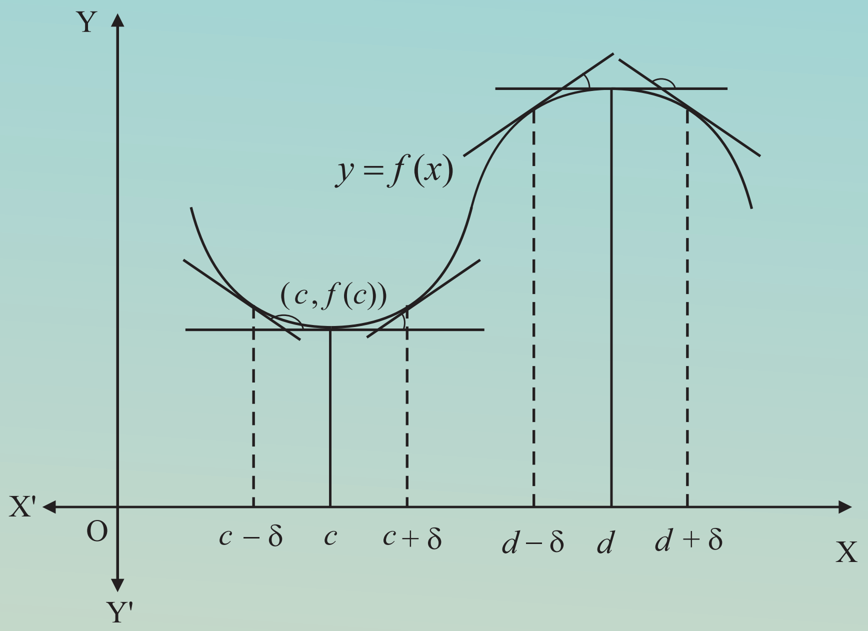
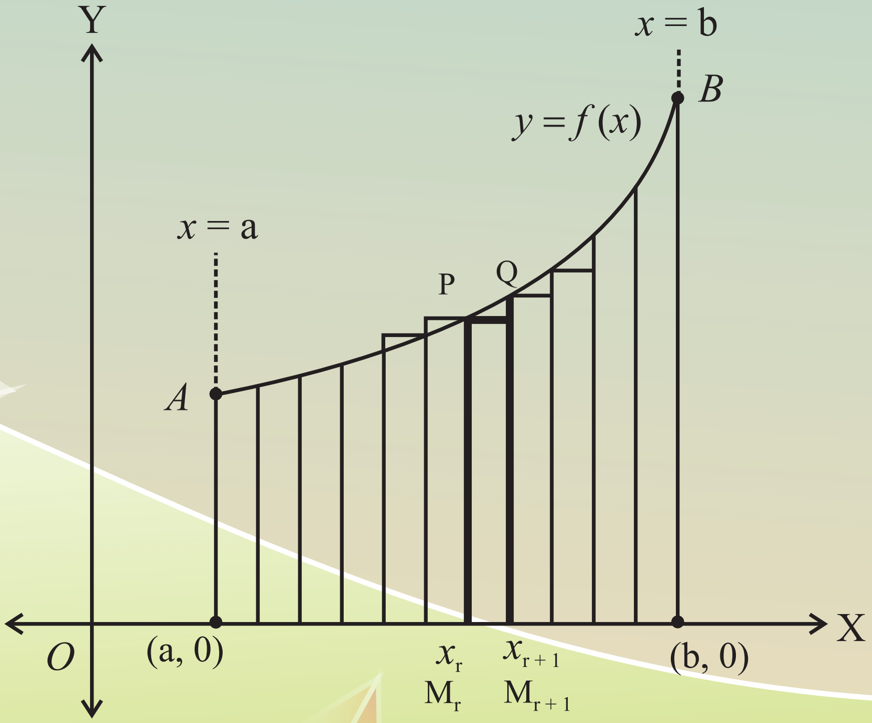
Mathematics & Statistics

Arts & Science Part 2

STANDARD XII





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Mathematics and Statistics

(Arts and Science)

Part - II

STANDARD - XII

Maharashtra State Bureau of Textbook Production and Curriculum Research, Pune - 411 004

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The Constitution of India

Preamble

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC and to secure to all its citizens:

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the unity and integrity of the Nation;

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949, do HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.

NATIONAL ANTHEM

Jana-gana-mana-adhināyaka jaya hē
Bhārata-bhāgya-vidhātā,

Panjāba-Sindhu-Gujarāta-Marāthā
Drāvida-Utkala-Banga

Vindhya-Himāchala-Yamunā-Gangā
uchchala-jaladhi-taranga

Tava subha nāmē jāgē, tava subha āsisa māgē,
gāhē tava jaya-gāthā,

Jana-gana-mangala-dāyaka jaya hē
Bhārata-bhāgya-vidhātā,

Jaya hē, Jaya hē, Jaya hē,
Jaya jaya jaya, jaya hē.

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give my parents, teachers and all elders respect, and treat everyone with courtesy.

To my country and my people, I pledge my devotion. In their well-being and prosperity alone lies my happiness.

PREFACE

Dear Students,

Welcome to Standard XII, an important milestone in your life.

Standard XII or Higher Secondary School Certificate opens the doors of higher education. Alternatively, you can pursue other career paths like joining the workforce. Either way, you will find that mathematics education helps you considerably. Learning mathematics enables you to think logically, constistently, and rationally. The curriculum for Standard XII Mathematics and Statistics for Science and Arts students has been designed and developed keeping both of these possibilities in mind.

The curriculum of Mathematics and Statistics for Standard XII for Science and Arts students is divided in two parts. Part I deals with topics like Mathematical Logic, Matrices, Vectors and Introduction to three dimensional geometry. Part II deals with Differentiation, Integration and their applications, Introduction to random variables and statistical methods.

The new text books have three types of exercises for focused and comprehensive practice. First, there are exercises on every important topic. Second, there are comprehensive exercises at the end of all chapters. Third, every chapter includes activities that students must attempt after discussion with classmates and teachers. Additional information has been provided on the E-balbharati website (www.ebalbharati.in).

We are living in the age of Internet. You can make use of modern technology with the help of the Q.R. code given on the title page. The Q.R. code will take you to links that provide additional useful information. Your learning will be fruitful if you balance between reading the text books and solving exercises. Solving more problems will make you more confident and efficient.

The text books are prepared by a subject committee and a study group. The books (Paper I and Paper II) are reviewed by experienced teachers and eminent scholars. The Bureau would like to thank all of them for their valuable contribution in the form of creative writing, constructive and useful suggestions for making the text books valuable. The Bureau hopes and wishes that the text books are very useful and well received by students, teachers and parents.

Students, you are now ready to study. All the best wishes for a happy learning experience and a well deserved success. Enjoy learning and be successful.

(Vivek Gosavi)
Director
Maharashtra State Bureau of Textbook
Production and Curriculum Research, Pune.

Pune
Date : 21 February 2020
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Mathematics and Statistics XII (Part II)
Arts and Science

Sr. No	Area / Topic	Sub Unit	Competency Statement
1.	Differentiation	Differentiation	The students will be able to • state and use standard formulas of derivative of standard functions • use chain rule of derivatives • find derivatives of the logarithm, implicit, inverse and parametric functions • find second and higher order derivatives.
2.	Applications of Derivatives	Applications of Derivatives	• find equations of tangents and normal to a curve • determine nature of the function- increasing or decreasing • find approximate values of the function • examine function for maximum and minimum values • verify mean value theorems
3.	Indefinite Integration	Indefinite Integration	• understand the relation between derivative and integral • use the method of substitution • solve integrals with the help of integration by parts • solve the integrals by the method of partial fractions
4.	Definite Integration	Definite Integration	• understand integral as a limit of sum • the properties of definite integral • state the properties of definite integral and use them to solve problems

5.	Application of Definite Integration	Application of Definite Integration	• find the area under the curve, bounded by the curves using definite integrals.
6.	Differential Equation	Differential Equation	• form a differential equation and find its order and degree • solve the first order and first degree differential equation by various methods • apply the differential equations to study the polpulation, growth and decay in amount of substance and physics.
7.	Probability Distribution	Probability Distribution	• understand the random variable and its types. • find probability mass function and its probability distribution. • find the expected value, variance and the standard deviation • find the probability density function of continuous random variable • find distribution function of c.r.v.
8	Binomial Distribution	Binomial Distribution	• understand random experiment with two or more outcomes. • determine probability distribution of random experiment with parameters n and p. • find mean, variance, expected value and standard deviation for the binomial distribution.

INDEX

1. DIFFERENTIATION

Let us Study

- Derivatives of Composite functions.
- Derivatives of Inverse functions
- Derivatives of Implicit functions.
- Higher order Derivatives.
- Geometrical meaning of Derivative.
- Logarithmic Differentiation
- Derivatives of Parametric functions.

Let us Recall

- The derivative of $f(x)$ with respect to x, at $x = a$ is given by $f'(a) = \lim_{h \to 0}\left[\frac{f(a+h) - f(a)}{h}\right]$
- The derivative can also be defined for $f(x)$ at any point x on the open interval as $f'(x) = \lim_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$. If the function is given as $y = f(x)$ then its derivative is written as $\frac{dy}{dx} = f'(x)$.
- For a differentiable function $y = f(x)$ if δx is a small increment in x and the corresponding increment in y is δy then $\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx}$.
- Derivatives of some standard functions.

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
c (Constant)	0
x^n	nx^{n-1}
$\frac{1}{x}$	$-\frac{1}{x^2}$
$\frac{1}{x^n}$	$-\frac{n}{x^{n+1}}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\cot x$	$-\operatorname{cosec}^2 x$
e^x	e^x
a^x	$a^x \log a$
$\log x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \log a}$

Table 1.1.1

Rules of Differentiation :

If u and v are differentiable functions of x such that

(i) $y = u \pm v$ then $\dfrac{dy}{dx} = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$ (ii) $y = uv$ then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

(iii) $y = \dfrac{u}{v}$ where $v \neq 0$ then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Introduction :

The history of mathematics presents the development of calculus as being accredited to Sir Isaac Newton (1642-1727) an English physicist and mathematician and Gottfried Wilhelm Leibnitz (1646-1716) a German physicist and mathematician. The Derivative is one of the fundamental ideas of calculus. It's all about rate of change in a function. We try to find interpretations of these changes in a mathematical way. The symbol δ will be used to represent the change, for example δx represents a small change in the variable x and it is read as "change in x" or "increment in x". δy is the corresponding change in y if y is a function of x.

We have already studied the basic concept, derivatives of standard functions and rules of differentiation in previous standard. This year, in this chapter we are going to study the geometrical meaning of derivative, derivatives of Composite, Inverse, Logarithmic, Implicit and Parametric functions and also higher order derivatives. We also add some more rules of differentiation.

Let us Learn

1.1.1 Derivatives of Composite Functions (Function of another function) :

So far we have studied the derivatives of simple functions like $\sin x$, $\log x$, e^x etc. But how about the derivatives of $\sin\sqrt{x}$, $\log\left(\sin(x^2+5)\right)$ or $e^{\tan x}$ etc ? These are known as composite functions. In this section let us study how to differentiate composite functions.

1.1.2 Theorem : If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x such that the composite function $y = f[g(x)]$ is a differentiable function of x then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$.

Proof : Given that $y = f(u)$ and $u = g(x)$. We assume that u is not a constant function. Let there be a small increment in the value of x say δx then δu and δy are the corresponding increments in u and y respectively.

As δx, δu, δy are small increments in x, u and y respectively such that $\delta x \neq 0$, $\delta u \neq 0$ and $\delta y \neq 0$.

We have $\dfrac{\delta y}{\delta x} = \dfrac{\delta y}{\delta u} \times \dfrac{\delta u}{\delta x}$.

Taking the limit as $\delta x \to 0$ on both sides we get,

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta u}\right) \times \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right)$$

As $\delta x \to 0$, we get, $\delta u \to 0$ ($\because$ u is a continuous function of x)

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta u \to 0}\left(\frac{\delta y}{\delta u}\right) \times \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right) \qquad \ldots\ldots \text{(I)}$$

Since y is a differentiable function of u and u is a differentiable function of x.

we have,

$$\lim_{\delta u \to 0}\left(\frac{\delta y}{\delta u}\right) = \frac{dy}{du} \text{ and } \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right) = \frac{du}{dx} \qquad \ldots\ldots \text{(II)}$$

From (I) and (II), we get

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{du} \times \frac{du}{dx} \qquad \ldots\ldots \text{(III)}$$

The R.H.S. of (III) exists and is finite, implies L.H.S.of (III) also exists and is finite

$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx}$. Then equation (III) becomes,

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}}$$

Note:

1. The derivative of a composite function can also be expressed as follows. $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x such that the composite function $y = f[g(x)]$ is defined then

$$\frac{dy}{dx} = f'[g(x)] \cdot g'(x).$$

2. If $y = f(v)$ is a differentiable function of v and $v = g(u)$ is a differentiable function of u and $u = h(x)$ is a differentiable function of x then

$$\frac{dy}{dx} = \frac{dy}{dv} \times \frac{dv}{du} \times \frac{du}{dx}.$$

3. If y is a differentiable function of u_1, u_i is a differentiable function of u_{i+1} for $i = 1, 2, \ldots, n-1$ and u_n is a differentiable function of x, then

$$\frac{dy}{dx} = \frac{dy}{du_1} \times \frac{du_1}{du_2} \times \frac{du_2}{du_3} \times \ldots \times \frac{du_{n-1}}{du_n} \times \frac{du_n}{dx}$$

This rule is also known as **Chain rule**.

1.1.3 Derivatives of some standard Composite Functions :

y	$\frac{dy}{dx}$
$[f(x)]^n$	$n[f(x)]^{n-1} \cdot f'(x)$
$\sqrt{f(x)}$	$\frac{f'(x)}{2\sqrt{f(x)}}$
$\frac{1}{[f(x)]^n}$	$-\frac{n \cdot f'(x)}{[f(x)]^{n+1}}$
$\sin[f(x)]$	$\cos[f(x)] \cdot f'(x)$
$\cos[f(x)]$	$-\sin[f(x)] \cdot f'(x)$
$\tan[f(x)]$	$\sec^2[f(x)] \cdot f'(x)$
$\sec[f(x)]$	$\sec[f(x)] \cdot \tan[f(x)] \cdot f'(x)$

y	$\frac{dy}{dx}$
$\cot[f(x)]$	$-\operatorname{cosec}^2[f(x)] \cdot f'(x)$
$\operatorname{cosec}[f(x)]$	$-\operatorname{cosec}[f(x)] \cdot \cot[f(x)] \cdot f'(x)$
$a^{f(x)}$	$a^{f(x)} \cdot \log a \cdot f'(x)$
$e^{f(x)}$	$e^{f(x)} \cdot f'(x)$
$\log[f(x)]$	$\frac{f'(x)}{f(x)}$
$\log_a[f(x)]$	$\frac{f'(x)}{f(x)\log a}$

Table 1.1.2

SOLVED EXAMPLES

Ex. 1 : Differentiate the following *w. r. t. x*.

(i) $y = \sqrt{x^2+5}$ (ii) $y = \sin(\log x)$ (iii) $y = e^{\tan x}$

(iv) $\log(x^5+4)$ (v) $5^{3\cos x - 2}$ (vi) $y = \frac{3}{(2x^2-7)^5}$

Solution : (i) $y = \sqrt{x^2+5}$

Method 1 :

Let $u = x^2 + 5$ then $y = \sqrt{u}$, where y is a differentiable function of u and u is a differentiable function of x then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \ldots\ldots (\mathrm{I})$$

Now, $y = \sqrt{u}$

Differentiate *w. r. t. u*

$$\frac{dy}{du} = \frac{d}{du}(\sqrt{u}) = \frac{1}{2\sqrt{u}} \text{ and } u = x^2+5$$

Differentiate *w. r. t. x*

$$\frac{du}{dx} = \frac{d}{dx}(x^2+5) = 2x$$

Now, equation (I) becomes,

$$\frac{dy}{dx} = \frac{1}{2\sqrt{u}} \times 2x = \frac{x}{\sqrt{x^2+5}}$$

Method 2 :

We have $y = \sqrt{x^2+5}$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x^2+5}\right)$$

[Treat x^2+5 as u in mind and use the formula of derivative of $\sqrt{u}$]

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+5}} \cdot \frac{d}{dx}(x^2+5)$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x^2+5}}(2x)$$

$$\frac{dy}{dx} = \frac{x}{\sqrt{x^2+5}}$$

(ii) $y = \sin(\log x)$

Method 1 :

Let $u = \log x$ then $y = \sin u$, where y is a differentiable function of u and u is a differentiable function of x then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \ldots\ldots (I)$$

Now, $y = \sin u$

Differentiate $w.\ r.\ t.\ u$

$$\frac{dy}{du} = \frac{d}{du}(\sin u) = \cos u \text{ and } u = \log x$$

Differentiate $w.\ r.\ t.\ x$

$$\frac{du}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

Now, equation (I) becomes,

$$\frac{dy}{dx} = \cos u \times \frac{1}{x} = \frac{\cos(\log x)}{x}$$

Method 2 :

We have $y = \sin(\log x)$

Differentiate $w.\ r.\ t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(\log x)]$$

[Treat $\log x$ as u in mind and use the formula of derivative of $\sin u$]

$$\frac{dy}{dx} = \cos(\log x) \times \frac{d}{dx}(\log x)$$

$$\frac{dy}{dx} = \cos(\log x) \cdot \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{\cos(\log x)}{x}$$

Note : Hence onwards let's use Method 2.

(iii) $y = e^{\tan x}$

Differentiate $w.\ r.\ t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}[e^{\tan x}]$$

$$\frac{dy}{dx} = e^{\tan x} \times \frac{d}{dx}(\tan x)$$

$$\frac{dy}{dx} = e^{\tan x} \cdot \sec^2 x = \sec^2 x \cdot e^{\tan x}$$

(iv) Let $y = \log(x^5 + 4)$

Differentiate $w.\ r.\ t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(x^5 + 4)\right]$$

$$\frac{dy}{dx} = \frac{1}{x^5 + 4} \times \frac{d}{dx}(x^5 + 4)$$

$$\frac{dy}{dx} = \frac{1}{x^5 + 4}(5x^4) = \frac{5x^4}{x^5 + 4}$$

(v) Let $y = 5^{3\cos x - 2}$

Differentiate $w.\ r.\ t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}\left[5^{3\cos x - 2}\right]$$

$$\frac{dy}{dx} = 5^{3\cos x - 2} \cdot \log 5 \times \frac{d}{dx}(3\cos x - 2)$$

$$\frac{dy}{dx} = -3\sin x \cdot 5^{3\cos x - 2} \cdot \log 5$$

(vi) Let $y = \dfrac{3}{(2x^2 - 7)^5}$

Differentiate $w.\ r.\ t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3}{(2x^2 - 7)^5}\right) = 3\frac{d}{dx}\left(\frac{1}{(2x^2 - 7)^5}\right)$$

$$= 3 \times \frac{-5}{(2x^2 - 7)^6} \times \frac{d}{dx}(2x^2 - 7)$$

$$= -\frac{15}{(2x^2 - 7)^6}(4x)$$

$$\frac{dy}{dx} = -\frac{60x}{(2x^2 - 7)^6}$$

Ex. 2 : Differentiate the following *w. r. t. x.*

(i) $y = \sqrt{\sin x^3}$ (ii) $y = \cot^2(x^3)$ (iii) $y = \log[\cos(x^5)]$

(iv) $y = (x^3 + 2x - 3)^4 (x + \cos x)^3$ (v) $y = (1 + \cos^2 x)^4 \times \sqrt{x + \sqrt{\tan x}}$

Solution :

(i) $y = \sqrt{\sin x^3}$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{\sin x^3}\right)$$

$$= \frac{1}{2\sqrt{\sin x^3}} \times \frac{d}{dx}(\sin x^3)$$

$$= \frac{1}{2\sqrt{\sin x^3}} \times \cos x^3 \times \frac{d}{dx}(x^3)$$

$$= \frac{1}{2\sqrt{\sin x^3}} \times \cos x^3 \times (3x^2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{3x^2 \cos x^3}{2\sqrt{\sin x^3}}$$

(ii) $y = \cot^2(x^3)$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cot^2(x^3)\right)$$

$$= \frac{d}{dx}[\cot(x^3)]^2$$

$$= 2\cot(x^3)\frac{d}{dx}[\cot(x^3)]$$

$$= 2\cot(x^3)[-\operatorname{cosec}^2(x^3)]\frac{d}{dx}(x^3)$$

$$= -2\cot(x^3)\operatorname{cosec}^2(x^3)(3x^2)$$

$$\therefore \quad \frac{dy}{dx} = -6x^2\cot(x^3)\operatorname{cosec}^2(x^3)$$

(iii) $y = \log[\cos(x^5)]$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left(\log[\cos(x^5)]\right)$$

$$= \frac{1}{\cos(x^5)} \cdot \frac{d}{dx}[\cos(x^5)]$$

$$= \frac{1}{\cos(x^5)}\left(-\sin(x^5)\right)\frac{d}{dx}(x^5)$$

$$\therefore \quad \frac{dy}{dx} = -\tan(x^5)(5x^4) = -5x^4\tan(x^5)$$

(iv) $y = (x^3 + 2x - 3)^4 (x + \cos x)^3$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left[(x^3 + 2x - 3)^4 (x + \cos x)^3\right]$$

$$= (x^3 + 2x - 3)^4 \cdot \frac{d}{dx}(x + \cos x)^3 + (x + \cos x)^3 \cdot \frac{d}{dx}(x^3 + 2x - 3)^4$$

$$= (x^3 + 2x - 3)^4 \cdot 3\,(x + \cos x)^2 \cdot \frac{d}{dx}(x + \cos x) + (x + \cos x)^3 \cdot 4(x^3 + 2x - 3)^3 \cdot \frac{d}{dx}(x^3 + 2x - 3)$$

$$= (x^3 + 2x - 3)^4 \cdot 3\,(x + \cos x)^2\,(1 - \sin x) + (x + \cos x)^3 \cdot 4(x^3 + 2x - 3)^3 (3x^2 + 2)$$

$$\therefore \quad \frac{dy}{dx} = 3(x^3 + 2x - 3)^4 (x + \cos x)^2\,(1 - \sin x) + 4\,(3x^2 + 2)\,(x^3 + 2x - 3)^3 (x + \cos x)^3$$

(v) $y = (1 + \cos^2 x)^4 \times \sqrt{x + \sqrt{\tan x}}$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}\left[(1+\cos^2 x)^4 \times \sqrt{x+\sqrt{\tan x}}\right]$$

$$= (1+\cos^2 x)^4 \frac{d}{dx}\left(\sqrt{x+\sqrt{\tan x}}\right) + \left(\sqrt{x+\sqrt{\tan x}}\right)\frac{d}{dx}(1+\cos^2 x)^4$$

$$= (1+\cos^2 x)^4 \cdot \frac{1}{2\sqrt{x+\sqrt{\tan x}}} \cdot \frac{d}{dx}\left(x+\sqrt{\tan x}\right) + \left(\sqrt{x+\sqrt{\tan x}}\right) \cdot 4(1+\cos^2 x)^3 \frac{d}{dx}[1+(\cos x)^2]$$

$$= (1+\cos^2 x)^4 \cdot \frac{1}{2\sqrt{x+\sqrt{\tan x}}}\left(1 + \frac{1}{2\sqrt{\tan x}} \cdot \frac{d}{dx}(\tan x)\right) + \left(\sqrt{x+\sqrt{\tan x}}\right) \cdot 4(1+\cos^2 x)^3 (2\cos x)\frac{d}{dx}(\cos x)$$

$$= (1+\cos^2 x)^4 \cdot \frac{1}{2\sqrt{x+\sqrt{\tan x}}}\left(1 + \frac{\sec^2 x}{2\sqrt{\tan x}}\right) + \left(\sqrt{x+\sqrt{\tan x}}\right) \cdot 4(1+\cos^2 x)^3 (2\cos x)(-\sin x)$$

$$= (1+\cos^2 x)^4 \cdot \frac{1}{2\sqrt{x+\sqrt{\tan x}}}\left(\frac{2\sqrt{\tan x} + \sec^2 x}{2\sqrt{\tan x}}\right) - \left(\sqrt{x+\sqrt{\tan x}}\right) \cdot 4(1+\cos^2 x)^3 (2\sin x\cos x)$$

$$\frac{dy}{dx} = \frac{(1+\cos^2 x)^4 (2\sqrt{\tan x} + \sec^2 x)}{4\sqrt{\tan x}\sqrt{x+\sqrt{\tan x}}} - 4\sin 2x(1+\cos^2 x)^3 \sqrt{x+\sqrt{\tan x}}$$

Ex. 3 : Differentiate the following *w. r. t. x*.

(i) $y = \log_3 (\log_5 x)$

(ii) $y = \log\left[e^{3x} \cdot \dfrac{(3x-4)^{\frac{2}{3}}}{\sqrt[3]{2x+5}}\right]$

(iii) $y = \log\left[\sqrt{\dfrac{1-\cos\left(\frac{3x}{2}\right)}{1+\cos\left(\frac{3x}{2}\right)}}\right]$

(iv) $y = \log\left[\dfrac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}-x}\right]$

(v) $y = (4)^{\log_2(\sin x)} + (9)^{\log_3(\cos x)}$

(vi) $y = a^{a^{\log_a(\cot x)}}$

Solution :

(i) $y = \log_3 (\log_5 x)$

$$= \log_3 \left(\frac{\log x}{\log 5}\right) \quad = \log_3 (\log x) - \log_3 (\log 5)$$

$$\therefore \quad y = \frac{\log(\log x)}{\log 3} - \log_3 (\log 5)$$

Differentiate $w.\ r.\ t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{\log(\log x)}{\log 3} - \log_3(\log 5)\right]$$

$$= \frac{1}{\log 3}\frac{d}{dx}[\log(\log x)] - \frac{d}{dx}[\log_3(\log 5)]$$

$$= \frac{1}{\log 3} \times \frac{1}{\log x}\frac{d}{dx}(\log x) - 0 \qquad \text{[Note that } \log_3(\log 5) \text{ is constant]}$$

$$= \frac{1}{\log 3} \times \frac{1}{\log x} \times \frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x \log x \log 3}$$

(ii) $y = \log\left[e^{3x} \cdot \dfrac{(3x-4)^{\frac{2}{3}}}{\sqrt[3]{2x+5}}\right] = \log\left[\dfrac{e^{3x} \cdot (3x-4)^{\frac{2}{3}}}{(2x+5)^{\frac{1}{3}}}\right]$

$$= \log\left[e^{3x} \cdot (3x-4)^{\frac{2}{3}}\right] - \log\left[(2x+5)^{\frac{1}{3}}\right]$$

$$= \log e^{3x} + \log(3x-4)^{\frac{2}{3}} - \log(2x+5)^{\frac{1}{3}}$$

$$\therefore \quad y = 3x + \frac{2}{3}\log(3x-4) - \frac{1}{3}\log(2x+5) \qquad [\because \log e = 1]$$

Differentiate $w.\ r.\ t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}\left[3x + \frac{2}{3}\log(3x-4) - \frac{1}{3}\log(2x+5)\right]$$

$$= 3\frac{d}{dx}(x) + \frac{2}{3} \cdot \frac{d}{dx}[\log(3x-4)] - \frac{1}{3} \cdot \frac{d}{dx}[\log(2x+5)]$$

$$= 3(1) + \frac{2}{3} \cdot \frac{1}{3x-4} \cdot \frac{d}{dx}(3x-4) - \frac{1}{3} \cdot \frac{1}{2x+5} \cdot \frac{d}{dx}(2x+5)$$

$$= 3(1) + \frac{2}{3} \cdot \frac{1}{3x-4} \cdot (3) - \frac{1}{3} \cdot \frac{1}{2x+5} \cdot (2)$$

$$\therefore \quad \frac{dy}{dx} = 3 + \frac{2}{3x-4} - \frac{2}{3(2x+5)}$$

(iii) $$y=\log\left[\sqrt{\frac{1-\cos\left(\frac{3x}{2}\right)}{1+\cos\left(\frac{3x}{2}\right)}}\right]=\log\left[\sqrt{\frac{2\sin^2\left(\frac{3x}{4}\right)}{2\cos^2\left(\frac{3x}{4}\right)}}\right]$$

$$\therefore\quad y=\log\left[\tan\left(\frac{3x}{4}\right)\right]$$

Differentiate *w. r. t. x*

$$\frac{dy}{dx}=\frac{d}{dx}\left\{\log\left[\tan\left(\frac{3x}{4}\right)\right]\right\}$$

$$=\frac{1}{\tan\left(\frac{3x}{4}\right)}\cdot\frac{d}{dx}\left[\tan\left(\frac{3x}{4}\right)\right]$$

$$=\cot\left(\frac{3x}{4}\right)\cdot\sec^2\left(\frac{3x}{4}\right)\cdot\frac{d}{dx}\left(\frac{3x}{4}\right)$$

$$=\frac{\cos\left(\frac{3x}{4}\right)}{\sin\left(\frac{3x}{4}\right)}\times\frac{1}{\cos^2\left(\frac{3x}{4}\right)}\times\frac{3}{4}$$

$$=\frac{3}{2\left[2\sin\left(\frac{3x}{4}\right)\cdot\cos\left(\frac{3x}{4}\right)\right]}=\frac{3}{2\sin\left(\frac{3x}{2}\right)}$$

$$\therefore\quad \frac{dy}{dx}=\frac{3}{2}\operatorname{cosec}\left(\frac{3x}{2}\right)$$

(iv) $$y=\log\left[\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}-x}\right]\quad=\log\left[\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}-x}\times\frac{\sqrt{x^2+a^2}+x}{\sqrt{x^2+a^2}+x}\right]$$

$$=\log\left[\frac{\left(\sqrt{x^2+a^2}+x\right)^2}{x^2+a^2-x^2}\right]$$

$$=\log\left[\frac{\left(\sqrt{x^2+a^2}+x\right)^2}{a^2}\right]$$

$$=\log\left(\sqrt{x^2+a^2}+x\right)^2-\log(a^2)$$

$$\therefore\quad y=2\log\left(\sqrt{x^2+a^2}+x\right)-\log(a^2)$$

Differentiate *w. r. t. x*

$$\frac{dy}{dx}=\frac{d}{dx}\left[2\log\left(\sqrt{x^2+a^2}+x\right)-\log(a^2)\right]$$

$$=2\frac{d}{dx}\left[\log\left(\sqrt{x^2+a^2}+x\right)\right]-\frac{d}{dx}[\log(a^2)]$$

$$=2\times\frac{1}{\sqrt{x^2+a^2}+x}\cdot\frac{d}{dx}\left[\sqrt{x^2+a^2}+x\right]-0$$

$$=\frac{2}{\sqrt{x^2+a^2}+x}\cdot\left[\frac{1}{2\sqrt{x^2+a^2}}\cdot\frac{d}{dx}(x^2+a^2)+1\right]$$

$$=\frac{2}{\sqrt{x^2+a^2}+x}\cdot\left[\frac{1}{2\sqrt{x^2+a^2}}(2x)+1\right]$$

$$=\frac{2}{\sqrt{x^2+a^2}+x}\cdot\left[\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right]$$

$$\frac{dy}{dx}=\frac{2}{\sqrt{x^2+a^2}}$$

(v) $y=(4)^{\log_2(\sin x)}+(9)^{\log_3(\cos x)}$

$=\left(2^2\right)^{\log_2(\sin x)}+\left(3^2\right)^{\log_3(\cos x)}$

$=\left(2\right)^{2\log_2(\sin x)}+\left(3\right)^{2\log_3(\cos x)}$

$=\left(2\right)^{\log_2(\sin^2 x)}+\left(3\right)^{\log_3(\cos^2 x)}$ $[\because a^{\log_a f(x)}=f(x)]$

$=\sin^2 x+\cos^2 x$

$\therefore\ y=1$

Differentiate $w.\ r.\ t.\ x$

$\frac{dy}{dx}=\frac{d}{dx}(1)=0$

(vi) $y=a^{a^{\log_a(\cot x)}}$

$y=a^{\cot x}$ $[\because a^{\log_a f(x)}=f(x)]$

Differentiate $w.\ r.\ t.\ x$

$\frac{dy}{dx}=\frac{d}{dx}(a^{\cot x})$

$=a^{\cot x}\log a\cdot\frac{d}{dx}(\cot x)$

$=a^{\cot x}\log a\,(-\operatorname{cosec}^2 x)$

$\frac{dy}{dx}=-\operatorname{cosec}^2 x\cdot a^{\cot x}\log a$

Ex. 4 : If $f(x)=\sqrt{7g(x)-3}$, $g(3)=4$ and $g'(3)=5$, find $f'(3)$.

Solution : Given that : $f(x)=\sqrt{7g(x)-3}$

Differentiate $w.\ r.\ t.\ x$

$$f'(x)=\frac{d}{dx}\left(\sqrt{7g(x)-3}\right)=\frac{1}{2\sqrt{7g(x)-3}}\frac{d}{dx}[7g(x)-3]$$

$$\therefore\ f'(x)=\frac{7g'(x)}{2\sqrt{7g(x)-3}}$$

For $x=3$, we get

$$f'(3)=\frac{7g'(3)}{2\sqrt{7g(3)-3}}=\frac{35}{2(5)}=\frac{7}{2}$$ [Since $g(3)=4$ and $g'(3)=5$]

Ex. 5 : If $F(x) = G\{3G[5G(x)]\}$, $G(0) = 0$ and $G'(0) = 3$, find $F'(0)$.

Solution : Given that : $F(x) = G\{3G[5G(x)]\}$

Differentiate *w. r. t.* x

$$F'(x) = \frac{d}{dx} G\{3G[5G(x)]\}$$

$$= G'\{3G[5G(x)]\} 3 \cdot \frac{d}{dx}[G[5G(x)]]$$

$$= G'\{3G[5G(x)]\} 3 \cdot G'[5G(x)]\ 5 \cdot \frac{d}{dx}[G(x)]$$

$$F'(x) = 15 \cdot G'\{3G[5G(x)]\} G'[5G(x)]\ G'(x)$$

For $x = 0$, we get

$$F'(0) = 15 \cdot G'\{3G[5G(0)]\} G'[5G(0)]\ G'(0)$$

$$= 15 \cdot G'[3G(0)] G'(0) \cdot (3) \qquad [\because G(0) = 0 \text{ and } G'(0) = 3]$$

$$= 15 \cdot G'(0)(3)(3) = 15 \cdot (3)(3)(3) = 405$$

Ex. 6 : Select the appropriate hint from the hint basket and fill in the blank spaces in the following paragraph. [Activity]

"Let $f(x) = \sin x$ and $g(x) = \log x$ then $f[g(x)] =$ _ _ _ _ _ _ _ _ _ _ _ _ _ _ and $g[f(x)] =$ _ _ _ _ _ _ _ _ _ . Now $f'(x) =$ _ _ _ _ _ _ _ _ _ _ _ and $g'(x) =$ _ _ _ _ _ _ _ _ _.

The derivative of $f[g(x)]$ *w. r. t.* x in terms of f and g is _ _ _ _ _ _ _ _ _ _ _ _ _.

Therefore $\frac{d}{dx}[f[g(x)]] =$ _ _ _ _ _ _ _ _ _ and $\left[\frac{d}{dx}[f[g(x)]]\right]_{x=1} =$ _ _ _ _ _ _ _ _ _ _ _ _ _.

The derivative of $g[f(x)]$ *w. r. t.* x in terms of f and g is _ _ _ _ _ _ _ _ _ _ _ _ _.

Therefore $\frac{d}{dx}[g[f(x)]] =$ _ _ _ _ _ _ _ _ _ _ and $\left[\frac{d}{dx}[g[f(x)]]\right]_{x=\frac{\pi}{3}} =$ _ _ _ _ _ _ _ _ _ _ _ _."

Hint basket : $\left\{ f'[g(x)] \cdot g'(x), \frac{\cos(\log x)}{x}, 1, g'[f(x)] \cdot f'(x), \cot x, \sqrt{3}, \sin(\log x), \log(\sin x), \cos x, \frac{1}{x} \right\}$

Solution : $\sin(\log x)$, $\log(\sin x)$, $\cos x$, $\frac{1}{x}$, $f'[g(x)] \cdot g'(x)$, $\frac{\cos(\log x)}{x}$, 1, $g'[f(x)] \cdot f'(x)$, $\cot x$, $\sqrt{3}$.

EXERCISE 1.1

(1) Differentiate *w. r. t.* x.

(i) $(x^3 - 2x - 1)^5$

(ii) $\left(2x^{\frac{3}{2}} - 3x^{\frac{4}{3}} - 5\right)^{\frac{5}{2}}$

(iii) $\sqrt{x^2 + 4x - 7}$

(iv) $\sqrt{x^2 + \sqrt{x^2 + 1}}$

(v) $\dfrac{3}{5\sqrt[3]{(2x^2 - 7x - 5)^5}}$

(vi) $\left(\sqrt{3x - 5} - \dfrac{1}{\sqrt{3x - 5}}\right)^5$

(2) Differentiate the following $w.r.t.\ x$

(i) $\cos(x^2+a^2)$ (ii) $\sqrt{e^{(3x+2)}+5}$

(iii) $\log\left[\tan\left(\frac{x}{2}\right)\right]$ (iv) $\sqrt{\tan\sqrt{x}}$

(v) $\cot^3[\log(x^3)]$ (vi) $5^{\sin^3 x+3}$

(vii) $\operatorname{cosec}(\sqrt{\cos x})$ (viii) $\log[\cos(x^3-5)]$

(ix) $e^{3\sin^2 x-2\cos^2 x}$ (x) $\cos^2[\log(x^2+7)]$

(xi) $\tan[\cos(\sin x)]$ (xii) $\sec[\tan(x^4+4)]$

(xiii) $e^{\log[(\log x)^2-\log x^2]}$ (xiv) $\sin\sqrt{\sin\sqrt{x}}$

(xv) $\log[\sec(e^{x^2})]$ (xvi) $\log_{e^2}(\log x)$

(xvii) $[\log[\log(\log x)]]^2$

(xviii) $\sin^2 x^2-\cos^2 x^2$

(3) Differentiate the following $w.r.t.\ x$

(i) $(x^2+4x+1)^3+(x^3-5x-2)^4$

(ii) $(1+4x)^5(3+x-x^2)^8$

(iii) $\frac{x}{\sqrt{7-3x}}$ (iv) $\frac{(x^3-5)^5}{(x^3+3)^3}$

(v) $(1+\sin^2 x)^2(1+\cos^2 x)^3$

(vi) $\sqrt{\cos x}+\sqrt{\cos\sqrt{x}}$

(vii) $\log(\sec 3x+\tan 3x)$ (viii) $\frac{1+\sin x^\circ}{1-\sin x^\circ}$

(ix) $\cot\left(\frac{\log x}{2}\right)-\log\left(\frac{\cot x}{2}\right)$

(x) $\frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$ (xi) $\frac{e^{\sqrt{x}}+1}{e^{\sqrt{x}}-1}$

(xii) $\log[\tan^3 x\cdot\sin^4 x\cdot(x^2+7)^7]$

(xiii) $\log\left(\sqrt{\frac{1-\cos 3x}{1+\cos 3x}}\right)$

(xiv) $\log\left(\sqrt{\frac{1+\cos\left(\frac{5x}{2}\right)}{1-\cos\left(\frac{5x}{2}\right)}}\right)$

(xv) $\log\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)$

(xvi) $\log\left[4^{2x}\left(\frac{x^2+5}{\sqrt{2x^3-4}}\right)^{\frac{3}{2}}\right]$

(xvii) $\log\left[\frac{e^{x^2}(5-4x)^{\frac{3}{2}}}{\sqrt[3]{7-6x}}\right]$

(xviii) $\log\left(\frac{a^{\cos x}}{(x^2-3)^3\log x}\right)$

(xix) $y=(25)^{\log_5(\sec x)}-(16)^{\log_4(\tan x)}$

(xx) $\frac{(x^2+2)^4}{\sqrt{x^2-5}}$

(4) A table of values of f, g, f' and g' is given

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	1	6	−3	4
4	3	4	5	−6
6	5	2	−4	7

(i) If $r(x)=f[g(x)]$ find $r'(2)$.

(ii) If $R(x)=g[3+f(x)]$ find $R'(4)$.

(iii) If $s(x)=f[9-f(x)]$ find $s'(4)$.

(iv) If $S(x)=g[g(x)]$ find $S'(6)$.

(5) Assume that $f'(3)=-1$, $g'(2)=5$, $g(2)=3$ and $y=f[g(x)]$ then $\left[\frac{dy}{dx}\right]_{x=2}=?$

(6) If $h(x)=\sqrt{4f(x)+3g(x)}$, $f(1)=4$, $g(1)=3$, $f'(1)=3$, $g'(1)=4$ find $h'(1)$.

(7) Find the x co-ordinates of all the points on the curve $y=\sin 2x-2\sin x$, $0\le x<2\pi$ where $\frac{dy}{dx}=0$.

(8) Select the appropriate hint from the hint basket and fill up the blank spaces in the following paragraph. [Activity]

"Let $f(x) = x^2 + 5$ and $g(x) = e^x + 3$ then

$f[g(x)] =$ _ _ _ _ _ _ _ _ _ and

$g[f(x)] =$ _ _ _ _ _ _ _ _ _.

Now $f'(x) =$ _ _ _ _ _ _ _ _ _ and

$g'(x) =$ _ _ _ _ _ _ _ _ _.

The derivative of $f[g(x)]$ *w. r. t.* x in terms of f and g is _ _ _ _ _ _ _ _ _.

Therefore $\frac{d}{dx}[f[g(x)]] =$ _ _ _ _ _ _ _ _ _ _ and

$\left[\frac{d}{dx}[f[g(x)]]\right]_{x=0} =$ _ _ _ _ _ _ _ _ _ _ _ _.

The derivative of $g[f(x)]$ *w. r. t.* x in terms of f and g is _ _ _ _ _ _ _ _ _ _ _ _.

Therefore $\frac{d}{dx}[g[f(x)]] =$ _ _ _ _ _ _ _ _ _ _

and $\left[\frac{d}{dx}[g[f(x)]]\right]_{x=-1} =$ _ _ _ _ _ _ _ _ _ _ _ _."

Hint basket : $\left\{f'[g(x)]\cdot g'(x),\ 2e^{2x}+6e^x,\ 8,\ g'[f(x)]\cdot f'(x),\ 2xe^{x^2+5},\ -2e^6,\ e^{2x}+6e^x+14,\ e^{x^2+5}+3,\ 2x,\ e^x\right\}$

1.2.1 Geometrical meaning of Derivative :

Consider a point P on the curve $f(x)$. At $x = a$, the coordinates of P are $(a, f(a))$. Let Q be another point on the curve, a little to the right of P i.e. to the right of $x = a$, with a value increased by a small real number h. Therefore the coordinates of Q are $((a+h), f(a+h))$. Now we can calculate the slope of the secant line PQ i.e. slope of the secant line connecting the points P $(a, f(a))$ and Q $((a+h), f(a+h))$, by using formula for slope.

