 \end{bmatrix}$ $\qquad$ 12) $X = -\frac{1}{3}\begin{bmatrix} 1 \\ 7 \\ -6 \end{bmatrix}$

13) $X = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

14) $\begin{bmatrix} 13 & 2 & -7 \\ -3 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$

15) $-\frac{1}{6}\begin{bmatrix} -4 & 2 & -2 \\ 3 & 0 & -3 \\ -2 & -2 & 2 \end{bmatrix}$

16) $\frac{1}{3}\begin{bmatrix} 0 & -2 & 1 \\ 6 & 1 & -5 \\ -3 & 0 & 3 \end{bmatrix}$

17) $\frac{1}{6}\begin{bmatrix} 4 & -2 & 2 \\ -3 & 0 & 3 \\ 2 & 2 & -2 \end{bmatrix}$

18) $\begin{bmatrix} 13 & 2 & -7 \\ -3 & -1 & 2 \\ -2 & 0 & 1 \end{bmatrix}$

19) Hint : Use the definition of the co-factors and the value of the determinant by considering.

$A = [a_{ij}]_{3 \times 3}$

20) $X = \frac{1}{6}\begin{bmatrix} 4 & 4 & 2 \\ 11 & 8 & -5 \\ 10 & 10 & 2 \end{bmatrix}$

Exercise 2.3

1) (i) 0, 1 (ii) 3, 1 (iii) Not solvable

2) (i) 4, –3 (ii) $\frac{1}{2}, \frac{1}{2}$ (iii) 1, 2 (iv) 2,–3

3) Rs. 5 for a pencil Rs. 8 for a pen and Rs.8 for an eraser.

4) The numbers are 1, –2, 3.

5) The cost price of one T.V. set is Rs.3000 and of one V.C.R. is Rs. 13,000.

The selling price of one T.V.Set is Rs.4000 and that of V.C.R. is Rs. 13,500.

Miscellaneous exercise - 2 (B)

I)

1	2	3	4	5	6	7	8	9	10	11	12
A	B	D	B	B	B	B	A	B	B	B	D

II) 1) (i) $-\frac{5}{11}, \frac{12}{11}$ (ii) $2-\frac{4}{a}, 0, -1+\frac{4}{a}$

(iii) $x = 3, y = 2, z = -2$ (iv) $x = 2,\ y = -3$

(v) $x = \frac{5}{2},\ y = \frac{3}{2}$ $, z = -2,$

2) (i) 1, 1, 1 (ii) $\frac{1}{3}, \frac{2}{3}, 1$

(iii) 1, 2, 1 (iv) 1, 2, 3

(v) 3, 2, 1 (vi) -1, 1, 2

3) The numbers are 1, 2, 3

4) Cost of a pencil, a pen and a book is respectively Rs.10, Rs.15 and Rs.25.

5) The costs are $3, \frac{5}{3}, \frac{4}{3}$

6) The numbers are $1, -1, 2$

7) 1750, 1500, 1750

8) Maths Rs.150, Phy. Rs.30, Chem. Rs. 30

3. Trigonometric Functions

Exercise 3.1

1) (i) $\frac{\pi}{3}, \frac{5\pi}{3}$ (ii) $\frac{\pi}{6}, \frac{11\pi}{6}$ (iii) $\frac{\pi}{6}, \frac{7\pi}{6}$ (iv) $0, \pi$

2) (i) $\frac{7\pi}{6}, \frac{11}{6}$ (ii) $\frac{3\pi}{4}, \frac{7\pi}{4}$ (iii) $\frac{4\pi}{3}, \frac{5\pi}{3}$

3) (i) $n\pi + (-1)^n \frac{\pi}{6}, n \in z$ (ii) $2n\pi \pm \frac{\pi}{6}, n \in z$

(iii) $n\pi + \frac{\pi}{6}, n \in z$ (iv) $n\pi + \frac{\pi}{2}, n \in z$

4) (i) $2n\pi \pm \frac{\pi}{4}, n \in z$ (ii) $n\pi + (-1)^n \frac{5\pi}{4}, n \in z$ (iii) $n\pi + \frac{3\pi}{4}, n \in z$

5) (i) $\frac{n\pi}{2} + (-1)^n \frac{\pi}{12}, n \in z$ (ii) $\frac{3n\pi}{2} + \frac{\pi}{2}, n \in z$ (iii) $\frac{n\pi}{4} + \frac{3\pi}{16}, n \in z$

6) (i) $n\pi \pm \frac{\pi}{6}, n \in z$ (ii) $n\pi \pm \frac{\pi}{6}, n \in z$ (iii) $\frac{n\pi}{3}, n \in z$

7) (i) $n\pi, n \in z$ (ii) $n\pi \text{ or } n\pi \pm \frac{\pi}{3}, n \in Z$

(iii) $2n\pi$ or $2n\pi \pm \frac{\pi}{2}, n \in z$

8) (i) and iv) have solutions (ii) and (iii) do not have solutions

Exercise 3.2

1) (i) $(1,1)$ (ii) $(0,4)$ (iii) $\left(-\frac{3}{4\sqrt{2}}, \frac{3}{4\sqrt{2}}\right)$ (iv) $\left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)$

2) (i) $\left(2, \frac{\pi}{4}\right)$ (ii) $\left(\frac{1}{2}, \frac{\pi}{2}\right)$ (iii) $\left(2, \frac{5\pi}{3}\right)$ (iv) $\left(3, \frac{\pi}{3}\right)$

3) (i) $2 : \sqrt{6} : 1 + \sqrt{3}$

10) (i) $\frac{4}{5}$ (ii) $\frac{1}{\sqrt{10}}$ (iii) $\frac{3}{\sqrt{10}}$ (iv) $\frac{1}{3}$ (v) 216 (vi) $\frac{3}{5}$

Exercise 3.3

1) (i) $\frac{\pi}{6}$ (ii) $\frac{\pi}{6}$ (iii) $-\frac{\pi}{4}$ (iv) $-\frac{\pi}{3}$ (v) $\frac{\pi}{4}$ (vi) $\frac{2\pi}{3}$

2) (i) $\frac{3\pi}{4}$ (ii) $\frac{2\pi}{3}$ (iii) $-\frac{\pi}{3}$ (iv) $-\frac{\pi}{12}$

Miscellaneous exercise - 3

I)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
B	A	A	A	D	C	A	B	A	C	B	D	A	B	D	A	B	A	B	B

II) i) $\left\{\frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$ ii) $\left\{\frac{3\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{15\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$

2) (i) $\left\{\frac{5\pi}{8}, \frac{7\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}\right\}$

(ii) $\left\{\frac{3\pi}{20}, \frac{7\pi}{20}, \frac{11\pi}{20}, \frac{15\pi}{20}, \frac{19\pi}{20}, \frac{23\pi}{20}, \frac{27\pi}{20}, \frac{31\pi}{20}, \frac{35\pi}{20}, \frac{39\pi}{20}\right\}$

(iii) $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

3) (i) and (ii) have solution, (iii) and iv) do not have solutions

4) (i) $n\pi + \frac{2\pi}{3}, n \in z$ (ii) $n\pi \pm \frac{\pi}{3}, n \in z$ (iii) $(2n+1)\pi$ or $2n\pi + \frac{\pi}{2}, n \in z$

iv) $2n\pi \pm \frac{\pi}{2}, n \in z$

10) $c = \sqrt{6}, A = 105^0, B = 15^0$

19) (i) $\frac{3\pi}{5}$ (ii) $\frac{\pi}{6}$

26) $\frac{\pi}{4}$ 27) $\frac{1}{\sqrt{3}}$

28) 0 29) $\frac{1}{6}$

♦♦♦♦♦

4. Pair of Straight Lines

Exercise 4.1

1) (i) $6x^2 + xy - y^2 = 0$

(ii) $x^2 - xy - 6y^2 + x + 7y - 2 = 0$

(iii) $xy - 3x - 2y + 6 = 0$

(iv) $6x^2 - 7xy - 3y^2 - 3x + 32y - 45 = 0$

(v) $3x^2 + 11xy + 6y^2 - 16x - 13y + 5 = 0$

2) (i) $y = 0,\ 7x + 3y = 0$

(ii) $\sqrt{5}\,x - 3y = 0,\ \sqrt{5}\,x + 3y = 0,$

(iii) $x = 0,\ x - 4y = 0$

(iv) $3x + 2y = 0,\ x - 4y = 0$

(v) $3x + \sqrt{3}\,y = 0,\ x = \sqrt{3}\,y = 0$

(vi) $(\operatorname{cosec}\alpha - \cot\alpha)x + y - 0,\ (\operatorname{cosec}\alpha + \cot\alpha)x - y = 0$