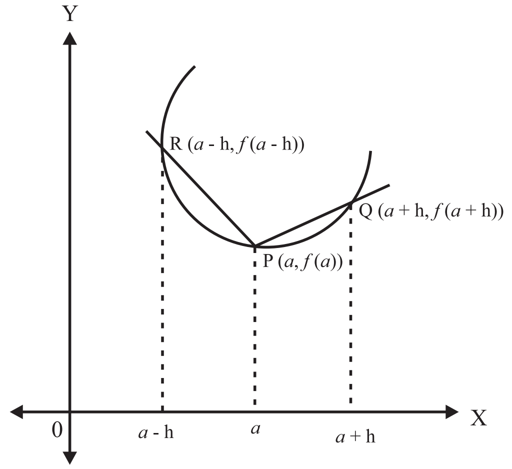


Fig. 1.2.1

$$\text{Slope of secant PQ} = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

Suppose we make h smaller and smaller then $a + h$ will approach a as h gets closer to zero, Q will approach P, that is as $h \to 0$, the secant coverges to the tangent at P.

$$\therefore \quad \lim_{Q\to P}(\text{Slope of secant PQ}) = \lim_{h\to 0}\left[\frac{f(a+h)-f(a)}{h}\right] = f'(a)$$

So we get, Slope of tangent at P $= f'(a)$ ..... [If limit exists]

Thus the derivative of a function $y = f(x)$ at any point $P(x_1, y_1)$ is the slope of the tangent at that point on the curve. If we consider the point $a - h$ to the left of a, $h > 0$, then with R $= ((a-h), f(a-h))$ we will find the slope of PR which will also converge to the slope of tangent at P.

For Example : If $y = x^2 + 3x + 2$ then slope of the tangent at (2,3) is given by

$$\text{Slope } m = \left[\frac{dy}{dx}\right]_{(2,3)} = \left[\frac{d}{dx}(x^2+3x+2)\right]_{(2,3)} = (2x+3)_{(2,3)} = 2(2)+3) \qquad \therefore \qquad m = 7$$

1.2.2 Derivatives of Inverse Functions :

We know that if $y = f(x)$ is a one-one and onto function then $x = f^{-1}(y)$ exists. If $f^{-1}(y)$ is differentiable then we can find its derivative. In this section let us discuss the derivatives of some inverse functions and the derivatives of inverse trigonometric functions.

Example 1 : Consider $f(x) = 2x - 2$ then its inverse is $f^{-1}(x) = \dfrac{x+2}{2}$. Let $g(x) = f^{-1}(x)$.

If we find the derivatives of these functions we see that $\dfrac{d}{dx}[f(x)] = 2$ and $\dfrac{d}{dx}[g(x)] = \dfrac{1}{2}$.

These derivatives are reciprocals of one another.

Example 2 : Consider $y = f(x) = x^2$. Let $g = f^{-1}$.

$\therefore\ g(y) = x = \sqrt{y}$

$\therefore\ g'(y) = \dfrac{1}{2\sqrt{y}}$ also $f'(x) = 2x$

Now $\dfrac{d}{dx}[g(f(x))] = \dfrac{f'(x)}{2\sqrt{f(x)}} = \dfrac{2x}{2\sqrt{x^2}} = 1$ and $g[f(x)] = x \therefore \dfrac{d}{dx}[g(f(x))] = \dfrac{d}{dx}(x) = 1$

At a point (x, x^2) on the curve, $f'(x) = 2x$ and $g'(y) = \dfrac{1}{2\sqrt{y}} = \dfrac{1}{2x} = \dfrac{1}{f'(x)}$.

1.2.3 Theorem : Suppose $y = f(x)$ is a differentiable function of x on an interval I and y is One-one, onto and $\dfrac{dy}{dx} \neq 0$ on I. Also if $f^{-1}(y)$ is differentiable on $f(I)$ then $\dfrac{d}{dy}[f^{-1}(y)] = \dfrac{1}{f'(x)}$ or $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$ where $\dfrac{dy}{dx} \neq 0$.

Proof : Given that $y = f(x)$ and $x = f^{-1}(y)$ are differentiable functions.

Let there be a small increment in the value of x say δx then correspondingly there will be an increment in the value of y say δy. As δx and δy are increments, $\delta x \neq 0$ and $\delta y \neq 0$.

We have, $\dfrac{\delta x}{\delta y} \times \dfrac{\delta y}{\delta x} = 1$

$\therefore\ \dfrac{\delta x}{\delta y} = \dfrac{1}{\frac{\delta y}{\delta x}}$, where $\dfrac{\delta y}{\delta x} \neq 0$

Taking the limit as $\delta x \to 0$, we get,

$$\lim_{\delta x \to 0}\left(\frac{\delta x}{\delta y}\right) = \lim_{\delta x \to 0}\left(\frac{1}{\frac{\delta y}{\delta x}}\right)$$

as $\delta x \to 0$, $\delta y \to 0$,

$$\lim_{\delta y \to 0}\left(\frac{\delta x}{\delta y}\right) = \frac{1}{\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)} \qquad \ldots\ldots (I)$$

Since $y = f(x)$ is a differentiable function of x.

we have, $\lim\limits_{\delta x\to 0}\left(\dfrac{\delta y}{\delta x}\right)=\dfrac{dy}{dx}$ and $\dfrac{dy}{dx}\neq 0$ . . . . . (II)

From (I) and (II), we get

$$\lim_{\delta y\to 0}\left(\frac{\delta x}{\delta y}\right)=\frac{1}{\frac{dy}{dx}} \qquad \text{. . . . . (III)}$$

As $\dfrac{dy}{dx}\neq 0$, $\dfrac{1}{\frac{dy}{dx}}$ exists and is finite. $\therefore$ $\lim\limits_{\delta y\to 0}\left(\dfrac{\delta x}{\delta y}\right)=\dfrac{dx}{dy}$ exists and is finite.

Hence, from (III) $\dfrac{dx}{dy}=\dfrac{1}{\frac{dy}{dx}}$ where $\dfrac{dy}{dx}\neq 0$

An alternative proof using derivatives of composite functions rule.

We know that $f^{-1}[f(x)]=x$ [Identity function]

Taking derivative on bothe sides we get,

$$\frac{d}{dx}\left[f^{-1}[f(x)]\right]=\frac{d}{dx}(x)$$

i.e. $\left(f^{-1}\right)'[f(x)]\,\dfrac{d}{dx}[f(x)] = 1$

i.e. $\left(f^{-1}\right)'[f(x)]\,f'(x)=1$

$\therefore$ $\left(f^{-1}\right)'[f(x)]=\dfrac{1}{f'(x)}$ . . . . . (I)

So, if $y=f(x)$ is a differentiable function of x and $x=f^{-1}(y)$ exists and is differentiable then

$$\left(f^{-1}\right)'[f(x)]=\left(f^{-1}\right)'(y)=\frac{dx}{dy} \text{ and } f'(x)=\frac{dy}{dx}$$

$\therefore$ (I) becomes

$$\frac{dx}{dy}=\frac{1}{\frac{dy}{dx}} \text{ where } \frac{dy}{dx}\neq 0$$

SOLVED EXAMPLES

Ex. 1 : Find the derivative of the function $y=f(x)$ using the derivative of the inverse function $x=f^{-1}(y)$ in the following

(i) $y=\sqrt[3]{x+4}$ (ii) $y=\sqrt{1+\sqrt{x}}$ (iii) $y=\ln x$

Solution :

(i) $y=\sqrt[3]{x+4}$

We first find the inverse of the function $y=f(x)$, i.e. x in term of y.

$y^3=x+4$ $\therefore$ $x=y^3-4$ $\therefore$ $x=f^{-1}(y)=y^3-4$

$$\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}=\frac{1}{\frac{d}{dy}\left(y^3-4\right)}=\frac{1}{3y^2}$$

$$=\frac{1}{3\left(\sqrt[3]{x+4}\right)^2}=\frac{1}{3\sqrt[3]{(x+4)^2}} \qquad \text{for } x\neq -4$$

(ii) $y = \sqrt{1+\sqrt{x}}$

We first find the inverse of the function $y = f(x)$, i.e. x in term of y.

$y^2 = 1 + \sqrt{x}$ i.e. $\sqrt{x} = y^2 - 1$, $\therefore\ x = f^{-1}(y) = (y^2 - 1)^2$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d}{dy}\left[(y^2-1)^2\right]} = \frac{1}{2(y^2-1)\frac{d}{dy}(y^2-1)}$$

$$= \frac{1}{2(y^2-1)(2y)} = \frac{1}{4\sqrt{1+\sqrt{x}}\left[\left(\sqrt{1+\sqrt{x}}\right)^2-1\right]}$$

$$= \frac{1}{4\sqrt{1+\sqrt{x}}\,(1+\sqrt{x}-1)} = \frac{1}{4\sqrt{x}\sqrt{1+\sqrt{x}}}$$

(iii) $y = \log x$

We first find the inverse of the function $y = f(x)$, i.e. x in term of y.

$y = \log x\ \therefore\ x = f^{-1}(y) = e^y$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\frac{d}{dy}(e^y)} = \frac{1}{e^y} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

Ex. 2 : Find the derivative of the inverse of function $y = 2x^3 - 6x$ and calculate its value at $x = -2$.

Solution : Given : $y = 2x^3 - 6x$

Diff. w. r. t. x we get,

$$\frac{dy}{dx} = 6x^2 - 6 = 6(x^2 - 1)$$

we have, $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

$$\therefore\ \frac{dx}{dy} = \frac{1}{6(x^2-1)}$$

at $x = -2$,

we get, $y = 2(-2)^3 - 6(-2)$

$= -16 + 12 = -4$

$$\left[\frac{dx}{dy}\right]_{y=-4} = \frac{1}{\left[\frac{dy}{dx}\right]_{x=-2}} = \frac{1}{6\left((-2)^2-1\right)} = \frac{1}{18}$$

Ex. 3 : Let f and g be the inverse functions of each other. The following table lists a few values of f, g and f'

x	$f(x)$	$g(x)$	$f'(x)$
-4	2	1	$\frac{1}{3}$
1	-4	-2	4

find $g'(-4)$.

Solution : In order to find $g'(-4)$, we should first find an expression for $g'(x)$ for any input x. Since f and g are inverses we can use the following identify which holds for any two diffetentiable inverse functions.

$g'(x) = \frac{1}{f'[g(x)]}$ ... [check, how?]

... $\left[\text{Hint} : f[g(x)] = x\right]$

$$\therefore\ g'(-4) = \frac{1}{f'\left[g(-4)\right]} = \frac{1}{f'(1)} = \frac{1}{4}$$

Ex. 4 : Let $f(x) = x^5 + 2x - 3$. Find $\left(f^{-1}\right)'(-3)$.

Solution : Given : $f(x) = x^5 + 2x - 3$

Diff. *w. r. t.* x we get,

$f'(x) = 5x^4 + 2$

Note that $y = -3$ corresponds to $x = 0$.

$$\therefore \quad \left(f^{-1}\right)'(-3) = \frac{1}{f'(0)}$$

$$= \frac{1}{5(0)+2} = \frac{1}{2}$$

1.2.4 Derivatives of Standard Inverse trigononmetric Functions :

We observe that inverse trigonometric functions are multi-valued functions and because of this, their derivatives depend on which branch of the function we are dealing with. We are not restricted to use these branches all the time. While solving the problems it is customary to select the branch of the inverse trigonometric function which is applicable to the kind of problem we are solving. We have to pay more attention towards the domain and range.

1. If $y = \sin^{-1} x$, $-1 \le x \le 1$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ then prove that $\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$, $|x| < 1$.

Proof : Given that $y = \sin^{-1} x$, $-1 \le x \le 1$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\therefore \quad x = \sin y \qquad \ldots. \text{(I)}$$

Differentiate *w. r. t.* y

$$\frac{dx}{dy} = \frac{d}{dy}(\sin y)$$

$$\frac{dx}{dy} = \cos y = \pm\sqrt{\cos^2 y} = \pm\sqrt{1-\sin^2 y}$$

$$\therefore \quad \frac{dx}{dy} = \pm\sqrt{1-x^2} \qquad \ldots. [\because \sin y = x]$$

But $\cos y$ is positive since y lies in 1st or 4th quadrant as $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\therefore \quad \frac{dx}{dy} = \sqrt{1-x^2}$$

We have $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, \ |x| < 1$$

2. If $y = \cos^{-1} x$, $-1 \le x \le 1$, $0 \le y \le \pi$ then prove that $\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$.

[As home work for students to prove.]

3. If $y = \cot^{-1} x, x \in R, 0 < y < \pi$ then $\dfrac{dy}{dx} = -\dfrac{1}{1+x^2}$.

Proof : Given that $y = \cot^{-1} x, x \in R, 0 < y < \pi$

$$\therefore \quad x = \cot y \qquad \ldots . (\text{I})$$

Differentiate *w. r. t. y*

$$\frac{dx}{dy} = \frac{d}{dy}(\cot y)$$

$$\frac{dx}{dy} = -\operatorname{cosec}^2 y = -(1 + \cot^2 y)$$

$$\therefore \quad \frac{dx}{dy} = -(1 + x^2) \qquad \ldots . [\because \cot y = x]$$

We have $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$

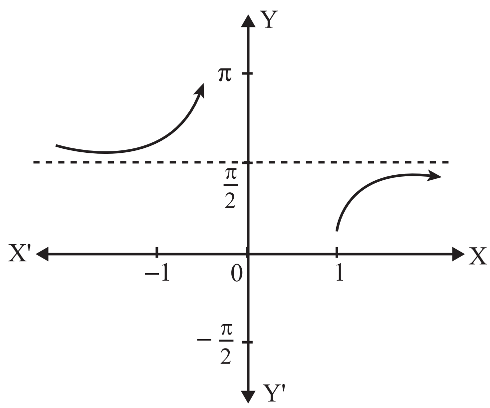
$$\therefore \quad \frac{dy}{dx} = \frac{1}{-(1+x^2)} \qquad \therefore \quad \frac{dy}{dx} = -\frac{1}{1+x^2}$$

4. If $y = \tan^{-1} x, x \in R, -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$ then $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$. [left as home work for students to prove.]

5. If $y = \sec^{-1} x$, such that $|x| \geq 1$ and $0 \leq y \leq \pi, y \neq \dfrac{\pi}{2}$ then $\dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}}$ if $x > 1$

$\dfrac{dy}{dx} = -\dfrac{1}{x\sqrt{x^2-1}}$ if $x < -1$

Proof : Given that $y = \sec^{-1} x, |x| \geq 1$ and $0 \leq y \leq \pi, y \neq \dfrac{\pi}{2}$



$y = \sec^{-1} x$

Fig. 1.2.2

$$\therefore \quad x = \sec y \qquad \ldots . (\text{I})$$

Differentiate *w. r. t. y*

$$\frac{dx}{dy} = \frac{d}{dy}(\sec y)$$

$$\frac{dx}{dy} = \sec y \cdot \tan y$$

$$\therefore \quad \frac{dx}{dy} = \pm \sec y \cdot \sqrt{\tan^2 y}$$

$$= \pm \sec y \cdot \sqrt{\sec^2 y - 1}$$

$$\therefore \quad \frac{dx}{dy} = \pm x \sqrt{x^2 - 1} \qquad \ldots \ldots [\because \sec y = x]$$

We use the sign $\pm$ because for y in 1st and 2nd quadrant. $\sec y \cdot \tan y > 0$.

Hence we choose $x \sqrt{x^2-1}$ if $x > 1$ and $-x \sqrt{x^2-1}$ if $x < -1$

In 1st quadrant both $\sec y$ and $\tan y$ are positive.

In 2nd quadrant both $\sec y$ and $\tan y$ are negative.

$\therefore$ $\sec y \cdot \tan y$ is positive in both first and second quadrant.

Also, for $x > 0$, $x\sqrt{x^2-1} > 0$

and for $x < 0$, $-x\sqrt{x^2-1} > 0$

$\frac{dx}{dy} = x\sqrt{x^2-1}$, when $x > 0$, $|x| > 1$ i.e. $x > 1$

$= -x\sqrt{x^2-1}$, when $x < 0$, $|x| > 1$ i.e. $x < -1$

$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$ if $x > 1$

$\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}}$ if $x < -1$

Note 1 : A function is increasing if its derivative is positive and is decreasing if its derivative is negative.

Note 2 : The derivative of $\sec^{-1} x$ is always positive because the graph of $\sec^{-1} x$ is always increasing.

6. If $y = -\operatorname{cosec} x$, such that $|x| \geq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, $y \neq 0$ then $\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2-1}}$ if $x > 1$

$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2-1}}$ if $x < -1$

[Left as home work for students to prove]

Note 3 : The derivative of $\operatorname{cosec}^{-1} x$ is always negative because the graph of $\operatorname{cosec}^{-1} x$ is always decreasing.

1.2.5 Derivatives of Standard Inverse trigonometric Functions :

y	$\frac{dy}{dx}$	Conditions
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$, $\lvert x\rvert < 1$	$-1 \leq x \leq 1$ $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$, $\lvert x\rvert < 1$	$-1 \leq x \leq 1$ $0 \leq y \leq \pi$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$x \in R$ $-\frac{\pi}{2} < y < \frac{\pi}{2}$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$	$x \in R$ $0 < y < \pi$

y	$\frac{dy}{dx}$	Conditions
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$ for $x > 1$ $-\frac{1}{x\sqrt{x^2-1}}$ for $x < -1$	$\lvert x\rvert \geq 1$ $0 \leq y \leq \pi$ $y \neq \frac{\pi}{2}$
$\operatorname{cosec}^{-1} x$	$-\frac{1}{x\sqrt{x^2-1}}$ for $x > 1$ $\frac{1}{x\sqrt{x^2-1}}$ for $x < -1$	$\lvert x\rvert \geq 1$ $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ $y \neq 0$

Table 1.2.1

1.2.6 Derivatives of Standard Inverse trigonometric Composite Functions :

y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
$\sin^{-1}[f(x)]$	$\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, \lvert f(x)\rvert < 1$	$\cot^{-1}[f(x)]$	$-\frac{f'(x)}{1+[f(x)]^2}$
$\cos^{-1}[f(x)]$	$-\frac{f'(x)}{\sqrt{1-[f(x)]^2}}, \lvert f(x)\rvert < 1$	$\sec^{-1}[f(x)]$	$\frac{f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$, for $\lvert f(x)\rvert > 1$
$\tan^{-1}[f(x)]$	$\frac{f'(x)}{1+[f(x)]^2}$	$\text{cosec}^{-1}[f(x)]$	$-\frac{f'(x)}{f(x)\sqrt{[f(x)]^2-1}}$, for $\lvert f(x)\rvert > 1$

Table 1.2.2

Some Important Formulae for Inverse Trigonometric Functions :

(1) $\sin^{-1}(\sin\theta)=\theta,\ \sin(\sin^{-1}x)=x$	(2) $\cos^{-1}(\cos\theta)=\theta,\ \cos(\cos^{-1}x)=x$	
(3) $\tan^{-1}(\tan\theta)=\theta,\ \tan(\tan^{-1}x)=x$	(4) $\cot^{-1}(\cot\theta)=\theta,\ \cot(\cot^{-1}x)=x$	
(5) $\sec^{-1}(\sec\theta)=\theta,\ \sec(\sec^{-1}x)=x$	(6) $\text{cosec}^{-1}(\text{cosec}\,\theta)=\theta,\ \text{cosec}(\text{cosec}^{-1}x)=x$	
(7) $\sin^{-1}(\cos\theta)=\sin^{-1}\left[\sin\left(\frac{\pi}{2}-\theta\right)\right]=\frac{\pi}{2}-\theta$	(8) $\cos^{-1}(\sin\theta)=\cos^{-1}\left[\cos\left(\frac{\pi}{2}-\theta\right)\right]=\frac{\pi}{2}-\theta$	
(9) $\tan^{-1}(\cot\theta)=\tan^{-1}\left[\tan\left(\frac{\pi}{2}-\theta\right)\right]=\frac{\pi}{2}-\theta$	(10) $\cot^{-1}(\tan\theta)=\cot^{-1}\left[\cot\left(\frac{\pi}{2}-\theta\right)\right]=\frac{\pi}{2}-\theta$	
(11) $\sec^{-1}(\text{cosec}\,\theta)=\sec^{-1}\left[\sec\left(\frac{\pi}{2}-\theta\right)\right]=\frac{\pi}{2}-\theta$		
(12) $\text{cosec}^{-1}(\sec\theta)=\text{cosec}^{-1}\left[\text{cosec}\left(\frac{\pi}{2}-\theta\right)\right]=\frac{\pi}{2}-\theta$		
(13) $\sin^{-1}(x)=\text{cosec}^{-1}\left(\frac{1}{x}\right)$	(14) $\text{cosec}^{-1}(x)=\sin^{-1}\left(\frac{1}{x}\right)$	
(15) $\cos^{-1}(x)=\sec^{-1}\left(\frac{1}{x}\right)$	(16) $\sec^{-1}(x)=\cos^{-1}\left(\frac{1}{x}\right)$	
(17) $\tan^{-1}(x)=\cot^{-1}\left(\frac{1}{x}\right)$	(18) $\cot^{-1}(x)=\tan^{-1}\left(\frac{1}{x}\right)$	
(19) $\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}$	(20) $\tan^{-1}x+\cot^{-1}x=\frac{\pi}{2}$	(21) $\sec^{-1}x+\text{cosec}^{-1}x=\frac{\pi}{2}$
(22) $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$	(23) $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$	

In above tables, x is a real variable with restrictions. **Table 1.2.3**

Some Important Substitutions :

Expression	Substitutions
$\sqrt{1-x^2}$	$x = \sin\theta$ or $x = \cos\theta$
$\sqrt{1-x^2}$	$x = \tan\theta$ or $x = \cot\theta$
$\sqrt{x^2+1}$	$x = \sec\theta$ or $x = \operatorname{cosec}\theta$
$\sqrt{\dfrac{a+x}{a-x}}$ or $\sqrt{\dfrac{a-x}{a+x}}$	$x = a\cos 2\theta$ or $x = a\cos\theta$
$\sqrt{\dfrac{1+x}{1-x}}$ or $\sqrt{\dfrac{1-x}{1+x}}$	$x = \cos 2\theta$ or $x = \cos\theta$
$\sqrt{\dfrac{a+x^2}{a-x^2}}$ or $\sqrt{\dfrac{a-x^2}{a+x^2}}$	$x^2 = a\cos 2\theta$ or $x^2 = a\cos\theta$

Expression	Substitutions
$\dfrac{2x}{1+x^2}$	$x = \tan\theta$
$\dfrac{1-x^2}{1+x^2}$	$x = \tan\theta$
$3x-4x^3$ or $1-2x^2$	$x = \sin\theta$
$4x^3-3x$ or $2x^2-1$	$x = \cos\theta$
$\dfrac{3x-x^3}{1-3x^2}$	$x = \tan\theta$
$\dfrac{2f(x)}{1+[f(x)]^2}$ or $\dfrac{1-[f(x)]^2}{1+[f(x)]^2}$	$f(x) = \tan\theta$

Table 1.2.4

SOLVED EXAMPLES

Ex. 1 : Using derivative prove that $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$.

Solution : Let $f(x) = \sin^{-1}x + \cos^{-1}x$ ..... (I)

We have to prove that $f(x) = \dfrac{\pi}{2}$

Differentiate (I) *w. r. t. x*

$$\frac{d}{dx}[f(x)] = \frac{d}{dx}[\sin^{-1}x + \cos^{-1}x]$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

$f'(x) = 0 \Rightarrow f(x)$ is a constant function.

Let $f(x) = c$. For any value of x, $f(x)$ must be c only. So conveniently we can choose $x = 0$,

$\therefore$ from (I) we get,

$$f(0) = \sin^{-1}(0) + \cos^{-1}(0) = 0 + \frac{\pi}{2} = \frac{\pi}{2} \Rightarrow c = \frac{\pi}{2} \therefore f(x) = \frac{\pi}{2}$$

Hence, $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$.

Ex. 2 : Differentiate the following *w. r. t. x*.

(i) $\sin^{-1}(x^3)$ (ii) $\cos^{-1}(2x^2 - x)$ (iii) $\sin^{-1}(2^x)$

(iv) $\cot^{-1}\left(\dfrac{1}{x^2}\right)$ (v) $\cos^{-1}\left(\sqrt{\dfrac{1+x}{2}}\right)$ (vi) $\sin^2(\sin^{-1}(x^2))$

Solution :

(i) Let $y = \sin^{-1}(x^3)$

Differentiate $w.r.t.x.$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}(x^3)\right)$$

$$= \frac{1}{\sqrt{1-(x^3)^2}} \cdot \frac{d}{dx}(x^3)$$

$$= \frac{1}{\sqrt{1-x^6}}(3x^2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{3x^2}{\sqrt{1-x^6}}$$

(iii) Let $y = \sin^{-1}(2^x)$

Differentiate $w.r.t.x.$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1}(2^x)\right)$$

$$= \frac{1}{\sqrt{1-(2^x)^2}} \cdot \frac{d}{dx}(2^x)$$

$$= \frac{1}{\sqrt{1-2^{2x}}}(2^x \log 2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{2^x \log 2}{\sqrt{1-4^x}}$$

(iv) Let $y = \cot^{-1}\left(\frac{1}{x^2}\right) = \tan^{-1}(x^2)$

Differentiate $w.r.t.x.$

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^{-1}(x^2))$$

$$= \frac{1}{1+(x^2)^2} \cdot \frac{d}{dx}(x^2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x}{1+x^4}$$

(vi) Let $y = \sin^2\left(\sin^{-1}(x^2)\right)$

$$= [\sin(\sin^{-1}(x^2))]^2 \quad = (x^2)^2$$

$$\therefore \quad y = x^4$$

Differentiate $w.r.t.x.$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) \qquad \therefore \frac{dy}{dx} = 4x^3$$

(ii) Let $y = \cos^{-1}(2x^2 - x)$

Hence $\cos y = 2x^2 - x$ . . . (I)

Differentiate $w.r.t.x.$

$$-\sin y \cdot \frac{dy}{dx} = 4x - 1$$

$$\frac{dy}{dx} = \frac{1-4x}{\sin y} \quad = \frac{1-4x}{\sqrt{1-\cos^2 y}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1-4x}{\sqrt{1-x^2(2x-1)^2}} \qquad \text{. . . from (I)}$$

Alternate Method :

If $y = \cos^{-1}(2x^2 - x)$

Differentiate $w.r.t.x.$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\cos^{-1}(2x^2-x)\right)$$

$$= \frac{-1}{\sqrt{1-(2x^2-x)^2}} \cdot \frac{d}{dx}(2x^2-x)$$

$$= \frac{-1}{\sqrt{1-x^2(2x-1)^2}} \cdot (4x-1)$$

$$\therefore \quad \frac{dy}{dx} = \frac{1-4x}{\sqrt{1-x^2(2x-1)^2}}$$

(v) Let $y = \cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)$

Differentiate $w.r.t.x.$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cos^{-1}\left(\sqrt{\frac{1+x}{2}}\right)\right]$$

$$= -\frac{1}{\sqrt{1-\left(\sqrt{\frac{1+x}{2}}\right)^2}} \cdot \frac{d}{dx}\left(\sqrt{\frac{1+x}{2}}\right)$$

$$= -\frac{1}{\sqrt{1-\frac{1+x}{2}}} \times \frac{1}{2\sqrt{\frac{1+x}{2}}} \times \frac{d}{dx}\left(\frac{1+x}{2}\right)$$

$$= -\frac{\sqrt{2}}{\sqrt{1-x}} \times \frac{1}{\sqrt{2}\sqrt{1+x}} \times \frac{1}{2}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2\sqrt{1-x^2}}$$

Ex. 3 : Differentiate the following *w. r. t. x*.

(i) $\cos^{-1}(4\cos^3 x - 3\cos x)$ (ii) $\cos^{-1}[\sin(4^x)]$ (iii) $\sin^{-1}\left(\sqrt{\dfrac{1-\cos x}{2}}\right)$

(iv) $\tan^{-1}\left(\dfrac{1-\cos 3x}{\sin 3x}\right)$ (v) $\cot^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$

Solution :

(i) Let $y = \cos^{-1}(4\cos^3 x - 3\cos x)$

$= \cos^{-1}(\cos 3x)$

$\therefore \quad y = 3x$

Differentiate *w. r. t. x*.

$$\frac{dy}{dx} = \frac{d}{dx}(3x)$$

$$\therefore \frac{dy}{dx} = 3$$

(ii) Let $y = \cos^{-1}[\sin(4^x)]$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 4^x\right)\right]$$

$$\therefore \quad y = \frac{\pi}{2} - 4^x$$

Differentiate *w. r. t. x*.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{2} - 4^x\right) = 0 - 4^x \log 4$$

$$\therefore \frac{dy}{dx} = -4^x \log 4$$

(iii) Let $y = \sin^{-1}\left(\sqrt{\dfrac{1-\cos x}{2}}\right)$

$$= \sin^{-1}\left(\sqrt{\frac{2\sin^2\left(\frac{x}{2}\right)}{2}}\right)$$

$$= \sin^{-1}\left[\sin\left(\frac{x}{2}\right)\right]$$

$$\therefore \quad y = \frac{x}{2}$$

Differentiate *w. r. t. x*.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{2}\right) \qquad \therefore \frac{dy}{dx} = \frac{1}{2}$$

(iv) Let $y = \tan^{-1}\left(\dfrac{1-\cos 3x}{\sin 3x}\right)$

$$= \tan^{-1}\left(\frac{2\sin^2\left(\frac{3x}{2}\right)}{2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)}\right)$$

$$= \tan^{-1}\left[\tan\left(\frac{3x}{2}\right)\right]$$

$$\therefore \quad y = \frac{3x}{2}$$

Differentiate *w. r. t. x*.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3x}{2}\right) \qquad \therefore \frac{dy}{dx} = \frac{3}{2}$$

(v) Let $y = \cot^{-1}\left(\dfrac{\cos x}{1+\sin x}\right) = \tan^{-1}\left(\dfrac{1+\sin x}{\cos x}\right)$

$$= \tan^{-1}\left(\frac{\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]^2}{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}\right) = \tan^{-1}\left(\frac{\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]^2}{\left[\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]}\right)$$

$$= \tan^{-1}\left(\frac{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}\right) = \tan^{-1}\left(\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right)$$ ....Divide Numerator & Denominator by $\cos\left(\dfrac{x}{2}\right)$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] \qquad \therefore y = \frac{\pi}{4}+\frac{x}{2}$$

Differentiate *w. r. t. x*.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}+\frac{x}{2}\right) = 0 + \frac{1}{2} \qquad \therefore \frac{dy}{dx} = \frac{1}{2}$$

Ex. 4 : Differentiate the following *w. r. t. x*.

(i) $\sin^{-1}\left(\dfrac{2\cos x + 3\sin x}{\sqrt{13}}\right)$ (ii) $\cos^{-1}\left(\dfrac{3\sin x^2 + 4\cos x^2}{5}\right)$ (iii) $\sin^{-1}\left(\dfrac{a\cos x - b\sin x}{\sqrt{a^2+b^2}}\right)$

Solution :

(i) Let $y = \sin^{-1}\left(\dfrac{2\cos x + 3\sin x}{\sqrt{13}}\right)$

$$= \sin^{-1}\left(\frac{2}{\sqrt{13}}\cos x + \frac{3}{\sqrt{13}}\cos x\right)$$

Put $\dfrac{2}{\sqrt{13}} = \sin\alpha,\ \dfrac{3}{\sqrt{13}} = \cos\alpha$

Also, $\sin^2\alpha + \cos^2\alpha = \dfrac{4}{3} + \dfrac{9}{13} = 1$

And $\tan\alpha = \dfrac{2}{3}\ \therefore\ \alpha = \tan^{-1}\left(\dfrac{2}{3}\right)$

$$y = \sin^{-1}(\sin\alpha\cos x + \cos\alpha\sin x)$$

$$y = \sin^{-1}(\sin x\cos\alpha + \cos x\sin\alpha)$$

$$y = \sin^{-1}[\sin(x+\alpha)]$$

$$\therefore\quad y = x + \tan^{-1}\left(\frac{2}{3}\right)$$

Differentiate *w. r. t. x*.

$$\frac{dy}{dx} = \frac{d}{dx}\left[x + \tan^{-1}\left(\frac{2}{3}\right)\right] = 1 + 0$$

$$\therefore\quad \frac{dy}{dx} = 1$$

(ii) Let $y = \cos^{-1}\left(\dfrac{3\sin x^2 + 4\cos x^2}{5}\right)$

$$= \cos^{-1}\left(\frac{3}{5}\sin x^2 + \frac{4}{5}\cos x^2\right)$$

Put $\dfrac{3}{5} = \sin\alpha,\ \dfrac{4}{5} = \cos\alpha$

Also, $\sin^2\alpha + \cos^2\alpha = \dfrac{9}{25} + \dfrac{16}{25} = 1$

And $\tan\alpha = \dfrac{3}{4}\ \therefore\ \alpha = \tan^{-1}\left(\dfrac{3}{4}\right)$

$$y = \cos^{-1}(\sin\alpha\sin x^2 + \cos\alpha\cos x^2)$$

$$y = \cos^{-1}(\cos x^2\cos\alpha + \sin x^2\sin\alpha)$$

$$y = \cos^{-1}[\cos(x^2 - \alpha)]$$

$$\therefore\quad y = x^2 - \tan^{-1}\left(\frac{3}{4}\right)$$

Differentiate *w. r. t. x*.

$$\frac{dy}{dx} = \frac{d}{dx}\left[x^2 - \tan^{-1}\left(\frac{3}{4}\right)\right] = 2x - 0$$

$$\therefore\quad \frac{dy}{dx} = 2x$$

(iii) Let $y = \sin^{-1}\left(\dfrac{a\cos x - b\sin x}{\sqrt{a^2+b^2}}\right) = \sin^{-1}\left(\dfrac{a}{\sqrt{a^2+b^2}}\cos x - \dfrac{b}{\sqrt{a^2+b^2}}\sin x\right)$

Put $\dfrac{a}{\sqrt{a^2+b^2}} = \sin\alpha,\ \dfrac{b}{\sqrt{a^2+b^2}} = \cos\alpha$

Also, $\sin^2\alpha + \cos^2\alpha = \dfrac{a^2}{a^2+b^2} + \dfrac{b^2}{a^2+b^2} = 1$ And $\tan\alpha = \dfrac{a}{b}\ \therefore\ \alpha = \tan^{-1}\left(\dfrac{a}{b}\right)$

$$y = \sin^{-1}(\sin\alpha\cos x - \cos\alpha\sin x)$$

But $\sin(\alpha - x) = \sin\alpha\cos x - \cos\alpha\sin x$

$$y = \sin^{-1}[\sin(\alpha - x)]$$

$$\therefore\quad y = \tan^{-1}\left(\frac{a}{b}\right) - x$$

Differentiate *w. r. t. x*.

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}\left(\frac{a}{b}\right) - x\right] = 0 - 1 \qquad \therefore\ \frac{dy}{dx} = -1$$

Ex. 5 : Differentiate the following *w. r. t. x*.

(i) $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ (ii) $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$ (iii) $\text{cosec}^{-1}\left(\dfrac{1}{3x-4x^3}\right)$

(iv) $\tan^{-1}\left(\dfrac{2e^x}{1-e^{2x}}\right)$ (v) $\cos^{-1}\left(\dfrac{1-9x^2}{1+9x^2}\right)$ (vi) $\cos^{-1}\left(\dfrac{2^x-2^{-x}}{2^x+2^{-x}}\right)$

(vii) $\tan^{-1}\left(\sqrt{\dfrac{3-x}{3+x}}\right)$ (viii) $\sin^{-1}\left(\dfrac{5\sqrt{1-x^2}-12x}{13}\right)$ (ix) $\sin^{-1}\left(\dfrac{2^{x+1}}{1+4^x}\right)$

Solution :

(i) Let $y = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

Put $x = \tan\theta \ \therefore\ \theta = \tan^{-1}x$

$\therefore \quad y = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right)$

$y = \sin^{-1}(\sin 2\theta) = 2\theta$

$\therefore \quad y = 2\tan^{-1}x$

Differentiate *w. r. t. x.*

$\dfrac{dy}{dx} = 2\dfrac{d}{dx}(\tan^{-1}x)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{2}{1+x^2}$

(ii) Let $y = \cos^{-1}\left(2x\sqrt{1-x^2}\right)$

Put $x = \sin\theta \ \therefore\ \theta = \sin^{-1}x$

$\therefore \quad y = \cos^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$y = \cos^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$

$y = \cos^{-1}(2\sin\theta\cos\theta) = \cos^{-1}(\sin 2\theta)$

$y = \cos^{-1}\left[\cos\left(\dfrac{\pi}{2}-2\theta\right)\right] = \dfrac{\pi}{2}-2\theta$

$\therefore \quad y = \dfrac{\pi}{2} - 2\sin^{-1}x$

Differentiate *w. r. t. x.*

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\pi}{2} - 2\sin^{-1}x\right)$

$\dfrac{dy}{dx} = 0 - \dfrac{2\times 1}{\sqrt{1-x^2}}$

$\therefore \quad \dfrac{dy}{dx} = -\dfrac{2}{\sqrt{1-x^2}}$

(iii) Let $y = \text{cosec}^{-1}\left(\dfrac{1}{3x-4x^3}\right)$

$y = \sin^{-1}(3x-4x^3)$

Put $x = \sin\theta \ \therefore\ \theta = \sin^{-1}x$

$y = \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$y = \sin^{-1}(\sin 3\theta) = 3\theta$

$\therefore \quad y = 3\sin^{-1}x$

Differentiate *w. r. t. x.*

$\dfrac{dy}{dx} = 3\dfrac{d}{dx}(\sin^{-1}x)$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{3}{\sqrt{1-x^2}}$

(iv) Let $y = \tan^{-1}\left(\dfrac{2e^x}{1-e^{2x}}\right)$

Put $e^x = \tan\theta \ \therefore\ \theta = \tan^{-1}(e^x)$

$y = \tan^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right)$

$y = \tan^{-1}(\tan 2\theta) = 2\theta$

$\therefore \quad y = 2\tan^{-1}(e^x)$

Differentiate *w. r. t. x.*

$\dfrac{dy}{dx} = 2\dfrac{d}{dx}\left[\tan^{-1}(e^x)\right]$

$\therefore \quad \dfrac{dy}{dx} = \dfrac{2}{1+(e^x)^2}\dfrac{d}{dx}(e^x) = \dfrac{2e^x}{1-e^{2x}}$

(v) Let $y = \cos^{-1}\left(\dfrac{1-9x^2}{1+9x^2}\right)$

$$y = \cos^{-1}\left(\frac{1-(3x)^2}{1+(3x)^2}\right)$$

Put $3x = \tan\theta \therefore \theta = \tan^{-1}(3x)$

$$\therefore \quad y = \cos^{-1}\left[\frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$$

$$y = \cos^{-1}(\cos 2\theta) = 2\theta$$

$$y = 2\tan^{-1}(3x)$$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = 2\frac{d}{dx}\left[\tan^{-1}(3x)\right]$$

$$\frac{dy}{dx} = \frac{2}{1+(3x)^2}\frac{d}{dx}(3x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{6}{1+9x^2}$$

(vi) Let $y = \cos^{-1}\left(\dfrac{2^x-2^{-x}}{2^x+2^{-x}}\right)$

$$y = \cos^{-1}\left(\frac{2^x(2^x-2^{-x})}{2^x(2^x+2^{-x})}\right) \quad \text{. . . [Multiply \& Devide by } 2^x]$$

$$y = \cos^{-1}\left(\frac{2^{2x}-1}{2^{2x}+1}\right) = \cos^{-1}\left[-\frac{1-(2^x)^2}{1+(2^x)^2}\right]$$

Put $2^x = \tan\theta \therefore \theta = \tan^{-1}(2^x)$

$$\therefore \quad y = \cos^{-1}\left[-\frac{1-\tan^2\theta}{1+\tan^2\theta}\right] = \cos^{-1}[-\cos 2\theta]$$

$$y = \cos^{-1}[\cos(\pi - 2\theta)] = \pi - 2\theta$$

$$y = \pi - 2\tan^{-1}(2^x)$$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = \frac{d}{dx}\left[\pi - 2\tan^{-1}(2^x)\right]$$

$$\frac{dy}{dx} = 0 - \frac{2}{1+(2^x)^2}\frac{d}{dx}(2^x) = -\frac{2\cdot 2^x\cdot \log 2}{1+2^{2x}}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{2^{x+1}\log 2}{1+2^{2x}}$$

(vii) Let $y = \tan^{-1}\left(\sqrt{\dfrac{3-x}{3+x}}\right)$

Put $x = 3\cos 2\theta \therefore \theta = \dfrac{1}{2}\cos^{-1}\left(\dfrac{x}{3}\right)$

$$\therefore \quad y = \tan^{-1}\left[\sqrt{\frac{3-3\cos 2\theta}{3+3\cos 2\theta}}\right] = \tan^{-1}\left[\sqrt{\frac{3(1-\cos 2\theta)}{3(1+\cos 2\theta)}}\right] = \tan^{-1}\left[\sqrt{\frac{2\sin^2\theta}{2\cos^2\theta}}\right]$$

$$y = \tan^{-1}\left(\sqrt{\tan^2\theta}\right) = \tan^{-1}(\tan\theta)$$

$$y = \theta = \frac{1}{2}\cos^{-1}\left(\frac{x}{3}\right)$$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{d}{dx}\left[\cos^{-1}\left(\frac{x}{3}\right)\right]$$

$$= \frac{1}{2}\left[-\frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}}\right]\frac{d}{dx}\left(\frac{x}{3}\right) = -\frac{1}{2}\times\frac{1}{\sqrt{\frac{9-x^2}{9}}}\times\frac{1}{3}$$

$$= -\frac{1}{2}\times\frac{1}{\frac{\sqrt{9-x^2}}{3}}\times\frac{1}{3}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{2\sqrt{9-x^2}}$$

(viii) Let $y = \sin^{-1}\left(\dfrac{5\sqrt{1-x^2}-12x}{13}\right)$

Put $x = \sin\theta \therefore \theta = \sin^{-1}x$

$$\therefore \quad y = \sin^{-1}\left(\frac{5\sqrt{1-\sin^2\theta}-12\sin\theta}{13}\right) = \sin^{-1}\left(\frac{5\sqrt{\cos^2\theta}-12\sin\theta}{13}\right) = \sin^{-1}\left(\frac{5\cos\theta-12\sin\theta}{13}\right)$$

$$\therefore \quad y = \sin^{-1}\left(\frac{5}{13}\cos\theta - \frac{12}{13}\sin\theta\right)$$

Put $\dfrac{5}{13} = \sin\alpha, \dfrac{12}{13} = \cos\alpha$

Also, $\sin^2\alpha + \cos^2\alpha = \dfrac{25}{169} + \dfrac{144}{169} = 1$

And $\tan\alpha = \dfrac{5}{12} \therefore \alpha = \tan^{-1}\left(\dfrac{5}{12}\right)$

$$y = \sin^{-1}(\sin\alpha\cos\theta - \cos\alpha\sin\theta) = \sin^{-1}[\sin(\alpha-\theta)] = (\alpha-\theta)$$

$$\therefore \quad y = \tan^{-1}\left(\frac{5}{12}\right) - \sin^{-1}x$$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}\left(\frac{5}{12}\right) - \sin^{-1}x\right] = 0 - \frac{1}{\sqrt{1-x^2}}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

(ix) Let $y = \sin^{-1}\left(\dfrac{2^{x+1}}{1+4^x}\right) = \sin^{-1}\left(\dfrac{2\cdot 2^x}{1+(2^x)^2}\right)$

Put $2^x = \tan\theta \therefore \theta = \tan^{-1}(2^x)$

$$\therefore \quad y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1}(2^x)$$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = 2\frac{d}{dx}\left[\tan^{-1}(2^x)\right] = \frac{2}{1+(2^x)^2}\cdot\frac{d}{dx}(2^x) = \frac{2}{1+2^{2x}}(2^x\cdot\log 2)$$

$$\therefore \quad \frac{dy}{dx} = -\frac{2^{x+1}\log 2}{1+4^x}$$

Ex. 6 : Differentiate the following *w. r. t. x.*

(i) $\tan^{-1}\left(\dfrac{4x}{1+21x^2}\right)$ (ii) $\tan^{-1}\left(\dfrac{7x}{1-12x^2}\right)$

(iii) $\cot^{-1}\left(\dfrac{b\sin x - a\cos x}{a\sin x + b\cos x}\right)$ (iv) $\tan^{-1}\left(\dfrac{5x+1}{3-x-6x^2}\right)$

Solution :

(i) Let $y = \tan^{-1}\left(\dfrac{4x}{1+21x^2}\right)$

$= \tan^{-1}\left(\dfrac{7x-3x}{1+(7x)(3x)}\right)$

$y = \tan^{-1}(7x) - \tan^{-1}(3x)$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(7x) - \tan^{-1}(3x)]$$

$$= \frac{d}{dx}[\tan^{-1}(7x)] - \frac{d}{dx}[\tan^{-1}(3x)]$$

$$= \frac{1}{1+(7x)^2}\cdot\frac{d}{dx}(7x) - \frac{1}{1+(3x)^2}\cdot\frac{d}{dx}(3x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{7}{1+49x^2} - \frac{3}{1+9x^2}$$

(ii) Let $y = \tan^{-1}\left(\dfrac{7x}{1-12x^2}\right)$

$= \tan^{-1}\left(\dfrac{3x+4x}{1-(3x)(4x)}\right)$

$y = \tan^{-1}(3x) + \tan^{-1}(4x)$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(3x) + \tan^{-1}(4x)]$$

$$= \frac{d}{dx}[\tan^{-1}(3x)] + \frac{d}{dx}[\tan^{-1}(4x)]$$

$$= \frac{1}{1+(3x)^2}\cdot\frac{d}{dx}(3x) + \frac{1}{1+(4x)^2}\cdot\frac{d}{dx}(4x)$$

$$\therefore \quad \frac{dy}{dx} = \frac{3}{1+9x^2} + \frac{4}{1+16x^2}$$

(iii) Let $y = \cot^{-1}\left(\dfrac{b\sin x - a\cos x}{a\sin x + b\cos x}\right) = \tan^{-1}\left(\dfrac{a\sin x + b\cos x}{b\sin x - a\cos x}\right) = \tan^{-1}\left(\dfrac{\frac{a}{b} + \cot x}{1 - \left(\frac{a}{b}\right)(\cot x)}\right)$

$= \tan^{-1}\left(\dfrac{a}{b}\right) + \tan^{-1}(\cot x) = \tan^{-1}\left(\dfrac{a}{b}\right) + \tan^{-1}\left[\tan\left(\dfrac{\pi}{2} - x\right)\right]$

$y = \tan^{-1}\left(\dfrac{a}{b}\right) + \dfrac{\pi}{2} - x$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = \frac{d}{dx}\left[\tan^{-1}\left(\frac{a}{b}\right) + \frac{\pi}{2} - x\right]$$

$$= \frac{d}{dx}\left[\tan^{-1}\left(\frac{a}{b}\right)\right] + \frac{d}{dx}\left(\frac{\pi}{2}\right) - \frac{d}{dx}(x)$$

$$= 0 + 0 - 1$$

$$\therefore \quad \frac{dy}{dx} = -1$$

(iv) Let $y = \tan^{-1}\left(\dfrac{5x+1}{3-x-6x^2}\right) = \tan^{-1}\left(\dfrac{5x+1}{1+2-x-6x^2}\right) = \tan^{-1}\left(\dfrac{5x+1}{1-(6x^2+x-2)}\right)$

$= \tan^{-1}\left(\dfrac{5x+1}{1-(6x^2+4x-3x-2)}\right) = \tan^{-1}\left(\dfrac{5x+1}{1-[2x(3x+2)-(3x+2)]}\right)$

$= \tan^{-1}\left(\dfrac{5x+1}{1-(3x+2)(2x-1)}\right) = \tan^{-1}\left(\dfrac{(3x+2)+(2x-1)}{1-(3x+2)(2x-1)}\right)$

$y = \tan^{-1}(3x+2) + \tan^{-1}(2x-1)$

Differentiate *w. r. t. x.*

$$\frac{dy}{dx} = \frac{d}{dx}[\tan^{-1}(3x+2) + \tan^{-1}(2x-1)]$$

$$= \frac{d}{dx}[\tan^{-1}(3x+2)] + \frac{d}{dx}[\tan^{-1}(2x-1)]$$

$$= \frac{1}{1+(3x+2)^2} \cdot \frac{d}{dx}(3x+2) + \frac{1}{1+(2x-1)^2} \cdot \frac{d}{dx}(2x-1)$$

$$\therefore \frac{dy}{dx} = \frac{3}{1+(3x+2)^2} + \frac{2}{1+(2x-1)^2}$$

EXERCISE 1.2

(1) Find the derivative of the function $y = f(x)$ using the derivative of the inverse function $x = f^{-1}(y)$ in the following

(i) $y = \sqrt{x}$ (ii) $y = \sqrt{2-\sqrt{x}}$

(iii) $y = \sqrt[3]{x-2}$ (iv) $y = \log(2x-1)$

(v) $y = 2x+3$ (vi) $y = e^x - 3$

(vii) $y = e^{2x-3}$ (viii) $y = \log_2\left(\frac{x}{2}\right)$

(2) Find the derivative of the inverse function of the following

(i) $y = x^2 \cdot e^x$ (ii) $y = x\cos x$

(iii) $y = x \cdot 7^x$ (iv) $y = x^2 + \log x$

(v) $y = x\log x$

(3) Find the derivative of the inverse of the following functions, and also find their value at the points indicated against them.