(vii) $(\sec\alpha - \tan\alpha)x + y = 0,\ (\sec\alpha + \tan\alpha)x - y = 0$

3) (i) $3x^2 + 8xy + 5y^2 = 0$

(ii) $x^2 + 2xy - 5y^2 = 0$

(iii) $3x^2 - xy = 0$

(iv) $4xy + 3y^2 = 0$

4) (i) -2 (ii) ± 2 (iii) 12

5) (i) $25a + 16b = 40h$ (ii) $9a + 6h + b = 0$

6) $ap^2 + 2hpq + bq^2 = 0$

7) $3x^2 - y^2 = 0$

Exercise 4.2

3) k = 4 4) i) 30° ii) $\tan^{-1}\left(\frac{3}{5}\right)$ iii) 45° iv) 60°

5) $23x^2 + 48xy + 3y^2 = 0$ 7) $x^2 - 3y^2 = 0$

Exercise 4.3

1) (i) $2x^2 + 3xy - 9y^2 - 5x - 24y - 7 = 0$ (ii) $x^2 + xy - y^2 - x - 8y - 11 = 0$

2) $h^2 - ab = -1 < 0$

3) $2x - 3y + 4 = 0$ and $x + y - 5 = 0$ are separate equations of lines.

4) $2x - y + 3 = 0$ and $x + y - 1 = 0$ are separate equations. $\theta = \tan^{-1}(3)$.

5) (i) $x - y - 3 = 0, x - 2y - 4 = 0$ (ii) $2x - y + 4 = 0, 5x + 3y - 1 = 0$

6) (i) -12 (ii) 15 (iii) -6

7) $p = -3, q = -8$

8) $p = 8, q = 1$

9) $36x^2 - 25xy - 252x + 350y - 784 = 0$

10) $7x - 8y = 0$

11) (1, 0)

Miscellaneous exercise - 4

I.

1	2	3	4	5	6	7	8	9	10	11	12	13	14
B	B	B	A	D	D	A	B	B	B	C	C	D	D

II. 1) (i) $x^2 - y^2 = 0$ (ii) $2x^2 + 3xy + y^2 - 7x - 4y + 3 = 0$ (iii) $6x^2 - 5xy + y^2 = 0$

(iv) $3x^2 - y^2 = 0$ (v) $xy - 2x - y + 2 = 0$ (vi) $xy - 2x - 3y + 6 = 0$

(vii) $8x^2 + 2xy - 3y^2 + 12x + 14y - 8 = 0$ (viii) $2x^2 + 2xy - y^2 = 0$

(ix) $x^2 - 81 = 0$ (x) $x^2 - 2xy - 2x + 6y - 3 = 0$ (xi) $2x^2 - 7xy + 3y^2 = 0$

3) (i) $2x - 3y = 0, 3x + 2y = 0$ (ii) $x - 2y = 0, x + 2y = 0$

(iii) $\sqrt{3}x + y = 0, \sqrt{3}x - y = 0$ (iv) $(\sqrt{3} - 1)x + y = 0, (\sqrt{3} + 1)x - y = 0$

4) (i) $5x^2 + 4xy - y^2 = 0$ (ii) $9x^2 - 3xy - 2y^2 = 0$ (iii) $x^2 + xy - y^2 = 0$

5) (i) 0 (ii) -1 (iii) 1 (iv) 8 (v) 1 (vi) 6 (vii) 5

6) $3x^2 + 2xy - 3y^2 = 0$

7) $x^2 - 3y^2 = 0$

8) $\dfrac{50}{\sqrt{3}}$

10) $x^2 - 2xy - y^2 = 0$

11) -4

13) (i) $0°$ (ii) $\tan^{-1}(3)$ (iii) $\tan^{-1}(3)$

14) $x^2 - 3y^2 = 0$

18) Area = $\sqrt{3}$ sq. unit, Perimeter = 6 unit

22) $e = 0$ or $bd = ae$ **26)** $a = 1, c = 0$.

♦♦♦♦♦

5. Vectors

Exercise 5.1

1) 25

2) (i) $2\bar{a}-2\bar{b}$ (ii) $\bar{a}+\bar{b}$ (iii) $\bar{b}-\bar{a}$

3) $\overrightarrow{OC}=2\bar{a}+2\bar{b}$, $\overrightarrow{OD}=-3\bar{a}+2\bar{b}$, $\overrightarrow{OE}=-2\bar{a}+\bar{b}$

5) Vectors do not form a triangle.

6) $\bar{c}=\frac{1}{2}\bar{a}+\frac{1}{2}\bar{b}$. $\bar{d}=\frac{1}{2}\bar{b}-\frac{1}{2}\bar{a}$.

7) $\frac{7}{\sqrt{5}}\hat{i}-\frac{14}{\sqrt{5}}\hat{j}$

8) (a) 6 (b) 4 (c) 2 (d) $2\sqrt{10}$ (e) $2\sqrt{13}$ (f) $2\sqrt{5}$

9) (a) $x=-3, y=4, z=5$ (b) (0, 1, 6)

10) $\frac{\sqrt{3}}{2}$ sq. units

11) Terminal Point is (3, 1, 7) **12)** $q=\frac{5}{2}$

13) Non coplanar **14)** $\bar{r}=2\bar{a}+2\bar{b}-3\bar{c}$

Exercise 5.2

1) (i) $\frac{1}{5}$ (–11, 4, –9) (ii) (–19, 8, –21)

2) M(6, –1, 5)

3) (i) C divides externally in the ration 3:1. (ii) $p=9, q=2$.

6) 15 : 4 and 10 : 9 respectively

9) C ≡ (–2, 0, 2)

10) OP : PD = 3 : 2

11) $\sqrt{107}$

12) G ≡ (4, –3, 2)

Exercise 5.3

1) $\pm\left(\frac{2}{\sqrt{17}}\hat{i}+\frac{2}{\sqrt{17}}\hat{j}+\frac{3}{\sqrt{17}}\hat{k}\right)$

6) (i) Parallel (ii) Orthogonal (iii) Orthogonal (iv) Neither parallel nor orthogonal

7) $\angle P=45^\circ$ **8)** (i) $\frac{1}{2}$ (ii) $\frac{1}{2}$

10) $\frac{\pi}{3}$

11) $0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

12) $\frac{2}{11}, \frac{-6}{11}, \frac{9}{11}$

13) (0, 5, 7) or (8, -3, 3)

14) –1, 1, 2 or 1, 2, 3.

Exercise 5.4

1) $-4\hat{i}+10\hat{j}+22\hat{k}$

2) $\pm\left(\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}+\frac{1}{3}\hat{k}\right)$

3) 60°

4) $\pm\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

5) (i) ± 6 ii) 1.6

7) $6i + 12j + 6k$

8) $\sqrt{146}$ sq. units

10) $\sqrt{42}$ sq. units

12) $\bar{b}=\frac{1}{3}\left(5\hat{i}+2\hat{j}+2\hat{k}\right)$

13) $2\hat{j}+\hat{k}$

14) $\frac{3\pi}{4}$

16) i) -3, 5, 11 ii) 4, -4, 4

17) $\left(\frac{-8}{5}, \frac{16}{5}, \frac{24}{5}\right)$

Exercise 5.5

1) 110

2) 23 cubic units

3) $p = 2$

6) (i) –12 ii) 16 iii) $\left|\bar{u}+\bar{v}\right|^2$

7) $\frac{16}{3}$ cubic units

9) (i) $6\hat{i}+3\hat{j}-6\hat{k}$ (ii) $-2\hat{i}+4\hat{j}$

Not same ; as $\bar{a}\times(\bar{b}\times\bar{c})$ lies in the plane of $\bar{b}$ and $\bar{c}$ whereas $(\bar{a}\times\bar{b})\times\bar{c}$ lies in the plane of $\bar{a}$ and $\bar{b}$.