(i) $y = x^5 + 2x^3 + 3x$, at $x = 1$

(ii) $y = e^x + 3x + 2$, at $x = 0$

(iii) $y = 3x^2 + 2\log x^3$, at $x = 1$

(iv) $y = \sin(x-2) + x^2$, at $x = 2$

(4) If $f(x) = x^3 + x - 2$, find $(f^{-1})'(0)$.

(5) Using derivative prove

(i) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

(ii) $\sec^{-1}x + \mathrm{cosec}^{-1}x = \frac{\pi}{2} \ldots [\text{for } |x| \geq 1]$

(6) Differentiate the following *w. r. t. x.*

(i) $\tan^{-1}(\log x)$ (ii) $\mathrm{cosec}^{-1}(e^{-x})$

(iii) $\cot^{-1}(x^3)$ (iv) $\cot^{-1}(4^x)$

(v) $\tan^{-1}(\sqrt{x})$ (vi) $\sin^{-1}\left(\sqrt{\frac{1+x^2}{2}}\right)$

(vii) $\cos^{-1}(1-x^2)$ (viii) $\sin^{-1}\left(x^{\frac{3}{2}}\right)$

(ix) $\cos^3[\cos^{-1}(x^3)]$ (x) $\sin^4[\sin^{-1}(\sqrt{x})]$

(7) Differentiate the following *w. r. t. x.*

(i) $\cot^{-1}[\cot(e^{x^2})]$

(ii) $\mathrm{cosec}^{-1}\left(\frac{1}{\cos(5^x)}\right)$

(iii) $\cos^{-1}\left(\sqrt{\frac{1+\cos x}{2}}\right)$

(iv) $\cos^{-1}\left(\sqrt{\frac{1-\cos(x^2)}{2}}\right)$

(v) $\tan^{-1}\left(\frac{1-\tan\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\right)$

(vi) $\mathrm{cosec}^{-1}\left(\frac{1}{4\cos^3 2x - 3\cos 2x}\right)$

(vii) $\tan^{-1}\left(\frac{1+\cos\left(\frac{x}{3}\right)}{\sin\left(\frac{x}{3}\right)}\right)$

(viii) $\cot^{-1}\left(\frac{\sin 3x}{1+\cos 3x}\right)$

(ix) $\tan^{-1}\left(\dfrac{\cos 7x}{1+\sin 7x}\right)$

(x) $\tan^{-1}\left(\sqrt{\dfrac{1+\cos x}{1-\cos x}}\right)$

(xi) $\tan^{-1}(\operatorname{cosec} x + \cot x)$

(xii) $\cot^{-1}\left(\dfrac{\sqrt{1+\sin\left(\frac{4x}{3}\right)}+\sqrt{1-\sin\left(\frac{4x}{3}\right)}}{\sqrt{1+\sin\left(\frac{4x}{3}\right)}-\sqrt{1-\sin\left(\frac{4x}{3}\right)}}\right)$

(8) Differentiate the following *w. r. t. x*.

(i) $\sin^{-1}\left(\dfrac{4\sin x+5\cos x}{\sqrt{41}}\right)$

(ii) $\cos^{-1}\left(\dfrac{\sqrt{3}\cos x-\sin x}{2}\right)$

(iii) $\sin^{-1}\left(\dfrac{\cos\sqrt{x}+\sin\sqrt{x}}{\sqrt{2}}\right)$

(iv) $\cos^{-1}\left(\dfrac{3\cos 3x-4\sin 3x}{5}\right)$

(v) $\cos^{-1}\left(\dfrac{3\cos(e^x)+2\sin(e^x)}{\sqrt{13}}\right)$

(vi) $\operatorname{cosec}^{-1}\left(\dfrac{10}{6\sin(2^x)-8\cos(2^x)}\right)$

(9) Differentiate the following *w. r. t. x*.

(i) $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ (ii) $\tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$

(iii) $\sin^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ (iv) $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

(v) $\cos^{-1}(3x-4x^3)$ (vi) $\cos^{-1}\left(\dfrac{e^x-e^{-x}}{e^x+e^{-x}}\right)$

(vii) $\cos^{-1}\left(\dfrac{1-9^x}{1+9^x}\right)$ (viii) $\sin^{-1}\left(\dfrac{4^{x+\frac{1}{2}}}{1+2^{4x}}\right)$

(ix) $\sin^{-1}\left(\dfrac{1-25x^2}{1+25x^2}\right)$ (x) $\sin^{-1}\left(\dfrac{1-x^3}{1+x^3}\right)$

(xi) $\tan^{-1}\left(\dfrac{2x^{\frac{5}{2}}}{1-x^5}\right)$ (xii) $\cot^{-1}\left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)$

(10) Differentiate the following *w. r. t. x*.

(i) $\tan^{-1}\left(\dfrac{8x}{1-15x^2}\right)$ (ii) $\cot^{-1}\left(\dfrac{1+35x^2}{2x}\right)$

(iii) $\tan^{-1}\left(\dfrac{2\sqrt{x}}{1+3x}\right)$ (iv) $\tan^{-1}\left(\dfrac{2^{x+2}}{1-3(4^x)}\right)$

(v) $\tan^{-1}\left(\dfrac{2^x}{1+2^{2x+1}}\right)$ (vi) $\cot^{-1}\left(\dfrac{a^2-6x^2}{5ax}\right)$

(vii) $\tan^{-1}\left(\dfrac{a+b\tan x}{b-a\tan x}\right)$

(viii) $\tan^{-1}\left(\dfrac{5-x}{6x^2-5x-3}\right)$

(ix) $\cot^{-1}\left(\dfrac{4-x-2x^2}{3x+2}\right)$

1.3.1 Logarithmic Differentiation

The complicated functions given by formulas that involve products, quotients and powers can often be simplified more quickly by taking the natural logarithms on both the sides. This enables us to use the laws of logarithms to simplify the functions and differentiate easily. Especially when the functions are of the form $y = [f(x)]^{g(x)}$ it is recommended to take logarithms on both the sides which simplifies to $\log y = g(x).\log[f(x)]$, now it becomes convenient to find the derivative. This process of finding the derivative is called logarithmic differentiation.

SOLVED EXAMPLES

Ex. 1 : Differentiate the following *w. r. t. x*.

(i) $\left(\dfrac{(x^2+3)^2\sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}\right)$ (ii) $\dfrac{e^{x^2}(\tan x)^{\frac{x}{2}}}{(1+x^2)^{\frac{3}{2}}\cos^3 x}$

(iii) $(x+1)^{\frac{3}{2}}(2x+3)^{\frac{5}{2}}(3x+4)^{\frac{2}{3}}$ for $x \geq 0$ (iv) $x^a + x^x + a^x$ (v) $(\sin x)^{\tan x} - x^{\log x}$

Solution :

(i) Let $y = \left(\dfrac{(x^2+3)^2 \sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}\right)$

Taking log of both the sides we get,

$$\log y = \log\left(\frac{(x^2+3)^2 \sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}\right) = \log\left[\frac{(x^2+3)^2 (x^3+5)^{\frac{2}{3}}}{(2x^2+1)^{\frac{3}{2}}}\right]$$

$$= \log\left[(x^2+3)^2 (x^3+5)^{\frac{2}{3}}\right] - \log(2x^2+1)^{\frac{3}{2}}$$

$$= \log(x^2+3)^2 + \log(x^3+5)^{\frac{2}{3}}] - \log(2x^2+1)^{\frac{3}{2}}$$

$$\log y = 2\log(x^2+3) + \frac{2}{3}\log(x^3+5) - \frac{3}{2}\log(2x^2+1)$$

Differentiate *w. r. t. x.*

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\left[2\log(x^2+3) + \frac{2}{3}\log(x^3+5) - \frac{3}{2}\log(2x^2+1)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = 2\cdot\frac{d}{dx}[\log(x^2+3)] + \frac{2}{3}\cdot\frac{d}{dx}[\log(x^3+5)] - \frac{3}{2}\cdot\frac{d}{dx}[\log(2x^2+1)]$$

$$= \frac{2}{x^2+3}\cdot\frac{d}{dx}(x^2+3) + \frac{2}{3(x^3+5)}\cdot\frac{d}{dx}(x^3+5) - \frac{3}{2(2x^2+1)}\cdot\frac{d}{dx}(2x^2+1)$$

$$\frac{dy}{dx} = y\left[\frac{2}{x^2+3}(2x) + \frac{2}{3(x^3+5)}(3x^2) - \frac{3}{2(2x^2+1)}(4x)\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{(x^2+3)^2 \sqrt[3]{(x^3+5)^2}}{\sqrt{(2x^2+1)^3}}\left[\frac{4x}{x^2+1} + \frac{2x^2}{(x^3+5)} - \frac{6x}{2x^2+1}\right]$$

(ii) Let $y = \dfrac{e^{x^2}(\tan x)^{\frac{x}{2}}}{(1+x^2)^{\frac{3}{2}}\cos^3 x}$

Taking log of both the sides we get,

$$\log y = \log\left(\frac{e^{x^2}(\tan x)^{\frac{x}{2}}}{(1+x^2)^{\frac{3}{2}}(\cos x)^3}\right) = \log\left[e^{x^2}(\tan x)^{\frac{x}{2}}\right] - \log\left[(1+x^2)^{\frac{3}{2}}(\cos x)^3\right]$$

$$= \log e^{x^2} + \log(\tan x)^{\frac{x}{2}} - \left[\log(1+x^2)^{\frac{3}{2}} + \log(\cos x)^3\right]$$

$$= x^2\log e + \frac{x}{2}\log(\tan x) - \frac{3}{2}\log(1+x^2) - 3\log(\cos x)$$

$$\therefore \quad \log y = x^2 + \frac{x}{2}\log(\tan x) - \frac{3}{2}\log(1+x^2) - 3\log(\cos x)$$

Differentiate $w.\ r.\ t.\ x.$

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\left[x^2 + \frac{x}{2}\log(\tan x) - \frac{3}{2}\log(1+x^2) - 3\log(\cos x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x^2) + \frac{x}{2}\cdot\frac{d}{dx}[\log(\tan x)] + \log(\tan x)\frac{d}{dx}\left(\frac{x}{2}\right) - \frac{3}{2}\frac{d}{dx}[\log(1+x^2)] - 3\frac{d}{dx}[\log(\cos x)]$$

$$= 2x + \frac{x}{2}\cdot\frac{1}{\tan x}\cdot\frac{d}{dx}(\tan x) + \log(\tan x)\frac{1}{2} - \frac{3}{2(1+x^2)}\cdot\frac{d}{dx}(1+x^2) - \frac{3}{\cos x}\cdot\frac{d}{dx}(\cos x)$$

$$= 2x + \frac{x}{2}\cdot(\cot x)(\sec^2 x) + \frac{1}{2}\log(\tan x) - \frac{3}{2(1+x^2)}\cdot(2x) - \frac{3}{\cos x}(-\sin x)$$

$$= 2x + \frac{x}{2}\times\frac{\cos x}{\sin x}\times\frac{1}{\cos^2 x} + \frac{1}{2}\log(\tan x) - \frac{3x}{1+x^2} + 3\tan x$$

$$\frac{dy}{dx} = y\left[2x + \frac{x}{2\sin x\cos x} + \frac{1}{2}\log(\tan x) - \frac{3x}{1+x^2} + 3\tan x\right]$$

$$\therefore \quad \frac{dy}{dx} = \frac{e^{x^2}(\tan x)^{\frac{x}{2}}}{(1+x^2)^{\frac{3}{2}}\cos^3 x}\left[2x + x\operatorname{cosec} 2x + \frac{1}{2}\log(\tan x) - \frac{3x}{1+x^2} + 3\tan x\right]$$

(iii) Let $y = (x+1)^{\frac{3}{2}}(2x+3)^{\frac{5}{2}}(3x+4)^{\frac{2}{3}}$

Taking log of both the sides we get,

$$\log y = \log\left[(x+1)^{\frac{3}{2}}(2x+3)^{\frac{5}{2}}(3x+4)^{\frac{2}{3}}\right]$$

$$= \log(x+1)^{\frac{3}{2}} + \log(2x+3)^{\frac{5}{2}} + \log(3x+4)^{\frac{2}{3}}$$

$$\log y = \frac{3}{2}\log(x+1) + \frac{5}{2}\log(2x+3) + \frac{2}{3}\log(3x+4)$$

Differentiate $w.\ r.\ t.\ x.$

$$\frac{d}{dx}(\log y) = \frac{d}{dx}\left[\frac{3}{2}\log(x+1) + \frac{5}{2}\log(2x+3) + \frac{2}{3}\log(3x+4)\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{3}{2}\cdot\frac{d}{dx}[\log(2x+1)] + \frac{5}{2}\cdot\frac{d}{dx}[\log(3x+2)] + \frac{2}{3}\cdot\frac{d}{dx}[\log(3x+4)]$$

$$= \frac{3}{2(2x+1)}\cdot\frac{d}{dx}(2x+1) + \frac{5}{2(3x+1)}\cdot\frac{d}{dx}(3x+2) + \frac{2}{3(3x+4)}\cdot\frac{d}{dx}(3x+4)$$

$$\frac{dy}{dx} = y\left[\frac{3}{2(2x+1)}(2) + \frac{5}{2(3x+1)}(3) + \frac{2}{3(3x+4)}(3)\right]$$

$$\therefore \quad \frac{dy}{dx} = (x+1)^{\frac{3}{2}}(2x+3)^{\frac{5}{2}}(3x+4)^{\frac{2}{3}}\left[\frac{3}{2x+1} + \frac{15}{2(3x+1)} + \frac{2}{3x+4}\right]$$

(iv) Let $y = x^a + x^x + a^x$

Here the derivatives of x^a and a^x can be found directly but we can not find the derivative of x^x without the use of logarithm. So the given function is split in to two functions, find their derivatives and then add them.

Let $u = x^a + a^x$ and $v = x^x$

$\therefore \quad y = u + v$, where u and v are differentiable functions of x.

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{.....(I)}$$

Now, $u = x^a + a^x$

Differentiate $w.\ r.\ t.\ x.$

$$\frac{du}{dx} = \frac{d}{dx}(x^a) + \frac{d}{dx}(a^x)$$

$$\frac{du}{dx} = ax^{a-1} + a^x \log a \qquad \text{.....(II)}$$

And, $v = x^x$

Taking log of both the sides we get,

$$\log v = \log x^x$$

$$\log v = x \log x$$

Differentiate $w.\ r.\ t.\ x.$

$$\frac{d}{dx}(\log v) = \frac{d}{dx}(x \log x)$$

$$\frac{1}{v}\frac{dv}{dx} = x\frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x)$$

$$\frac{dv}{dx} = v\left[x \times \frac{1}{x} + \log x\,(1)\right]$$

$$\frac{dv}{dx} = x^x\,[1 + \log x] \qquad \text{.....(III)}$$

Substituting (II) and (III) in (I) we get,

$$\frac{dy}{dx} = ax^{a-1} + a^x \log a + x^x\,[1 + \log x]$$

(v) Let $y = (\sin x)^{\tan x} - x^{\log x}$

Let $u = (\sin x)^{\tan x}$ and $v = x^{\log x}$

$\therefore \quad y = u - v$, where u and v are differentiable functions of x.

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} \qquad \text{.....(I)}$$

Now, $u = (\sin x)^{\tan x}$, taking log of both the sides we get,

$\log u = \log (\sin x)^{\tan x} \qquad \therefore \qquad \log u = \tan x \log (\sin x)$

Differentiate $w.\ r.\ t.\ x.$

$$\frac{d}{dx}(\log u) = \frac{d}{dx}[\tan x \log (\sin x)]$$

$$\frac{1}{u}\frac{du}{dx} = \tan x \frac{d}{dx}[\log (\sin x)] + \log (\sin x) \frac{d}{dx}(\tan x)$$

$$= \tan x \cdot \frac{1}{\sin x} \cdot \frac{d}{dx} \cdot (\sin x) + \log (\sin x) \cdot (\sec^2)$$

$$\frac{du}{dx} = u\left[\tan x \cdot \frac{1}{\sin x} \cdot (\cos x) + \sec^2 x \cdot \log(\sin x)\right]$$

$$\frac{du}{dx} = (\sin x)^{\tan x}[\tan x \cdot \cot x + \sec^2 x \cdot \log(\sin x)]$$

$$\frac{du}{dx} = (\sin x)^{\tan x}[1 + \sec^2 x \cdot \log(\sin x)] \quad \ldots\ldots (\text{II})$$

And, $\quad v = x^{\log x}$

Taking log on both the sides we get,

$$\log v = \log(x^{\log x})$$

$$\log v = \log x \log x = (\log x)^2$$

Differentiate *w. r. t. x.*

$$\frac{d}{dx}(\log v) = \frac{d}{dx}[(\log x)^2]$$

$$\frac{1}{v}\frac{dv}{dx} = 2\log x \frac{d}{dx}(\log x)$$

$$\frac{dv}{dx} = u\left[\frac{2\log x}{x}\right] = \frac{2x^{\log x}\log x}{x} \quad \ldots\ldots (\text{III})$$

Substituting (II) and (III) in (I) we get,

$$\frac{dy}{dx} = (\sin x)^{\tan x}[1 + \sec^2 x \cdot \log(\sin x)] - \frac{2x^{\log x}\log x}{x}$$

1.3.2 Implicit Functions

Functions can be represented in a variety of ways. Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y solely in terms of the variable x. It is not always possible to solve for one variable explicitly in terms of another. Those cases where it is possible to solve for one variable in terms of another to obtain $y = f(x)$ or $x = g(y)$ are said to be in **explicit** form.

If an equation in x and y is given but x is not an explicit function of y and y is not an explicit function of x then either of the variables is an **Implicit function** of the other.

1.3.3 Derivatives of Implicit Functions

1. Differentiate both sides of the equation with respect to x (independent variable), treating y as a differentiable function of x.

2. Collect the terms containing $\frac{dy}{dx}$ on one side of the equation and solve for $\frac{dy}{dx}$.

SOLVED EXAMPLES

Ex. 1 : Find $\frac{dy}{dx}$ if

(i) $x^5 + xy^3 + x^2y + y^4 = 4$

(ii) $y^3 + \cos(xy) = x^2 - \sin(x + y)$

(iii) $x^2 + e^{xy} = y^2 + \log(x + y)$

Solution :

(i) Given that : $x^5 + xy^3 + x^2y + y^4 = 4$

Differentiate *w. r. t. x.*

$$\frac{d}{dx}(x^5) + \frac{d}{dx}(xy^3) + \frac{d}{dx}(x^2y) + \frac{d}{dx}(y^4) = \frac{d}{dx}(4)$$

$$5x^4 + x\frac{d}{dx}(y^3) + y^3\frac{d}{dx}(x) + x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2) + 4y^3\frac{d}{dx}(y) = 0$$

$$5x^4 + x(3y^2)\frac{dy}{dx} + y^3(1) + x^2\frac{dy}{dx} + y(2x) + 4y^3\frac{dy}{dx} = 0$$

$$x^2\frac{dy}{dx} + 3xy^2\frac{dy}{dx} + 4y^3\frac{dy}{dx} = -5x^4 - 2xy - y^3$$

$$(x^2 + 3xy^2 + 4y^3)\frac{dy}{dx} = -(5x^4 + 2xy + y^3)$$

$$\therefore \quad \frac{dy}{dx} = -\frac{5x^4 + 2xy + y^3}{x^2 + 3xy^2 + 4y^3}$$

(ii) Given that : $y^3 + \cos(xy) = x^2 - \sin(x + y)$

Differentiate *w. r. t. x.*

$$\frac{d}{dx}(y^3) + \frac{d}{dx}[\cos(xy)] = \frac{d}{dx}(x^2) - \frac{d}{dx}[\sin(x + y)]$$

$$3y^2\frac{d}{dx}(y) - \sin(xy)\frac{d}{dx}(xy) = 2x - \cos(x + y)\frac{d}{dx}(x + y)$$

$$3y^2\frac{dy}{dx} - \sin(xy)\left[x\frac{dy}{dx} + y(1)\right] = 2x - \cos(x + y)\left[1 + \frac{dy}{dx}\right]$$

$$3y^2\frac{dy}{dx} - x\sin(xy)\frac{dy}{dx} - y\sin(xy) = 2x - \cos(x + y) - \cos(x + y)\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} - x\sin(xy)\frac{dy}{dx} + \cos(x + y)\frac{dy}{dx} = 2x + y\sin(xy) - \cos(x + y)$$

$$[3y^2 - x\sin(xy) + \cos(x + y)]\frac{dy}{dx} = 2x + y\sin(xy) - \cos(x + y)$$

$$\therefore \quad \frac{dy}{dx} = \frac{2x + y\sin(xy) - \cos(x + y)}{3y^2 - x\sin(xy) + \cos(x + y)}$$

(iii) Given that : $x^2 + e^{xy} = y^2 + \log(x+y)$

Recall that $\frac{d}{dx} g(f(x)) = g'(f(x)) \cdot \frac{d}{dx} f(x)$

Differentiate *w. r. t. x.*

$$\frac{d}{dx}(x^2) + \frac{d}{dx}[e^{xy}] = \frac{d}{dx}(y^2) + \frac{d}{dx}[\log(x+y)]$$

$$2x + e^{xy}\frac{d}{dx}(xy) = 2y\frac{dy}{dx} + \frac{1}{x+y}\frac{d}{dx}(x+y)$$

$$2x + e^{xy}\left[x\frac{dy}{dx} + y(1)\right] = 2y\frac{dy}{dx} + \frac{1}{x+y}\left[1 + \frac{dy}{dx}\right]$$

$$2x + xe^{xy}\frac{dy}{dx} + ye^{xy} = 2y\frac{dy}{dx} + \frac{1}{x+y} + \frac{1}{x+y}\cdot\frac{dy}{dx}$$

$$2x + ye^{xy} - \frac{1}{x+y} = 2y\frac{dy}{dx} - xe^{xy}\frac{dy}{dx} + \frac{1}{x+y}\cdot\frac{dy}{dx}$$

$$2x + ye^{xy} - \frac{1}{x+y} = \left[2y - xe^{xy} + \frac{1}{x+y}\right]\frac{dy}{dx}$$

$$\frac{2x(x+y) + ye^{xy}(x+y) - 1}{x+y} = \left[\frac{2y(x+y) - xe^{xy}(x+y) + 1}{x+y}\right]\frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = \frac{2x(x+y) + ye^{xy}(x+y) - 1}{2y(x+y) - xe^{xy}(x+y) + 1}$$

Ex. 2 : Find $x^m \cdot y^n = (x+y)^{m+n}$, then prove that $\frac{dy}{dx} = \frac{y}{x}$.

Solution : Given that : $x^m \cdot y^n = (x+y)^{m+n}$

Taking log on both the sides, we get

$$\log[x^m \cdot y^n] = \log[(x+y)^{m+n}]$$

$$m\log x + n\log y = (m+n)\log(x+y)$$

Differentiate *w. r. t. x.*

$$m\frac{d}{dx}(\log x) + n\frac{d}{dx}(\log y) = (m+n)\frac{d}{dx}[\log(x+y)]$$

$$\frac{m}{x} + \frac{n}{y}\cdot\frac{dy}{dx} = \frac{m+n}{x+y}\cdot\frac{d}{dx}(x+y)$$

$$\frac{m}{x} + \frac{n}{y}\cdot\frac{dy}{dx} = \frac{m+n}{x+y}\cdot\left[1 + \frac{dy}{dx}\right]$$

$$\frac{m}{x} + \frac{n}{y}\cdot\frac{dy}{dx} = \frac{m+n}{x+y} + \frac{m+n}{x+y}\cdot\frac{dy}{dx}$$

$$\frac{n}{y}\cdot\frac{dy}{dx} - \frac{m+n}{x+y}\cdot\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\left[\frac{n}{y}-\frac{m+n}{x+y}\right]\frac{dy}{dx}=\frac{m+n}{x+y}-\frac{m}{x}$$

$$\left[\frac{n(x+y)-(m+n)y}{y(x+y)}\right]\frac{dy}{dx}=\left[\frac{(m+n)x-m(x+y)}{x(x+y)}\right]$$

$$\left[\frac{nx+ny-my-ny}{y}\right]\frac{dy}{dx}=\frac{mx+nx-mx-my}{x}$$

$$\left[\frac{nx-my}{y}\right]\frac{dy}{dx}=\frac{nx-my}{x}$$

$$\therefore\quad \frac{dy}{dx}=\frac{y}{x}$$

Ex. 3 : If $\sin\left(\frac{px^m-qy^m}{px^m+qy^m}\right)=r$, then show that $\frac{dy}{dx}=\frac{y}{x}$, where r is a constant.

Solution : Given that : $\sin\left(\frac{px^m-qy^m}{px^m+qy^m}\right)=r$

$$\frac{px^m-qy^m}{px^m+qy^m}=\sin^{-1}r$$

$$\frac{px^m-qy^m}{px^m+qy^m}=t \qquad \ldots\ldots [\text{ Let } t=\sin^{-1}r\ ]$$

$$px^m-qy^m=ptx^m+qty^m$$

$$px^m-ptx^m=qy^m+qty^m$$

$$p(1-t)x^m=q(1+t)y^m$$

$$y^m=\left(\frac{p(1-t)}{q(1+t)}\right)x^m$$

$$y^m=s\cdot x^m \qquad \ldots\ldots (\mathrm{I}) \qquad \ldots\ldots\left[\text{Let } s=\left(\frac{p(1-t)}{q(1-t)}\right)\right]$$

Differentiate $w.\ r.\ t.\ x$

$$\frac{d}{dx}(y^m)=s\frac{d}{dx}(x^m)$$

$$my^{m-1}\frac{dy}{dx}=s\cdot mx^{m-1}$$

$$\frac{dy}{dx}=s\cdot\frac{x^{m-1}}{y^{m-1}}x^{m-1}$$

$$\frac{dy}{dx}=\frac{y^m}{x^m}\times\frac{x^{m-1}}{y^{m-1}} \qquad \ldots\ldots [\text{ From (I)}]$$

$$\therefore\quad \frac{dy}{dx}=\frac{y}{x}$$

Ex. 4 : If $\sec^{-1}\left(\frac{x^3+y^3}{x^3-y^3}\right) = 2a$, then show that $\frac{dy}{dx} = \frac{x^2\tan^2 a}{y^2}$, where a is a constant.

Solution : Given that : $\sec^{-1}\left(\frac{x^3+y^3}{x^3-y^3}\right) = 2a$ . . . . . [We will not eliminate a, as answer contains a]

$$\therefore \quad \cos^{-1}\left(\frac{x^3-y^3}{x^3+y^3}\right) = 2a$$

$$\frac{x^3-y^3}{x^3+y^3} = \cos 2a$$

$$x^3 - y^3 = x^3\cos 2a + y^3\cos 2a$$

$$x^3 - x^3\cos 2a = y^3\cos 2a + y^3$$

$$x^3(1-\cos 2a) = y^3(1+\cos 2a)$$

$$y^3 = \left(\frac{1-\cos 2a}{1+\cos 2a}\right)x^3$$

$$y^3 = \left(\frac{2\sin^2 a}{2\cos^2 a}\right)x^3$$

$$y^3 = (\tan^2 a)x^3 \qquad \text{. . . . . . (I)}$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}(y^3) = (\tan^2 a)\frac{d}{dx}(x^3)$$

$$3y^2\frac{dy}{dx} = (\tan^2 a)\,3x^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2\tan^2 a}{y^2}$$

Ex. 5 : If $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots \infty}}}$, then show that $\frac{dy}{dx} = \frac{\sec^2 x}{2y-1}$.

Solution : Given that : $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots \infty}}}$ . . . . (I)

Squaring both sides, we get

$y^2 = \tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots \infty}}$, which is same as

$$y^2 = \tan x + \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \ldots \infty}}}$$

$$y^2 = \tan x + y \qquad \text{. . . . . . . [From (I)]}$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(\tan x) + \frac{dy}{dx}$$

$$2y\frac{dy}{dx}-\frac{dy}{dx}=\sec^2 x$$

$$(2y-1)\frac{dy}{dx}=\sec^2 x$$

$$\therefore \quad \frac{dy}{dx}=\frac{\sec^2 x}{2y-1}$$

Ex. 6 : If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, then show that $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$.

Solution : Given that : $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$ . . . . (I)

Put $x=\sin\alpha$, $y=\sin\beta$

$\therefore \quad \alpha=\sin^{-1}x,\ \beta=\sin^{-1}y$

Equation (I) becomes,

$$\sqrt{1-\sin^2\alpha}+\sqrt{1-\sin^2\beta}=a(\sin\alpha-\sin\beta)$$

$$\cos\alpha+\cos\beta=a(\sin\alpha-\sin\beta)$$

$$2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)=2a\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\cos\left(\frac{\alpha-\beta}{2}\right)=a\sin\left(\frac{\alpha+\beta}{2}\right)\Rightarrow\cot\left(\frac{\alpha-\beta}{2}\right)=a$$

$$\frac{\alpha-\beta}{2}=\cot^{-1}a \qquad \therefore \qquad \alpha-\beta=2\cot^{-1}a$$

$$\therefore \quad \sin^{-1}x-\sin^{-1}y=2\cot^{-1}a$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}(\sin^{-1}x)-\frac{d}{dx}(\sin^{-1}y)=\frac{d}{dx}(2\cot^{-1}a)$$

$$\frac{1}{\sqrt{1-x^2}}-\frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx}=0$$

$$\therefore \quad \frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$$

EXERCISE 1.3

(1) Differentiate the following *w. r. t. x*

(i) $\frac{(x+1)^2}{(x+2)^3(x+3)^4}$

(ii) $\sqrt[3]{\frac{4x-1}{(2x+3)(5-2x)^2}}$

(iii) $(x^2+3)^{\frac{3}{2}}\cdot\sin^3 2x\cdot 2^{x^2}$

(iv) $\frac{(x^2+2x+2)^{\frac{3}{2}}}{(\sqrt{x}+3)^3(\cos x)^x}$

(v) $\frac{x^5\cdot\tan^3 4x}{\sin^2 3x}$

(vi) $x^{\tan^{-1}x}$

(vii) $(\sin x)^x$

(viii) $\sin x^x$

(2) Differentiate the following $w.\ r.\ t.\ x$.

(i) $x^e + x^x + e^x + e^e$ (ii) $x^{x^x} + e^{x^x}$

(iii) $(\log x)^x - (\cos x)^{\cot x}$

(iv) $x^{e^x} + (\log x)^{\sin x}$ (v) $e^{\tan x} + (\log x)^{\tan x}$

(vi) $(\sin x)^{\tan x} + (\cos x)^{\cot x}$

(vii) $10^{x^x} + x^{x^{10}} + x^{10^x}$

(viii) $\left[(\tan x)^{\tan x}\right]^{\tan x}$ at $x = \dfrac{\pi}{4}$

(3) Find $\dfrac{dy}{dx}$ if

(i) $\sqrt{x} + \sqrt{y} = \sqrt{a}$ (ii) $x\sqrt{x} + y\sqrt{y} = a\sqrt{a}$

(iii) $x + \sqrt{xy} + y = 1$

(iv) $x^3 + x^2y + xy^2 + y^3 = 81$

(v) $x^2y^2 - \tan^{-1}\sqrt{x^2+y^2} = \cot^{-1}\sqrt{x^2+y^2}$

(vi) $xe^y + ye^x = 1$ (vii) $e^{x+y} = \cos(x-y)$

(viii) $\cos(xy) = x + y$ (ix) $e^{e^{x-y}} = \dfrac{x}{y}$

(x) $x + \sin(x+y) = y - \cos(x-y)$

(4) Show that $\dfrac{dy}{dx} = \dfrac{y}{x}$ in the following, where a and p are constants.

(i) $x^7y^5 = (x+y)^{12}$

(ii) $x^py^4 = (x+y)^{p+4}$, $p \in N$

(iii) $\sec\left(\dfrac{x^5+y^5}{x^5-y^5}\right) = a^2$

(iv) $\tan^{-1}\left(\dfrac{3x^2-4y^2}{3x^2+4y^2}\right) = a^2$

(v) $\cos^{-1}\left(\dfrac{7x^4+5y^4}{7x^4-5y^4}\right) = \tan^{-1}a$

(vi) $\log\left(\dfrac{x^{20}-y^{20}}{x^{20}+y^{20}}\right) = 20$

(vii) $e^{\frac{x^7-y^7}{x^7+y^7}} = a$

(viii) $\sin\left(\dfrac{x^3-y^3}{x^3+y^3}\right) = a^3$

(5) (i) If $\log(x+y) = \log(xy) + p$, where p is constant then prove that $\dfrac{dy}{dx} = -\dfrac{y^2}{x^2}$.

(ii) If $\log_{10}\left(\dfrac{x^3-y^3}{x^3+y^3}\right) = 2$,

show that $\dfrac{dy}{dx} = -\dfrac{99x^2}{101y^2}$.

(iii) If $\log_5\left(\dfrac{x^4+y^4}{x^4-y^4}\right) = 2$,

show that $\dfrac{dy}{dx} = -\dfrac{12x^3}{13y^3}$.

(iv) If $e^x + e^y = e^{x+y}$, then

show that $\dfrac{dy}{dx} = -e^{y-x}$.

(v) If $\sin^{-1}\left(\dfrac{x^5-y^5}{x^5+y^5}\right) = \dfrac{\pi}{6}$,

show that $\dfrac{dy}{dx} = \dfrac{x^4}{3y^4}$.

(vi) If $x^y = e^{x-y}$, then

show that $\dfrac{dy}{dx} = \dfrac{\log x}{(1+\log x)^2}$.

(vii) If $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \infty}}}$,

then show that $\dfrac{dy}{dx} = \dfrac{\sin x}{1-2y}$.

(viii) If $y = \sqrt{\log x + \sqrt{\log x + \sqrt{\log x + \ldots \infty}}}$,

then show that $\dfrac{dy}{dx} = \dfrac{1}{x(2y-1)}$.

(ix) If $y = x^{x^{x^{\cdot^{\cdot^{\cdot\infty}}}}}$, then

show that $\dfrac{dy}{dx} = \dfrac{y^2}{x(1-\log y)}$.

(x) If $e^y = y^x$, then

show that $\dfrac{dy}{dx} = \dfrac{(\log y)^2}{\log y - 1}$.

1.4.1 Derivatives of Parametric Functions

Consider the equations $x = f(t)$, $y = g(t)$. These equations may imply a functional relation between the variables x and y. Given the value of t in some domain $[a, b]$, we can find x and y.

For example $x = a\cos t$ and $y = a\sin t$. The functional relation between these two functions is that, $x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2(\cos^2 t + \sin^2 t) = a^2$ represents the equation of a circle of radius a with center at the origin. And the domain of t is $[0, 2\pi]$. We can find x and y for any $t \in [0, 2\pi]$.

If two variables x and y are defined separately as functions by an inter mediating varibale t, then that inter mediating variable is known as parameter. Let us discuss the derivatives of parametric functions.

1.4.2 Theorem : If $x = f(t)$ and $y = g(t)$ are differentiable functions of t so that y is a differentiable function of x and if $\frac{dx}{dt} \neq 0$ then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$.

Proof : Given that $x = f(t)$ and $y = g(t)$.

Let there be a small increment in the value of t say δt then δx and δy are the corresponding increments in x and y respectively.

As δt, δx, δy are small increments in t, x and y respectively such that $\delta t \neq 0$ and $\delta x \neq 0$.

Consider, the incrementary ratio $\frac{\delta y}{\delta x}$, and note that $\delta x \to 0 \Rightarrow \delta t \to 0$.

i.e. $\frac{\delta y}{\delta x} = \frac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}}$, since $\frac{\delta x}{\delta t} \neq 0$

Taking the limit as $\delta t \to 0$ on both sides we get,

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta t \to 0}\left(\frac{\frac{\delta y}{\delta t}}{\frac{\delta x}{\delta t}}\right)$$

As $\delta t \to 0$, $\delta x \to 0$

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{\lim_{\delta t \to 0}\left(\frac{\delta y}{\delta t}\right)}{\lim_{\delta t \to 0}\left(\frac{\delta x}{\delta t}\right)} \qquad \text{.....(I)}$$

Since x and y are differentiable function of t. we have,

$$\lim_{\delta t \to 0}\left(\frac{\delta x}{\delta t}\right) = \frac{dx}{dt} \text{ and } \lim_{\delta t \to 0}\left(\frac{\delta y}{\delta t}\right) = \frac{dy}{dt} \text{ exist and are finite} \qquad \text{.....(II)}$$

From (I) and (II), we get

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{.....(III)}$$

The R.H.S. of (III) exists and is finite, implies L.H.S.of (III) also exist and finite

$$\therefore \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx}$$

Thus the equation (III) becomes,

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}} \qquad \text{where } \frac{dx}{dt} \neq 0$$

SOLVED EXAMPLES

Ex. 1 : Find $\dfrac{dy}{dx}$ if

(i) $x = at^4, y = 2at^2$

(ii) $x = t - \sqrt{t}, y = t + \sqrt{t}$

(iii) $x = \cos(\log t),\ y = \log(\cos t)$

(iv) $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$

(v) $x = \sqrt{1 - t^2},\ y = \sin^{-1} t$

Solution :

(i) Given, $y = 2at^2$

Differentiate *w. r. t.* t

$$\frac{dy}{dt} = 2a\frac{d}{dt}(t^2) = 2a(2t) = 4at. \ldots\ldots (\text{I})$$

And, $x = at^4$

Differentiate *w. r. t.* t

$$\frac{dx}{dt} = a\frac{d}{dt}(t^4) = a(4t^3) = 4at^3. \ldots\ldots (\text{II})$$

Now, $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{4at}{4at^3}$ ...[From (I) and (II)]

$\therefore \quad \dfrac{dy}{dx} = \dfrac{1}{t^2}$

(ii) Given, $y = t + \sqrt{t}$

Differentiate *w. r. t.* t

$$\frac{dy}{dt} = \frac{d}{dt}(t + \sqrt{t}) = 1 + \frac{1}{2\sqrt{t}}$$

$$\frac{dy}{dt} = \frac{2\sqrt{t} + 1}{2\sqrt{t}} \quad \ldots\ldots (\text{I})$$

And, $x = t - \sqrt{t}$

Differentiate *w. r. t.* t

$$\frac{dx}{dt} = \frac{d}{dt}(t - \sqrt{t}) = 1 - \frac{1}{2\sqrt{t}}$$

$$\frac{dx}{dt} = \frac{2\sqrt{t} - 1}{2\sqrt{t}} \quad \ldots\ldots (\text{II})$$

Now, $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\frac{2\sqrt{t}+1}{2\sqrt{t}}}{\frac{2\sqrt{t}-1}{2\sqrt{t}}}$ ...[From (I) and (II)]

$\therefore \quad \dfrac{dy}{dx} = \dfrac{2\sqrt{t} + 1}{2\sqrt{t} - 1}$

(iii) Given, $y = \log(\cos t)$

Differentiate *w. r. t.* t

$$\frac{dy}{dt} = \frac{d}{dt}[\log(\cos t)] = \frac{1}{\cot t}\cdot\frac{d}{dt}(\cos t) = \frac{1}{\cot t}(-\sin t) \qquad \therefore \frac{dy}{dt} = -\tan t \quad \ldots\ldots (\text{I})$$

And, $x = \cos(\log t)$

Differentiate *w. r. t.* t

$$\frac{dx}{dt} = \frac{d}{dt}[\cos(\log t)] = -\sin(\log t)\cdot\frac{d}{dt}(\log t) = -\frac{\sin(\log t)}{t} \qquad \therefore \frac{dx}{dt} = -\frac{\sin(\log t)}{t} \quad \ldots\ldots (\text{II})$$

Now, $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{-\tan t}{-\frac{\sin(\log t)}{t}}$ ...[From (I) and (II)]

$\therefore \quad \dfrac{dy}{dx} = \dfrac{t \cdot \tan t}{\sin(\log t)}$

(iv) Given, $y = a(1 - \cos\theta)$

Differentiate w. r. t. θ

$$\frac{dy}{d\theta} = a\frac{d}{d\theta}[(1 - \cos\theta)] = a[0 - (-\sin\theta)]$$

$$\frac{dy}{dt} = a\sin\theta \qquad \text{.....(I)}$$

And, $x = a(\theta + \sin\theta)$

Differentiate w. r. t. θ

$$\frac{dx}{dt} = a\frac{d}{d\theta}(\theta + \sin\theta) = a(1 + \cos\theta)$$

$$\frac{dx}{dt} = a(1 + \cos\theta) \qquad \text{.....(II)}$$

Now, $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{a\sin\theta}{a(1 + \cos\theta)}$ ...[From (I) and (II)]

$$\therefore \quad \frac{dy}{dx} = \frac{2\sin\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)} = \tan\left(\frac{\theta}{2}\right)$$

(v) Given, $y = \sin^{-1} t$

Differentiate w. r. t. t

$$\frac{dy}{dt} = \frac{d}{dt}(\sin^{-1} t) = \frac{1}{\sqrt{1 - t^2}}$$

$$\frac{dy}{dt} = \frac{1}{\sqrt{1 - t^2}} \qquad \text{.....(I)}$$

And, $x = \sqrt{1 - t^2}$

Differentiate w. r. t. t

$$\frac{dx}{dt} = \frac{d}{dt}\left(\sqrt{1 - t^2}\right) = \frac{1}{2\sqrt{1 - t^2}}\cdot\frac{d}{dt}(1 - t^2)$$

$$\frac{dx}{dt} = \frac{1}{2\sqrt{1 - t^2}}\cdot(-2t) = -\frac{t}{\sqrt{1 - t^2}} \qquad \text{.....(II)}$$

Now, $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{\frac{1}{\sqrt{1 - t^2}}}{-\frac{t}{\sqrt{1 - t^2}}}$ ... [From (I) and (II)]

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{t}$$

Ex. 2 : Find $\dfrac{dy}{dx}$ if (i) $x = \sec^2\theta, y = \tan^3\theta$, at $\theta = \dfrac{\pi}{3}$ (ii) $x = t + \dfrac{1}{t}, y = \dfrac{1}{t^2}$, at $t = \dfrac{1}{2}$

(iii) $x = 3\cos t - 2\cos^3 t, y = 3\sin t - 2\sin^3 t$, at $t = \dfrac{\pi}{6}$

Solution :

(i) Given, $y = \tan^3\theta$

Differentiate w. r. t. θ

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(\tan\theta)^3 = 3\tan^2\theta\frac{d}{d\theta}(\tan\theta) \qquad \therefore \quad \frac{dy}{d\theta} = 3\tan^2\theta\cdot\sec^2\theta \qquad \text{.....(I)}$$

And, $x = \sec^2\theta$

Differentiate w. r. t. θ

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(\sec^2\theta) = 2\sec\theta\cdot\frac{d}{d\theta}(\sec\theta)$$

$$\frac{dx}{d\theta} = 2\sec\theta\cdot\sec\theta\tan\theta = 2\sec^2\theta\cdot\tan\theta \qquad \text{.....(II)}$$

Now, $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{3\tan^2\theta\cdot\sec^2\theta}{2\sec^2\theta\cdot\tan\theta}$ ...[From (I) and (II)]

$$\therefore \quad \frac{dy}{dx} = \frac{3}{2}\tan\theta$$

At $\theta = \dfrac{\pi}{3}$, we get

$$\left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{3}} = \frac{3}{2}\tan\left(\frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$$

(ii) Given, $y = \frac{1}{t^2}$

Differentiate *w. r. t. t*

$$\frac{dy}{dt} = \frac{d}{dt}\left(\frac{1}{t^2}\right)$$

$$\frac{dy}{dt} = -\frac{2}{t^3} \qquad \text{. . . . . (I)}$$

And, $x = t + \frac{1}{t}$

Differentiate *w. r. t. t*

$$\frac{dx}{dt} = \frac{d}{dt}\left(t + \frac{1}{t}\right) = 1 - \frac{1}{t^2}$$

$$\frac{dx}{dt} = -\frac{t^2 - 1}{t^2} \qquad \text{. . . . . (II)}$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{2}{t^3}}{\frac{t^2-1}{t^2}}$ . . . [From (I) and (II)]

$\therefore$ $$\frac{dy}{dx} = -\frac{2}{t(t^2 - 1)}$$

At $t = \frac{1}{2}$, we get

$$\left(\frac{dy}{dx}\right)_{t=\frac{1}{2}} = -\frac{2}{\left(\frac{1}{2}\right)\left[\left(\frac{1}{2}\right)^2 - 1\right]}$$

$$= -\frac{2}{\left(\frac{1}{2}\right)\left(\frac{1}{4} - 1\right)}$$

$$\left(\frac{dy}{dx}\right)_{t=\frac{1}{2}} = -\frac{2}{\left(\frac{1}{2}\right)\left(-\frac{3}{4}\right)}$$

$$\left(\frac{dy}{dx}\right)_{t=\frac{1}{2}} = \frac{16}{3}$$

(iii) Given, $y = 3 \sin t - 2 \sin^3 t$

Differentiate *w. r. t. t*

$$\frac{dy}{dt} = \frac{d}{dt}(3 \sin t - 2 \sin^3 t)$$

$$= 3\frac{d}{dt}(\sin t) - 2(\sin t)^3$$

$$= 3 \cos t - 2(3) \sin^2 t \frac{d}{dt}(\sin t)$$

$$= 3 \cos t - 6 \sin^2 t (\cos t)$$

$$= 3 \cos t (1 - 2 \sin^2 t)$$

$$\frac{dy}{dt} = 3 \cos t \cos 2t \qquad \text{. . . . . (I)}$$

And, $x = 3 \cos t - 2 \cos^3 t$

Differentiate *w. r. t. t*

$$\frac{dx}{dt} = \frac{d}{dt}(3 \cos t - 2 \cos^3 t)$$

$$= 3\frac{d}{dt}(\cos t) - 2\frac{d}{dt}(\cos^3 t)$$

$$= 3(-\sin t) - 2(3) \cos^2 t \frac{d}{dt}(\cos t)$$

$$= -3 \sin t - 6 \cos^2 t(-\sin t)$$

$$= 6 \cos^2 t \sin t - 3 \sin t$$

$$= 3 \sin t (2 \cos^2 t - 1)$$

$$\frac{dx}{dt} = 3 \sin t \cos 2t \qquad \text{. . . . . (II)}$$

Now, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3 \cos t \cos 2t}{3 \sin t \cos 2t}$ . . . . . [From (I) and (II)]

$\therefore$ $$\frac{dy}{dx} = -\cot t$$

At $t = \frac{\pi}{6}$, we get

$$\left(\frac{dy}{dx}\right)_{t=\frac{\pi}{6}} = -\cot\left(\frac{\pi}{6}\right) = \sqrt{3}$$

Ex. 3 : If $x^2+y^2=t+\frac{1}{t}$ and $x^4+y^4=t^2+\frac{1}{t^2}$, then show that $x^3y\frac{dy}{dx}=1$.

Solution :

Given that, $x^4+y^4=t^2+\frac{1}{t^2}$ . . . (I)

And $x^2+y^2=t+\frac{1}{t}$

Squaring both sides,

$$(x^2+y^2)^2=\left(t+\frac{1}{t}\right)^2$$

$$x^4+2x^2y^2+y^4=t^2+2+\frac{1}{t^2}$$

$$x^4+2x^2y^2+y^4=x^4+y^4+2 \quad \text{. . . [From (I)]}$$

$$2x^2y^2=2 \quad \therefore \quad x^2y^2=1 \quad \text{. . . (II)}$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}(x^2y^2)=\frac{d}{dx}(1)$$

$$x^2\frac{d}{dx}(y^2)+y^2\frac{d}{dx}(x^2)=0$$

$$x^2(2y)\frac{dy}{dx}+y^2(2x)=0$$

$$2x^2y\frac{dy}{dx}=-2xy^2 \Rightarrow \frac{dy}{dx}=-\frac{2xy^2}{2x^2y}$$

$$\frac{dy}{dx}=-\frac{x\left(-\frac{1}{x^2}\right)}{x^2y} \quad \text{. . . [From (II)]}$$

$$\therefore \quad \frac{dy}{dx}=\frac{1}{x^3y} \quad \therefore \quad x^3y\frac{dy}{dx}=1$$

Ex. 4 : If $x=a\left(t-\frac{1}{t}\right)$ and $y=b\left(t+\frac{1}{t}\right)$, then show that $\frac{dy}{dx}=\frac{b^2x}{a^2y}$.

Solution :

Given that, $x=a\left(t-\frac{1}{t}\right)$ and $y=b\left(t+\frac{1}{t}\right)$

i.e. $\frac{x}{a}=t-\frac{1}{t}$ . . . (I) and $\frac{y}{b}=t+\frac{1}{t}$ . . . (II)

Square of (I) $-$ Square of (II) gives,

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\left(t-\frac{1}{t}\right)^2-\left(t+\frac{1}{t}\right)^2$$

$$=t^2-2+\frac{1}{t^2}-t^2-2-\frac{1}{t^2}$$

$$\therefore \quad \frac{x^2}{a^2}-\frac{y^2}{a^2}=-4$$

Differentiate *w. r. t. x*

$$\frac{1}{a^2}\cdot\frac{d}{dx}(x^2)-\frac{1}{b^2}\cdot\frac{d}{dx}(y^2)=\frac{d}{dx}(-4)$$

$$\frac{1}{a^2}(2x)-\frac{1}{b^2}(2y)\cdot\frac{d}{dx}=0$$

$$\frac{1}{a^2}(2x)-\frac{1}{b^2}(2y)\cdot\frac{dy}{dx}=0$$

$$\frac{2y}{b^2}\cdot\frac{dy}{dx}=\frac{2x}{a^2} \Rightarrow \frac{dy}{dx}=\frac{b^2x}{a^2y}$$

$$\therefore \quad \frac{dy}{dx}=\frac{b^2x}{a^2y}$$

Ex. 5 : If $x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$, then show that $\frac{dy}{dx}=-\frac{y}{x}$.

Solution : Given that, $x=\sqrt{a^{\sin^{-1}t}}$ and $y=\sqrt{a^{\cos^{-1}t}}$

i.e. $x=\sqrt{a^{\sin^{-1}t}}$ . . . (I) and $y=\sqrt{a^{\cos^{-1}t}}$ . . . (II)

Differentiate (I) *w. r. t. t*

$$\frac{dx}{dt}=\frac{d}{dt}\left(\sqrt{a^{\sin^{-1}t}}\right)=\frac{1}{2\sqrt{a^{\sin^{-1}t}}}\cdot\frac{d}{dt}(a^{\sin^{-1}t})$$

$$=\frac{1}{2\sqrt{a^{\sin^{-1}t}}}\cdot a^{\sin^{-1}t}\cdot\log a\frac{d}{dt}(\sin^{-1}t)$$

$$= \frac{a^{\sin^{-1} t} \cdot \log a}{2\sqrt{a^{\sin^{-1} t}}} \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dx}{dt} = \frac{\sqrt{a^{\sin^{-1} t}} \cdot \log a}{2\sqrt{1-x^2}} = \frac{x \log a}{2\sqrt{1-x^2}} \quad \ldots \text{(III)} \ldots \text{[From (I)]}$$

Now $y = \sqrt{a^{\cos^{-1} t}}$

Differentiate (II) *w. r. t.* t

$$\frac{dy}{dt} = \frac{d}{dt}\left(\sqrt{a^{\cos^{-1} t}}\right) = \frac{1}{2\sqrt{a^{\cos^{-1} t}}} \cdot \frac{d}{dt}\left(a^{\cos^{-1} t}\right)$$

$$= \frac{1}{2\sqrt{a^{\cos^{-1} t}}} \cdot a^{\cos^{-1} t} \cdot \log a \frac{d}{dt}\left(\cos^{-1} t\right)$$

$$= \frac{a^{\cos^{-1} t} \cdot \log a}{2\sqrt{a^{\cos^{-1} t}}}\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$\frac{dy}{dt} = \frac{-\sqrt{a^{\cos^{-1} t}} \cdot \log a}{2\sqrt{1-x^2}} = -\frac{y \log a}{2\sqrt{1-x^2}} \quad \ldots \text{(IV)} \ldots \text{[From (II)]}$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{y \log a}{2\sqrt{1-x^2}}}{\frac{x \log a}{2\sqrt{1-x^2}}} \quad \ldots\ldots \text{[From (III) and (IV)]}$$

$$\therefore \frac{dy}{dx} = -\frac{y}{x}$$

1.4.3 Differentiation of one function with respect to another function :

If y is differentiable function of x, then the derivative of y with respect to x is $\frac{dy}{dx}$.

Similarly, if $u = f(x)$, $v = g(x)$ differentiable function of x, such that $\frac{du}{dx} = f'(x)$ and $\frac{dv}{dx} = g'(x)$

then the derivative of u with respect to v is $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{f'(x)}{g'(x)}$.

SOLVED EXAMPLES

Ex. 1 : Find the derivative of 7^x *w. r. t.* x^7.

Solution : Let : $u = 7^x$ and $v = x^7$, then we have to find $\frac{du}{dv}$.

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} \quad \ldots \text{(I)}$$

Now, $u = 7^x$

Differentiate *w. r. t.* x

$$\frac{du}{dx} = \frac{d}{dx}(7^x) = 7^x \log 7 \quad \ldots \text{(II)}$$

And, $v = x^7$

Differentiate *w. r. t.* x

$$\frac{dv}{dx} = \frac{d}{dx}(x^7) = 7x^6 \quad \ldots \text{(III)}$$

Substituting (II) and (III) in (I) we get,

$$\therefore \frac{du}{dv} = \frac{7^x \log 7}{7x^6}$$

Ex. 2 : Find the derivative of $\cos^{-1}x$ *w. r. t.* $\sqrt{1-x^2}$.

Solution : Let $u = \cos^{-1}x$ and $v = \sqrt{1-x^2}$, then we have to find $\frac{du}{dv}$.

i.e. $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$ . . . (I)

Now, $u = \cos^{-1}x$

Differentiate *w. r. t.* x

$$\frac{du}{dx} = \frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}} \quad \text{. . . (II)}$$

And, $v = \sqrt{1-x^2}$

Differentiate *w. r. t.* x

$$\frac{dv}{dx} = \frac{d}{dx}\left(\sqrt{1-x^2}\right) = \frac{1}{2\sqrt{1-x^2}}\cdot\frac{d}{dx}(1-x^2) = \frac{1}{2\sqrt{1-x^2}}\cdot(-2x)$$

$$\frac{dv}{dx} = -\frac{x}{\sqrt{1-x^2}} \quad \text{. . . (III)}$$

Substituting (II) and (III) in (I) we get,

$$\frac{du}{dv} = \frac{-\frac{1}{\sqrt{1-x^2}}}{-\frac{x}{\sqrt{1-x^2}}} \qquad \therefore \qquad \frac{du}{dv} = \frac{1}{x}$$

Ex. 3 : Find the derivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ *w. r. t.* $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

Solution : Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then we have to find $\frac{du}{dv}$.

i.e. $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$ . . . (I)

Now, $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Put $x = \tan\theta$ $\therefore$ $\theta = \tan^{-1}x$

$$u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right) = \left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left[\frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\right] = \tan^{-1}\left[\tan\left(\frac{\theta}{2}\right)\right]$$

$$u = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

Differentiate *w. r. t.* x

$$\frac{du}{dx} = \frac{1}{2}\frac{d}{dx}(\tan^{-1}x) = \frac{1}{2(1+x^2)} \quad \text{. . . (II)}$$

And, $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$

$v = 2\tan^{-1}x$

Differentiate $w.\ r.\ t.\ x$

$$\frac{dv}{dx} = 2\frac{d}{dx}(\tan^{-1}x) = \frac{2}{1+x^2} \qquad \ldots \text{(III)}$$

Substituting (II) and (III) in (I) we get,

$$\frac{du}{dv} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{1+x^2}} = \frac{1}{4}$$

EXERCISE 1.4

(1) Find $\frac{dy}{dx}$ if

(i) $x = at^2, y = 2at$

(ii) $x = a\cot\theta, y = b\operatorname{cosec}\theta$

(iii) $x = \sqrt{a^2+m^2}, y = \log(a^2+m^2)$

(iv) $x = \sin\theta, y = \tan\theta$

(v) $x = a(1-\cos\theta), y = b(\theta - \sin\theta)$

(vi) $x = \left(t+\frac{1}{t}\right)^a, y = a^{t+\frac{1}{t}}$,

where $a > 0$, $a \neq 1$ and $t \neq 0$.

(vii) $x = \cos^{-1}\left(\frac{2t}{1+t^2}\right), y = \sec^{-1}(\sqrt{1+t^2})$

(viii) $x = \cos^{-1}(4t^3-3t), y = \tan^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$

(2) Find $\frac{dy}{dx}$ if

(i) $x = \operatorname{cosec}^2\theta, y = \cot^3\theta$, at $\theta = \frac{\pi}{6}$

(ii) $x = a\cos^3\theta, y = a\sin^3\theta$, at $\theta = \frac{\pi}{3}$

(iii) $x = t^2+t+1, y = \sin\left(\frac{\pi t}{2}\right) + \cos\left(\frac{\pi t}{2}\right)$, at $t = 1$

(iv) $x = 2\cos t + \cos 2t, y = 2\sin t - \sin 2t$, at $t = \frac{\pi}{4}$

(v) $x = t + 2\sin(\pi t), y = 3t - \cos(\pi t)$, at $t = \frac{1}{2}$

(3) (i) If $x = a\sqrt{\sec\theta - \tan\theta}, y = a\sqrt{\sec\theta+\tan\theta}$, then show that $\frac{dy}{dx} = -\frac{y}{x}$.

(ii) If $x = e^{\sin 3t}, y = e^{\cos 3t}$, then show that $\frac{dy}{dx} = -\frac{y\log x}{x\log y}$.

(iii) If $x = \frac{t+1}{t-1}, y = \frac{t-1}{t+1}$, then show that $y^2 + \frac{dy}{dx} = 0$.

(iv) If $x = a\cos^3 t, y = a\sin^3 t$, then show that $\frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$.

(v) If $x = 2\cos^4(t+3), y = 3\sin^4(t+3)$, show that $\frac{dy}{dx} = -\sqrt{\frac{3y}{2x}}$.

(vi) If $x = \log(1+t^2), y = t - \tan^{-1}t$, show that $\frac{dy}{dx} = \frac{\sqrt{e^x-1}}{2}$.

(vii) If $x = \sin^{-1}(e^t), y = \sqrt{1-e^{2t}}$, show that $\sin x + \frac{dy}{dx} = 0$.

(viii) If $x = \frac{2bt}{1+t^2}, y = a\left(\frac{1-t^2}{1+t^2}\right)$, show that $\frac{dx}{dy} = -\frac{b^2y}{a^2x}$.

(4) (i) Differentiate $x \sin x$ *w. r. t.* $\tan x$.

(ii) Differentiate $\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$ *w. r. t.* $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$.

(iii) Differentiate $\tan^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right)$ *w. r. t.* $\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$.

(iv) Differentiate $\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ *w. r. t.* $\tan^{-1} x$.

(v) Differentiate 3^x *w. r. t.* $\log_x 3$.

(vi) Differentiate $\tan^{-1}\left(\dfrac{\cos x}{1+\sin x}\right)$ *w. r. t.* $\sec^{-1} x$.

(vii) Differentiate x^x *w. r. t.* $x^{\sin x}$.

(viii) Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ *w. r. t.* $\tan^{-1}\left(\dfrac{2x\sqrt{1-x^2}}{1-2x^2}\right)$.

1.5.1 Higher order derivatives :

If $f(x)$ is differentiable function of x on an open interval I, then its derivative $f'(x)$ is also a function on I, so $f'(x)$ may have a derivative of its own, denoted as $\big(f'(x)\big)' = f''(x)$. This new function $f''(x)$ is called the **second derivative** of $f(x)$. By Leibniz notation, we write the second detivative of

$$y = f(x) \text{ as } y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

By method of first principle

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \frac{dy}{dx} \text{ and}$$

$$f''(x) = \lim_{h\to 0}\left(\frac{f'(x+h)-f'(x)}{h}\right) = \frac{d^2y}{dx^2}$$

Further if $f''(x)$ is a differentiable function of x then its derivative is denoted as $\dfrac{d}{dx}[f''(x)] = f'''(x)$.

Now the new function $f'''(x)$ is called the **third derivative** of $f(x)$. We write the third of $y = f(x)$ as $y''' = f'''(x) = \dfrac{d}{dx}\left(\dfrac{d^2y}{dx^2}\right) = \dfrac{d^3y}{dx^3}$. The **fourth derivative**, is usually denoted by $f^{(4)}(x)$. Therefore $f^{(4)}(x) = \dfrac{d^4y}{dx^4}$.

In general, the n^{th} derivative of $f(x)$, is denoted by $f^{(n)}(x)$ and it obtained by differentiating $f(x)$, n times. So, we can write the n^{th} derivative of $y = f(x)$ as $y^{(n)} = f^{(n)}(x) = \dfrac{d^ny}{dx^n}$. These are called higher order derivatives.

Note : The higher order derivatives may also be denoted by $y_2, y_3, \ldots, y_n$.

For example: Consider $f(x) = x^3 - x$

Differentiate *w. r. t. x*

$$f'(x) = \frac{d}{dx}[f(x)] = 3x^2 - 1$$

Differentiate *w. r. t. x*

$$f''(x) = \frac{d}{dx}[f'(x)] = 6x$$

Differentiate *w. r. t. x*

$$f'''(x) = \frac{d}{dx}[f''(x)] = 6$$

Which is the slope of the line represented by $f''(x)$. Hence forth all its next derivatives are zero.

Note : From the above example we can deduce one important result that, if $f(x)$ is a polynomial of degree n, then its n^{th} order derivative is a constant and all the onward detivatives are zeros.

SOLVED EXAMPLES

Ex. 1 : Find the second order derivative of the following :

(i) $x^3 + 7x^2 - 2x - 9$ (ii) $x^2 e^x$ (iii) $e^{2x} \sin 3x$

(iv) $x^2 \log x$ (v) $\sin(\log x)$

Solution :

(i) Let $y = x^3 + 7x^2 - 2x - 9$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 + 7x^2 - 2x - 9)$$

$$\frac{dy}{dx} = 3x^2 + 14x - 2$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(3x^2 + 14x - 2)$$

$$\frac{d^2y}{dx^2} = 6x + 14$$

(ii) Let $y = x^2 e^x$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 e^x)$$

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(e^x) + e^x \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = x^2 e^x + 2x e^x = e^x (x^2 + 2x)$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}[e^x (x^2 + 2x)]$$

$$\frac{d^2y}{dx^2} = e^x \frac{d}{dx}(x^2 + 2x) + (x^2 + 2x)\frac{d}{dx}(e^x)$$

$$= e^x (2x + 2) + (x^2 + 2x)(e^x)$$

$$= (x^2 + 4x + 2)\, e^x$$

$$\frac{d^2y}{dx^2} = (x^2 + 4x + 2)\, e^x$$

(iii) Let $y = e^{2x} \sin 3x$

Differentiate $w.r.t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}(e^{2x} \sin 3x) = e^{2x} \frac{d}{dx}(\sin 3x) + \sin 3x \frac{d}{dx}(e^{2x})$$

$$\frac{dy}{dx} = e^{2x}(\cos 3x)(3) + \sin 3x\,(e^{2x})(2)$$

$$\frac{dy}{dx} = e^{2x}(3 \cos 3x + 2 \sin 3x)$$

Differentiate $w.r.t.\ x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[e^{2x}(3 \cos 3x + 2 \sin 3x)\right]$$

$$\frac{d^2y}{dx^2} = e^{2x} \frac{d}{dx}(3 \cos 3x + 2 \sin 3x) + (3 \cos 3x + 2 \sin 3x) \frac{d}{dx}(e^{2x})$$

$$= e^{2x}\left[3(-\sin 3x)(3) + 2(\cos 3x)(3)\right] + (3 \cos 3x + 2 \sin 3x)\, e^{2x}(2)$$

$$= e^{2x}\left[-9 \sin 3x + 6 \cos 3x + 6 \cos 3x + 4 \sin 3x\right]$$

$$\frac{d^2y}{dx^2} = e^{2x}\left[12 \cos 3x - 5 \sin 3x\right]$$

(iv) Let $y = x^2 \log x$

Differentiate $w.r.t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2 \log x)$$

$$\frac{dy}{dx} = x^2 \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x^2)$$

$$\frac{dy}{dx} = x^2 \cdot \frac{1}{x} + \log x\,(2x)$$

$$\frac{dy}{dx} = x(1 + 2 \log x)$$

Differentiate $w.r.t.\ x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[x(1 + 2 \log x)\right]$$

$$\frac{d^2y}{dx^2} = x \frac{d}{dx}(1 + 2 \log x) + (1 + 2 \log x) \frac{d}{dx}(x)$$

$$= x \cdot \frac{2}{x} + (1 + 2 \log x)(1)$$

$$\frac{d^2y}{dx^2} = 3 + 2 \log x$$

(v) Let $y = \sin(\log x)$

Differentiate $w.r.t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}[\sin(\log x)]$$

$$\frac{dy}{dx} = \cos(\log x) \frac{d}{dx}(\log x)$$

$$\frac{dy}{dx} = \frac{\cos(\log x)}{x}$$

Differentiate $w.r.t.\ x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\frac{\cos(\log x)}{x}\right]$$

$$\frac{d^2y}{dx^2} = \frac{x \frac{d}{dx}[\cos(\log x)] - \cos(\log x) \frac{d}{dx}(x)}{x^2}$$

$$= \frac{x\,[-\sin(\log x)] \frac{d}{dx}(\log x) - \cos(\log x)(1)}{x^2}$$

$$= \frac{-\frac{x \sin(\log x)}{x} - \cos(\log x)}{x^2}$$

$$\frac{d^2y}{dx^2} = -\frac{\sin(\log x) + \cos(\log x)}{x^2}$$

Ex. 2 : Find $\frac{d^2y}{dx^2}$ if, (i) $x = \cot^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$ and $x = \operatorname{cosec}^{-1}\left(\frac{1+t^2}{2t}\right)$ (ii) $x = a\cos^3\theta$, $y = b\sin^3\theta$ at $\theta = \frac{\pi}{4}$

Solution :

(i) $x = \cot^{-1}\left(\frac{\sqrt{1-t^2}}{t}\right)$ and $x = \operatorname{cosec}^{-1}\left(\frac{1+t^2}{2t}\right)$

Put $t = \sin\theta$ $\quad \therefore\ \theta = \sin^{-1} t$

$$x = \cot^{-1}\left(\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right) = \cot^{-1}\left(\frac{\sqrt{\sin^2\theta}}{\sin\theta}\right)$$

$x = \cot^{-1}(\cot\theta) = \theta \qquad \therefore\ x = \sin^{-1} t$

Differentiate *w. r. t. t*

$$\frac{dx}{dt} = \frac{d}{dt}(\sin^{-1} t) = \left(\frac{1}{\sqrt{1-t^2}}\right) \quad \ldots \text{(I)}$$

$$y = \operatorname{cosec}^{-1}\left(\frac{1+t^2}{2t}\right) = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$$

Put $t = \tan\theta$ $\quad \therefore\ \theta = \tan^{-1} t$

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$\therefore\ y = 2\tan^{-1} t$

Differentiate *w. r. t. t*

$$\frac{dy}{dt} = 2\frac{d}{dt}(\tan^{-1} t) = \frac{2}{1+t^2} \quad \ldots \text{(II)}$$

We know that,

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2}{1+t^2}}{\frac{1}{\sqrt{1-t^2}}} \ldots \text{[From (I) and (II)]} \qquad \therefore\ \frac{dy}{dx} = \left(\frac{2\sqrt{1-t^2}}{1+t^2}\right)$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}\cdot\frac{dy}{dx} = \frac{d}{dx}\cdot\left(\frac{2\sqrt{1-t^2}}{1+t^2}\right)$$

$$\frac{d^2y}{dx^2} = 2\frac{d}{dt}\cdot\left(\frac{\sqrt{1-t^2}}{1+t^2}\right) \times \frac{dt}{dx}$$

$$= 2 \times \left[\frac{(1+t^2)\frac{d}{dt}\left(\sqrt{1-t^2}\right) - \sqrt{1-t^2}\,\frac{d}{dt}(1+t^2)}{(1+t^2)^2}\right] \times \frac{1}{\frac{dx}{dt}}$$

$$= 2 \times \left[\frac{(1+t^2)\frac{1}{2\sqrt{1-t^2}}\frac{d}{dt}\left(\sqrt{1-t^2}\right) - \sqrt{1-t^2}\,(2t)}{(1+t^2)^2}\right] \times \frac{1}{\frac{1}{\sqrt{1-t^2}}} \text{ [From (I)]}$$

$$= 2 \times \left[\frac{(1+t^2)\frac{1}{2\sqrt{1-t^2}}(-2t) - 2t\left(\sqrt{1-t^2}\right)}{(1+t^2)^2}\right] \times \sqrt{1-t^2}$$

$$= 2 \times \left[\frac{(1+t^2)\frac{-t}{2\sqrt{1-t^2}} - 2t\left(\sqrt{1-t^2}\right)}{(1+t^2)^2}\right] \times \sqrt{1-t^2}$$

$$= 2 \times \left[\frac{-t(1+t^2) - 2t(1-t^2)}{(1+t^2)^2}\right] = 2 \times \left[\frac{-t - t^3 - 2t + 2t^3}{(1+t^2)^2}\right]$$

$$= 2 \times \left[\frac{t^3 - 3t}{(1+t^2)^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{2t(t^2-3)}{(1+t^2)^2}$$

(ii) $x = a\cos^3\theta,\ y = b\sin^3\theta$ at $\theta = \dfrac{\pi}{4}$

Solution :

Given that : $x = a\cos^3\theta$

Differentiate $w.r.t.\ \theta$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos^3\theta) = a\,(3)\,(\cos^2\theta)\,\frac{d}{d\theta}(\cos\theta)$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \qquad \ldots \text{(I)}$$

$y = b\sin^3\theta$

Differentiate $w.r.t.\ \theta$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\sin^3\theta) = b\,(3)\,(\sin^2\theta)\,\frac{d}{d\theta}(\sin\theta)$$

$$\frac{dy}{d\theta} = 3b\sin^2\theta\cos\theta \qquad \ldots \text{(II)}$$

We know that,

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3b\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} \qquad \ldots \text{[From (I) and (II)]}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{b}{a}\cdot\tan\theta$$

Differentiate $w.r.t.\ x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{b}{a}\cdot\frac{d}{dx}(\tan\theta)$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a}\cdot\frac{d}{d\theta}\cdot(\tan\theta)\times\frac{d\theta}{dx}$$

$$= -\frac{b}{a}(\sec^2\theta)\times\frac{1}{\frac{dx}{d\theta}}$$

$$= -\frac{b}{a}(\sec^2\theta)\times\frac{1}{-3a\cos^2\theta\sin\theta} \qquad \ldots \text{[From (I)]}$$

$$\frac{d^2y}{dx^2} = \frac{b}{3a^2}\times\frac{\sec^2\theta}{\cos^2\theta\sin\theta}$$

$$\frac{d^2y}{dx^2} = \frac{b\sec^4\theta}{3a^2\sin\theta}$$

When $\theta = \dfrac{\pi}{4}$

$$\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{4}} = \frac{b\sec^4\left(\frac{\pi}{4}\right)}{3a^2\sin\left(\frac{\pi}{4}\right)} = \frac{b\,(\sqrt{2})^4}{3a^2\left(\frac{1}{\sqrt{2}}\right)}$$

$$\left(\frac{d^2y}{dx^2}\right)_{\theta=\frac{\pi}{4}} = \frac{4\sqrt{2}b}{3a^2}$$

Ex. 3 : If $ax^2 + 2hxy + by^2 = 0$ then show that $\dfrac{d^2y}{dx^2} = 0.$

Solution : Given that $ax^2 + 2hxy + by^2 = 0$ . . . (I)

$$ax^2 + hxy + hxy + by^2 = 0$$

$$x(ax + hy) + y(hx + by) = 0$$

$$y(hx + by) = -\, x(ax + hy)$$

$$\frac{y}{x} = -\frac{ax + hy}{hx + by} \qquad \text{. . . (II)}$$

Differentiate (I) *w. r. t. x*

$$a\frac{d}{dx}(x^2) + 2h\frac{d}{dx}(xy) + b\frac{d}{dx}(y^2) = 0$$

$$a\,(2x) + 2h\left[a\frac{dy}{dx} + y(1)\right] + b\,(2y)\frac{dy}{dx} = 0$$

$$2\left[ax + hx\frac{dy}{dx} + hy + by\frac{dy}{dx}\right] = 0$$

$$(hx + by)\frac{dy}{dx} = -\,ax - hy$$

$$\frac{dy}{dx} = -\frac{ax + hy}{hx + by}$$

From (II), we get

$$\therefore \quad \frac{dy}{dx} = \frac{y}{x} \qquad \text{. . . (III)}$$

Differentiate (III), *w. r. t. x*

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{y}{x}\right)$$

$$\frac{d^2y}{dx^2} = \frac{x\frac{dy}{dx} - y(1)}{x^2} = \frac{x\left(\frac{y}{x}\right) - y}{x^2} \text{. . . [From (II)]}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{y - y}{x^2} = 0$$

Ex. 4 : If $y = \cos\,(m\cos^{-1}x)$ then show that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + m^2y = 0.$

Solution : Given that $y = \cos\,(m\cos^{-1}x)$

$$\therefore \quad \cos^{-1}y = m\cos^{-1}x$$

Differentiate (I) *w. r. t. x*

$$\frac{d}{dx}(\cos^{-1}y) = m\frac{d}{dx}(\cos^{-1}x)$$

$$-\frac{1}{\sqrt{1 - y^2}}\cdot\frac{dy}{dx} = -\frac{m}{\sqrt{1 - x^2}}$$

$$\sqrt{1-x^2}\cdot\frac{dy}{dx}=m\sqrt{1-y^2}$$

Squaring both sides

$$(1-x^2)\cdot\left(\frac{dy}{dx}\right)^2=m^2(1-y^2)$$

Differentiate *w. r. t. x*

$$(1-x^2)\frac{d}{dx}\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2\frac{d}{dx}(1-x^2)=m^2\frac{d}{dx}(1-y^2)$$

$$(1-x^2)\cdot 2\left(\frac{dy}{dx}\right)\cdot\frac{d}{dx}\cdot\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)^2(-2x)=m^2(-2y)\frac{dy}{dx}$$

$$2(1-x^2)\cdot\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}-2x\left(\frac{dy}{dx}\right)^2=-2m^2y\frac{dy}{dx}$$

Dividing throughout by $2\dfrac{dy}{dx}$ we get,

$$(1-x^2)\cdot\frac{d^2y}{dx^2}-x\frac{dy}{dx}=-m^2y$$

$$\therefore\quad(1-x^2)\cdot\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0$$

Ex. 5 : If $x=\sin t$, $y=e^{mt}$ then show that $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-m^2y=0$.

Solution : Given that $x=\sin t \quad \therefore \quad t=\sin^{-1}x$

and $y=e^{mt} \quad \therefore \quad y=e^{m\sin^{-1}x}$ . . . (I)

Differentiate *w. r. t. x*

$$\frac{dy}{dx}=\frac{d}{dx}\left(e^{m\sin^{-1}x}\right)=e^{m\sin^{-1}x}\cdot m\frac{d}{dx}(\sin^{-1}x)$$

$$\frac{dy}{dx}=\frac{m\cdot e^{m\sin^{-1}x}}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\frac{dy}{dx}=my \qquad \text{. . . [From (I)]}$$

Squaring both sides

$$(1-x^2)\cdot\left(\frac{dy}{dx}\right)^2=m^2y^2$$

Differentiate *w. r. t. x*

$$(1-x^2)\frac{d}{dx}\left(\frac{dy}{dx}\right)^2+\left(\frac{dy}{dx}\right)^2\frac{d}{dx}(1-x^2)=m^2\frac{d}{dx}(y^2)$$

$$(1-x^2)\cdot 2\left(\frac{dy}{dx}\right)\cdot\frac{d}{dx}\cdot\left(\frac{dy}{dx}\right)+\left(\frac{dy}{dx}\right)^2(-2x)=m^2(2y)\frac{dy}{dx}$$

$$2(1-x^2)\cdot\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}-2x\left(\frac{dy}{dx}\right)^2=2m^2y\,\frac{dy}{dx}$$

Dividing throughout by $2\dfrac{dy}{dx}$ we get,

$$(1-x^2)\cdot\frac{d^2y}{dx^2}-x\,\frac{dy}{dx}=m^2y$$

$$\therefore \qquad (1-x^2)\cdot\frac{d^2y}{dx^2}-x\,\frac{dy}{dx}-m^2y=0$$

1.5.2 Successive differentiation (or n^{th} order derivative) of some standard functions :

Successive Differentiation is the process of differentiating a given function successively for n times and the results of such differentiation are called successive derivatives. The higher order derivatives are of utmost importance in scientific and engineering applications.

There is no general formula to find n^{th} derivative of a function. Because each and every function has it's own specific general formula for it's n^{th} derivative. But there are algorithms to find it.

So, here is the algorithm, for some standard functions.

Let us use the method of mathematical induction whereever applicable.

Step 1 :- Use simple differentiation to get 1^{st}, 2^{nd} and 3^{rd} order derivatives.

Step 2 :- Observe the changes in the coefficients, the angles, the power of the function and the signs of each term etc.

Step 3 :- Express the n^{th} derivative with the help of the patterns of changes that you have observed. This will be your general formula for the n^{th} derivative of the given standard function.