Miscellaneous exercise - 5

I.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
C	B	B	B	A	D	C	A	B	B	A	B	A	A	A	B	C	B	A	A

II. 1) (i) $\bar{b}-\frac{1}{2}\bar{a}$ (ii) $\bar{b}-3\bar{a}$ (iii) $\frac{3}{2}\bar{a}-\bar{b}$ (iv) $2\bar{a}-\bar{b}$

2) $-\frac{1}{2}\bar{a}-\frac{1}{2}\bar{b}+\bar{c}$

4) $\overline{AB}=-2\hat{i}+5\hat{j}+\hat{k}$ and $\overline{AD}=4\hat{i}-2\hat{j}+3\hat{k}$

5) 3

6) $\sqrt{2}$

7) (i) Right angled triangle ii) Isosceles triangle

8) (i) $2j\pm 2\sqrt{3}\,\hat{k}$ ii) $\pm 5\sqrt{2}\,\hat{i}+5\sqrt{2}\,\hat{k}$

9) $\frac{1}{\sqrt{17}}(3i+2j+2k)$ and $\frac{1}{\sqrt{21}}(-i-2j+4k)$

11) $\pm\frac{1}{\sqrt{17}}(i+4j)$

12) $\hat{i}+4\hat{j}-4\hat{k}=1(2\hat{i}-\hat{j}+3\hat{k})+2(\hat{i}-2\hat{j}+4\hat{k})+3(-\hat{i}+3\hat{j}-5\hat{k})$

14) $7(\hat{i}+\hat{j}+\hat{k})$

15) (-4, 9, 6)

20) OP : PD = 3 : 2

21) $3\hat{i}+2\hat{k}$

22) $-\frac{3}{2}$

24) $\bar{a}_1=6\hat{i}+2\hat{k}$ and $\bar{a}_2=-\hat{i}-2\hat{j}+3\hat{k}$

25) $\pm\left(\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}\right)$

26) $\cos\theta=\frac{7}{5\sqrt{2}}$

27) $\cos\alpha=\frac{2}{3}, \cos\beta=\frac{1}{3}$ *and* $\cos\gamma=\frac{2}{3}$ $\cos\alpha=\frac{1}{4}, \cos\beta=\cos\gamma=\frac{2}{3}$

28) $2\hat{i}-\hat{j}$

30) $\cos^{-1}\left(\frac{1}{6}\right)$

31) $\left(\frac{19}{9}, \frac{28}{9}, \frac{41}{9}\right)$

33) $\frac{bc\hat{i} + ac\hat{j} + ab\hat{k}}{\sqrt{b^2c^2 + a^2c^2 + a^2b^2}}$ and area $= \frac{1}{2}\sqrt{b^2c^2 + a^2c^2 + a^2b^2}$

34) a) meaningful, scalar b) meaningless c) meaningful, vector
d) meaningless e) meaningless f) meaningful, scalar
g) meaningless h) meaningful, vector i) meaningful, scalar
j) meaningful scalar k) meaningless l) meaningless

36) (i) No ii) No iii) Yes

37) $\sqrt{286}$ sq. units.

40) $a = \pm\frac{1}{\sqrt{3}}$

41) $2a^3$ cu. units.

44) 2 cubic units, $\frac{1}{3}$ cubic units

♦♦♦♦♦

6. Line and Plane

Exercise 6.1

1) $\bar{r} = \left(-2\hat{i} + \hat{j} + \hat{k}\right) + \lambda\left(4\hat{i} - \hat{j} + 2\hat{k}\right)$

2) $\bar{r} = \left(3\hat{i} + 4\hat{j} - 7\hat{k}\right) + \lambda\left(3\hat{i} - 5\hat{j} + 8\hat{k}\right)$

3) $\bar{r} = \left(5\hat{i} + 4\hat{j} + 3\hat{k}\right) + \lambda\left(-3\hat{i} + 4\hat{j} + 2\hat{k}\right)$

4) $\bar{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(2\hat{i} + \hat{j} - 3\hat{k}\right)$

5) $\bar{r} = \left(-\hat{i} - \hat{j} + 2\hat{k}\right) + \lambda\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$

6) $\frac{x+1}{2} = \frac{y-2}{3} = \frac{z-1}{1}$

7) $\frac{x-2}{-1} = \frac{y-2}{1} = \frac{z-1}{-1}$

8) $\frac{x+2}{3} = \frac{y-3}{-2} = \frac{z-4}{-2}$

9) (–11, –4, 5)

10) $\bar{r} = \left(3\hat{i} - \hat{j} + 2\hat{k}\right) + \lambda\left(-2\hat{i} - 3\hat{j} - 2\hat{k}\right)$

Exercise 6.2

1) $\sqrt{35}$

2) $(1, 2, 3)$, $\sqrt{14}$

3) $\frac{1}{\sqrt{3}}$

4) $2\sqrt{29}$

5) $2\sqrt{6}$, $(3, -4, -2)$

6) $\left(\frac{99}{53}, \frac{-187}{53}, \frac{95}{53}\right)$

7) a) do not intersect b) do not intersect

8) $\frac{9}{2}$

Miscellaneous exercise - 6A

1) $\bar{r} = \left(3\hat{i} + 4\hat{j} - 7\hat{k}\right) + \lambda\left(6\hat{i} - \hat{j} + \hat{k}\right)$

2) $\bar{r} = \left(3\hat{i} + 2\hat{j} + \hat{k}\right) + \lambda\left(2\hat{i} + 2\hat{j} - 3\hat{k}\right)$

3) $\bar{r} = \left(-2\hat{i} + 4\hat{j} - 5\hat{k}\right) + \lambda\left(3\hat{i} + 5\hat{j} + 6\hat{k}\right)$

4) $\bar{r} = \left(-5\hat{i} - 4\hat{j} - 5\hat{k}\right) + \lambda\left(3\hat{i} + 5\hat{j} + 6\hat{k}\right)$

5) $\bar{r} = \lambda\left(5\hat{i} - 2\hat{j} + 3\hat{k}\right)$

6) $x = 3$, $y = -2$

7) $\frac{x-3}{-2} = \frac{y-2}{1}; z = 1$

8) $x - 1 = y - 1 = z - 2$

9) $\frac{x-2}{2} = \frac{y-1}{-7} = \frac{z-3}{4}$

10) $\bar{r} = \lambda\left(-\hat{i} + \hat{k}\right)$

11) $-\frac{10}{11}$

12) $60°$

13) $45°$

14) $45°$

15) $(2, 3, -1)$

16) i) intersect ii) intersect

17) -1

18) $\frac{x+1}{3}=\frac{y+1}{2}=\frac{z-2}{1}$, $\bar{r}=\left(-\hat{i}-\hat{j}+2\hat{k}\right)+\lambda\left(3\hat{i}+2\hat{j}+\hat{k}\right)$

19) $\frac{2}{\sqrt{13}},\frac{3}{\sqrt{13}},0$

20) $\frac{x}{7}=\frac{y}{-12}=\frac{z}{5}$

21) $\bar{r}=\left(2\hat{j}+\frac{5}{3}\hat{k}\right)+\lambda\left(3\hat{i}+4\hat{k}\right)$

22) (2, 0, 5), (0, 4, 1)

Exercise 6.3

1) $\bar{r}\cdot\left(2\hat{i}+\hat{j}-2\hat{k}\right)=126$

2) 1

3) $\left(\frac{18}{7},\frac{54}{7},\frac{-27}{7}\right)$

4) $\bar{r}\cdot\left(\frac{3}{13}\hat{i}+\frac{4}{13}\hat{j}+\frac{12}{13}\hat{k}\right)=6$, (i) 6 (ii) $\left(\frac{3}{13},\frac{4}{13},\frac{12}{13}\right)$

5) $\bar{r}\cdot\left(4\hat{i}+5\hat{j}+6\hat{k}\right)=15$

6) $2y+5z=19$

7) $z=6$

8) $\bar{r}\cdot\left(\hat{i}\right)=1$

9) $\bar{r}\cdot\left(-4\hat{i}-\hat{j}+5\hat{k}\right)=26$

10) $5x-2y-3z=38$

11) $\bar{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=1$

Exercise 6.4

1) $60°$

2) $\sin^{-1}\left(\frac{5}{7\sqrt{6}}\right)$

3) $\bar{r}\cdot\left(-\hat{i}+2\hat{j}-\hat{k}\right)=7$

4) 2

5) $\frac{1}{13}$

Miscellaneous exercise - 6B

I.

1	2	3	4	5	6	7	8	9	10
B	A	A	C	D	B	B	C	D	B

11	12	13	14	15	16	17	18	19	20
A	D	D	D	A	A	B	B	A	B

II. 1) $\bar{r}\cdot\left(2\hat{i}+\hat{j}+2\hat{k}\right)=15$

2) 1

3) (2, 3, 6)