SOLVED EXAMPLES

Ex. 1 : Find the n^{th} derivative of the following :

(i) x^m (ii) $\dfrac{1}{ax+b}$ (iii) $\log x$

(iv) $\sin x$ (v) $\cos(ax+b)$ (vi) $e^{ax}\sin(bx+c)$

Solution :

(i) Let $y=x^m$

Differentiate *w. r. t. x*

$$\frac{dy}{dx}=\frac{d}{dx}(x^m)=mx^{m-1}$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)=m\,\frac{d}{dx}x^{m-1}$$

$$\frac{d^2y}{dx^2}=m\cdot(m-1)\,x^{m-2}$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=m\cdot(m-1)\,\frac{d}{dx}(x^{m-2})$$

$$\frac{d^3y}{dx^3}=m\cdot(m-1)\cdot(m-2)\,x^{m-3}$$

In general n^{th} order derivative will be

$$\frac{d^ny}{dx^n}=m\cdot(m-1)\cdot(m-2)\ldots[m-(n-1)]\,x^{m-n}$$

$$\frac{d^ny}{dx^n}=m\cdot(m-1)\cdot(m-2)\ldots[m-n+1]\,x^{m-n}$$

case (i) :- If $m > 0$ and $m > n$, then

$$\frac{d^n y}{dx^n} = \frac{m \cdot (m-1) \cdot (m-2) \ldots [m-(n-1)] \cdot (m-n) \ldots 2 \cdot 1}{(m-n) \cdot [m-n-1] \ldots 2 \cdot 1} x^{m-n}$$

$$\frac{d^n y}{dx^n} = \frac{m!.\, x^{m-n}}{(m-n)!}$$

case (ii) :- If $m > 0$ and $m = n$, then

$$\frac{d^n y}{dx^n} = \frac{n!.\, x^{m-n}}{(n-n)!} = \frac{n!.\, x^0}{0!} = n!$$

case (iii) :- If $m > 0$ and $m < n$, then

$$\frac{d^n y}{dx^n} = 0$$

(ii) Let $y = \dfrac{1}{ax+b}$

Differentiate $w.\,r.\,t.\,x$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{ax+b}\right) = \frac{-1}{(ax+b)^2} \cdot \frac{d}{dx}(ax+b)$$

$$\frac{dy}{dx} = \frac{(-1) \cdot a}{(ax+b)^2}$$

Differentiate $w.\,r.\,t.\,x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = (-1)(a)\,\frac{d}{dx}\left(\frac{1}{(ax+b)^2}\right)$$

$$\frac{d^2 y}{dx^2} = (-1)(a)\,\frac{-2}{(ax+b)^3} \cdot \frac{d}{dx}(ax+b)$$

$$\frac{d^2 y}{dx^2} = \frac{(-1)^2 \cdot 2 \cdot 1 \cdot a^2}{(ax+b)^3}$$

Differentiate $w.\,r.\,t.\,x$

$$\frac{d}{dx}\left(\frac{d^2 y}{dx^2}\right) = (-1)^2 \cdot 2 \cdot 1 \cdot a^2 \cdot \frac{d}{dx}\left(\frac{1}{(ax+b)^3}\right)$$

$$\frac{d^3 y}{dx^3} = (-1)^2 \cdot 2 \cdot 1 \cdot a^2 \cdot \frac{-3}{(ax+b)^4} \cdot \frac{d}{dx}(ax+b)$$

$$\frac{d^3 y}{dx^3} = \frac{(-1)^3 \cdot 3 \cdot 2 \cdot 1 \cdot a^3}{(ax+b)^4}$$

In general n^{th} order derivative will be

$$\frac{d^n y}{dx^n} = \frac{(-1)^n \cdot n \cdot (n-1) \ldots 2 \cdot 1 \cdot a^n}{(ax+b)^{n+1}}$$

$$\frac{d^n y}{dx^n} = \frac{(-1)^n \cdot n! \cdot a^n}{(ax+b)^{n+1}}$$

(iii) Let $y = \log x$

Differentiate $w.\,r.\,t.\,x$

$$\frac{dy}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$$

Differentiate $w.\,r.\,t.\,x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\right)$$

$$\frac{d^2 y}{dx^2} = \frac{-1}{x^2} = \frac{(-1)^1}{x^2}$$

Differentiate $w.\,r.\,t.\,x$

$$\frac{d}{dx}\left(\frac{d^2 y}{dx^2}\right) = (-1)^1\,\frac{d}{dx}\left(\frac{1}{x^2}\right)$$

$$\frac{d^3 y}{dx^3} = (-1)^1\left(\frac{-2}{x^3}\right) = \frac{(-1)^2 \cdot 1 \cdot 2}{x^3}$$

In general n^{th} order derivative will be

$$\frac{d^n y}{dx^n} = \frac{(-1)^{n-1} \cdot 1 \cdot 2 \cdot 3 \ldots (n-1)}{x^n}$$

$$\frac{d^n y}{dx^n} = \frac{(-1)^{n-1} \cdot (n-1)!}{x^n}$$

(iv) Let $y = \sin x$

Differentiate $w.\ r.\ t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x) = \cos x$$

$$\frac{dy}{dx} = \sin\left(\frac{\pi}{2} + x\right)$$

Differentiate $w.\ r.\ t.\ x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} + x\right)\right]$$

$$\frac{d^2y}{dx^2} = \cos\left(\frac{\pi}{2} + x\right)\frac{d}{dx}\left(\frac{\pi}{2} + x\right)$$

$$\frac{d^2y}{dx^2} = \sin\left(\frac{\pi}{2} + \frac{\pi}{2} + x\right)(1)$$

$$\frac{d^2y}{dx^2} = \sin\left(\frac{2\pi}{2} + x\right)$$

Differentiate $w.\ r.\ t.\ x$

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left[\sin\left(\frac{2\pi}{2} + x\right)\right]$$

$$\frac{d^3y}{dx^3} = \cos\left(\frac{2\pi}{2} + x\right)\frac{d}{dx}\left(\frac{2\pi}{2} + x\right)$$

$$= \sin\left(\frac{\pi}{2} + \frac{2\pi}{2} + x\right)(1)$$

$$\frac{d^3y}{dx^3} = \sin\left(\frac{3\pi}{2} + x\right)$$

In general n^{th} order derivative will be

$$\frac{d^ny}{dx^n} = \sin\left(\frac{n\pi}{2} + x\right)$$

(v) Let $y = \cos(ax + b)$

Differentiate $w.\ r.\ t.\ x$

$$\frac{dy}{dx} = \frac{d}{dx}[\cos(ax + b)]$$

$$= -\sin(ax + b)\frac{d}{dx}(ax + b)$$

$$= \cos\left(\frac{\pi}{2} + ax + b\right)(a)$$

$$\frac{dy}{dx} = a\cos\left(\frac{\pi}{2} + ax + b\right)$$

Differentiate $w.\ r.\ t.\ x$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[a\cos\left(\frac{\pi}{2} + ax + b\right)\right]$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = a\frac{d}{dx}\left[\cos\left(\frac{\pi}{2} + ax + b\right)\right]$$

$$\frac{d^2y}{dx^2} = a\left[-\sin\left(\frac{\pi}{2} + ax + b\right)\right]\frac{d}{dx}\left(\frac{\pi}{2} + ax + b\right)$$

$$= a\cos\left(\frac{\pi}{2} + \frac{\pi}{2} + ax + b\right)(a)$$

$$\frac{d^2y}{dx^2} = a^2\cos\left(\frac{2\pi}{2} + ax + b\right)$$

Differentiate $w.\ r.\ t.\ x$

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left[a^2\cos\left(\frac{2\pi}{2} + ax + b\right)\right]$$

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = a^2\frac{d}{dx}\left[\cos\left(\frac{2\pi}{2} + ax + b\right)\right]$$

$$\frac{d^3y}{dx^3} = a^2\left[-\sin\left(\frac{2\pi}{2} + ax + b\right)\right]\frac{d}{dx}\left(\frac{2\pi}{2} + ax + b\right)$$

$$= a^2\cos\left(\frac{\pi}{2} + \frac{2\pi}{2} + ax + b\right)(a)$$

$$\frac{d^3y}{dx^3} = a^3\cos\left(\frac{3\pi}{2} + ax + b\right)$$

In general n^{th} order derivative will be

$$\frac{d^ny}{dx^n} = a^n\cos\left(\frac{n\pi}{2} + ax + b\right)$$

(vi) Let $y = e^{ax} \sin(bx + c)$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}[e^{ax}\sin(bx+c)] = e^{ax}\frac{d}{dx}[\sin(bx+c)] + [\sin(bx+c)]\frac{d}{dx}\left(e^{ax}\right)$$

$$= e^{ax}\cos(bx+c)\frac{d}{dx}(bx+c) + \sin(bx+c)\cdot e^{ax}\cdot\frac{d}{dx}(ax)$$

$$= e^{ax}[\,b\cos(bx+c) + a\sin(bx+c)]$$

$$= e^{ax}\sqrt{a^2+b^2}\left[\frac{b}{\sqrt{a^2+b^2}}\cos(bx+c) + \frac{a}{\sqrt{a^2+b^2}}\sin(bx+c)\right]$$

Let $\dfrac{b}{\sqrt{a^2+b^2}} = \sin\alpha$, $\dfrac{b}{\sqrt{a^2+b^2}} = \cos\alpha$, $\alpha = \tan^{-1}\left(\dfrac{b}{a}\right)$ . . . (I)

$$\frac{dy}{dx} = e^{ax}\sqrt{a^2+b^2}\,[\,\sin\alpha\cdot\cos(bx+c) + \sin(bx+c)\cdot\cos\alpha]$$

$$\frac{dy}{dx} = e^{ax}(a^2+b^2)^{\frac{1}{2}}\cdot\sin(bx+c+\alpha)$$

Differentiate *w. r. t. x*

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left[e^{ax}(a^2+b^2)^{\frac{1}{2}}\cdot\sin(bx+c+\alpha)\right]$$

$$= (a^2+b^2)^{\frac{1}{2}}\cdot\frac{d}{dx}[e^{ax}\cdot\sin(bx+c+\alpha)]$$

$$= (a^2+b^2)^{\frac{1}{2}}\left[e^{ax}\frac{d}{dx}[\sin(bx+c+\alpha)] + [\sin(bx+c+\alpha)]\frac{d}{dx}[e^{ax}]\right]$$

$$= (a^2+b^2)^{\frac{1}{2}}\left[e^{ax}\cos(bx+c+\alpha)\frac{d}{dx}(bx+c+\alpha) + \sin(bx+c+\alpha)\cdot e^{ax}\frac{d}{dx}(ax)\right]$$

$$= e^{ax}(a^2+b^2)^{\frac{1}{2}}\,[b\cos(bx+c+\alpha) + a\sin(bx+c+\alpha)]$$

$$= e^{ax}(a^2+b^2)^{\frac{1}{2}}\sqrt{a^2+b^2}\left[\frac{b}{\sqrt{a^2+b^2}}\cos(bx+c+\alpha) + \frac{a}{\sqrt{a^2+b^2}}\sin(bx+c+\alpha)\right]$$

$$\frac{d^2y}{dx^2} = e^{ax}(a^2+b^2)^{\frac{2}{2}}\,[\sin\alpha\cos(bx+c+\alpha) + \sin(bx+c+\alpha)\cos\alpha] \quad \text{. . . [from (I)]}$$

$$\frac{d^2y}{dx^2} = e^{ax}(a^2+b^2)^{\frac{2}{2}}\cdot\sin(bx+c+2\alpha)$$

Similarly,

$$\frac{d^3y}{dx^3} = e^{ax}(a^2+b^2)^{\frac{3}{2}}\cdot\sin(bx+c+3\alpha)$$

In general n^{th} order derivative will be

$$\frac{d^ny}{dx^n} = e^{ax}(a^2+b^2)^{\frac{n}{2}}\cdot\sin(bx+c+n\alpha) \text{ where } \alpha = \tan^{-1}\left(\frac{b}{a}\right).$$

EXERCISE 1.5

(1) Find the second order derivative of the following :

(i) $2x^5 - 4x^3 - \frac{2}{x^2} - 9$ (ii) $e^{2x} \cdot \tan x$

(iii) $e^{4x} \cdot \cos 5x$ (iv) $x^3 \log x$

(v) $\log (\log x)$ (iv) x^x

(2) Find $\frac{d^2y}{dx^2}$ of the following :

(i) $x = a\,(\theta - \sin\theta),\ y = a\,(1 - \cos\theta)$

(ii) $x = 2at^2,\ y = 4at$

(iii) $x = \sin\theta,\ y = \sin^3\theta$ when $\theta = \frac{\pi}{2}$

(iv) $x = a\cos\theta,\ y = b\sin\theta$ at $\theta = \frac{\pi}{4}$

(3) (i) If $x = at^2$ and $y = 2at$ then show that $xy\frac{d^2y}{dx^2} + a = 0$

(ii) If $y = e^{m\tan^{-1}x}$, show that $(1 + x^2)\frac{d^2y}{dx^2} + (2x - m)\frac{dy}{dx} = 0$

(iii) If $x = \cos t,\ y = e^{mt}$ show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - m^2y = 0$

(iv) If $y = x + \tan x$, show that $\cos^2 x \cdot \frac{d^2y}{dx^2} - 2y + 2x = 0$

(v) If $y = e^{ax} \cdot \sin(bx)$, show that $y_2 - 2ay_1 + (a^2 + b^2)\,y = 0$

(vi) If $\sec^{-1}\left(\frac{7x^3 - 5y^3}{7x^3 + 5y^3}\right) = m$, show that $\frac{d^2y}{dx^2} = 0$.

(vii) If $2y = \sqrt{x+1} + \sqrt{x-1}$, show that $4(x^2 - 1)\,y_2 + 4x\,y_1 - y = 0$.

(viii) If $y = \log\left(x + \sqrt{x^2 + a^2}\right)^m$, show that $(x^2 + a^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$

(ix) If $y = \sin(m\cos^{-1}x)$ then show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$

(x) If $y = \log(\log 2x)$, show that $x\,y_2 + y_1\,(1 + x\,y_1) = 0$.

(xi) If $x^2 + 6xy + y^2 = 10$, show that $\frac{d^2y}{dx^2} = \frac{80}{(3x+y)^3}$.

(xii) If $x = a\sin t - b\cos t,\ y = a\cos t + b\sin t$, show that $\frac{d^2y}{dx^2} = -\frac{x^2+y^2}{y^3}$.

(4) Find the n^{th} derivative of the following :

(i) $(ax + b)^m$ (ii) $\frac{1}{x}$

(iii) e^{ax+b} (iv) a^{px+q}

(v) $\log(ax + b)$ (vi) $\cos x$

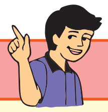
(vii) $\sin(ax + b)$ (viii) $\cos(3 - 2x)$

(ix) $\log(2x + 3)$

(x) $\frac{1}{3x - 5}$

(xi) $y = e^{ax} \cdot \cos(bx + c)$

(xii) $y = e^{8x} \cdot \cos(6x + 7)$



Let us Remember

- If a function $f(x)$ is differentiable at $x = a$ then it is continuous at $x = a$, but the converse is not true.

- **Chain Rule :** If y is differerentiable function of u and u is differerentiable function of x then y is differerentiable function of x and $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

- If $y = f(x)$ is a differentiable of x such that the inverse function $x = f^{-1}(y)$ exists then

$$\frac{dy}{dx} = \frac{1}{\frac{dy}{dx}}, \qquad \text{where } \frac{dy}{dx} \neq 0$$

- **Derivatives of Inverse Trigonometric functions :**

$f(x)$	$\sin^{-1} x$	$\cos^{-1} x$	$\tan^{-1} x$	$\cot^{-1} x$	$\sec^{-1} x$	$\operatorname{cosec}^{-1} x$
$f'(x)$	$\dfrac{1}{\sqrt{1-x^2}}$, $\lvert x\rvert < 1$	$-\dfrac{1}{\sqrt{1-x^2}}$, $\lvert x\rvert < 1$	$\dfrac{1}{1+x^2}$ $x \in R$	$-\dfrac{1}{1+x^2}$ $x \in R$	$\dfrac{1}{x\sqrt{x^2-1}}$ $\lvert x\rvert < 1$	$-\dfrac{1}{x\sqrt{x^2-1}}$ $\lvert x\rvert < 1$

- This is a simple shortcut to find the derivative of (function) $^{(\text{function})}$

$$\frac{d}{dx} f^{g} = f^{g}\left[\frac{g}{f} \cdot f' + (\log f) \cdot g'\right]$$

- If $y = f(t)$ and $y = g(t)$ is a differentiable of t such that y is a function of x then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \qquad \text{where } \frac{dx}{dt} \neq 0$$

- Implict function of the form $x^m y^n = (x+y)^{m+n}$, $m, n \in R$ always have the first order derivative $\dfrac{dy}{dx} = \dfrac{y}{x}$ and second order derivative $\dfrac{d^2y}{dx^2} = 0$

MISCELLANEOUS EXERCISE 1

(I) Choose the correct option from the given alternatives :

(1) Let $f(1) = 3, f'(1) = -\dfrac{1}{3}, g(1) = -4$ and $g'(1) = -\dfrac{8}{3}$. The derivative of $\sqrt{[f(x)]^2 + [g(x)]^2}$ *w. r. t.* x at $x = 1$ is

(A) $-\dfrac{29}{15}$ (B) $\dfrac{7}{3}$ (C) $\dfrac{31}{15}$ (D) $\dfrac{29}{15}$

(2) If $y = \sec(\tan^{-1} x)$ then $\dfrac{dy}{dx}$ at $x = 1$, is equal to :

(A) $\dfrac{1}{2}$ (B) 1 (C) $\dfrac{1}{\sqrt{2}}$ (D) $\sqrt{2}$

(3) If $f(x) = \sin^{-1}\left(\dfrac{4^{x+\frac{1}{2}}}{1+2^{4x}}\right)$, which of the following is not the derivative of $f(x)$

(A) $\dfrac{2 \cdot 4^x \log 4}{1 + 4^{2x}}$ (B) $\dfrac{4^{x+1} \log 2}{1 + 4^{2x}}$ (C) $\dfrac{4^{x+1} \log 4}{1 + 4^{4x}}$ (D) $\dfrac{2^{2(x+1)} \log 2}{1 + 2^{4x}}$

(4) If $x^y = y^x$, then $\dfrac{dy}{dx} = \ldots$

(A) $\dfrac{x(x \log y - y)}{y(y \log x - x)}$ (B) $\dfrac{y(y \log x - x)}{x(x \log y - y)}$ (C) $\dfrac{y^2(1 - \log x)}{x^2(1 - \log y)}$ (D) $\dfrac{y(1 - \log x)}{x(1 - \log y)}$

(5) If $y = \sin(2 \sin^{-1} x)$, then $\dfrac{dy}{dx} = \ldots$

(A) $\dfrac{2 - 4x^2}{\sqrt{1 - x^2}}$ (B) $\dfrac{2 + 4x^2}{\sqrt{1 - x^2}}$ (C) $\dfrac{4x^2 - 1}{\sqrt{1 - x^2}}$ (D) $\dfrac{1 - 2x^2}{\sqrt{1 - x^2}}$

(6) If $y = \tan^{-1}\left(\dfrac{x}{1 + \sqrt{1 - x^2}}\right) + \sin\left[2 \tan^{-1}\left(\sqrt{\dfrac{1 - x}{1 + x}}\right)\right]$, then $\dfrac{dy}{dx} = \ldots$

(A) $\dfrac{x}{\sqrt{1 - x^2}}$ (B) $\dfrac{1 - 2x}{\sqrt{1 - x^2}}$ (C) $\dfrac{1 - 2x}{2\sqrt{1 - x^2}}$ (D) $\dfrac{1 - 2x^2}{\sqrt{1 - x^2}}$

(7) If y is a function of x and $\log(x + y) = 2xy$, then the value of $y'(0) = \ldots$

(A) 2 (B) 0 (C) -1 (D) 1

(8) If g is the inverse of a function f and $f'(x) = \dfrac{1}{1 + x^7}$, then the value of $g'(x)$ is equal to :

(A) $1 + x^7$ (B) $\dfrac{1}{1 + [g(x)]^7}$ (C) $1 + [g(x)]^7$ (D) $7x^6$

(9) If $x\sqrt{y + 1} + y\sqrt{x + 1} = 0$ and $x \neq y$ then $\dfrac{dy}{dx} = \ldots$

(A) $\dfrac{1}{(1 + x)^2}$ (B) $-\dfrac{1}{(1 + x)^2}$ (C) $(1 + x)^2$ (D) $-\dfrac{x}{x + 1}$

(10) If $y = \tan^{-1}\left(\sqrt{\dfrac{a - x}{a + x}}\right)$, where $-a < x < a$ then $\dfrac{dy}{dx} = \ldots$

(A) $\dfrac{x}{\sqrt{a^2 - x^2}}$ (B) $\dfrac{a}{\sqrt{a^2 - x^2}}$ (C) $-\dfrac{1}{2\sqrt{a^2 - x^2}}$ (D) $\dfrac{1}{2\sqrt{a^2 - x^2}}$

(11) If $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ then $\left[\frac{d^2y}{dx^2}\right]_{\theta=\frac{\pi}{4}} = \ldots$

(A) $\frac{8\sqrt{2}}{a\pi}$ (B) $-\frac{8\sqrt{2}}{a\pi}$ (C) $\frac{a\pi}{8\sqrt{2}}$ (D) $\frac{4\sqrt{2}}{a\pi}$

(12) If $y = a\cos(\log x)$ and $A\frac{d^2y}{dx^2} + B\frac{dy}{dx} + C = 0$, then the values of A, B, C are ...

(A) $x^2, -x, -y$ (B) x^2, x, y (C) $x^2, x, -y$ (D) $x^2, -x, y$

(II) Solve the following :

(1)

$f(x) = -x,$	for $-2 \le x < 0$	$g(x) = 6 - 3x,$	for $0 \le x \le 2$
$= 2x,$	for $0 \le x \le 2$	$= \frac{2x-4}{3},$	for $2 < x \le 7$
$= \frac{18-x}{4},$	for $2 < x \le 7$		

Let $u(x) = f[g(x)]$, $v(x) = g[f(x)]$ and $w(x) = g[g(x)]$.

Find each derivative at $x = 1$, if it exists i.e. find $u'(1)$, $v'(1)$ and $w'(1)$. if it doesn't exist then explain why ?

(2) The values of $f(x)$, $g(x)$, $f'(x)$ and $g'(x)$ are given in the following table.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	3	2	-3	4
2	2	-1	-5	-4

Match the following.

A Group - Function	B Group - Derivative
(A) $\frac{d}{dx}[f(g(x))]$ at $x = -1$	1. -16
(B) $\frac{d}{dx}[g(f(x) - 1)]$ at $x = -1$	2. 20
(C) $\frac{d}{dx}[f(f(x) - 3)]$ at $x = 2$	3. -20
(D) $\frac{d}{dx}[g(g(x))]$ at $x = 2$	4. 15
	5. 12

(3) Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	$\frac{1}{3}$
1	3	-4	$-\frac{1}{3}$	$-\frac{8}{3}$

(i) The derivative of $f[g(x)]$ *w. r. t.* x at $x = 0$ is ......

(ii) The derivative of $g[f(x)]$ *w. r. t.* x at $x = 0$ is ......

(iii) The value of $\left[\frac{d}{dx}[x^{10} + f(x)]^{-2}\right]_{x=1}$ is ......

(iv) The derivative of $f[(x + g(x)]$ *w. r. t.* x at $x = 0$ is ......

(4) Differentiate the following *w. r. t.* x

(i) $\sin\left[2\tan^{-1}\left(\sqrt{\frac{1-x}{1+x}}\right)\right]$

(ii) $\sin^2\left[\cot^{-1}\left(\sqrt{\frac{1+x}{1-x}}\right)\right]$

(iii) $\tan^{-1}\left[\frac{\sqrt{x}\,(3-x)}{1-3x}\right]$

(iv) $\cos^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{2}\right)$

(v) $\tan^{-1}\left(\frac{x}{1+6x^2}\right)+\cot^{-1}\left(\frac{1-10x^2}{7x}\right)$

(vi) $\tan^{-1}\left[\sqrt{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}}\right]$

(5) (i) If $\sqrt{y+x}+\sqrt{y-x}=c$, then show that $\frac{dy}{dx}=\frac{y}{x}-\sqrt{\frac{y^2}{x^2}-1}$.

(ii) If $x\sqrt{1-y^2}+y\sqrt{1-x^2}=1$, then show that $\frac{dy}{dx}=-\sqrt{\frac{1-y^2}{1-x^2}}$.

(iii) If $x\sin(a+y)+\sin a\cos(a+y)=0$, then show that $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$.

(iv) If $\sin y=x\sin(a+y)$, then show that $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$.

(v) If $x=e^{\frac{x}{y}}$, then show that $\frac{dy}{dx}=\frac{x-y}{x\log x}$.

(vi) If $y=f(x)$ is a differentiable function then show that $\frac{d^2x}{dy^2}=-\left(\frac{dy}{dx}\right)^{-3}\cdot\frac{d^2y}{dx^2}$.

(6) (i) Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ *w. r. t.* $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$.

(ii) Differentiate $\log\left(\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\right)$ *w. r. t.* $\cos(\log x)$.

(iii) Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ *w. r. t.* $\cos^{-1}\left(\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}\right)$.

(7) (i) If $y^2=a^2\cos^2 x+b^2\sin^2 x$, show that $y+\frac{d^2y}{dx^2}=\frac{a^2b^2}{y^3}$.

(ii) If $\log y=\log(\sin x)-x^2$, show that $\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+(4x^2+3)\,y=0$.

(iii) If $x=a\cos\theta$, $y=b\sin\theta$, show that $a^2\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]+b^2=0$.

(iv) If $y=A\cos(\log x)+B\sin(\log x)$, show that $x^2y_2+xy_1+y=0$.

(v) If $y=A\,e^{mx}+B\,e^{nx}$, show that $y_2-(m+n)\,y_1+(mn)\,y=0$.

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2. APPLICATIONS OF DERIVATIVES

Let us Study

- Applications of Drivatives to Tangents and Normals
- Derivative as a rate measure
- Approximations
- Rolle's Theorem and Lagrange's Mean Value Theorem.
- Increasing and Decreasing Functions
- Maxima and Minima

Let us Recall

- Continuous functions.
- Derivatives of Composite, Inverse Trigonometric, Logarithmic, Parametric functions.
- Relation between derivative and slope.
- Higher Order Derivatives.

2.1.1 Introduction :

In the previous chapter we have studied the derivatives of various functions such as composite functions, Inverse Trigonometric functions, Logarithmic functions etc. and also the relation between Derivative and slope of the tangent. In this chapter we are going to study various applications of differentiation such as application to (i) Geometry, (ii) Rate measure (iii) Approximations (iv) Rolle's Theorem and Lagrange's Mean Value Therorem (v) Increasing and Decreasing functions and (vi) Maxima and Minima.

Let us Learn

2.1.2 Application of Derivative in Geometry :

In the previous chapter we have studied the relation between derivative and slope of a line or slope of a tangent to the curve at a given point on it.

Let $y = f(x)$ be a continuous function of x representing a curve in XY- plane and $P\ (x_1, y_1)$ be any point on the curve.

Then $\left[\dfrac{dy}{dx}\right]_{(x_1, y_1)} = [f'(x)]_{(x_1, y_1)}$ represents slope, also called gradient, of the tangent to the curve at $P(x_1, y_1)$. The normal is perpendicular to the tangent. Hence, the slope of the normal at P will be the negative of reciprocal of the slope of tangent at P. Let m and m' be the slopes of tangent and normal respectively, then $m = \left[\dfrac{dy}{dx}\right]_{(x_1, y_1)}$ and $m' = -\dfrac{1}{\left[\frac{dy}{dx}\right]_{(x_1, y_1)}}$ if $\left[\dfrac{dy}{dx}\right]_{(x_1, y_1)} \neq 0.$

Equation of tangent at $P(x_1, y_1)$ is given by $y - y_1 = m(x - x_1)$ i.e. $y - y_1 = \left[\dfrac{dy}{dx}\right]_{(x_1, y_1)}(x - x_1)$

and equation of normal at $P(x_1, y_1)$ is given by

$y - y_1 = m'(x - x_1)$ where $m' = -\dfrac{1}{\left[\frac{dy}{dx}\right]_{(x_1, y_1)}}$

SOLVED EXAMPLES

Ex. 1 : Find the equations of tangent and normal to the curve at the given point on it.

(i) $y = 2x^3 - x^2 + 2$ at $\left(\dfrac{1}{2}, 2\right)$

(ii) $x^3 + 2x^2y - 9xy = 0$ at $(2, 1)$

(iii) $x = 2\sin^3\theta, y = 3\cos^3\theta$ at $\theta = \dfrac{\pi}{4}$

Solution :

(i) Given that : $y = 2x^3 - x^2 + 2$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3 - x^2 + 2) = 6x^2 - 2x$$

Slope of tangent at $\left(\dfrac{1}{2}, 2\right) = m = 6\left(\dfrac{1}{2}\right)^2 - 2\left(\dfrac{1}{2}\right)$

$$\therefore \quad m = \frac{1}{2}$$

Slope of normal at $\left(\dfrac{1}{2}, 2\right) = m' = -2$

Equation of tangent is given by

$$y - 2 = \frac{1}{2}\left(x - \frac{1}{2}\right) \Rightarrow 2y - 4 = \frac{2x - 1}{2}$$

$$4y - 8 = 2x - 1 \Rightarrow 2x - 4y + 7 = 0$$

Equation of normal is given by

$$y - 2 = -2\left(x - \frac{1}{2}\right) \Rightarrow y - 2 = -2x + 1$$

$$2x + y - 3 = 0$$

(ii) Given that : $x^3 + 2x^2y - 9xy = 0$

Differentiate *w. r. t. x*

$$3x^2 + 2\left(x^2\frac{dy}{dx} + y\frac{d}{dx}(x^2)\right) - 9\left(x\frac{dy}{dx} + y\frac{d}{dx}(x)\right) = 0$$

$$3x^2 + 2x^2\frac{dy}{dx} + 4xy - 9x\frac{dy}{dx} - 9y = 0$$

$$(2x^2 - 9x)\frac{dy}{dx} = 9y - 4xy - 3x^2 \therefore \frac{dy}{dx} = \frac{9y - 4xy - 3x^2}{2x^2 - 9x}$$

Slope of tangent at $(2, 1)$

$$\left(\frac{dy}{dx}\right)_{(1, 2)} = m = \frac{9(1) - 4(2)(1) - 3(4)}{2(4) - 9(1)} = \frac{9 - 8 - 12}{8 - 9}$$

$$m = \frac{-11}{-1} \quad \therefore m = 11$$

Slope of normal at $(2, 1) = m' = -\dfrac{1}{11}$

Equation of tangent is given by

$$y - 1 = 11(x - 2) \Rightarrow 11x - y - 21 = 0$$

Equation of normal is given by

$$y - 1 = -\frac{1}{11}(x - 2) \Rightarrow 11y - 11 = -x + 2$$

$$x + 11y - 13 = 0$$

(iii) Given that : $y = 3\cos^3\theta$

Differentiate *w. r. t.* θ

$$\frac{dy}{d\theta} = 3\frac{d}{d\theta}(\cos\theta)^3 = 9\cos^2\theta\frac{d}{d\theta}(\cos\theta)$$

$$\therefore \quad \frac{dy}{d\theta} = -9\cos^2\theta\sin\theta$$

Now, $x = 2\sin^3\theta$

Differentiate *w. r. t.* θ

$$\frac{dx}{d\theta} = 2\frac{d}{d\theta}(\sin\theta)^3 = 6\sin^2\theta\frac{d}{d\theta}(\sin\theta)$$

$$\therefore \quad \frac{dx}{d\theta} = 6\sin^2\theta\cos\theta$$

We know that

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{9\cos^2\theta\sin\theta}{6\sin^2\theta\cos\theta} = -\frac{3}{2}\cot\theta$$

Slope of tangent at $\theta = \frac{\pi}{4}$ is

$$\left(\frac{dy}{dx}\right)_{\theta=\frac{\pi}{4}} = m = -\frac{3}{2}\cot\left(\frac{\pi}{4}\right) = -\frac{3}{2}$$

Slope of normal at $\left(\theta = \frac{\pi}{4}\right) = m' = \frac{2}{3}$

When, $\theta = \frac{\pi}{4}$

$$x = 2\sin^3\left(\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{1}{\sqrt{2}}$$

$$y = 3\cos^3\left(\frac{\pi}{4}\right) = 3\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{3}{2\sqrt{2}}$$

$\therefore$ The point is $P = \left(\frac{1}{\sqrt{2}}, \frac{3}{2\sqrt{2}}\right)$

Equation of tangent at P is given by

$$y - \frac{3}{2\sqrt{2}} = -\frac{3}{2}\left(x - \frac{1}{\sqrt{2}}\right) \Rightarrow y - \frac{3}{2\sqrt{2}} = -\frac{3x}{2} + \frac{3}{2\sqrt{2}}$$

$$\frac{3x}{2} + y - \frac{3}{\sqrt{2}} = 0 \qquad \text{i.e. } 3x + 2y - 3\sqrt{2} = 0$$

Equation of normal is given by

$$y - \frac{3}{2\sqrt{2}} = \frac{2}{3}\left(x - \frac{1}{\sqrt{2}}\right) \Rightarrow y - \frac{3}{2\sqrt{2}} = \frac{2x}{2} - \frac{2}{3\sqrt{2}}$$

$$\frac{2x}{3} - y - \frac{2}{3\sqrt{2}} + \frac{3}{2\sqrt{2}} = 0$$

i.e. $4\sqrt{2}x - 6\sqrt{2}y + 5 = 0$ . . . [Multiply by $6\sqrt{2}$]

Ex. 2 : Find points on the curve given by $y = x^3 - 6x^2 + x + 3$ where the tangents are parallel to the line $y = x + 5$.

Solution : Equation of curve is $y = x^3 - 6x^2 + x + 3$

Differentiate *w. r. t. x*

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 - 6x^2 + x + 3) = 3x^2 - 12x + 1$$

Given that the tangent is parallel to $y = x + 5$ whose slope is 1.

$\therefore$ Slope of tangent $= \frac{dy}{dx} = 1 \Rightarrow 3x^2 - 12x + 1 = 1$

$3x(x - 4) = 0$ so, $x = 0$ or $x = 4$

When $x = 0$, $y = (0)^3 - 6(0)^2 + (0) + 3 = 3$

When $x = 4$, $y = (4)^3 - 6(4)^2 + (4) + 3 = -25$

So the required points on the curve are $(0, 3)$ and $(4, -25)$.

2.1.3 Derivative as a Rate measure :

If $y = f(x)$ is the given function then a change in x from x_1 to x_2 is generally denoted by $\delta x = x_2 - x_1$ and the corresponding change in y is denoted by $\delta y = f(x_2) - f(x_1)$. The difference quotient $\frac{\delta y}{\delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ is called the **average rate of change** with respect to x. This can also be interpreted geometrically as the slope of the secant line joining the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ on the graph of function $y = f(x)$.

Consider the average rate of change over smaller and smaller intervals by letting x_2 to approach x_1 and therefore letting δx to approach 0. The limit of these average rates of change is called the **instantaneous rate of change** of y with respect to x at $x = x_1$, which is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$. Therefore instantaneous rate of change is given by

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{x_2 \to x_1}\left(\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right)$$

We recognize this limit as being the derivative of $f(x)$ at $x = x_1$, i.e. $f'(x_1)$. We know that one interpretation of the derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect x when $x = a$. The other interpretation is $f(x)$ at $f'(a)$ is the slope of the tangent to $y = f(x)$ at $(a, f(a))$.

SOLVED EXAMPLES

Ex. 1 : A stone is dropped in to a quiet lake and waves in the form of circles are generated, radius of the circular wave increases at the rate of 5 cm/ sec. At the instant when the radius of the circular wave is 8 cm, how fast the area enclosed is increasing ?

Solution : Let R be the radius and A be the area of the circular wave.

$\therefore \quad A = \pi \cdot R^2$

Differentiate *w. r. t. t*

$$\frac{dA}{dt} = \pi \frac{d}{dt}(R^2)$$

$$\frac{dA}{dt} = 2\pi R \frac{dR}{dt} \qquad \ldots \text{(I)}$$

Given that $\frac{dR}{dt} = 5$ cm/sec.

Thus when $R = 8$ cm, from (I) we get,

$$\frac{dA}{dt} = 2\pi(8)(5) = 80\pi$$

Hence when the radius of the circular wave is 8 cm, the area of the circular wave is increasing at the rate of 80π cm^2/ sec.

Ex. 2 : The volume of the spherical ball is increasing at the rate of 4π cc/sec. Find the rate at which the radius and the surface area are changing when the volume is 288π cc.

Solution : Let R be the radius, S be the surface area and V be the volume of the spherical ball.

$$V = \frac{4}{3}\pi R^3 \qquad \ldots \text{(I)}$$

Differentiate *w. r. t. t*

$$\frac{dV}{dt} = \frac{4\pi}{3} \cdot \frac{d}{dt}(R^3)$$

$$4\pi = \frac{4\pi}{3} \cdot 3R^2 \frac{dR}{dt} \qquad \ldots \left[\text{Given } \frac{dV}{dt} = 4\pi \text{ cc/sec }\right]$$

$$\frac{dR}{dt} = \frac{1}{R^2} \qquad \ldots \text{(II)}$$

When volume is 288π cc.

i.e. $\frac{4}{3}\pi \cdot R^3 = 288\pi$ we get, $R^3 = 216 \Rightarrow R = 6$ ... [From (I)]

From (II) we get, $\frac{dR}{dt} = \frac{1}{36}$

So, the radius of the spherical ball is increasing at the rate of $\frac{1}{36}$ cc/sec.

Now, $S = 4\pi R^2$

Differentiate *w. r. t. t.*

$$\frac{dS}{dt} = 4\pi \frac{d}{dt}(R^2) = 8\pi R \frac{dR}{dt}$$

So, when $R = 6$ cm

$$\left[\frac{dS}{dt}\right]_{R=6} = 8\pi(6)\frac{1}{36} = \frac{4\pi}{3}$$

$\therefore$ Surface area is increasing at the rate of $\frac{4\pi}{3}$ cm^2/ sec.

Ex. 3 : Water is being poured at the rate of 36 m^3/sec in to a cylindrical vessel of base radius 3 meters. Find the rate at which water level is rising.

Solution : Let R be the radius of the base, H be the height and V be the volume of the cylindrical vessel at any time t. R, V and H are functions of t.

$$V = \pi R^2 H$$

$$V = \pi(3)^2 H = 9\pi H \quad \ldots [\text{ Given : } R = 3]$$

Differentiate $w. r. t. t$

$$\frac{dV}{dt} = 9\pi \frac{dH}{dt}$$

$$\frac{dH}{dt} = \frac{1}{9\pi} \cdot \frac{dV}{dt} \quad \ldots (\text{I})$$

Given that,

$$\frac{dV}{dt} = 36 \text{ m}^3/\text{sec} \quad \ldots (\text{II})$$

From (I) we get, $\quad \dfrac{dH}{dt} = \dfrac{1}{9\pi} \cdot (36) = \dfrac{4}{\pi}$

$\therefore$ Water level is rising at the rate of $\dfrac{4}{\pi}$ meter/sec.

Ex. 4 : A man of height 180 cm is moving away from a lamp post at the rate of 1.2 meters per second. If the height of the lamp post is 4.5 meters, find the rate at which (i) his shadow is lengthening. (ii) the tip of the shadow is moving.

Solution : Let OA be the lamp post, MN be the man, $MB = x$ be the length of shadow and $OM = y$ be the distance of the man from the lamp post at time t. Given that man is moving away from the lamp post at the rate of 1.2 meter/sec. x and y are functions of t.

Hence $\dfrac{dy}{dt} = 1.2$. The rate at which shadow is lengthening $= \dfrac{dx}{dt}$.

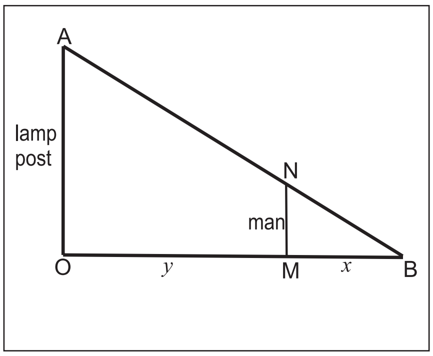
B is the tip of the shadow and it is at a distance of $(x + y)$ from the post.

$$\frac{x}{1.8} = \frac{x+y}{4.5} \quad \text{i.e. } 45x = 18x + 18y \quad \text{i.e. } 27x = 18y$$

$$\therefore \quad x = \frac{2y}{3}$$

Differentiate $w. r. t. t$

$$\frac{dx}{dt} = \frac{2}{3} \times \frac{dy}{dt} = \frac{2}{3} \times 1.2 = 0.8 \text{ meter/sec.}$$



rate at which the tip of the shadow is moving is given by

$$\frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt}$$

$$\therefore \quad \frac{d}{dt}(x+y) = 0.8 + 1.2 = 2 \text{ meter/sec.}$$

Shadow is lengthening at the rate of 0.8 meter/ sec. and its tip is moving at the rate of 2 meters/sec.

2.1.4 Velocity, Acceleration and Jerk :

If $s = f(t)$ is the desplacement function of a particle that moves along a straight line, then $f'(t)$ is the rate of change of the displacement s with respect to the time t. In other words, $f'(t)$ is the **velocity** of the particle. The **speed** of the particle is the absolute value of the velocity, that is, $|f'(t)|$.

The rate of change of velocity with respect to time is valled the **acceleration** of the particle denoted by $a(t)$. Thus the acceleration function is the derivative of the velocity function and is therefore the second derivative of the position function $s = f(t)$.

Thus, $a = \dfrac{dy}{dt} = \dfrac{d^2s}{dt^2}$ i.e. $a(t) = v'(t) = s''(t)$.

Let us consider the third derivative of the position function $s = f(t)$ of an object that moves along a straight line. $s'''(t) = v''(t) = a'(t)$ is derivative of the acceleration function and is called the **Jerk** (j).

Thus, $j = \dfrac{da}{dt} = \dfrac{d^3s}{dt^3}$. Hence the jerk j is the rate of change of acceleration. It is aptly named because a jerk means a sudden change in acceleration, which causes an abrupt movement in a vehicle.

SOLVED EXAMPLES

Ex. 1 : A car is moving in such a way that the distance it covers, is given by the equation $s = 4t^2 + 3t$ where s is in meters and t is in seconds. What would be the velocity and the acceleration of the car at time $t = 20$ second ?

Solution : Let v be the velocity and a be the acceleration of the car.

Distance traveled by the car is given by

$s = 4t^2 + 3t$

Differentiate *w. r. t. t.*

$\therefore$ Velocity of the car is given by

$$v = \frac{ds}{dt} = \frac{d}{dt}(4t^2 + 3t) = 8t + 3 \qquad \ldots \text{(I)}$$

and Acceleration of the car is given by

$$a = \frac{d}{dt}\left(\frac{dv}{dt}\right) = \frac{d}{dt}(8t + 3) = 8 \qquad \ldots \text{(II)}$$

Put $t = 20$ in (I),

$\therefore$ Velocity of the car, $v_{t=20} = 8(20) + 3 = 163$ m/sec.

Put $t = 20$ in (II),

$\therefore$ Acceleration of the car, $a_{t=20} = 8$ m/sec^2.

Note : In this problem, the acceleration is independent of time. Such a motion is said to be uniformly accelerated motion.

Ex. 2 : The displacement of a particle at time t is given by $s = 2t^3 - 5t^2 + 4t - 3$. Find the time when the acceleration is 14 ft/ sec^2, the velocity and the displacement at that time.

Solution : Displacement of a particle is given by

$$s = 2t^3 - 5t^2 + 4t - 3 \qquad \ldots \text{(I)}$$

Differentiate *w. r. t. t.*

Velocity, $v = \dfrac{ds}{dt} = \dfrac{d}{dt}(2t^3 - 5t^2 + 4t - 3)$

$$\therefore \quad v = 6t^2 - 10t + 4 \qquad \ldots \text{(II)}$$

Acceleration, $a = \dfrac{dv}{dt} = \dfrac{d}{dt}(6t^2 - 10t + 4)$

$$\therefore \quad a = 12t - 10 \qquad \ldots \text{(III)}$$

Given : Acceleration = 14 ft/ sec^2.

$\therefore$ $12t - 10 = 14 \Rightarrow 12t = 24 \Rightarrow t = 2$

So, the particle reaches an acceleration of 14 ft/ sec^2 in 2 seconds.

Velocity, when $t = 2$ is

$\therefore$ $v_{t=2} = 6(2)^2 - 10(2) + 4 = 8$ ft/ sec.

Displacement when $t = 2$ is

$\therefore$ $s_{t=2} = 2(2)^3 - 5(2)^2 + 4(2) - 3 = 1$ foot.

Hence the velocity is 8 ft/ sec and the displacement is 1 foot after 2 seconds.

EXERCISE 2.1

(1) Find the equations of tangents and normals to the curve at the point on it.

(i) $y = x^2 + 2e^x + 2$ at $(0, 4)$

(ii) $x^3 + y^3 - 9xy = 0$ at $(2, 4)$

(iii) $x^2 - \sqrt{3}xy + 2y^2 = 5$ at $(\sqrt{3}, 2)$

(iv) $2xy + \pi \sin y = 2\pi$ at $\left(1, \frac{\pi}{2}\right)$

(v) $x \sin 2y = y \cos 2x$ at $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

(vi) $x = \sin\theta$ and $y = \cos 2\theta$ at $\theta = \frac{\pi}{6}$

(vii) $x = \sqrt{t},\ y = t - \frac{1}{\sqrt{t}}$ at $t = 4$.

(2) Find the point on the curve $y = \sqrt{x-3}$ where the tangent is perpendicular to the line $6x + 3y - 5 = 0$.

(3) Find the points on the curve $y = x^3 - 2x^2 - x$ where the tangents are parallel to $3x - y + 1 = 0$.

(4) Find the equations of the tangents to the curve $x^2 + y^2 - 2x - 4y + 1 = 0$ which are parallel to the X-axis.

(5) Find the equations of the normals to the curve $3x^2 - y^2 = 8$, which are parallel to the line $x + 3y = 4$.

(6) If the line $y = 4x - 5$ touches the curve $y^2 = ax^3 + b$ at the point $(2, 3)$ find a and b.

(7) A particle moves along the curve $6y = x^2 + 2$. Find the points on the curve at which y-coordinate is changing 8 times as fast as the X-coordinate.

(8) A spherical soap bubble is expanding so that its radius is increasing at the rate of 0.02 cm/sec. At what rate is the surface area is increasing, when its radius is 5 cm?

(9) The surface area of a spherical balloon is increasing at the rate of 2 cm^2/ sec. At what rate the volume of the balloon is increasing when radius of the balloon is 6 cm?