4) i) $\frac{1}{2}$ ii) $\frac{3}{13},\frac{4}{13},\frac{12}{13}$

5) $\bar{r}\cdot\left(16\hat{i}+4\hat{k}\right)=20$

6) $y+2=0$

7) $6x+8y+7z=148$

8) $\bar{r}\cdot\left(\hat{i}+2\hat{j}\right)=5$

9) $\bar{r}\cdot\left(bc\hat{i}+ca\hat{j}+ab\hat{k}\right)=abc$

10) $\bar{r}\cdot\left(-3\hat{i}+3\hat{j}+4\hat{k}\right)=35$

11) $\bar{r}\cdot\left(5\hat{i}-4\hat{j}+\hat{k}\right)=0$

12) $x+y+z=6,\ x-2y+z=0$

13) $x+y+z=3$

14) $90°$

15) $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

16) $\bar{r}\cdot\left(-\hat{i}+2\hat{j}-\hat{k}\right)=7$

17) 0 unit

18) 19 units

19) $\bar{r}\cdot\left(\hat{i}-\hat{k}\right)=0$

20) $\bar{r}\cdot\left(\hat{i}-4\hat{k}\right)=-5$

21) $\bar{r}\cdot\left(\hat{k}\right)=0$

♦♦♦♦♦

7. Linear Programming

Exercise 7.3

1) maximize $z = 30x + 20y$ subject to $10x + 6y \leq 60$, $5x + 4y \leq 35$, $x \geq 0$, $y \geq 0$

2) maximize $z = 3x + 2y$ subject to $2x + y \geq 14$, $2x + 3y \geq 22$, $x + y \geq 1$, $x \geq 0$, $y \geq 0$

3) maximize $p = 350x + 400y$ subject to $3x + 2y \leq 120$, $2x + 5y \leq 160$, $x \geq 0$, $y \geq 0$

4) maximize $z = 10x + 15y$ subject to $2x + 3y \leq 36$, $5x + 2y \leq 50$, $2x + 6y \leq 60$, $x \geq 0$, $y \geq 0$

5) maximize $p = 13.5x + 55y$ subject to $x + 2y \leq 10$, $3x + 4y \leq 12$, $x \geq 0$, $y \geq 0$

6) maximize $z = 500x + 750y$ subject to $2x + 3y \leq 40$, $x + 4y \leq 70$, $x \geq 0$, $y \geq 0$

7) minimize $z = 4.5x + 3.5y$ subject to $4x + 6y \geq 18$, $14x + 12y \geq 28$, $7x + 8y \geq 14$, $x \geq 0$, $y \geq 0$

8) maximize $z = x_1 + x_2$ subject to $\frac{x_1}{60} + \frac{x_2}{90} \leq 1$, $5x_1 + 8x_2 \leq 600$, $x \geq 0$, $x_2 \geq 0$

9) minimize $C = 20\,x_1 + 6x_2$ s. t $x_1 > 4$, $x_2 < 2$, $x_1 + x_2 \geq 5$, $x \geq 0$, $x_2 \geq 0$.

Exercise 7.4

1) Maximum at (4, 2), 60

2) Maximum at (0, 6), maximum value = 36

3) Maximum at (4.5, 2.5), 59

4) Maximum at (2, 3), maximum value = 95

5) Maximum at (4, 5), maximum $z = 37$

6) Maximum at (0, 5), 5

7) Maximum at (1.5, 4), 52

8) Maximum at (2, 0.5), 22.5

Miscellaneous exercise - 7

I.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A	C	B	C	A	D	C	B	A	B	B	B	A	C	C

5) (i) $x_1 = 4.5$, $x_2 = 3$ max $z = 40.5$.

(ii) $x = 3$, $y = 18$ min $z = 48$.

(iii) infinite number of optimum solutions on the line $3x + 5y = 10$ between $A\left(\frac{45}{16}, \frac{5}{16}\right)$ and B(0, 2).

6) (i) $x = 4, y = 3$ maximize $z = 25$.

(ii) $x = 10, y = 15$maximize $z = 1350$.

(iii) $x = 3, y = 18$ maximize $z = 48$.

7) maximize $z = 140x + 210y$ s.t. $3x + 3y \leq 36$, $5x + 2y \leq 50$, $2x + 6y \leq 60$

$x, y \geq 0$ where x = no. of tables = 3

y = no. of chairs = 9

maximize z = maximum profit = 2310/-

8) Maximize $z = 180x + 220y$ s.t. $6x + 4y \leq 120$, $3x + 10y \leq 180$, $x \geq 0, y \geq 0$.

Ans. $x = 10, y = 15$.

9) Minimize $z = 4x + 6y$ s.t. $x + 2y \geq 80$, $3x + y \geq 75$, $x \geq 0, y \geq 0$.

Ans. $x = 14, y = 33$.

10) Maximize $z = 2000x + 3000y$ s.t. $3x + 3y \leq 36$, $5x + 2y \leq 50$, $2x + 6y \leq 60$, $x \geq 0, y \geq 0$.

Ans. $x = 3, y = 9$.

11) Minimize $z = 800x + 640y$ s.t. $4x + 2y \geq 16$, $12x + 2y \geq 24$, $2x + 6y \geq 18$, $x \geq 0, y \geq 0$.

Ans. Minimum cost ₹3680/- when $x = 3, y = 2$.

12) Maximize $z = 75x + 125y$ s.t. $4x + 2y \leq 208$, $2x + 4y \leq 152$, $x \geq 0, y \geq 0$.

Ans. $x = 44, y = 16$.

13) Maximize $z = -3x + 4y$ s.t. $x + y \leq 450$, $2x+y \leq 600$, $x \geq 0$, $y \geq 0$

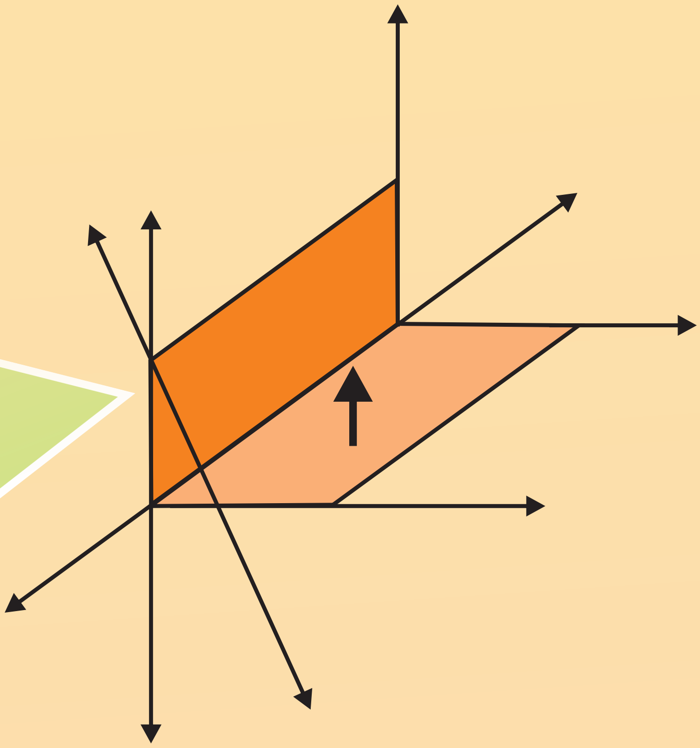
maximum profit = Rs. 1800/- at (0, 450)

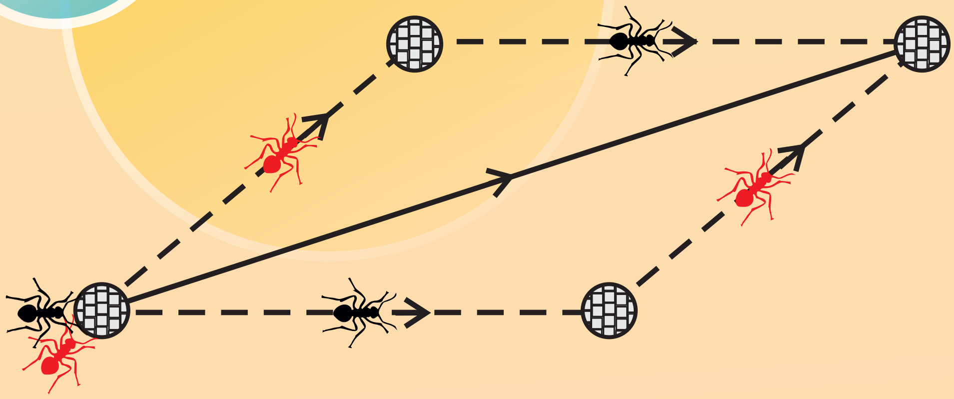
14) Maximize $z = 20x + 30y$ s.t. $2x + 2y \leq 210$, $3x + 4y \leq 300$, $x \geq 0$, $y \geq 0$

maximum profit = Rs. 2400/- at (30, 60)

✦✦✦✦✦

Notes





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