(10) If each side of an equilateral triangle increases at the rate of $\sqrt{2}$ cm/ sec, find the rate of increase of its area when its side of length 3 cm .

(11) The volume of a sphere increase at the rate of 20 cm^3/ sec. Find the rate of change of its surface area when its radius is 5 cm.

(12) The edge of a cube is decreasing at the rate of 0.6 cm/sec. Find the rate at which its volume is decreasing when the edge of the cube is 2 cm.

(13) A man of height 2 meters walks at a uniform speed of 6 km/hr away from a lamp post of 6 meters high. Find the rate at which the length of the shadow is increasing.

(14) A man of height 1.5 meters walks toward a lamp post of height 4.5 meters, at the rate of $\left(\frac{3}{4}\right)$ meter/sec. Find the rate at which (i) his shadow is shortening. (ii) the tip of the shadow is moving.

(15) A ladder 10 meter long is leaning against a vertical wall. If the bottom of the ladder is pulled horizontally away from the wall at the rate of 1.2 meters per second, find how fast the top of the ladder is sliding down the wall when the bottom is 6 meters away from the wall.

(16) If water is poured into an inverted hollow cone whose semi-vertical angel is 30°, so that its depth (measured along the axis) increases at the rate of 1 cm/ sec. Find the rate at which the volume of water increasing when the depth is 2 cm.

2.2.1 Approximations

If $f(x)$ is a differentiable function of x, then its derivative at $x = a$ is given by

$$f'(a) = \lim_{h \to 0} \left[\frac{f(a+h) - f(a)}{h}\right]$$

Here we use $\doteqdot$ sign for approximation.

For a sufficiently small h we have,

$$f'(a) \doteqdot \left[\frac{f(a+h) - f(a)}{h}\right]$$

i.e. $\quad h f'(a) \doteqdot f(a+h) - f(a)$

$\therefore \quad f(a+h) \doteqdot f(a) + h f'(a)$

This is the formula to find the approximate value of the function at $x = a + h$, when $f'(a)$ exists. Let us solve some problems by using this formula.

SOLVED EXAMPLES

Ex. 1 : Find the approximate value of $\sqrt{64.1}$.

Solution :

Let $f(x) = \sqrt{x}$ $\quad \ldots$ (I)

Differentiate *w. r. t. x*.

$f'(x) = \dfrac{1}{2\sqrt{x}}$ $\quad \ldots$ (II)

Let $a = 64$, $h = 0.1$

For $x = a = 64$, from (I) we get

$f(a) = f(64) = \sqrt{64} = 8$ $\quad \ldots$ (III)

For $x = a = 64$, from (II) we get

$f'(a) = f'(64) = \dfrac{1}{2\sqrt{64}} = \dfrac{1}{16}$

$\therefore \quad f'(a) = 0.0625$ $\quad \ldots$ (IV)

We have, $f(a+h) \doteqdot f(a) + h f'(a)$

$f(64 + 0.1) \doteqdot f(64) + (0.1)\cdot f'(64)$

$f(64.1) \doteqdot 8 + (0.1)\cdot(0.0625)$ . . . [From (III) and (IV)]

$\doteqdot 8 + 0.00625$

$\therefore \quad f(64.1) = \sqrt{64.1} \doteqdot 8.00625$

Ex. 2 : Find the approximate value of $(3.98)^3$.

Solution :

Let $f(x) = x^3$ $\quad \ldots$ (I)

Differentiate *w. r. t. x*.

$f'(x) = 3x^2$ $\quad \ldots$ (II)

Let $a = 4$, $h = -0.02$

For $x = a = 4$, from (I) we get

$f(a) = f(4) = (4)^3 = 64$ $\quad \ldots$ (III)

For $x = a = 4$, from (II) we get

$f'(a) = f'(4) = 3(4)^2 = 48$ $\quad \ldots$ (IV)

We have, $f(a+h) \doteqdot f(a) + h f'(a)$

$f[4 + (-0.02)] \doteqdot f(4) + (-0.02)\cdot f'(4)$

$f(3.98) \doteqdot 64 + (-0.02).(48)$ . . . [From (III) and (IV)]

$f(3.98) \doteqdot 64 - 0.96$

$\therefore \quad f(3.98) = (3.98)^3 \doteqdot 63.04$

Ex. 3 : Find the approximate value of $\sin(30° \, 30')$. Given that $1° = 0.0175^c$ and $\cos 30° = 0.866$.

Solution : Let $f(x) = \sin x$ $\quad \ldots$ (I)

Differentiate $w.\ r.\ t.\ x$.

$f'(x) = \cos x$

Now, $30° \, 30' = 30° + 30' = 30° + \left(\frac{1}{2}\right)^{\circ}$

$$= \frac{\pi}{6} + \frac{0.1750^c}{2}$$

$$30° \, 30' = \frac{\pi}{6} + 0.00875 \quad \ldots \text{(II)}$$

Let $a = \frac{\pi}{6}$, $h = 0.00875$

For $x = a = \frac{\pi}{6}$, from (I) we get

$$f(a) = f\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} = 0.5 \ldots \text{(III)}$$

For $x = a = \frac{\pi}{6}$, from (II) we get

$$f'(a) = f'\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = 0.866 \ldots \text{(IV)}$$

We have, $f(a+h) \doteqdot f(a) + h f'(a)$

$$f\left(\frac{\pi}{6} + 0.00875^c\right) \doteqdot f\left(\frac{\pi}{6}\right) + (0.00875) \cdot f'\left(\frac{\pi}{6}\right)$$

$$f(30° \, 30') \doteqdot 0.5 + (0.00875) \cdot (0.866) \ldots$$

... [From (III) and (IV)]

$$\doteqdot 0.5 + 0.075775$$

$$\therefore \quad f(30° \, 30') = \sin(30° \, 30') \doteqdot 0.575775$$

Ex. 4 : Find the approximate value of $\tan^{-1}(0.99)$, Given that $\pi \doteqdot 3.1416$.

Solution : Let $f(x) = \tan^{-1} x$ $\quad \ldots$ (I)

Differentiate $w.\ r.\ t.\ x$.

$$f'(x) = \frac{1}{1 + x^2} \quad \ldots \text{(II)}$$

Let $a = 1$, $h = -0.01$

For $x = a = 1$, from (I) we get

$$f(a) = f(1) = \tan^{-1}(1) = \frac{\pi}{4} \quad \ldots \text{(III)}$$

For $x = a = 1$, from (II) we get

$$f'(a) = f'(1) = \frac{1}{1 + 1^2} = 0.5 \quad \ldots \text{(IV)}$$

We have, $f(a+h) \doteqdot f(a) + h f'(a)$

$$f[(1) + (-0.01)] \doteqdot f(1) + (-0.01) \cdot f'(1)$$

$$f(0.99) \doteqdot \frac{\pi}{4} - (0.01) \cdot (0.5) \ldots$$ [From (III) and (IV)]

$$\doteqdot \frac{\pi}{4} - 0.005$$

$$\doteqdot \frac{3.1416}{4} - 0.005$$

$$\doteqdot 0.7854 - 0.005 = 0.7804$$

$$\therefore \quad f(0.99) = \tan^{-1}(0.99) \doteqdot 0.7804$$

Ex. 5 : Find the approximate value of $e^{1.005}$. Given that $e = 2.7183$.

Solution : Let $f(x) = e^x$ $\quad \ldots$ (I)

Differentiate $w.\ r.\ t.\ x$.

$f'(x) = e^x$ $\quad \ldots$ (II)

Let $a = 1$, $h = 0.005$

For $x = a = 1$, from (I) we get

$f(a) = f(1) = e^1 = 2.7183$ $\quad \ldots$ (III)

For $x = a = 1$, from (II) we get

$f'(a) = f'(1) = e^1 = 2.7183$ $\quad \ldots$ (IV)

We have, $f(a+h) \doteqdot f(a) + h f'(a)$

$$f(1 + 0.005) \doteqdot f(1) + (0.005) \cdot f'(1)$$

$$f(1.005) \doteqdot 2.7183 + (0.005)(2.7183) \ldots$$

... [From (III) and (IV)]

$$f(1.005) \doteqdot 2.7183 + 0.0135915$$

$$\doteqdot 2.7318915$$

$$f(1.005) = e^{1.005} \doteqdot 2.73189$$

Ex. 6 : Find the approximate value of $\log_{10}(998)$. Given that $\log_{10} e = 0.4343$.

Solution : Let $f(x) = \log_{10} x = \dfrac{\log x}{\log 10}$

$\therefore \quad f(x) = (\log_{10} e)\cdot\log x \qquad \ldots \text{(I)}$

Differentiate *w. r. t. x.*

$f'(x) = \dfrac{\log_{10} e}{x} = \dfrac{0.4343}{x} \qquad \ldots \text{(II)}$

Let $a = 1000$, $h = -2$

For $x = a = 1000$, from (I) we get

$f(a) = f(1000) = \log_{10} 1000$

$\therefore \quad f(a) = 3\log_{10}10 = 3 \qquad \ldots \text{(III)}$

For $x = a = 1000$, from (II) we get

$f'(a) = f'(1000) = \dfrac{0.4343}{1000}$

$\therefore \quad f'(a) = 0.0004343 \qquad \ldots \text{(IV)}$

We have, $f(a+h) \doteqdot f(a) + h f'(a)$

$f[1000 + (-2)] \doteqdot f(1000) + (-2) f'(1000)$

$f(998) \doteqdot 3 - (2)(0.0004343) \ldots$ [From (III) and (IV)]

$\doteqdot 3 - 0.0008686$

$f(998) = \log(998) \doteqdot 2.9991314$

Ex. 7 : Find the approximate value of $f(x) = x^3 + 5x^2 - 2x + 3$ at $x = 1.98$.

Solution : Let $f(x) = x^3 + 5x^2 - 2x + 3 \qquad \ldots \text{(I)}$

Differentiate *w. r. t. x.*

$f'(x) = 3x^2 + 10x - 2 \qquad \ldots \text{(II)}$

Let $a = 2$, $h = -0.02$

For $x = a = 2$, from (I) we get

$f(a) = f(2) = (2)^3 + 5(2)^2 - 2(2) + 3$

$\therefore \quad f(a) = 27 \qquad \ldots \text{(III)}$

For $x = a = 2$, from (II) we get

$f'(a) = f'(2) = 3(2)^2 + 10(2) - 2$

$\therefore \quad f'(a) = 30 \qquad \ldots \text{(IV)}$

We have, $f(a+h) \doteqdot f(a) + h f'(a)$

$f[(2) + (-0.02)] \doteqdot f(2) + (-0.02)\cdot f'(2)$

$f(1.98) \doteqdot 27 - (0.02)\cdot(30) \ldots$ [From (III) and (IV)]

$\doteqdot 27 - 0.6$

$f(1.98) \doteqdot 26.4$

EXERCISE 2.2

(1) Find the approximate value of given functions, at required points.

(i) $\sqrt{8.95}$ (ii) $\sqrt[3]{28}$ (iii) $\sqrt[5]{31.98}$

(iv) $(3.97)^4$ (v) $(4.01)^3$

(2) Find the approximate value of

(i) $\sin(61°)$ given that $1° = 0.0174^c$, $\sqrt{3} = 1.732$

(ii) $\sin(29°\,30')$ given that $1° = 0.0175^c$, $\sqrt{3} = 1.732$

(iii) $\cos(60°\,30')$ given that $1° = 0.0175^c$, $\sqrt{3} = 1.732$

(iv) $\tan(45°\,40')$ given that $1° = 0.0175^c$.

(3) Find the approximate value of

(i) $\tan^{-1}(0.999)$ (ii) $\cot^{-1}(0.999)$

(iii) $\tan^{-1}(1.001)$

(4) Find the approximate value of

(i) $e^{0.995}$ (ii) $e^{2.1}$ given that $e^2 = 7.389$

(iii) $3^{2.01}$ given that $\log 3 = 1.0986$

(5) Find the approximate value of

(i) $\log_e(101)$ given that $\log_e 10 = 2.3026$

(ii) $\log_e(9.01)$ given that $\log 3 = 1.0986$

(iii) $\log_{10}(1016)$ given that $\log_{10} e = 0.4343$

(6) Find the approximate value of

(i) $f(x) = x^3 - 3x + 5$ at $x = 1.99$

(ii) $f(x) = x^3 + 5x^2 - 7x + 10$ at $x = 1.12$

2.3.1 Rolle's Theorem or Rolle's Lemma :

If a real-valued function f is continous on $[a, b]$, differentiable on the open interval (a, b) and $f(a) = f(b)$, then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.

Rolle's Theorem essentially states that any real-valued differentiable function that attains equal values at two distinct points on it, must have at least one stationary point somewhere in between them, that is, a point where the first derivative (the slope of the tangent line to the graph of the function) is zero.

Geometrical Significance :

Let $f(x)$ be a real valued function defined on $[a, b]$ and it is continuous on $[a, b]$. This means that we can draw the graph $f(x)$ between the values $x = a$ and $x = b$. Also $f(x)$ is differentiable on (a, b) which means the graph of $f(x)$ has a tangent at each point of (a, b). Now the existence of real number $c \in (a, b)$ such that $f'(c) = 0$ shows that the tangent to the curve at $x = c$ has slope zero, that is, tangent is parallel to X-axis since $f(a) = f(b)$.

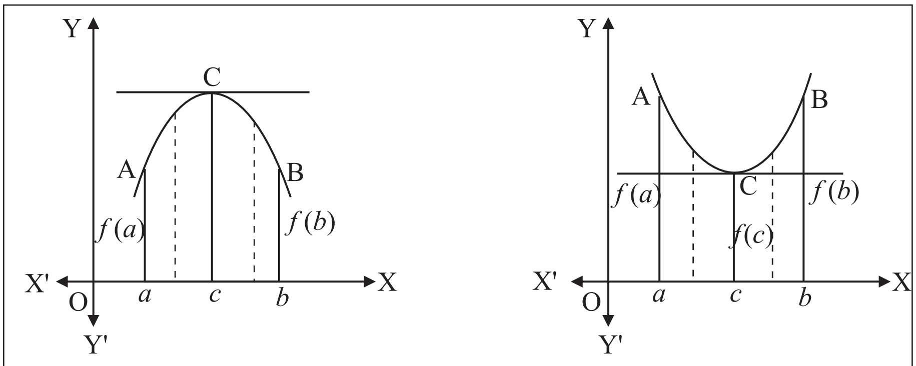


Fig. 2.3.1

SOLVED EXAMPLES

Ex. 1 : Check whether conditions of Rolle's theorem are satisfied by the following functions.

(i) $f(x) = 2x^3 - 5x^2 + 3x + 2,\ x \in \left[0, \frac{3}{2}\right]$ (ii) $f(x) = x^2 - 2x + 3,\ x \in [1, 4]$

Solution :

(i) Given that $f(x) = 2x^3 - 5x^2 + 3x + 2$ . . . (I)

$f(x)$ is a polynomial which is continuous on $\left[0, \frac{3}{2}\right]$ and it is differentiable on $\left(0, \frac{3}{2}\right)$.

Let $a = 0$, and $b = \frac{3}{2}$,

For $x = a = 0$ from (I) we get,

$f(a) = f(0) = 2(0)^3 - 5(0)^2 + 3(0) + 2 = 2$

For $x = b = \left(\frac{3}{2}\right)$ from (I) we get,

$$f(b) = f\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - 5\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) + 2 = \frac{54}{8} - \frac{45}{4} + \frac{9}{2} + 2$$

$$f(b) = f\left(\frac{3}{2}\right) = \frac{54 - 90 + 36}{8} + 2 = 2$$

So, here $f(a) = f(b)$ i.e. $f(0) = f\left(\frac{3}{2}\right) = 2$

Hence conditions of Rolle's Theorem are satified.

(ii) Given that $f(x) = x^2 - 2x + 3$ . . . (I)

$f(x)$ is a polynomial which is continuous on [1, 4] and it is differentiable on (1, 4).

Let $a = 1$, and $b = 4$

For $x = a = 1$ from (I) we get,

$f(a) = f(1) = (1)^2 - 2(1) + 3 = 2$

For $x = b = 4$ from (I) we get,

$f(b) = f(4) = (4)^2 - 2(4) + 3 = 11$

So, here $f(a) \neq f(b)$ i.e. $f(1) \neq f(4)$

Hence conditions of Rolle's theorem are not satisfied.

Ex. 2 : Verify Rolle's theorem for the function $f(x) = x^2 - 4x + 10$ on [0, 4].

Solution :

Given that $f(x) = x^2 - 4x + 10$ . . . (I)

$f(x)$ is a polynomial which is continuous on [0, 4] and it is differentiable on (0, 4).

Let $a = 0$, and $b = 4$

For $x = a = 0$ from (I) we get,

$f(a) = f(0) = (0)^2 - 4(0) + 10 = 10$

For $x = b = 4$ from (I) we get,

$f(b) = f(4) = (4)^2 - 4(4) + 10 = 10$

So, here $f(a) = f(b)$ i.e. $f(0) = f(4) = 10$

All the conditions of Rolle's theorem are satisfied.

To get the value of c, we should have

$f'(c) = 0$ for some $c \in (0, 4)$

Differentiate (I) *w. r. t. x.*

$f'(x) = 2x - 4 = 2(x - 4)$

Now, for $x = c$,

$f'(c) = 0 \Rightarrow 2(c - 2) = 0 \Rightarrow c = 2$

Also $c = 2 \in (0, 4)$

Thus Rolle's theorem is verified.

Ex. 3 : Given an interval $[a, b]$ that satisfies hypothesis of Rolle's theorem for the function $f(x) = x^3 - 2x^2 + 3$. It is known that $a = 0$. Find the value of b.

Solution :

Given that $f(x) = x^3 - 2x^2 + 3$ . . . (I)

Let $g(x) = x^3 - 2x^2 = x^2(x - 2)$

From (I), $f(x) = g(x) + 3$

We see that $g(x)$ becomes zero for $x = 0$ and $x = 2$.

We observe that for $x = 0$,

$$f(0) = g(0) + 3 = 3$$

and for $x = 2$,

$$f(2) = g(2) + 3 = 3$$

$\therefore$ We can write that $f(0) = f(2) = 3$

It is obvious that the function $f(x)$ is everywhere continuous and differentiable as a cubic polynomial. Consequently, it satisfies all the conditions of Rolle's theorem on the interval [0, 2].

So $b = 2$.

Ex. 4 : Verify Rolle's theorem for the function $f(x) = e^x (\sin x - \cos x)$ on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Solution : Given that, $f(x) = e^x (\sin x - \cos x)$ . . . (I)

We know that e^x, $\sin x$ and $\cos x$ are continuous and differentiable on their domains. Therefore $f(x)$ is continuous and differentiable on $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$ and $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ respectively.

Let $a = \frac{\pi}{4}$, and $b = \frac{5\pi}{4}$

For $x = a = \frac{\pi}{4}$ from (I) we get,

$$f(a) = f\left(\frac{\pi}{4}\right) = e^{\frac{\pi}{4}}\left[\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right)\right] = e^{\frac{\pi}{4}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$$

For $x = b = \left(\frac{5\pi}{4}\right)$ from (I) we get,

$$f(a) = f\left(\frac{5\pi}{4}\right) = e^{\frac{5\pi}{4}}\left[\sin\left(\frac{5\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right)\right] = e^{\frac{5\pi}{4}}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = 0$$

$\therefore \quad f(a) = f(b) \quad$ i.e. $f\left(\frac{\pi}{4}\right) = f\left(\frac{5\pi}{4}\right)$.

All the conditions of Rolle's theorem are satisfied.

To get the value of c, we should have $f'(c) = 0$ for some $c \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$.

Differentiate (I) *w. r. t. x.*

$f'(x) = e^x (\cos x + \sin x) + (\sin x - \cos x)\, e^x = 2e^x \sin x$

Now, for $x = c$, $f'(c) = 0 \Rightarrow 2e^c \sin c = 0$. As $e^c \neq 0$ for any $c \in R$

$\sin c = 0 \Rightarrow c = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

It is clearly seen that $\pi \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right] \therefore c = \pi$

Thus Rolle's theorem is verified.

2.3.2 Lagrange's Mean Value Theorem (LMVT) :

If a real-valued function f is continous on a closed $[a, b]$ and differentiable on the open interval (a, b) then there exists at least one c in the open interval (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Lagrange's mean value theorem states, that for any real-valued diffenentiable function which is continuous at the two end points, there is at least one point at which the tangent is parallel to the the secant through its end points.

Geometrical Significance :

Draw the curve $y = f(x)$ (see Figure 2.3.2) and take the end points $A\ (a, f(a))$ and $B\ (b, f(b))$ on the curve, then

Slope of the chord $AB = \dfrac{f(b) - f(a)}{b - a}$

Since by statement of Lagrange's Mean Value.

Theorem $\quad f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$f'(c) =$ Slope of the chord AB.

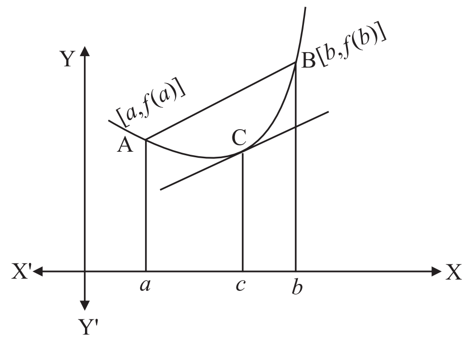


Fig. 2.3.2

This shows that the tangent to the curve $y = f(x)$ at the point $x = c$ is parallel to the chord AB.

SOLVED EXAMPLES

Ex. 1 : Verify Lagrange's mean value theorem for the function $f(x) = \sqrt{x+4}$ on the interval [0, 5].

Solution : Given that $f(x) = \sqrt{x+4}$ . . . (I)

The function $f(x)$ is continuous on the closed interval [0, 5] and differentiable on the open interval (0, 5), so the LMVT is applicable to the function.

Differentiate (I) *w. r. t. x.*

$$f'(x) = \frac{1}{2\sqrt{x+4}} \quad \text{. . . (II)}$$

Let $a = 0$ and $b = 5$

From (I), $\quad f(a) = f(0) = \sqrt{0+4} = 2$

$\qquad f(b) = f(5) = \sqrt{5+4} = 3$

Let $c \in (0, 5)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{1}{2\sqrt{c+4}} = \frac{3-2}{5-0} = \frac{1}{5}$$

$$\therefore \sqrt{c+4} = \frac{5}{2} \Rightarrow c + 4 = \frac{25}{4} \ \therefore\ c = \frac{9}{4} \in (0, 5)$$

Thus Lagrange's Mean Value Theorem is verified.

Ex. 2 : Verify Lagrange's mean value theorem for the function $f(x) = x + \dfrac{1}{x}$ on the interval [1, 3].

Solution : Given that $f(x) = x + \dfrac{1}{x}$ . . . (I)

The function $f(x)$ is continuous on the closed interval [1, 3] and differentiable on the open interval (1, 3), so the LMVT is applicable to the function.

Differentiate (I) *w. r. t. x.*

$$f'(x) = 1 - \frac{1}{x^2} \quad \text{. . . (II)}$$

Let $a = 1$ and $b = 3$

From (I), $\quad f(a) = f(1) = 1 + \dfrac{1}{1} = 2$

$\qquad f(b) = f(3) = 3 + \dfrac{1}{3} = \dfrac{10}{3}$

Let $c \in (1, 3)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$1 - \frac{1}{c^2} = \frac{\frac{10}{3} - 2}{3 - 1}$$

$$1 - \frac{1}{c^2} = \frac{\frac{4}{3}}{2} = \frac{2}{3}$$

$$\therefore\ c^2 = 3 \Rightarrow c = \pm\sqrt{3}$$

$$\therefore\ c = \sqrt{3} \in (1, 3) \text{ and } c = -\sqrt{3} \notin (1, 3)$$

EXERCISE 2.3

(1) Check the validity of the Rolle's theorem for the following functions.

(i) $f(x) = x^2 - 4x + 3, x \in [1, 3]$

(ii) $f(x) = e^{-x} \sin x, x \in [0, \pi]$

(iii) $f(x) = 2x^2 - 5x + 3, x \in [1, 3]$

(iv) $f(x) = \sin x - \cos x + 3, x \in [0, 2\pi]$

(v) $f(x) = x^2$ if $0 \le x \le 2$

$= 6 - x$ if $2 \le x \le 6$

(vi) $f(x) = x^{\frac{2}{3}}, x \in [-1, 1]$

(2) Given an interval $[a, b]$ that satisfies hypothesis of Rolle's thorem for the function $f(x) = x^4 + x^2 - 2$. It is known that $a = -1$. Find the value of b.

(3) Verify Rolle's theorem for the following functions.

(i) $f(x) = \sin x + \cos x + 7, x \in [0, 2\pi]$

(ii) $f(x) = \sin\left(\frac{x}{2}\right), x \in [0, 2\pi]$

(iii) $f(x) = x^2 - 5x + 9, x \in [1, 4]$

(4) If Rolle's theorem holds for the function $f(x) = x^3 + px^2 + qx + 5, x \in [1, 3]$ with $c = 2 + \frac{1}{\sqrt{3}}$, find the values of p and q.

(5) Rolle's theorem holds for the function $f(x) = (x - 2) \log x, x \in [1, 2]$, show that the equation $x \log x = 2 - x$ is satisfied by at least one value of x in $(1, 2)$.

(6) The function $f(x) = x(x + 3)e^{-\frac{x}{2}}$ satisfies all the conditions of Rolle's theorem on $[-3, 0]$. Find the value of c such that $f'(c) = 0$.

(7) Verify Lagrange's mean value theorem for the following functions.

(i) $f(x) = \log x$, on $[1, e]$

(ii) $f(x) = (x - 1)(x - 2)(x - 3)$ on $[0, 4]$

(iii) $f(x) = x^2 - 3x - 1, x \in \left[-\frac{11}{7}, \frac{13}{7}\right]$

(iv) $f(x) = 2x - x^2, x \in [0, 1]$

(v) $f(x) = \frac{x - 1}{x - 3}$ on $[4, 5]$

2.4.1 Increasing and decreasing functions :

Increasing functions :

Definition : A function f is said to be a monotonically (or strictly) increasing function on an interval (a, b) if for any $x_1, x_2 \in (a, b)$ with if $x_1 < x_2$, we have $f(x_1) < (x_2)$.

Consider an increasing function $y = f(x)$ in (a, b). Let $h > 0$ be a small increment in x then,

$$x < x + h \qquad [x = x_1, x + h = x_2]$$

$$f(x) < f(x + h) \qquad [f(x_1) < f(x_2)]$$

$$\therefore \quad f(x + h) > f(x)$$

$$\therefore \quad f(x + h) - f(x) > 0$$

$$\therefore \quad \frac{f(x + h) - f(x)}{h} > 0$$

$$\therefore \quad \lim_{h \to 0}\left[\frac{f(x + h) - f(x)}{h}\right] \ge 0$$

$$\therefore \quad f'(x) \ge 0$$

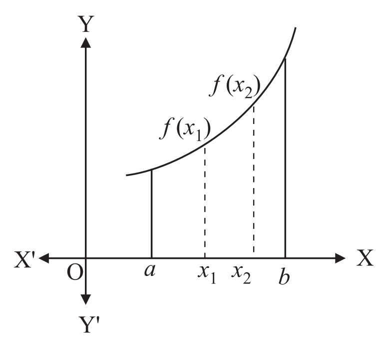


Fig. 2.4.1

If $f'(a) > 0$, then in a small δ-neighborhood of a i.e. $(a - \delta, a + \delta)$, we have f strictly increasing if

$$\frac{f(a+h)-f(a)}{h} > 0 \qquad \text{for } |h| < \delta$$

Hence if $0 < h < \delta$, $f(a+h) - f(a) > 0$ and $f(a-h) - f(a) < 0$

Thus for $0 < h < \delta$, $f(a-h) < f(a) < f(a+h)$

Decreasing functions :

Definition : A function f is said to be a monotonically (strictly) decreasing function on an interval (a, b) if for any $x_1, x_2 \in (a, b)$ with $x_1 < x_2$, we have $f(x_1) > (x_2)$.

Consider a decreasing function $y = f(x)$ in (a, b). Let $h > 0$ be a small increment in x then,

$$
\begin{aligned}
x + h &> x \qquad [x = x_1, x + h = x_2] \\
f(x) &< f(x+h) \qquad [f(x_1) < f(x_2)] \\
\therefore \quad f(x+h) &< f(x) \\
\therefore \quad f(x+h) - f(x) &< 0 \\
\therefore \quad \frac{f(x+h)-f(x)}{h} &< 0 \\
\therefore \quad \lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right] &\le 0 \\
\therefore \quad f'(x) &\le 0
\end{aligned}
$$

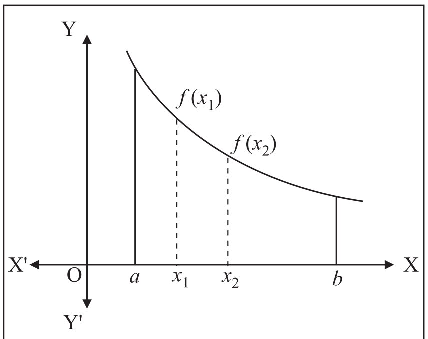


Fig. 2.4.2

If $f'(a) < 0$, then in a small δ-neighborhood of a i.e. $(a - \delta, a + \delta)$, we have f strictly decreasing

because $$\frac{f(a+h)-f(a)}{h} < 0 \qquad \text{for } |h| < \delta$$

Hence for $0 < h < \delta$, $f(a-h) > f(a) > f(a+h)$

Note : Whenever $f'(x) = 0$, at that point the tangent is parallel to X-axis, we cannot deduce that whether $f(x)$ is increasing or decreasing at that point.

SOLVED EXAMPLES

Ex. 1: Show that the function $f(x) = x^3 + 10x + 7$ for $x \in R$ is strictly increasing.

Solution : Given that $f(x) = x^3 + 10x + 7$

Differentiate *w. r. t. x*.

$f'(x) = 3x^2 + 10$

Here, $3x^2 \ge 0$ for all $x \in R$ and $10 > 0$.

$\therefore \quad 3x^2 + 10 > 0 \Rightarrow f'(x) > 0$

Thus $f(x)$ is a strictly increasing function.

Ex. 2: Test whether the function $f(x) = x^3 + 6x^2 + 12x - 5$ is increasing or decreasing for all $x \in R$.

Solution : Given that $f(x) = x^3 + 6x^2 + 12x - 5$

Differentiate *w. r. t. x*.

$f'(x) = 3x^2 + 12x + 12 = 3(x^2 + 4x + 4)$

$f'(x) = 3(x+2)^2$

$3(x+2)^2$ is always positive for $x \neq -2$

$\therefore f'(x) \ge 0$ for all $x \in R$

Hence $f(x)$ is an increasing function for all $x \in R$.

Ex. 3: Find the values of x, for which the funciton $f(x) = x^3 + 12x^2 + 36x + 6$ is (i) monotonically increasing. (ii) monotonically decreasing.

Solution : Given that $f(x) = x^3 + 12x^2 + 36x + 6$

Differentiate *w. r. t. x*.

$f'(x) = 3x^2 + 24x + 36$

$= 3(x^2 + 8x + 12)$

$f'(x) = 3(x + 2)(x + 6)$

(i) $f(x)$ is monotonically increasing if $f'(x) > 0$

i.e. $3(x + 2)(x + 6) > 0$, $(x + 2)(x + 6) > 0$

then either $(x + 2) < 0$ and $(x + 6) < 0$ or $(x + 2) > 0$ and $(x + 6) > 0$

Case (I) : $x + 2 < 0$ and $x + 6 < 0$

$x < -2$ and $x < -6$

Thus for every $x < -6$, $(x + 2)(x + 6) > 0$, hence f is monotonically increasing.

Case (II) : $x + 2 > 0$ and $x + 6 > 0$

$x > -2$ and $x > -6$

Thus for every $x > -2$, $(x + 2)(x + 6) > 0$ and f is monotonically increasing.

$\therefore$ From Case (I) and Case (II), $f(x)$ is monotonically increasing if and only if $x < -6$ or $x > -2$.

Hence, $x \in (\infty, -6)$ or $x \in (-2, \infty) \Rightarrow f$ is monotonically increasing.

(ii) $f(x)$ is said to be monotonically decreasing if $f'(x) = 0$

i.e. $3(x + 2)(x + 6) < 0$, $(x + 2)(x + 6) < 0$

then either $(x + 2) < 0$ and $(x + 6) > 0$ or $(x + 2) > 0$ and $(x + 6) < 0$

Case (I) : $x + 2 < 0$ and $x + 6 > 0$

$x < -2$ and $x > -6$

Thus for $x \in (-6, -2)$, f is monotonically decreasing.

Case (II) : $x + 2 > 0$ and $x + 6 < 0$

$x > -2$ and $x < -6$

$\therefore$ This case does not arise. . . . [check. why ?]

2.4.2 Maxima and Minima :

Maxima of a function $f(x)$: A function $f(x)$ is said to have a maxima at $x = c$ if the value of the function at $x = c$ is greater than any other value of $f(x)$ in a δ-neighborhood of c. That is for a small $\delta > 0$ and for $x \in (c - \delta, c + \delta)$ we have $f(c) > f(x)$. The value $f(c)$ is called a Maxima of $f(x)$. Thus the function $f(x)$ will have maxima at $x = c$ if $f(x)$ is increasing in $c - \delta < x < c$ and decreasing in $c < x < c + \delta$.

Minima of a function $f(x)$: A function $f(x)$ is said to have a minima at $x = c$ if the value of the function at $x = c$ is less than any other value of $f(x)$ in a δ-neighborhood of c. That is for a small $\delta > 0$ and for $x \in (c - \delta, c + \delta)$ we have $f(c) < f(x)$. The value $f(c)$ is called a Minima of $f(x)$. Thus the function $f(x)$ will have minima at $x = c$ if $f(x)$ is decreasing in $c - \delta < x < c$ and increasing in $c < x < c + \delta$.

If $f'(c) = 0$ then at $x = c$ the function is neither increasing nor decrasing, such a point on the curve is called **turning point** or **stationary point** of the function. Any point at which the tangent to the graph is horizontal is a turning point. We can locate the turn points by looking for points at which $\frac{dy}{dx} = 0$. At these points if the function has Maxima or Minima then these are called extreme values of the function.

Note : The maxima and the minima of a function are not necessarily the greatest and the least values of the function in the whole domain. Actually these are the greatest and the least values of the function in a small interval. Hence the maxima or the minima defined above are known as **local (or relative) maximum** and **the local (or relative) minimum** of the function $f(x)$.

To find the extreme values of the function let us use following tests.

2.4.3 First derivative test :

A function $f(x)$ has a maxima at $x = c$ if

(i) $f'(c) = 0$

(ii) $f'(c - h) > 0$ [$f(x)$ is increasing for values of $x < c$]

(iii) $f'(c + h) < 0$ [$f(x)$ is decreasing for values of $x > c$]

where h is a small positive number.

A function $f(x)$ has a minima at $x = c$ if

(i) $f'(c) = 0$

(ii) $f'(c - h) < 0$ [$f(x)$ is decreasing for values of $x < c$]

(iii) $f'(c + h) > 0$ [$f(x)$ is increasing for values of $x > c$]

where h is a small positive number.

Note : If $f'(c) = 0$ and $f'(c - h) > 0, f'(c + h) > 0$ or $f'(c - h) <0, f'(c + h) < 0$ then $f(c)$ in neither maxima nor minima. In such a case $x = c$ is called a **point of inflexion**. e.g. $f(x) = x^3$, $f(x) = x^5$ in $[-2, 2]$.

SOLVED EXAMPLES

Ex. 1 : Find the local maxima or local minima of $f(x) = x^3 - 3x$.

Solution : Given that $f(x) = x^3 - 3x$ $\quad$ . . . (I)

Differentiate (I) *w. r. t. x*.

$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$ $\quad$ . . . (II)

For extreme values, $f'(x) = 0$

$3x^2 - 3 = 0$ $\quad$ i.e. $3(x^2 - 1) = 0$

i.e. $x^2 - 1 = 0 \Rightarrow 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

The turning points are $x = 1$ and $x = -1$

Let's consider the turning point, $x = 1$

Let $x = 1 - h$ for a small, $h > 0$, from (II) we get,

$f'(1 - h) = 3[(1 - h)^2 - 1] = 3(1 - 2h + h^2 - 1) = 3h(h - 2)$

$\therefore$ $f'(1 - h) < 0$ . . . [since, $h > 0, h - 2 < 0$]

$\therefore$ $f'(x)$ for $x = 1 - h \Rightarrow f(x)$ is decreasing for, $x > 1$.

Now for $x = 1 + h$ for a small, $h > 0$, from (II) we get,

$f'(1 + h) = 3[(1 + h)^2 - 1] = 3(1 + 2h + h^2 - 1) = 3(h^2 + 2h)$

$\therefore$ $f'(1 + h) > 0$ . . . [since, $h > 0, h^2 + 2h > 0$]

$\therefore$ $f'(x) < 0$ for $x = 1 + h \Rightarrow f(x)$ is increasing for, $x < 1$.

$\therefore$ $f'(x) < 0$ for $1 - h < x < 1$

$\therefore$ $f'(x) > 0$ for $1 < x < 1 + h$.

$\therefore$ $x = 1$ is the point of local minima.

Minima of $f(x)$, is $f(1) = 1^3 - 3(1) = -2$

Now, let's consider the turning point, $x = -1$

Let $x = -1 - h$ for a small, $h > 0$, from (II) we get,

$\therefore$ $f'(-1 - h) = 3[(-1 - h)^2 - 1] = 3(1 + 2h + h^2 - 1) = 3(h^2 + 2h)$

$\therefore$ $f'(-1 - h) > 0$ . . . [since, $h > 0, h^2 + 2h > 0$]

$\therefore$ $f'(x) > 0$ for $x = -1 - h \Rightarrow f(x)$ is increasing for, $x < -1$.

Now for $x = -1 + h$ for a small, $h > 0$, from (II) we get,

$\therefore$ $f'(-1 + h) = 3[(-1 + h)^2 - 1] = 3(1 - 2h + h^2 - 1) = -3h(2 - h)$

$\therefore$ $f'(-1 + h) < 0$ . . . [since, $h > 0, 2 - h > 0$]

$\therefore$ $f'(x) < 0$ for $x = -1 + h \Rightarrow f(x)$ is decreasing for, $x > -1$.

$\therefore$ $f'(x) > 0$ for $-1 - h < x < -1$

$\therefore$ $f'(x) > 0$ for $-1 < x < -1 + h$.

$\therefore$ $x = -1$ is the point of local maxima.

Maxima of $f(x)$, is $f(-1) = (-1)^3 - 3(-1) = -1 + 3 = 2$

Hence, Maxima of $f(x) = 2$ and Minima of $f(x) = -2$

2.4.4 Second derivative test :

A function $f(x)$ has a maxima at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$

A function $f(x)$ has a minima at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$

Note : If $f''(c) = 0$ then second derivative test fails so, you may try using first derivative test.

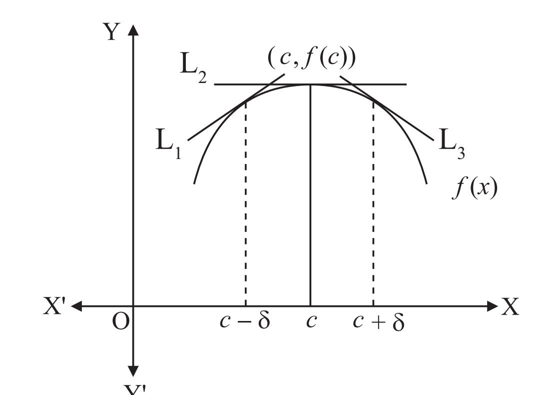


Fig. 2.4.4 (a)

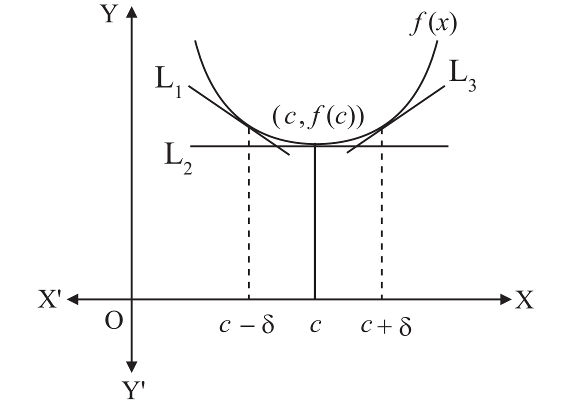


Fig. 2.4.4 (b)

Maxima at A : Consider the slopes of the tangents (See Fig 2.4.4a) Slope of L_1 is +ve, slope of $L_2 = 0$ and slope of L_3 is −ve. Thus the slope is seen to be decreasing if there is a maximum at A.

Minima at A : Consider the slopes of the tangents (See Fig 2.4.4b) slope of L_1 is −ve, slope of $L_2 = 0$ and slope of L_3 is +ve. Thus the slope is seen to be increasing if there is a minima at A.

SOLVED EXAMPLES

Ex. 1: Find the local maximum and local minimum value of $f(x) = x^3 - 3x^2 - 24x + 5$.

Solution : Given that $f(x) = x^3 - 3x^2 - 24x + 5$ . . . (I)

Differentiate (I) *w. r. t. x*.

$f'(x) = 3x^2 - 6x - 24$ . . . (II)

For extreme values, $f'(x) = 0$

$3x^2 - 6x - 24$ i.e. $3(x^2 - 2x - 8) = 0$

i.e. $x^2 - 2x - 8 = 0$ i.e. $(x + 2)(x - 4) = 0$

$\Rightarrow x + 2 = 0$ or $x - 4 = 0 \Rightarrow x = -2$ or $x = 4$

The stationary points are $x = -2$ and $x = 4$.

Differentiate (II) *w. r. t. x*.

$f''(x) = 6x - 6$ . . . (III)

For $x = -2$, from (III) we get,

$f''(-2) = 6(-2) - 6 = -18 < 0$

$\therefore$ At $x = -2$, $f(x)$ has a maximum value.

For maximum of $f(x)$, put $x = -2$ in (I)

$f(-2) = (-2)^3 - 3(-2)^2 - 24(-2) + 5 = 33.$

For $x = 4$, from (III) we get

$f''(4) = 6(4) - 6 = 18 > 0$

$\therefore$ At $x = 4$, $f(x)$ has a minimum value.

For minima of $f(x)$, put $x = 4$ in (I)

$f(4) = (4)^3 - 3(4)^2 - 24(4) + 5 = -75$

$\therefore$ Local maximum of $f(x)$ is 33 when $x = -2$

and

Local minimum of $f(x)$ is −75 when $x = 4$.

Ex. 2: A wire of length 120 cm is bent in the form of a rectangle. Find its dimensions if the area of the rectangle is maximum.

Solution : Let x cm and y cm be the length and the breadth of the rectangle. Perimeter of rectangle = 120 cm.

$\therefore$ $2(x + y) = 120$ so, $x + y = 60$

$\therefore$ $y = 60 - x$ . . . (I)

Let A be the area of the rectangle

$\therefore$ $A = xy = x(60 - x) = 60x - x^2$. . . [From (I)]

Differentiate *w. r. t. x*.

$$\frac{dA}{dx} = 60 - 2x \quad \text{. . . (II)}$$

For maximum area $\frac{dA}{dx} = 0$

i.e. $60 - 2x = 0 \Rightarrow x = 30$

Differentiate (II) *w. r. t. x*.

$$\frac{d^2A}{dx^2} = -2 \quad \text{. . . (III)}$$

For, $x = 30$ from (III) we get,

$$\left(\frac{d^2A}{dx^2}\right)_{x=30} = -2 < 0$$

When, $x = 30$, Area of the rectangle is maximum.

Put $x = 30$ in (I) we get $y = 60 - 30 = 30$

$\therefore$ Area of the rectangle is maximum if length = breadth = 30 cm.

Ex. 3: A Rectangular sheet of paper has it area 24 sq. meters. The margin at the top and the bottom are 75 cm each and at the sides 50 cm each. What are the dimensions of the paper, if the area of the printed space is maximum?

Solution : Let x m and y m be the width and the length of the rectangular sheet of paper respectively. Area of the paper = 24 sq. m.

$\therefore$ $xy = 24 \Rightarrow y = \frac{24}{x}$ . . . (I)

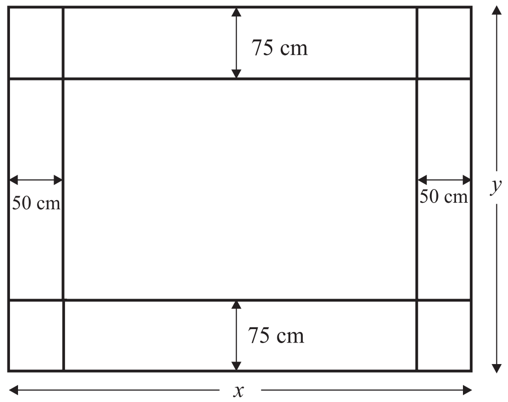


Fig. 2.4.5

After leaving the margins, length of the printing space is $(x - 1)$ m and breadth of the printing space is $(y - 1.5)$ m.

Let A be the area of the printing space

$$A = (x-1)(y-1.5) = (x-1)\left(\frac{24}{x} - 1.5\right)$$

$$= 24 - 1.5x - \frac{24}{x} + 1.5 \quad \text{. . . [From (I)]}$$

$$A = 25.5 - 1.5x - \frac{24}{x} \quad \text{. . . (II)}$$

Differentiate *w. r. t. x*.

$$\frac{dA}{dx} = -1.5 + \frac{24}{x^2} \quad \text{. . . (III)}$$

For maximum printing space $\frac{dA}{dx} = 0$

i.e. $-1.5x + \frac{24}{x^2} = 0 \Rightarrow 1.5x^2 = 24 \Rightarrow x = \pm 4, x \neq -4$

$\therefore$ $x = 4$

Differentiate (III) *w. r. t. x*.

$$\frac{d^2A}{dx^2} = -\frac{48}{x^3} \quad \text{. . . (IV)}$$

For, $x = 4$, from (IV) we get,

$$\left(\frac{d^2A}{dx^2}\right)_{x=4} = -\frac{48}{(4)^3} < 0$$

When, $x = 4$ Area of the rectangular printing space is maximum.

Put $x = 4$ in (I) we get $y = \frac{24}{4} = 6$

$\therefore$ Area of the printing space is maximum when width printing space = 4 m. and length of the printing space = 6 m.

Ex. 4: An open box is to be cut out of piece of square card coard of side 18 cm by cutting of equal squares from the corners and turning up the sides. Find the maximum volume of the box.

Solution : Let the side of each of the small squares cut be x cm, so that each side of the box to be made is $(18 - 2x)$ cm. and height x cm.

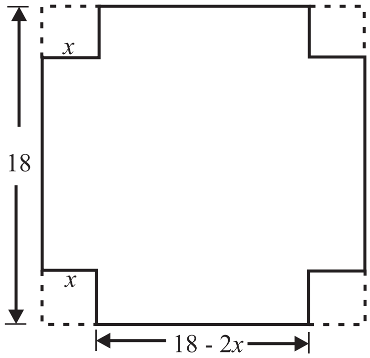


Fig. 2.4.6

Let V be the volume of the box.

V = Area of the base × Height

$= (18 - 2x)^2 x = (324 - 72x + 4x^2) x$

$V = 4x^3 - 72x^2 + 324x$ . . . (I)

Differentiate *w. r. t.* x

$\frac{dV}{dx} = 12x^2 - 144x + 324$ . . . (II)

For maximum volume $\frac{dV}{dx} = 0$

i.e. $12x^2 - 144x + 324 = 0 \Rightarrow x^2 - 12x + 27 = 0$

$(x - 3)(x - 9) = 0 \Rightarrow x - 3 = 0$ or $x - 9 = 0$

$\therefore$ $x = 3$ or $x = 9$, but $x \neq 9$ $\therefore$ $x = 3$

Differentiate (II) *w. r. t.* x

$\frac{d^2V}{dx^2} = 24x - 144$ . . . (III)

For, $x = 3$ from (III) we get,

$\left(\frac{d^2V}{dx^2}\right)_{x=3} = 24(3) - 144 = -72 < 0$

Volume of the box is maximum when $x = 3$.

Maximum volume of the box

$= (18 - 6)^2 (3) = 432$ c.c.

Ex. 5: Two sides of a triangle are given, find the angle between them such that the area of the triangle is maximum.

Solution : Let ABC be a triangle. Let the given sides be $AB = c$ and $AC = b$.

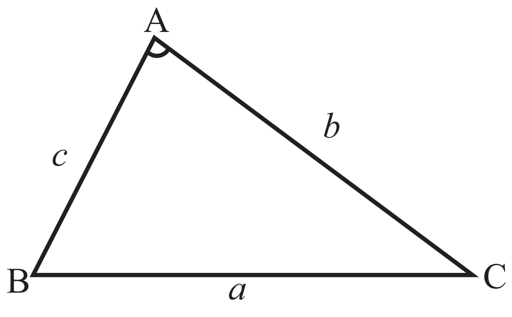


Fig. 2.4.7

Let Δ be the area of the triangle.

$\Delta = \frac{1}{2} bc \sin A$ . . . (I)

Differentiate *w. r. t.* A.

$\frac{d\Delta}{dA} = \frac{bc}{2} \cos A$ . . . (II)

For maximum area $\frac{d\Delta}{dA} = 0$

i.e. $\frac{bc}{2} \cos A = 0 \Rightarrow \cos A = 0 \Rightarrow A = \frac{\pi}{2}$

Differentiate (II) *w. r. t.* A.

$\frac{d^2\Delta}{dA^2} = -\frac{bc}{2} \sin A$ . . . (III)

For, $A = \frac{\pi}{2}$ from (III) we get,

$\left(\frac{d^2\Delta}{dA^2}\right)_{A=\frac{\pi}{2}} = -\frac{bc}{2} \sin\left(\frac{\pi}{2}\right) = \frac{bc}{a} < 0$

When, $A = \frac{\pi}{2}$ Area of the triangle is maximum.

Hence, the area of the triangle is maximum when the angle between the given sides $\frac{\pi}{2}$.

Note : $\sin A$ is maximum (=1), when $A = \frac{\pi}{2}$

Ex. 6: The slant side of a right circular cone is l. Show that the semi-vertical angle of the cone of maximum volume is $\tan^{-1}(\sqrt{2})$.

Solution : Let x be the height of the cone and r be the radius of the base.

So, $r^2 = l^2 - x^2$ . . . (I)

Let V be the volume of the cone.

$$V = \frac{1}{3}\pi r^2 x = \frac{\pi}{3}(l^2 - x^2)\,x$$

$$\therefore \quad V = \frac{\pi}{3}(l^2 x - x^3)$$

Differentiate *w. r. t.* x

$$\frac{dV}{dx} = \frac{\pi}{3}(l^2 - 3x^2) \quad . . . \text{(II)}$$

For maximum volume $\frac{dV}{dx} = 0$

i.e. $\frac{\pi}{3}(l^2 - 3x^2) = 0 \Rightarrow x^2 = \frac{l^2}{3}$

$x = \pm\frac{l}{\sqrt{3}} \Rightarrow x = \frac{l}{\sqrt{3}}$ or $x = -\frac{l}{\sqrt{3}}$ is the stataionary point but, $x \neq -\frac{l}{\sqrt{3}}$ $\therefore$ $x = \frac{l}{\sqrt{3}}$

Differentiate (II) *w. r. t.* x

$$\left(\frac{d^2V}{dx^2}\right) = -2\pi x \quad . . . \text{(III)}$$

For, $x = \frac{l}{\sqrt{3}}$ from (III) we get,

$$\left(\frac{d^2V}{dx^2}\right)_{x=\frac{l}{\sqrt{3}}} = -\frac{2\pi l}{\sqrt{3}} < 0$$

Volume of the cone is maximum when height of the cone is $x = \frac{l}{\sqrt{3}}$.

Put $x = \frac{l}{\sqrt{3}}$ in (I) we get, $r = \sqrt{l^2 - \left(\frac{l}{\sqrt{3}}\right)^2} = \frac{l\sqrt{2}}{\sqrt{3}}$

Let α be the semi-vertical angle.

Then $\tan\alpha = \frac{r}{x} = \frac{\frac{l\sqrt{2}}{\sqrt{3}}}{\frac{l}{\sqrt{3}}} = \sqrt{2}$

$$\therefore \quad \alpha = \tan^{-1}(\sqrt{2})$$

Ex. 7: Find the height of a covered box of fixed volume so that the total surface area of the box is minimum whose base is a rectangle with one side three times as long as the other.

Solution : Given that, box has a rectangular base with one side three times as long as other.

Let x and $3x$ be the sides of the rectangular base.

Let h be the height of the box and V be its volume.

$V = (x)(3x)(h) = 3x^2h \ldots$ [Observe that V is constant]

Differentiate *w. r. t. x*.

$$\frac{dV}{dx} = 3x^2 \frac{dh}{dx} + h \frac{d}{dx}(3x^2)$$

$$\therefore \quad 3x^2 \frac{dh}{dx} + 6xh = 0 \Rightarrow \frac{dh}{dx} = -\frac{2h}{x} \qquad \ldots \text{(I)}$$

Let S be the surface area of the box.

$$\therefore \quad S = (2 \times 3x^2) + (2 \times 3xh) + (2 \times xh) = 6x^2 + 8xh$$

Differentiate *w. r. t. x*.

$$\frac{dS}{dx} = 12x + 8\left(x\frac{dh}{dx} + h\frac{d}{dx}(x)\right)$$

$$\frac{dS}{dx} = 12x + 8\left[x\left(-\frac{2h}{x}\right) + h\right] \qquad \ldots [\text{ from (I) }]$$

$$= 12x + 8(-2h + h)$$

$$\therefore \quad \frac{dS}{dx} = 12x - 8h \qquad \ldots \text{(II)}$$

For minimum surface area

$$\frac{dS}{dx} = 0 \Rightarrow 12x - 8h = 0 \Rightarrow h = \frac{3x}{2}$$

Differentiate (II) *w. r. t. x*.

$$\left(\frac{d^2S}{dx^2}\right) = 12 - 8\frac{dh}{dx} = 12 - 8\left(-\frac{2h}{x}\right) = 12 + \frac{16h}{x} \qquad \ldots \text{(III)} \ldots [\text{ from (I) }]$$

Both x and h are positive, from (III) we get,

$$\left(\frac{d^2S}{dx^2}\right) = 12 + \frac{16h}{x} > 0$$

Surface area of the box is minimum if height $= \frac{3}{2} \times$ shorter side of base.

EXERCISE 2.4

(1) Test whether the following functions are increasing or decreasing.

(i) $f(x) = x^3 - 6x^2 + 12x - 16,\ x \in R$

(ii) $f(x) = 2 - 3x + 3x^2 - x^3,\ x \in R$

(iii) $f(x) = x - \frac{1}{x},\ x \in R$ and $x \neq 0$

(2) Find the values of x for which the following functions are strictly increasing -

(i) $f(x) = 2x^3 - 3x^2 - 12x + 6$

(ii) $f(x) = 3 + 3x - 3x^2 + x^3$

(iii) $f(x) = x^3 - 6x^2 - 36x + 7$

(3) Find the values of x for which the following functions are strictly decreasing -

(i) $f(x) = 2x^3 - 3x^2 - 12x + 6$

(ii) $f(x) = x + \frac{25}{x}$

(iii) $f(x) = x^3 - 9x^2 + 24x + 12$

(4) Find the values of x for which the function $f(x) = x^3 - 12x^2 - 144x + 13$

(a) Increasing (b) Decreasing

(5) Find the values of x for which

$f(x) = 2x^3 - 15x^2 - 144x - 7$ is

(a) strictly increasing

(b) strictly decreasing

(6) Find the values of x for which $f(x) = \dfrac{x}{x^2 + 1}$ is

(a) strictly increasing

(b) strictly decreasing

(7) Show that $f(x) = 3x + \dfrac{1}{3x}$ increasing in $\left(\dfrac{1}{3}, 1\right)$ and decreasing in $\left(\dfrac{1}{9}, \dfrac{1}{3}\right)$.

(8) Show that $f(x) = x - \cos x$ is increasing for all x.

(9) Find the maximum and minimum of the following functions -

(i) $y = 5x^3 + 2x^2 - 3x$

(ii) $f(x) = 2x^3 - 21x^2 + 36x - 20$

(iii) $f(x) = x^3 - 9x^2 + 24x$

(iv) $f(x) = x^2 + \dfrac{16}{x^2}$

(v) $f(x) = x \log x$ (vi) $f(x) = \dfrac{\log x}{x}$

(10) Divide the number 30 in to two parts such that their product is maximum.

(11) Divide that number 20 in to two parts such that sum of their squares is minimum.

(12) A wire of length 36 meters is bent in the form of a rectangle. Find its dimensions if the area of the rectangle is maximum.

(13) A ball is thrown in the air. Its height at any time t is given by $h = 3 + 14t - 5t^2$. Find the maximum height it can reach.

(14) Find the largest size of a rectangle that can be inscribed in a semi circle of radius 1 unit, So that two vertices lie on the diameter.

(15) An open cylindrical tank whose base is a circle is to be constructed of metal sheet so as to contain a volume of πa^3 cu. cm of water. Find the dimensions so that sheet required is minimum.

(16) The perimeter of a triangle is 10 cm. If one of the side is 4 cm. What are the other two sides of the triangle for its maximum area ?

(17) A box with a square base is to have an open top. The surface area of the box is 192 sq.cm. What should be its dimensions in order that the volume is largest ?

(18) The profit function $P(x)$ of a firm, selling x items per day is given by

$P(x) = (150 - x)x - 1625$. Find the number of items the firm should manufacture to get maximum profit. Find the maximum profit.

(19) Find two numbers whose sum is 15 and when the square of one multiplied by the cube of the other is maximum.

(20) Show that among rectangles of given area, the square has the least perimeter.

(21) Show that the height of a closed right circular cylinder of a given volume and least surface area is equal to its diameter.

(22) Find the volume of the largest cylinder that can be inscribed in a sphere of radius 'r' cm.

(23) Show that $y = \log(1 + x) - \dfrac{2x}{2 + x}$, $x > -1$ is an increasing function on its domain.

(24) Prove that $y = \dfrac{4 \sin \theta}{2 + \cos \theta} - \theta$ is an increasing function of $\theta \in \left[0, \dfrac{\pi}{2}\right]$.

Let us Remember

- Equations of tangent and Normal at $P(x_1, y_1)$ respectively are given by

$$y - y_1 = m(x - x_1) \text{ where } m = \left[\frac{dy}{dx}\right]_{(x_1, y_1)}$$

$$y - y_1 = m'(x - x_1) \text{ where } m' = -\frac{1}{\left[\frac{dy}{dx}\right]_{(x_1, y_1)}}, \text{ if } \left[\frac{dy}{dx}\right]_{(x_1, y_1)} \neq 0$$

- Approximate value of the function $f(x)$ at $x = a + h$ is given by $f(a + h) \doteqdot f(a) + h f'(a)$
- **Rolle's theorem :** If real-valued function f is continous on a closed $[a, b]$, differentiable on the open interval (a, b) and $f(a) = f(b)$, then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.
- **Lagrange's Mean Value Theorem (LMVT) :** If a real-valued function f is continous on a closed $[a, b]$ and differentiable on the open interval (a, b) then there exists at least one c in the open interval (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$
- **Increasing and decreasing functions :**

  (i) A function f is monotonically increasing if $f'(x) > 0$.

  (ii) A function f is monotonically decreasing if $f'(x) < 0$.

  (iii) A function f is increasing if $f'(x) \geq 0$.

  (iv) A function f is decreasing if $f'(x) \leq 0$.

- **(i) First Derivative test :**

  A function $f(x)$ has a maxima at $x = c$ if

  (i) $f'(c) = 0$

  (ii) $f'(c - h) > 0$ [$f(x)$ is increasing for values of $x < c$]

  (iii) $f'(c + h) < 0$ [$f(x)$ is decreasing for values of $x > c$]

  where h is a small positive number.

  A function $f(x)$ has a minima at $x = c$ if

  (i) $f'(c) = 0$

  (ii) $f'(c - h) < 0$ [$f(x)$ is decreasing for values of $x < c$]

  (iii) $f'(c + h) > 0$ [$f(x)$ is increasing for values of $x > c$]

  where h is a small positive number.

  (ii) Second Derivative test :

  A function $f(x)$ has a maxima at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$.

  A function $f(x)$ has a minimum at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$.

MISCELLANEOUS EXERCISE 2

(I) Choose the correct option from the given alternatives :

(1) If the function $f(x) = ax^3 + bx^2 + 11x - 6$ satisfies conditions of Rolle's theorem in [1, 3] and $f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$, then values of a and b are respectively.

(A) 1, −6 (B) −2, 1 (C) −1, −6 (D) −1, 6

(2) If $f(x) = \frac{x^2 - 1}{x^2 + 1}$, for every real x, then the minimum value of f is -

(A) 1 (B) 0 (C) −1 (D) 2

(3) A ladder 5 m in length is resting against vertical wall. The bottom of the ladder is pulled along the ground away from the wall at the rate of 1.5 m/ sec. The length of the higher point of ladder when the foot of the ladder is 4.0 m away from the wall decreases at the rate of

(A) 1 (B) 2 (C) 2.5 (D) 3

(4) Let $f(x)$ and $g(x)$ be differentiable for $0 < x < 1$ such $f(0) = 0$, $g(0) = 0$, $f(1) = 6$. Let there exist a real number c in (0, 1) such that $f'(c) = 2g'(c)$, then the value of $g(1)$ must be

(A) 1 (B) 3 (C) 2.5 (D) −1

(5) Let $f(x) = x^3 - 6x^2 + 9x + 18$, then $f(x)$ is strictly decreasing in -

(A) $(-\infty, 1)$ (B) $[3, \infty)$ (C) $(-\infty, 1] \cup [3, \infty)$ (D) $(1, 3)$

(6) If $x = -1$ and $x = 2$ are the extreme points of $y = \alpha \log x + \beta x^2 + x$ then

(A) $\alpha = -6, \beta = \frac{1}{2}$ (B) $\alpha = -6, \beta = -\frac{1}{2}$ (C) $\alpha = 2, \beta = -\frac{1}{2}$ (D) $\alpha = 2, \beta = \frac{1}{2}$

(7) The normal to the curve $x^2 + 2xy - 3y^2 = 0$ at (1, 1)

(A) Meets the curve again in second quadrant. (B) Does not meet the curve again.

(C) Meets the curve again in third quadrant. (D) Meets the curve again in fourth quadrant.

(8) The equation of the tangent to the curve $y = 1 - e^{\frac{x}{2}}$ at the point of intersection with Y-axis is

(A) $x + 2y = 0$ (B) $2x + y = 0$ (C) $x - y = 2$ (D) $x + y = 2$

(9) If the tangent at (1, 1) on $y^2 = x(2 - x)^2$ meets the curve again at P then P is

(A) (4, 4) (B) (−1, 2) (C) (3, 6) (D) $\left(\frac{9}{4}, \frac{3}{8}\right)$

(10) The appoximate value of tan (44° 30′) given that 1° = 0.0175.

(A) 0.8952 (B) 0.9528 (C) 0.9285 (D) 0.9825

(II) (1) If the curves $ax^2 + by^2 = 1$ and $a' x^2 + b' y^2 = 1$ intersect orthogonally, then prove that $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$.

(2) Determine the area of the triangle formed by the tangent to the graph of the function $y = 3 - x^2$ drawn at the point (1, 2) and the cordinate axes.

(3) Find the equation of the tangent and normal drawn to the curve $y^4 - 4x^4 - 6xy = 0$ at the point $M(1, 2)$.

(4) A water tank in the form of an inverted cone is being emptied at the rate of 2 cubic feet per second. The height of the cone is 8 feet and the radius is 4 feet. Find the rate of change of the water level when the depth is 6 feet.

(5) Find all points on the ellipse $9x^2 + 16y^2 = 400$, at which the y-coordinate is decreasing and the x-coordinate is increasing at the same rate.

(6) Verify Rolle's theorem for the function $f(x) = \dfrac{2}{e^x + e^{-x}}$ on $[-1, 1]$.

(7) The position of a particle is given by the function $s(t) = 2t^2 + 3t - 4$. Find the time $t = c$ in the interval $0 \leq t \leq 4$ when the instantaneous velocity of the particle equals to its average velocity in this interval.

(8) Find the approximate value of the function $f(x) = \sqrt{x^2 + 3x}$ at $x = 1.02$.

(9) Find the approximate value of $\cos^{-1}(0.51)$ given $\pi = 3.1416$, $\dfrac{2}{\sqrt{3}} = 1.1547$.

(10) Find the intervals on which the function $y = x^x$, $(x > 0)$ is increasing and decreasing.

(11) Find the intervals on the which the function $f(x) = \dfrac{x}{\log x}$, is increasing and decreasing.

(12) An open box with a square base is to be made out of a given quantity of sheet of area a^2, Show the maximum volume of the box is $\dfrac{a^3}{6\sqrt{3}}$.

(13) Show that of all rectangles inscribed in a given circle, the square has the maximum area.

(14) Show that a closed right circular cyclinder of given surface area has maximum volume if its height equals the diameter of its base.

(15) A window is in the form of a rectangle surmounted by a semi-circle. If the perimeter be 30 m, find the dimensions so that the greatest possible amount of light may be admitted.

(16) Show that the height of a right circular cylinder of greatest volume that can be inscribed in a right circular cone is one-third of that of the cone.

(17) A wire of length l is cut in to two parts. One part is bent into a circle and the other into a square. Show that the sum of the areas of the circle and the square is least, if the radius of the circle is half the side of the square.

(18) A rectangular Sheet of paper of fixed perimeter with the sides having their length in the ratio 8 : 15 converted in to an open rectangular box by folding after removing the squares of equal area from all corners. If the total area of the removed squares is 100, the resulting box has maximum valume. Find the lengths of the rectangular sheet of paper.

(19) Show that the altitude of the right circular cone of maximum volume that can be inscribed in a shpere of radius r is $\dfrac{4r}{3}$.

(20) Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\dfrac{2R}{\sqrt{3}}$. Also find the maximum volume.

(21) Find the maximum and minimum values of the function $f(x) = \cos^2 x + \sin x$.

❖ ❖ ❖

3. INDEFINITE INTEGRATION

Let us Study

- Definition and Properties
- Different Techniques : **1.** by substitution **2.** by parts **3.** by partial fraction

Introduction :

In differential calculus, we studied differentiation or derivatives of some functions. We saw that derivatives are used for finding the slopes of tangents, maximum or minimum values of the function.

Now we will try to find the function whose derivative is known, or given $f(x)$. We will find $g(x)$ such that $g'(x) = f(x)$. Here the integration of $f(x)$ with respect to x is $g(x)$ or $g(x)$ is called the primitive of $f(x)$. For example, we know that the derivative of x^3 *w. r. t.* x is $3x^2$. So $\frac{d}{dx}x^3 = 3x^2$; and integral of $3x^2$ *w. r. t.* x is x^3. This is shown with the sign of integration namely '$\int$'. We write $\int 3x^2 \cdot dx = x^3$.

In this chapter we restrict ourselves only to study the methods of integration. The theory of integration is developed by Sir Isaac Newton and Gottfried Leibnitz.

$\int f(x) \cdot dx = g(x)$, read as an integral of $f(x)$ with respect to x, is $g(x)$. Since the derivative of constant function with respect to x is zero (0), we can also write

$\int f(x) \cdot dx = g(x) + c$, where c is an arbitary constant and c can take infinitely many values.

For example :

$f(x) = x^2 + c$ represents familly of curves for different values of c.

$f'(x) = 2x$ gives the slope of the tangent to $f(x) = x^2 + c$.

In the figure we have shown the curves

$y = x^2$, $y = x^2 + 4$, $y = x^2 - 5$.

Note that at the points (2, 4), (2, 8) (2, −1) respectivelly on those curves, the slopes of tangents are 2 (2) = 4.

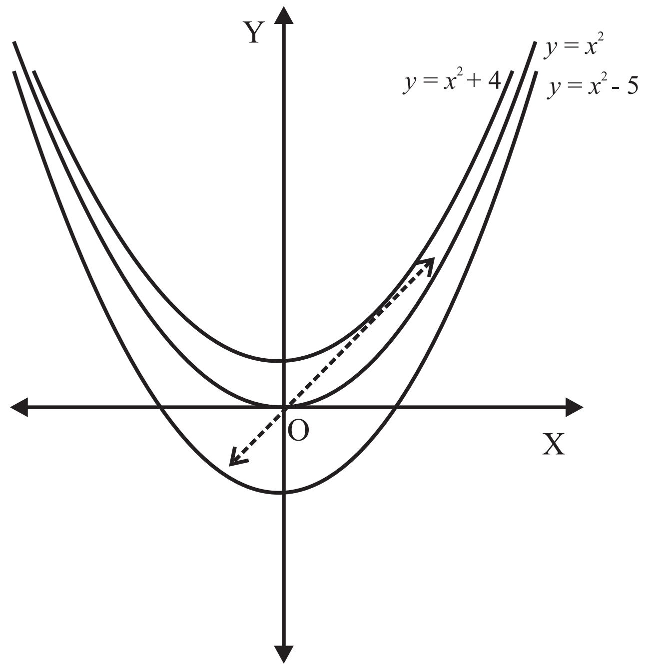


Fig. 3.1.1

3.1.1 Elementary Integration Formulae

(i) $\dfrac{d}{dx}\left(\dfrac{x^{n+1}}{n+1}\right) = \dfrac{(n+1)\,x^{n}}{(n+1)}$, $n \neq -1$

$= x^n \quad \Rightarrow \quad \therefore \quad \int x^n \cdot dx = \dfrac{x^{n+1}}{n+1} + c$

$\dfrac{d}{dx}\left(\dfrac{(ax+b)^{n+1}}{(n+1)\cdot a}\right) = \dfrac{(n+1)\,(ax+b)^{n}}{(n+1)}$

$= (ax+b)^n \quad \Rightarrow \quad \therefore \quad \int (ax+b)^n \cdot dx = \dfrac{(ax+b)^{n+1}}{n+1} \cdot \dfrac{1}{a} + c$

This result can be extended for n replaced by any rational $\dfrac{p}{q}$.

(ii) $\dfrac{d}{dx}\left(\dfrac{a^x}{\log a}\right) = a^x,\ a > 0 \quad \Rightarrow \quad \therefore \quad \int a^x \cdot dx = \dfrac{a^x}{\log a} + c$

$\int A^{ax+b} \cdot dx = \dfrac{A^{ax+b}}{\log A} \cdot \dfrac{1}{a} + c\ ,\ A > 0$

(iii) $\dfrac{d}{dx}\, e^x = e^x \quad \Rightarrow \quad \int e^x \cdot dx = e^x + c$

$\int e^{ax+b} \cdot dx = e^{ax+b} \cdot \dfrac{1}{a} + c$

(iv) $\dfrac{d}{dx} \sin x = \cos x \quad \Rightarrow \quad \int \cos x \cdot dx = \sin x + c$

$\int \cos\,(ax+b) \cdot dx = \sin\,(ax+b) \cdot \dfrac{1}{a} + c$

(v) $\dfrac{d}{dx} \cos x = -\sin x \quad \Rightarrow \quad \int \sin x \cdot dx = -\cos x + c$

$\int \sin\,(ax+b) \cdot dx = -\cos\,(ax+b) \cdot \dfrac{1}{a} + c$

(vi) $\dfrac{d}{dx} \tan x = \sec^2 x \quad \Rightarrow \quad \int \sec^2 x \cdot dx = \tan x + c$

$\int \sec^2 (ax+b) \cdot dx = \tan\,(ax+b) \cdot \dfrac{1}{a} + c$

(vii) $\dfrac{d}{dx} \sec x = \sec x \cdot \tan x \quad \Rightarrow \quad \int \sec x \cdot \tan x \cdot dx = \sec x + c$

$\int \sec\,(ax+b) \cdot \tan\,(ax+b) \cdot dx = \sec\,(ax+b) \cdot \dfrac{1}{a} + c$

(viii) $\dfrac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cdot \cot x \quad \Rightarrow \quad \int \operatorname{cosec} x \cdot \cot x \cdot dx = -\operatorname{cosec} x + c$

$\int \operatorname{cosec}\,(ax+b) \cdot \cot\,(ax+b) \cdot dx = -\operatorname{cosec}\,(ax+b) \cdot \dfrac{1}{a} + c$

(ix) $\dfrac{d}{dx} \cot x = -\operatorname{cosec}^2 x \quad \Rightarrow \quad \int \operatorname{cosec}^2 x \cdot dx = -\cot x + c$

$\int \operatorname{cosec}^2 (ax+b) \cdot dx = -\cot\,(ax+b) \cdot \dfrac{1}{a} + c$

(x) $\dfrac{d}{dx} \log x = \dfrac{1}{x},\ x > 0 \quad \Rightarrow \quad \int \dfrac{1}{x}\, dx = \log x + c,\ x \neq 0.$

$\therefore$ also $\int \dfrac{1}{(ax+b)} \cdot dx = \log\,(ax+b) \cdot \dfrac{1}{a} + c$

We assume that the trigonometric functions and logarithmic functions are defined on the respective domains.

3.1.2

Theorem 1 : If f and g are real valued integrable functions of x, then

$$\int [f(x) + g(x)] \cdot dx = \int f(x) \cdot dx + \int g(x) \cdot dx$$

Theorem 2 : If f and g are real valued integrable functions of x, then

$$\int [f(x) - g(x)] \cdot dx = \int f(x) \cdot dx - \int g(x) \cdot dx$$

Theorem 3 : If f and g are real valued integrable functions of x, and k is constant, then

$$\int k[f(x)] \cdot dx = k \int f(x) \cdot dx$$

Proof : 1. Let $\int f(x) \cdot dx = g_1(x) + c_1$ and $\int g(x) \cdot dx = g_2(x) + c_2$ then

$$\frac{d}{dx}[(g_1(x) + c_1)] = f(x) \quad \text{and} \quad \frac{d}{dx}[(g_2(x) + c_2)] = g(x)$$

$$\therefore \quad \frac{d}{dx}[(g_1(x) + c_1) + (g_2(x) + c_2)]$$

$$= \frac{d}{dx}[(g_1(x) + c_1)] + \frac{d}{dx}[(g_2(x) + c_2)]$$

$$= f(x) + g(x)$$

By definition of integration.

$$\int f(x) + g(x) = (g_1(x) + c_1) + (g_2(x) + c_2)$$

$$= \int f(x) \cdot dx + \int g(x) \cdot dx$$

Note : Students can construct the proofs of the other two theorems (Theorem 2 and Theorem 3).

SOLVED EXAMPLES

Ex. : Evaluate the following :

1. $\int (x^3 + 3^x) \cdot dx$

Solution : $\int (x^3 + 3^x) \cdot dx$

$$= \int x^3 \cdot dx + \int 3^x \cdot dx$$

$$= \frac{x^4}{4} + \frac{3^x}{\log 3} + c$$

2. $$\int\left(\sin x+\frac{1}{x}+\frac{1}{\sqrt[3]{x}}\right)\cdot dx$$

Solution : $$\int\left(\sin x+\frac{1}{x}+\frac{1}{\sqrt[3]{x}}\right)\cdot dx$$

$$= \int\sin x\cdot dx+\int\frac{1}{x}\cdot dx+\int\frac{1}{\sqrt[3]{x}}\cdot dx$$

$$= \int\sin x\cdot dx+\int\frac{1}{x}\cdot dx+\int x^{-\frac{1}{3}}\cdot dx$$

$$= -\cos x+\log x+\frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1}+c$$

$$= -\cos x+\log x+\frac{x^{\frac{2}{3}}}{\frac{2}{3}}+c$$

3. $$\int(\tan x+\cot x)^2\cdot dx$$

Solution : $$\int(\tan x+\cot x)^2\cdot dx$$

$$= \int(\tan^2 x+2\tan x\cdot\cot x+\cot^2 x)\cdot dx$$

$$= \int(\tan^2 x+2+\cot^2 x)\cdot dx$$

$$= \int(\sec^2 x-1+2+\operatorname{cosec}^2 x-1)\cdot dx$$

$$= \int(\sec^2 x+\operatorname{cosec}^2 x)\cdot dx$$

$$= \int\sec^2 x\cdot dx+\int\operatorname{cosec}^2 x\cdot dx$$

$$= \tan x+(-\cot x)+c$$

$$= \tan x-\cot x+c$$

4. $$\int\frac{\sqrt{x}+1}{x+\sqrt{x}}\cdot dx$$

Solution : $$\int\frac{\sqrt{x}+1}{x+\sqrt{x}}\cdot dx$$

$$= \int\frac{\sqrt{x}+1}{\sqrt{x}(\sqrt{x}+1)}\cdot dx$$

$$= \int\frac{1}{\sqrt{x}}\cdot dx$$

$$= 2\cdot\int\frac{1}{2\sqrt{x}}\cdot dx$$

$$= 2\sqrt{x}+c$$

5. $$\int\frac{e^{4\log x}-e^{5\log x}}{x^5}\cdot dx$$

Solution : $$\int\frac{e^{4\log x}-e^{5\log x}}{x^5}\cdot dx$$

$$= \int\frac{e^{\log x^4}-e^{\log x^5}}{x^5}\cdot dx,\qquad \because a^{\log_a f(x)}=f(x)$$

$$= \int\left(\frac{x^4-x^5}{x^5}\right)\cdot dx$$

$$= \int\left(\frac{1}{x}-1\right)\cdot dx$$

$$= \log(x)-x+c$$

6. $$\int\frac{2x+3}{5x-1}\cdot dx$$

Solution : $$\frac{\mathrm{N}}{\mathrm{D}}=\mathrm{Q}+\frac{\mathrm{R}}{\mathrm{D}}$$

$$\begin{array}{r|l} & \frac{2}{5} \\ \hline (5x-1) & 2x+3 \\ & -\;\; 2x-\frac{2}{5} \\ & -\quad + \\ \hline & 3+\frac{2}{5}=\frac{17}{5} \end{array}$$

$$\therefore\quad 2x+3=\frac{2}{5}(5x-1)+3+\frac{2}{5}$$

$$\mathrm{I} = \int\left[\frac{2}{5}+\frac{\frac{17}{5}}{5x-1}\right]\cdot dx$$

$$= \frac{2}{5}x+\frac{17}{5}\log(5x-1)\cdot\frac{1}{5}+c$$

$$= \frac{2}{5}x+\frac{17}{25}\log(5x-1)+c$$

7. $\displaystyle\int \frac{1}{\sqrt{3x+1}-\sqrt{3x-5}}\cdot dx$

Solution : $\displaystyle\int \frac{1}{\sqrt{3x+1}-\sqrt{3x-5}}\cdot dx$

$$= \int\left(\frac{1}{\sqrt{3x+1}-\sqrt{3x-5}}\right)\cdot\left(\frac{\sqrt{3x+1}+\sqrt{3x-5}}{\sqrt{3x+1}+\sqrt{3x-5}}\right)\cdot dx$$

$$= \int \frac{\sqrt{3x+1}+\sqrt{3x-5}}{3x+1-3x+5}\cdot dx$$

$$= \int \frac{\sqrt{3x+1}+\sqrt{3x-5}}{6}\cdot dx$$

$$= \frac{1}{6}\cdot\int\left((3x+1)^{\frac{1}{2}}+(3x-5)^{\frac{1}{2}}\right)\cdot dx$$

$$= \frac{1}{6}\left\{\int(3x+1)^{\frac{1}{2}}\cdot dx+\int(3x-5)^{\frac{1}{2}}\cdot dx\right\}$$

$$= \frac{1}{6}\cdot\left\{\frac{(3x+1)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\cdot 3}+\frac{(3x-5)^{\frac{1}{2}+1}}{\left(\frac{1}{2}+1\right)\cdot 3}\right\}+c$$

$$= \frac{1}{18}\cdot\left\{\frac{2}{3}(3x+1)^{\frac{3}{2}}+\frac{2}{3}(3x-5)^{\frac{3}{2}}\right\}+c$$

$$= \frac{1}{27}\cdot\left\{(3x+1)^{\frac{3}{2}}+(3x-5)^{\frac{3}{2}}\right\}+c$$

8. $\displaystyle\int \frac{2x-7}{\sqrt{3x-2}}\cdot dx$

Solution : Express $(2x-7)$ in terms of $(3x-2)$

$$2x-7 = \frac{2}{3}(3x-2)+\frac{4}{3}-7$$

$$= \frac{2}{3}(3x-2)-\frac{17}{3}$$

$$\text{I} = \int\left[\frac{\frac{2}{3}(3x-2)-\frac{17}{3}}{\sqrt{3x-2}}\right]\cdot dx$$

$$= \int\left[\frac{\frac{2}{3}(3x-2)}{\sqrt{3x-2}}-\frac{\frac{17}{3}}{\sqrt{3x-2}}\right]\cdot dx$$

$$= \frac{2}{3}\int\sqrt{3x-2}\cdot dx-\frac{17}{3}\int\frac{1}{\sqrt{3x-2}}\cdot dx$$

$$= \frac{2}{3}\int(3x-2)^{\frac{1}{2}}\cdot dx-\frac{17}{3}\int\frac{1}{\sqrt{3x-2}}\cdot dx$$

$$= \frac{2}{3}\cdot\frac{(3x-2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\cdot\frac{1}{3}-\frac{17}{3}\cdot 2\cdot\left(\sqrt{3x-2}\right)\cdot\frac{1}{3}+c$$

$$= \frac{4}{27}\cdot(3x-2)^{\frac{3}{2}}-\frac{34}{9}\cdot(3x-2)^{\frac{1}{2}}+c$$

9. $\displaystyle\int \frac{x^3}{x-1}\cdot dx$

Solution :

$$\text{I} = \int\frac{x^3-1+1}{x-1}\cdot dx$$

$$= \int\left(\frac{x^3-1}{x-1}+\frac{1}{x-1}\right)\cdot dx$$

$$= \int\left(\frac{(x-1)(x^2+x+1)}{(x-1)}+\frac{1}{x-1}\right)\cdot dx$$

$$= \int\left(x^2+x+1+\frac{1}{x-1}\right)\cdot dx$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+x+\log(x-1)+c$$

10. $\displaystyle\int \frac{3^x-4^x}{5^x}\cdot dx$

Solution :

$$\text{I} = \int\left(\frac{3^x}{5^x}-\frac{4^x}{5^x}\right)\cdot dx$$

$$= \int\left[\left(\frac{3}{5}\right)^x-\left(\frac{4}{5}\right)^x\right]\cdot dx$$

$$= \frac{\left(\frac{3}{5}\right)^x}{\log\frac{3}{5}}-\frac{\left(\frac{4}{5}\right)^x}{\log\frac{4}{5}}+c$$

11. $\int \cos^3 x \cdot dx$

Solution : $\cos 3A = 4\cos^3 A - 3\cos A$

$$I = \int \frac{1}{4}(\cos 3x + 3\cos x)\cdot dx$$

$$= \frac{1}{4}\left(\sin 3x \cdot \frac{1}{3} + 3\cdot \sin x\right) + c$$

$$= \frac{1}{12}\sin 3x + \frac{3}{4}\sin x + c$$

12. $\int \sqrt{1+\sin 3x}\cdot dx$

Solution :

$$I = \int \sqrt{\cos^2\frac{3x}{2} + \sin^2\frac{3x}{2} + 2\sin\frac{3x}{2}\cdot\cos\frac{3x}{2}}\cdot dx$$

$$= \int \sqrt{\left(\cos\frac{3x}{2} + \sin\frac{3x}{2}\right)^2}\cdot dx$$

$$= \int \left(\cos\frac{3x}{2} + \sin\frac{3x}{2}\right)\cdot dx$$

$$= \sin\frac{3x}{2}\cdot\frac{1}{\frac{3}{2}} - \cos\frac{3x}{2}\cdot\frac{1}{\frac{3}{2}} + c$$

$$= \frac{2}{3}\left(\sin\frac{3x}{2} - \cos\frac{3x}{2}\right) + c$$

13. $\int \sin^4 x \cdot dx$

Solution :

$$I = \int (\sin^2 x)^2 \cdot dx$$

$$= \int \left(\frac{1}{2}(1-\cos 2x)\right)^2 \cdot dx$$

$$= \frac{1}{4}\cdot\int (1 - 2\cos 2x + \cos^2 2x)\cdot dx$$

$$= \frac{1}{4}\cdot\int \left[1 - 2\cos 2x + \frac{1}{2}(1+\cos 4x)\right]\cdot dx$$

$$= \frac{1}{4}\cdot\int \left(1 - 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right)\cdot dx$$

$$= \frac{1}{4}\cdot\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right)\cdot dx$$

$$= \frac{1}{4}\cdot\left[\frac{3}{2}x - 2\sin 2x\cdot\frac{1}{2} + \frac{1}{2}\sin 4x\cdot\frac{1}{4}\right] + c$$

$$= \frac{1}{4}\cdot\left[\frac{3}{2}x - \sin 2x + \frac{1}{8}\sin 4x\right] + c$$

14. $\int \sin 5x \cdot \cos 7x \cdot dx$

Solution : We know that

$$2\sin A\cdot\cos B = \sin(A+B) + \sin(A-B)$$

$$I = \frac{1}{2}\int 2\sin 5x\cdot\cos 7x\cdot dx$$

$$= \frac{1}{2}\int [\sin(5x+7x) + \sin(5x-7x)]\cdot dx$$

$$= \frac{1}{2}\int [\sin(12x) + \sin(-2x)]\cdot dx$$

$$= \frac{1}{2}\int (\sin 12x - \sin 2x)\cdot dx$$

$$= \frac{1}{2}\cdot\left[-\cos 12x\cdot\frac{1}{12} + \cos 2x\cdot\frac{1}{2}\right] + c$$

$$I = -\frac{1}{24}\cos 12x + \frac{1}{4}\cos 2x + c$$

15. $\int \frac{\sin^3 x - \cos^3 x}{\sin^2 x\cdot\cos^2 x}\cdot dx$

Solution : $I = \int \left(\frac{\sin^3 x}{\sin^2 x\cdot\cos^2 x} - \frac{\cos^3 x}{\sin^2 x\cdot\cos^2 x}\right)\cdot dx$

$$= \int \left(\frac{\sin x}{\cos^2 x} - \frac{\cos x}{\sin^2 x}\right)\cdot dx$$

$$= \int \left(\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} - \frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}\right)\cdot dx$$

$$= \int (\sec x\cdot\tan x - \operatorname{cosec} x\cdot\cot x)\cdot dx$$

$$= \sec x - (-\operatorname{cosec} x) + c$$

$$I = \sec x + \operatorname{cosec} x + c$$

16. $\displaystyle\int \frac{1}{1-\sin x} \cdot dx$

Solution :

$$\begin{aligned}
\mathrm{I} &= \int \left(\frac{1}{1-\sin x}\right)\left(\frac{1+\sin x}{1+\sin x}\right) \cdot dx \\
&= \int \frac{1+\sin x}{1-\sin^2 x} \cdot dx \\
&= \int \frac{1+\sin x}{\cos^2 x} \cdot dx \\
&= \int \left(\frac{1}{\cos^2 x} + \frac{\sin x}{\cos^2 x}\right) \cdot dx \\
&= \int (\sec^2 x + \sec x \cdot \tan x) \cdot dx \\
&= \tan x + \sec x + c
\end{aligned}$$

17. $\displaystyle\int \left(\frac{\cos x}{1-\cos x}\right) \cdot dx$

Solution :

$$\begin{aligned}
\mathrm{I} &= \int \left(\frac{\cos x}{1-\cos x}\right)\left(\frac{1+\cos x}{1+\cos x}\right) \cdot dx \\
&= \int \frac{\cos x\,(1+\cos x)}{1-\cos^2 x} \cdot dx \\
&= \int \left(\frac{\cos x + \cos^2 x)}{\sin^2 x}\right) \cdot dx \\
&= \int \left(\frac{\cos x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\right) \cdot dx \\
&= \int (\operatorname{cosec} x \cdot \cot x + \cot^2 x) \cdot dx \\
&= \int (\operatorname{cosec} x \cdot \cot x + \operatorname{cosec}^2 x - 1) \cdot dx \\
&= (-\operatorname{cosec} x) + (-\cot x) - x + c \\
&= -\operatorname{cosec} x - \cot x - x + c
\end{aligned}$$

Activity :

18. $\displaystyle\int \frac{\cos x - \cos 2x}{1-\cos x} \cdot dx$

Solution :

$$\begin{aligned}
& \int \frac{\cos x - \cos 2x}{1-\cos x} \cdot dx \\
&= \int \frac{\cos x - (\ldots\ldots\ldots\ldots)}{1-\cos x} \cdot dx \\
&= \int \frac{\cos x - \ \ldots\ldots\ldots\ldots}{1-\cos x} \cdot dx \\
&= \int \frac{\cos x\,(1-\cos x) + \ \ldots\ldots\ldots\ldots}{1-\cos x} \cdot dx \\
&= \int \left[\cos x + \frac{\ldots\ldots\ldots\ldots}{1-\cos x}\right] \cdot dx \\
&= \int [\cos x + (1+\cos x)] \cdot dx \\
&= \int (1 + 2\cos x) \cdot dx \\
&= x + 2\sin x + c
\end{aligned}$$

19. $\int \sin^{-1}(\cos 3x)\cdot dx$

Solution :

$$I = \int \sin^{-1}\left(\sin \frac{\pi}{2} - 3x\right)\cdot dx$$

$$= \int\left(\frac{\pi}{2} - 3x\right)\cdot dx$$

$$= \frac{\pi}{2}x - 3\frac{x^2}{2} + c$$

20. $\int \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)\cdot dx$

Solution :

$$I = \int \cot^{-1}\left(\frac{1+\cos 2x}{\sin 2x}\right)\cdot dx$$

$$= \int \cot^{-1}\left(\frac{2\cos^2 x}{2\sin x\cdot\cos x}\right)\cdot dx$$

$$= \int \cot^{-1}(\cot x)\cdot dx$$

$$= \int x\cdot dx = \frac{x^2}{2} + c$$

21. $\int \tan^{-1}\sqrt{\frac{1-\sin x}{1+\sin x}}\cdot dx$

Solution :

$$I = \int \tan^{-1}\sqrt{\frac{1-\cos\left(\frac{\pi}{2}-x\right)}{1+\cos\left(\frac{\pi}{2}-x\right)}}\cdot dx$$

$$= \int \tan^{-1}\sqrt{\frac{2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}}\cdot dx$$

$$= \int \tan^{-1}\sqrt{\tan^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\cdot dx$$

$$= \int \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]\cdot dx$$

$$= \int\left(\frac{\pi}{4}-\frac{x}{2}\right)\cdot dx$$

$$= \frac{\pi}{4}x - \frac{1}{2}\cdot\frac{x^2}{2} + c$$

$$= \frac{\pi}{4}x - \frac{x^2}{4} + c$$

EXERCISE 3.1

I. Integrate the following functions *w. r. t. x* :

(i) $x^3 + x^2 - x + 1$ (ii) $x^2\left(1-\frac{2}{x}\right)^2$

(iii) $3\sec^2 x - \frac{4}{x} + \frac{1}{x\sqrt{x}} - 7$

(iv) $2x^3 - 5x + \frac{3}{x} + \frac{4}{x^5}$ (v) $\frac{3x^3 - 2x + 5}{x\sqrt{x}}$

II. Evaluate :

(i) $\int \tan^2 x\cdot dx$ (ii) $\int \frac{\sin 2x}{\cos x}\cdot dx$

(iii) $\int \frac{\sin x}{\cos^2 x}\cdot dx$ (iv) $\int \frac{\cos 2x}{\sin^2 x}\cdot dx$

(v) $\int \frac{\cos 2x}{\sin^2 x\cdot\cos^2 x}\cdot dx$ (vi) $\int \frac{\sin x}{1+\sin x}\cdot dx$

(vii) $\int \frac{\tan x}{\sec x + \tan x}\cdot dx$ (viii) $\int \sqrt{1+\sin 2x}\cdot dx$

(ix) $\int \sqrt{1-\cos 2x}\cdot dx$ (x) $\int \sin 4x\cdot\cos 3x\cdot dx$

III. Evaluate :

(i) $\int \frac{x}{x+2}\cdot dx$ (ii) $\int \frac{4x+3}{2x+1}\cdot dx$

(iii) $\int \frac{5x+2}{3x-4}\cdot dx$ (iv) $\int \frac{x-2}{\sqrt{x+5}}\cdot dx$

(v) $\int \frac{2x-7}{\sqrt{4x-1}}\cdot dx$ (vi) $\int \frac{\sin 4x}{\cos 2x}\cdot dx$

(vii) $\int \sqrt{1+\sin 5x}\cdot dx$ (viii) $\int \cos^2 x\cdot dx$

(ix) $\int \frac{2}{\sqrt{x}-\sqrt{x+3}}\cdot dx$

(x) $\int \frac{3}{\sqrt{7x-2}-\sqrt{7x-5}}\cdot dx$

IV. $f'(x) = x - \frac{3}{x^3}$, $f(1) = \frac{11}{2}$ then find $f(x)$.

3.2 Methods of integration :

We have evaluated the integrals which can be reduced to standard forms by algebric or trigonometric simplifications. This year we are going to study three special methods of reducing an integral to a standard form, namely –

1. Integration by substitution
2. Integration by parts
3. Integration by partial fraction

3.2.1 Integration by substitution :

Theorem 1 : If $x = \phi(t)$ is a differentiable function of t, then $\int f(x) \cdot dx = \int f[\phi(t)] \cdot \phi'(t)\, dt$.

Proof : $x = \phi(t)$ is a differentiable function of t.

$$\therefore \quad \frac{dx}{dt} = \phi'(t)$$

Let $\int f(x)\, dx = g(x) \Rightarrow \frac{d}{dx}[g(x)] = f(x)$

By Chain rule,

$$\begin{aligned} \frac{d}{dt}[g(x)] &= \frac{d}{dx}[g(x)] \cdot \frac{dx}{dt} \\ &= f(x) \cdot \frac{dx}{dt} \\ &= f[\phi(t)] \cdot \phi'(t) \end{aligned}$$

By definition of integration,

$$g(x) = \int f[\phi(t)] \cdot \phi'(t) \cdot dt$$

$$\therefore \quad \int f(x) \cdot dx = \int f[\phi(t)] \cdot \phi'(t) \cdot dt$$

For example 1 : $\int 3x^2 \sin(x^3) \cdot dx$

Let $x^3 = t$

$\therefore \quad 3x^2\, dx = dt$

$= \int \sin t \cdot dt$

$= -\cos t + c$

$= -\cos(x^3) + c$

Corollary I :

If $\int f(x)\cdot dx = g(x) + c$

then $\int f(ax+b)\cdot dx = g(ax+b)\dfrac{1}{a} + c$

Proof : Let $I = \int f(ax+b)\cdot dx$

put $ax + b = t$

Differentiating both the sides

$$a\cdot dx = 1\cdot dt \Rightarrow dx = \frac{1}{a}dt$$

$$\begin{aligned} I &= \int f(t)\cdot\frac{1}{a}\cdot dt \\ &= \frac{1}{a}\cdot\int f(t)\cdot dt \\ &= \frac{1}{a}\cdot g(t) + c \\ &= \frac{1}{a}\cdot g(ax+b) + c \end{aligned}$$

$$\therefore \quad \int f(ax+b)\cdot dx = g(ax+b)\frac{1}{a} + c$$

For example : $\int \sec^2(5x-4)\cdot dx$

$$= \frac{1}{5}\tan(5x-4) + c$$

Corollary II :

$$\int [f(x)]^n\cdot f'(x)\cdot dx = \frac{[f(x)]^{n+1}}{n+1} + c,\ n \neq -1$$

Proof : Let $I = \int [f(x)]^{n+1}\cdot f'(x)\cdot dx$

put $f(x) = t$

Differentiating both the sides

$f'(x)\cdot dx = dt$

$$\begin{aligned} I &= \int [t]^n\cdot dt \\ &= \frac{t^{n+1}}{n+1} + c\ , \qquad n \neq -1 \\ &= \frac{[f(x)]^{n+1}}{n+1} + c \end{aligned}$$

$$\therefore \quad \int [f(x)]^n\cdot f'(x)\cdot dx = \frac{[f(x)]^{n+1}}{n+1} + c$$

For example : $\displaystyle\int \frac{(\sin^{-1}x)^3}{\sqrt{1-x^2}}\cdot dx$

$$= \int [(\sin^{-1}x)^3]\cdot\left(\frac{1}{\sqrt{1-x^2}}\right)\cdot dx$$

$$= \frac{(\sin^{-1}x)^4}{4} + c$$

Corollary III :

$$\int \frac{f'(x)}{f(x)}\cdot dx = \log(f(x)) + c$$

Proof : Consider $\displaystyle\int \frac{f'(x)}{f(x)}\cdot dx$

put $f(x) = t$

Differentiating both the sides

$f'(x)\cdot dx = dt$

$$\begin{aligned} I &= \int \frac{1}{t}\cdot dt \\ &= \log(t) + c \\ &= \log\big(f(x)\big) + c \end{aligned}$$

$$\therefore \quad \int \frac{f'(x)}{f(x)}\cdot dx = \log\big(f(x)\big) + c$$

For example : $\int \cot x\cdot dx$

$$= \int \frac{\cos x}{\sin x}\cdot dx$$

$$\frac{d}{dx}\sin x = \cos x$$

$$= \int\left(\frac{\frac{d}{dx}\sin x}{\sin x}\cdot dx\right)$$

$$= \log(\sin x) + c$$

Corollary IV :

$$\int \frac{f'(x)}{\sqrt{f(x)}} \cdot dx = 2\sqrt{f(x)} + c$$

Proof : Consider $\int \frac{f'(x)}{\sqrt{f(x)}} \cdot dx$

put $f(x) = t$

Differentiating both the sides

$f'(x) \cdot dx = dt$

$$\begin{aligned} I &= \int \frac{1}{\sqrt{t}} \cdot dt \\ &= 2 \cdot \int \frac{1}{2\sqrt{t}} \cdot dt \\ &= 2\sqrt{t} + c \\ &= 2\sqrt{f(x)} + c \end{aligned}$$

$$\therefore \int \frac{f'(x)}{\sqrt{f(x)}} \cdot dx = 2\sqrt{f(x)} + c$$

For example : $\int \frac{1}{x\sqrt{\log x}} \cdot dx$

$$= \int \left(\frac{\frac{1}{x}}{\sqrt{\log x}} \right) \cdot dx$$

$$= \int \left(\frac{\frac{d}{dx} \log x}{\sqrt{\log x}} \cdot dx \right)$$

$$= 2\sqrt{\log x} + c$$

Using corollary III, $\int \frac{f'(x)}{f(x)} \cdot dx = \log \left(f(x) \right) + c$ we find the integrals of some trigonometric functions.

3.2.2 Integrals of trignometric functions :

1. $\int \tan x \cdot dx$

Solution :

$$\begin{aligned} I &= \int \tan x \cdot dx \\ &= \int \frac{\sin x}{\cos x} \cdot dx \\ &= -\int \frac{-\sin x}{\cos x} \cdot dx \\ &= -\log(\cos x) + c \\ &= \log(\sec x) + c \end{aligned}$$

Activity :

2. $\int \cot(5x - 4) \cdot dx$

Solution :

$$I = \int \frac{\ldots\ldots\ldots\ldots}{\sin(5x-4)} \cdot dx$$

$$= \frac{1}{\ldots\ldots} \int \frac{5\cos(5x-4)}{\ldots\ldots\ldots\ldots} \cdot dx$$

$$\frac{d}{dx}(\ldots\ldots\ldots\ldots) = \ldots\ldots\ldots\ldots$$

$$= \frac{1}{5} \log[\sec(5x-4)] + c$$

3. $\int \sec x \cdot dx = \log(\sec x + \tan x) + c$

Solution : Let $I = \int \sec x \cdot dx$

$$= \int \frac{(\sec x)(\sec x + \tan x)}{\sec x + \tan x} \cdot dx$$

$$= \int \frac{\sec^2 x + \sec x \cdot \tan x}{\sec x + \tan x} \cdot dx$$

$$= \int \frac{\sec x \cdot \tan x + \sec^2 x}{\sec x + \tan x} \cdot dx$$

$\because \frac{d}{dx}(\sec x + \tan x) = \sec x \cdot \tan x + \sec^2 x$

$\therefore \int \sec x \cdot dx = \log(\sec x + \tan x) + c$

Also,

$$\int \sec x \cdot dx = \log\left[\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right] + c$$

Activity :

4. $\int \operatorname{cosec} x \cdot dx = \log(\operatorname{cosec} x - \cot x) + c$

Solution : Let $I = \int \operatorname{cosec} x \cdot dx$

$$= \int \frac{(\operatorname{cosec} x)(\ldots\ldots\ldots\ldots)}{(\ldots\ldots\ldots\ldots)} \cdot dx$$

$$= \int \frac{\ldots\ldots\ldots\ldots\ldots\ldots}{\ldots\ldots\ldots} \cdot dx$$

$$= \int \frac{-\operatorname{cosec} x \cdot \cot x + \operatorname{cosec}^2 x}{\ldots\ldots\ldots} \cdot dx$$

$\frac{d}{dx}(\operatorname{cosec} x - \cot x)$

$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$= \log(\operatorname{cosec} x - \cot x) + c$

$\therefore \int \operatorname{cosec} x \cdot dx = \log(\operatorname{cosec} x - \cot x) + c$

Also,

$$\int \operatorname{cosec} x \cdot dx = \log\left(\tan\frac{x}{2}\right) + c$$

SOLVED EXAMPLES

Ex. : Evaluate the following functions :

1. $\int \frac{\cot(\log x)}{x} \cdot dx$

Solution : Let $I = \int \frac{\cot(\log x)}{x} \cdot dx$

put $\log x = t$

$\therefore \frac{1}{x} \cdot dx = 1 \cdot dt$

$= \int \cot t \cdot dt$

$= \log(\sin t) + c$

$= \log(\sin \log x) + c$

2. $\int \frac{\cos\sqrt{x}}{\sqrt{x}} \cdot dx$

Solution : Let $I = \int \frac{\cos\sqrt{x}}{\sqrt{x}} \cdot dx$

put $\sqrt{x} = t$

$\therefore \frac{1}{2\sqrt{x}} \cdot dx = 1 \cdot dt$

$\therefore \frac{1}{\sqrt{x}} \cdot dx = 2 \cdot dt$

$= 2 \cdot \int \cos t \cdot dt$

$= 2 \cdot \sin t + c$

$= 2 \cdot \sin\sqrt{x} + c$

3. $\displaystyle\int \frac{\sec^8 x}{\operatorname{cosec} x} \cdot dx$

Solution : $\displaystyle I = \int \sec^7 x \cdot \sec x \cdot \frac{1}{\operatorname{cosec} x} \cdot dx$

$$= \int \sec^7 x \cdot \frac{1}{\cos x} \cdot \sin x \cdot dx$$

$$= \int \sec^7 x \cdot \tan x \cdot dx$$

$$= \int \sec^6 x \cdot \sec x \cdot \tan x \cdot dx$$

put $\sec x = t$

$\therefore \quad \sec x \cdot \tan x \cdot dx = dt$

$$= \int t^6 \cdot dt$$

$$= \frac{t^7}{7} + c$$

$$= \frac{\sec^7 x}{7} + c$$

4. $\displaystyle\int \frac{1}{x + \sqrt{x}} \cdot dx$

Solution : $\displaystyle I = \int \frac{1}{x + \sqrt{x}} \cdot dx$

$$= \int \frac{1}{\sqrt{x}\,(\sqrt{x} + 1)} \cdot dx$$

put $\sqrt{x} + 1 = t$

$\therefore \quad \dfrac{1}{2\sqrt{x}} \cdot dx = 1 \cdot dt$

$\therefore \quad \dfrac{1}{\sqrt{x}} \cdot dx = 2 \cdot dt$

$$= \int \frac{1}{t} \cdot 2 \cdot dt$$

$$= 2 \cdot \int \frac{1}{t} \cdot dt$$

$$= 2 \cdot \log (\, t \,) + c$$

$$= 2 \cdot \log (\sqrt{x} + 1\,) + c$$

5. $\displaystyle\int 5^{5^x} \cdot 5^x \cdot dx$

Solution : $\displaystyle I = \int 5^{5^x} \cdot 5^x \cdot dx$

put $5^x = t$

$\therefore \quad 5^x \cdot \log 5 \cdot dx = 1\ dt$

$\quad 5^x \cdot dx = \dfrac{1}{\log 5} \cdot dt$

$$= \int 5^t \cdot \frac{1}{\log 5} \cdot dt$$

$$I = \frac{1}{\log 5} \cdot \int 5^t \cdot dt$$

$$= \frac{1}{\log 5} \cdot 5^t \cdot \frac{1}{\log 5} + c$$

$$= \left(\frac{1}{\log 5}\right)^2 \cdot 5^{5^x} + c$$

6. $\displaystyle\int \frac{1}{1 + e^{-x}} \cdot dx$

Solution : $\displaystyle I = \int \frac{1}{1 + e^{-x}} \cdot dx$

$$= \int \frac{1}{1 + \frac{1}{e^x}} \cdot dx$$

$$= \int \frac{1}{\frac{e^x + 1}{e^x}} \cdot dx$$

$$= \int \frac{e^x}{e^x + 1} \cdot dx$$

$\because \quad \dfrac{d}{dx}\left(\, e^x + 1 \,\right) \cdot dx = e^x$

$$= \log [\, e^x + 1 \,] + c$$

7. $\displaystyle\int \frac{e^x\,(1 + x)}{\cos (x \cdot e^x)} \cdot dx$

Solution : put $x \cdot e^x = t$

Differentiating both sides

$(x \cdot e^x + e^x \cdot 1) \cdot dx = 1\ dt$

$e^x\,(1 + x) \cdot dx = 1\ dt$

$$I = \int \frac{1}{\cos t} \cdot dt$$

$$= \int \sec t \cdot dt$$

$$= \log (\sec t + \tan t\,) + c$$

$$= \log (\sec (xe^x) + \tan (xe^x)) + c$$

8. $\displaystyle\int \frac{1}{3x + 7x^{-n}} \cdot dx$

Solution : Consider $\displaystyle\int \frac{1}{3x + 7x^{-n}} \cdot dx$

$$= \int \frac{1}{3x + \frac{7}{x^n}} \cdot dx = \int \frac{1}{\frac{3x^{n+1} + 7}{x^n}} \cdot dx$$

$$= \int \frac{x^n}{3x^{n+1} + 7} \cdot dx$$

put $3x^{n+1} + 7 = t$

Differentiate $w. r. t. x$

$3(n + 1)\, x^n \cdot dx = dt$

$\therefore \quad x^n \cdot dx = \dfrac{1}{3(n+1)}\, dt$

$$= \int \frac{\frac{1}{3(n+1)}}{t} \cdot dt$$

$$= \frac{1}{3(n+1)} \cdot \log(t) + c$$

$$= \frac{1}{3(n+1)} \cdot \log\left(3x^{n+1} + 7\right) + c$$

9. $\displaystyle\int (3x + 2)\sqrt{x - 4} \cdot dx$

Solution : put $x - 4 = t$

$\therefore \quad x = 4 + t$

Differentiate

$1 \cdot dx = 1 \cdot dt$

$$= \int [\,3(4 + t) + 2\,] \cdot \sqrt{t} \cdot dt$$

$$= \int (14 + 3t) \cdot t^{\frac{1}{2}} \cdot dt$$

$$= \int \left(14t^{\frac{1}{2}} + 3t^{\frac{3}{2}}\right) \cdot dt$$

$$= 14\,\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + 3\,\frac{t^{\frac{5}{2}}}{\frac{5}{2}} \cdot dx$$

$$= \frac{28}{3}(x - 4)^{\frac{3}{2}} + \frac{6}{5}(x - 4)^{\frac{5}{2}} + c$$

10. $\displaystyle\int \frac{\sin(x + a)}{\cos(x - b)} \cdot dx$

Solution :

$$= \int \frac{\sin[(x - b) + (a + b)]}{\cos(x - b)} \cdot dx$$

$$= \int \frac{\sin(x - b) \cdot \cos(a + b) + \cos(x - b) \cdot \sin(a + b)}{\cos(x - b)} \cdot dx$$

$$= \int \left[\frac{\sin(x - b) \cdot \cos(a + b)}{\cos(x - b)} + \frac{\cos(x - b) \cdot \sin(a + b)}{\cos(x - b)}\right]$$

$$= \int [\cos(a + b) \cdot \tan(x - b) + \sin(a + b)] \cdot dx$$

$$= \cos(a + b) \cdot \log\left(\sec(x - b)\right) + x \cdot \sin(a + b) + c$$

11. $\displaystyle\int \frac{e^x+1}{e^x-1}\cdot dx$

Solution :

$$I = \int \frac{e^x-1+2}{e^x-1}\cdot dx$$

$$= \int\left(\frac{e^x-1}{e^x-1}+\frac{2}{e^x-1}\right)\cdot dx$$

$$= \int\left(1+\frac{2}{e^x-1}\right)\cdot dx$$

$$= \int dx + \int \frac{2}{e^x(1-e^{-x})}\cdot dx$$

$$= \int 1dx + 2\int \frac{e^{-x}}{1-e^{-x}}\cdot dx$$

put $(1-e^{-x}) = t$

Differentiate $w.\ r.\ t.\ x$

$-(e^{-x})\ (-1)\cdot dx = 1\ dt$

$e^{-x}\cdot dx = 1\ dt$

$$I = \int 1dx + 2\int \frac{1}{t}\cdot dt$$

$$= x + 2\cdot \log(\,t\,) + c$$

$$= x + 2\log(1-e^{-x}) + c$$

$$\therefore \int \frac{e^x+1}{e^x-1}\cdot dx = x + 2\log(1-e^{-x}) + c$$

12. $\displaystyle\int \frac{1}{1-\tan x}\cdot dx$

Solution :

$$I = \int \frac{1}{1-\dfrac{\sin x}{\cos x}}\cdot dx$$

$$= \int \frac{\cos x}{\cos x - \sin x}$$

$$= \int \frac{\cos x}{\sqrt{2}\left(\dfrac{1}{\sqrt{2}}\cos x - \dfrac{1}{\sqrt{2}}\cos x\right)}\cdot dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{\cos x}{\cos\dfrac{\pi}{4}\cos x - \sin\dfrac{\pi}{4}\sin x}\cdot dx$$

$$= \frac{1}{\sqrt{2}}\int \frac{\cos x}{\cos\left(x+\dfrac{\pi}{4}\right)}\cdot dx$$

put $x+\dfrac{\pi}{4} = t \quad \therefore x = t-\dfrac{\pi}{4}$

Differentiating both sides

$1\cdot dx = 1\cdot dt$

$$= \frac{1}{\sqrt{2}}\int \frac{\cos\left(t-\dfrac{\pi}{4}\right)}{\cos t}\cdot dt$$

$$= \frac{1}{\sqrt{2}}\int \frac{\cos t\cdot\cos\dfrac{\pi}{4}+\sin t\cdot\sin\dfrac{\pi}{4}}{\cos t}\cdot dt$$

$$= \frac{1}{\sqrt{2}}\int\left[\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\tan t\right]\cdot dt$$

$$= \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}[\,t+\log(\sec t\,)] + c$$

$$= \frac{1}{2}\left[x+\frac{\pi}{4}+\log\sec\left(x+\frac{\pi}{4}\right)\right] + c$$

To evaluate the integrals of type $\displaystyle\int \frac{a\cos x + b\sin x}{c\cos x + d\sin x}\cdot dx$, express the Numerator as Nr = λ (Dr) + μ (Dr)' , find the constants λ & μ by compairing the co-efficients of like terms and then integrate the function.

EXERCISE 3.2 (A)

I. Integrate the following functions *w. r. t. x* :

1. $\dfrac{(\log x)^n}{x}$
2. $\dfrac{(\sin^{-1} x)^{\frac{3}{2}}}{\sqrt{1-x^2}}$
3. $\dfrac{1+x}{x\cdot\sin(x+\log x)}$
4. $\dfrac{x\cdot\sec^2(x^2)}{\sqrt{\tan^3(x^2)}}$
5. $\dfrac{e^{3x}}{e^{3x}+1}$
6. $\dfrac{(x^2+2)}{(x^2+1)}\cdot a^{x+\tan^{-1}x}$
7. $\dfrac{e^x\cdot\log(\sin e^x)}{\tan(e^x)}$
8. $\dfrac{e^{2x}+1}{e^{2x}-1}$
9. $\sin^4 x\cdot\cos^3 x$
10. $\dfrac{1}{4x+5x^{-11}}$
11. $x^9\cdot\sec^2(x^{10})$
12. $e^{3\log x}\cdot(x^4+1)^{-1}$
13. $\dfrac{\sqrt{\tan x}}{\sin x\cdot\cos x}$
14. $\dfrac{(x-1)^2}{(x^2+1)^2}$
15. $\dfrac{2\sin x\cdot\cos x}{3\cos^2 x+4\sin^2 x}$
16. $\dfrac{1}{\sqrt{x}+\sqrt{x^3}}$
17. $\dfrac{10x^9+10^x\cdot\log 10}{10^x+x^{10}}$
18. $\dfrac{x^{n-1}}{\sqrt{1+4x^n}}$
19. $(2x+1)\sqrt{x+2}$
20. $x^5\cdot\sqrt{a^2+x^2}$
21. $(5-3x)(2-3x)^{-\frac{1}{2}}$
22. $\dfrac{7+4x+5x^2}{(2x+3)^{\frac{3}{2}}}$
23. $\dfrac{x^2}{\sqrt{9-x^6}}$
24. $\dfrac{1}{x(x^3-1)}$
25. $\dfrac{1}{x\cdot\log x\cdot\log(\log x)}$

II. Integrate the following functions w. r. t. x :

1. $\dfrac{\cos 3x-\cos 4x}{\sin 3x+\sin 4x}$
2. $\dfrac{\cos x}{\sin(x-a)}$
3. $\dfrac{\sin(x-a)}{\cos(x+b)}$
4. $\dfrac{1}{\sin x\cdot\cos x+2\cos^2 x}$
5. $\dfrac{\sin x+2\cos x}{3\sin x+4\cos x}$
6. $\dfrac{1}{2+3\tan x}$
7. $\dfrac{4e^x-25}{2e^x-5}$
8. $\dfrac{20+12e^x}{3e^x+4}$
9. $\dfrac{3e^{2x}+5}{4e^{2x}-5}$
10. $\cos^8 x\cdot\cot x$
11. $\tan^5 x$
12. $\cos^7 x$
13. $\tan 3x\cdot\tan 2x\cdot\tan x$
14. $\sin^5 x\cdot\cos^8 x$
15. $3^{\cos^2 x}\cdot\sin 2x$
16. $\dfrac{\sin 6x}{\sin 10x\cdot\sin 4x}$
17. $\dfrac{\sin x\cdot\cos^3 x}{1+\cos^2 x}$

3.2.3 Some Special Integrals

1. $\displaystyle\int\frac{1}{x^2+a^2}\cdot dx=\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$
2. $\displaystyle\int\frac{1}{x^2-a^2}\cdot dx=\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+c$
3. $\displaystyle\int\frac{1}{a^2-x^2}\cdot dx=\frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c$
4. $\displaystyle\int\frac{1}{\sqrt{a^2-x^2}}\cdot dx=\sin^{-1}\left(\frac{x}{a}\right)+c$
5. $\displaystyle\int\frac{1}{\sqrt{x^2-a^2}}\cdot dx=\log\left(x+\sqrt{x^2-a^2}\right)+c$
6. $\displaystyle\int\frac{1}{\sqrt{x^2+a^2}}\cdot dx=\log\left(x+\sqrt{x^2+a^2}\right)+c$
7. $\displaystyle\int\frac{1}{x\sqrt{x^2-a^2}}\cdot dx=\frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+c$

While evaluating an integral there is no unique substitution, we can use some standard substitutions and try.

No.	Function	Substitution
1.	$\sqrt{a^2 - x^2}$	$x = a \cdot \sin\theta$ ($x = a \cdot \cos\theta$ can also be used.)
2.	$\sqrt{a^2 + x^2}$	$x = a \cdot \tan\theta$
3.	$\sqrt{x^2 - a^2}$	$x = a \cdot \sec\theta$
4.	$\sqrt{\dfrac{a-x}{a+x}}$	$x = a \cdot \cos 2\theta$

1. $$\int \frac{1}{x^2 + a^2} \cdot dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Proof :

Let $\quad I = \displaystyle\int \frac{1}{x^2 + a^2} \cdot dx$

put $x = a \cdot \tan\theta \Rightarrow \tan\theta = \dfrac{x}{a}$

i.e. $\theta = \tan^{-1}\left(\dfrac{x}{a}\right)$

$\therefore dx = a \cdot \sec^2\theta \cdot d\theta$

$$I = \int \frac{1}{a^2 \cdot \tan^2\theta + a^2} \cdot a \cdot \sec^2\theta \cdot d\theta$$

$$= \int \frac{a \cdot \sec^2\theta}{a^2(\tan^2\theta + 1)} \cdot d\theta$$

$$= \int \frac{\sec^2\theta}{a \cdot \sec^2\theta} \cdot d\theta$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c$$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\therefore \int \frac{1}{x^2 + a^2} \cdot dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

e.g. $$\int \frac{1}{x^2 + 5^2} \cdot dx = \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + c$$

Alternatively

Cosider,

$$\frac{d}{dx}\left[\frac{1}{a} \cdot \tan^{-1}\left(\frac{x}{a}\right) + c\right]$$

$$= \frac{d}{dx}\left[\frac{1}{a} \cdot \tan^{-1}\left(\frac{x}{a}\right)\right] + \frac{d}{dx}c$$

$$= \frac{1}{a} \cdot \frac{1}{1 + \left(\frac{x}{a}\right)^2} \cdot \frac{d}{dx}\left(\frac{x}{a}\right) + 0$$

$$= \frac{1}{a} \cdot \frac{1}{1 + \frac{x^2}{a^2}} \cdot \frac{1}{a}$$

$$= \frac{1}{a^2} \cdot \frac{1}{\frac{a^2 + x^2}{a^2}}$$

$$= \frac{1}{x^2 + a^2}$$

Therefore,

by definition of integration

$$\therefore \int \frac{1}{x^2 + a^2} \cdot dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

2. $\displaystyle\int \frac{1}{x^2 - a^2} \cdot dx = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + c$

Proof :

Let $\displaystyle I = \int \frac{1}{x^2 - a^2} \cdot dx$

$\displaystyle = \int \frac{1}{(x+a)(x-a)} \cdot dx$

$\displaystyle = \int \frac{1}{2a} \cdot \left[\frac{1}{x-a} - \frac{1}{x+a}\right] \cdot dx$

$\displaystyle = \frac{1}{2a} \cdot \int \left[\frac{1}{x-a} - \frac{1}{x+a}\right] \cdot dx$

$\displaystyle = \frac{1}{2a} \cdot [\log(x-a) - \log(x+a)] + c$

$\displaystyle = \frac{1}{2a} \cdot \log\left(\frac{x-a}{x+a}\right) + c$

$\displaystyle \therefore \quad \int \frac{1}{x^2 - a^2} \cdot dx = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + c$

e.g. $\displaystyle \int \frac{1}{x^2 - 9} \cdot dx = \frac{1}{2(3)} \log\left(\frac{x-3}{x+3}\right) + c$

Activity :

3. $\displaystyle\int \frac{1}{a^2 - x^2} \cdot dx = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right) + c$

Proof : Consider,

$\displaystyle I = \int \frac{1}{a^2 - x^2} \cdot dx$

$\displaystyle = \int \frac{1}{(\ldots\ldots)(\ldots\ldots)} \cdot dx$

$\displaystyle = \int \frac{1}{2a} \cdot \left[\frac{1}{\ldots\ldots} - \frac{1}{\ldots\ldots}\right] \cdot dx$

$\displaystyle = \frac{1}{2a} \cdot \int \left[\frac{\ldots\ldots}{\ldots\ldots} - \frac{1}{a+x}\right] \cdot dx$

$\displaystyle = \frac{1}{2a} \cdot [\log(a+x) - \log(a-x)] + c$

$\displaystyle = \frac{1}{2a} \cdot \log\left(\frac{\ldots\ldots}{\ldots\ldots}\right) + c$

$\displaystyle \therefore \quad \int \frac{1}{a^2 - x^2} \cdot dx = \frac{1}{2a} \log\left(\frac{a+x}{a-x}\right) + c$

e.g. $\displaystyle \int \frac{1}{16 - x^2} \cdot dx = \frac{1}{2(4)} \log\left(\frac{4+x}{4-x}\right) + c$

4. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

Proof :

Let $\displaystyle I = \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx$

put $x = a\sin\theta \Rightarrow \quad \sin\theta = \dfrac{x}{a}$

$\displaystyle \therefore \quad \theta = \sin^{-1}\left(\frac{x}{a}\right)$

$dx = a \cdot \cos\theta \, d\theta$

$\displaystyle I = \int \frac{1}{\sqrt{a^2 - a^2\sin^2\theta}} \cdot a \cdot \cos\theta \, d\theta$

$\displaystyle I = \int \frac{a \cdot \cos\theta}{a\sqrt{1 - a^2\sin^2\theta}} \cdot d\theta$

$\displaystyle = \int \frac{\cos\theta}{\cos\theta} \cdot d\theta$

$\displaystyle = \int 1 \cdot d\theta$

$= \theta + c$

$\displaystyle \therefore \quad \int \frac{1}{\sqrt{a^2 - x^2}} \cdot dx = \sin^{-1}\left(\frac{x}{a}\right) + c$

e.g. $\displaystyle \int \frac{1}{\sqrt{81 - x^2}} \cdot dx = \sin^{-1}\left(\frac{x}{9}\right) + c$

5. $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}} \cdot dx = \log\left(x+\sqrt{x^2-a^2}\right)+c$

Proof : Let $\displaystyle I = \int \frac{1}{\sqrt{x^2-a^2}} \cdot dx$

put $x = a\sec\theta \Rightarrow \theta = \sec^{-1}\left(\frac{x}{a}\right)$

$\therefore \quad dx = a\cdot\sec\theta\cdot\tan\theta\cdot d\theta$

$$I = \int \frac{1}{\sqrt{a^2\sec^2\theta - a^2}} \cdot a\cdot\sec\theta\cdot\tan\theta\cdot d\theta$$

$$= \int \frac{a\cdot\sec\theta\cdot\tan\theta}{\sqrt{a^2(\sec^2\theta-1)}} \cdot d\theta$$

$$= \int \frac{a\cdot\sec\theta\cdot\tan\theta}{\sqrt{a^2\cdot\tan^2\theta}} \cdot d\theta$$

$$= \int \frac{a\cdot\sec\theta\cdot\tan\theta}{a\cdot\tan\theta} \cdot d\theta$$

$$= \int \sec\theta\cdot d\theta$$

$$= \log(\sec\theta+\tan\theta)+c$$

$$= \log(\sec\theta+\sqrt{\sec^2\theta-1})+c$$

$$= \log\left(\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right)+c_1$$

$$= \log\left(\frac{x}{a}+\sqrt{\frac{x^2}{a^2}-1}\right)+c_1$$

$$= \log\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)+c_1$$

$$= \log\left(x+\sqrt{x^2-a^2}\right)-\log a+c_1$$

$$= \log\left(x+\sqrt{x^2-a^2}\right)+c$$

where $c = c_1 - \log a$

$$\therefore \quad \int \frac{1}{\sqrt{x^2-a^2}} \cdot dx = \log\left(x+\sqrt{x^2-a^2}\right)+c$$

e.g. $\displaystyle\int \frac{1}{\sqrt{x^2-16}} \cdot dx = \log\left(x+\sqrt{x^2-16}\right)+c$

Activity :

6. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}} \cdot dx = \log\left(x+\sqrt{a^2-x^2}\right)+c$

Proof : use substitution $x = a\cdot\tan\theta$

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Activity :

7. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}} \cdot dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+c$

Proof : Let $\displaystyle I = \int \frac{1}{x\sqrt{x^2-a^2}} \cdot dx$

put $x = a\sec\theta \Rightarrow \theta = \sec^{-1}\left(\frac{x}{a}\right)$

$\therefore \quad dx = a\cdot\sec\theta\cdot\tan\theta\cdot d\theta$

$$I = \int \frac{1}{a\sec\theta\sqrt{\ldots\ldots - a^2}} \cdot \ldots\ldots\ldots$$

$$= \int \frac{\tan\theta}{\sqrt{a^2(\ldots\ldots\ldots)}} \cdot d\theta$$

$$= \frac{1}{a}\int 1\cdot d\theta$$

$$= \frac{1}{a}\cdot\theta+c$$

$$= \frac{1}{a}\cdot\sec^{-1}\left(\frac{x}{a}\right)+c$$

$$\therefore \quad \int \frac{1}{x\sqrt{x^2-a^2}} \cdot dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+c$$

e.g. $\displaystyle\int \frac{1}{x\sqrt{x^2-64}} \cdot dx = \frac{1}{8}\sec^{-1}\left(\frac{x}{8}\right)+c$

3.2.4

In order to evaluate the integrals of type $\int \frac{1}{ax^2+bx+c}\cdot dx$ and $\int \frac{1}{\sqrt{ax^2+bx+c}}\cdot dx$ we can use the following steps.

(1) Write ax^2+bx+c as, $a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$, $a>0$ and take a or $\sqrt{a}$ out of the integral sign.

(2) $\left(x^2+\frac{b}{a}x\right)$ or $\left(\frac{b}{a}x-x^2\right)$ is expressed by the method of completing square by adding and subtracting $\left(\frac{1}{2}\text{ coefficient of } x\right)^2$.

(3) Express the quadractic expression as a sum or difference of two squares i.e. $\left((x+\beta)^2 \pm \alpha^2\right)$ or $\left(\alpha^2-(x+\beta)^2\right)$

(4) We know that $\int f(x)\, dx = g(x)+c \quad \Rightarrow \quad \int f(x+\beta)\, dx = g(x+\beta)+c$

$$\int f(\alpha x+\beta)\, dx = \frac{1}{\alpha} g(\alpha x+\beta)+c$$

(5) Use the standard integral formula and express the result in terms of x.

3.2.5

In order to evaluate the integral of type $\int \frac{1}{a\sin^2 x+b\cos^2 x+c}\cdot dx$ we can use the following steps.

(1) Divide the numerator and denominator by $\cos^2 x$ or $\sin^2 x$.

(2) In denominator replace $\sec^2 x$ by $1+\tan^2 x$ and /or $\operatorname{cosec}^2 x$ by $1+\cot^2 x$, if exists.

(3) Put $\tan x = t$ or $\cot x = t$ so that the integral reduces to the form $\int \frac{1}{at^2+bt+c}\cdot dt$

(4) Use the standard integral formula and express the result in terms of x.

3.2.6

To evaluate the integral of the form $\int \frac{1}{a\sin x+b\cos x+c}\cdot dx$, we use the standard substitution $\tan\frac{x}{2}=t$.

If $\tan\frac{x}{2}=t$ then (i) $\sec^2\frac{x}{2}\cdot\frac{1}{2}\cdot dx = 1\cdot dt$

$$\text{i.e. } dx = \frac{2}{\sec^2\frac{x}{2}}\cdot dt = \frac{2}{1+\tan^2\frac{x}{2}}\cdot dt = \frac{2\,dt}{1+t^2}$$

(ii) $$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

(iii) $\cos x = \dfrac{1-\tan^2\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}} = \dfrac{1-t^2}{1+t^2}$

We put $\tan x = t$ for the integral of the type $\displaystyle\int \frac{1}{a\sin 2x + b\cos 2x + c} \cdot dx$

therefore

$$dx = \frac{1}{1+t^2} \cdot dt$$

$$\sin 2x = \frac{2t}{1+t^2} \cdot dt$$

and

$$\cos 2x = \frac{1-t^2}{1+t^2} \cdot dt$$

With this substitution the integral reduces to the form $\displaystyle\int \frac{1}{ax^2+bx+c} \cdot dx$. Now use the standard integral formula and express the result in terms of x.

SOLVED EXAMPLES

Ex. : Evaluate :

1. $\displaystyle\int \frac{1}{4x^2+11} \cdot dx$

Solution : $\displaystyle I = \int \frac{1}{4\left(x^2+\frac{11}{4}\right)} \cdot dx$

$$= \frac{1}{4} \cdot \int \frac{1}{x^2+\left(\frac{\sqrt{11}}{2}\right)^2} \cdot dx$$

$$\because \int \frac{1}{x^2+a^2} \cdot dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$I = \frac{1}{4} \cdot \left[\frac{1}{\left(\frac{\sqrt{11}}{2}\right)}\right] \cdot \tan^{-1}\left[\frac{x}{\left(\frac{\sqrt{11}}{2}\right)}\right] + c$$

$$= \frac{1}{2\sqrt{11}}\tan^{-1}\left(\frac{2x}{\sqrt{11}}\right) + c$$

2. $\displaystyle\int \frac{1}{a^2-b^2x^2} \cdot dx$

Solution : $\displaystyle I = \int \frac{1}{b^2\left(\frac{a^2}{b^2}-x^2\right)} \cdot dx$

$$= \frac{1}{b^2} \cdot \int \frac{1}{\left(\frac{a}{b}\right)^2 - x^2} \cdot dx$$

$$\because \int \frac{1}{a^2-x^2} \cdot dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) + c$$

$$I = \frac{1}{b^2} \cdot \frac{1}{2\left(\frac{a}{b}\right)} \cdot \log\left(\frac{\frac{a}{b}+x}{\frac{a}{b}-x}\right) + c$$

$$= \frac{1}{b^2} \cdot \frac{1}{2\left(\frac{a}{b}\right)} \cdot \log\left(\frac{\frac{a}{b}+x}{\frac{a}{b}-x}\right) + c$$

$$= \frac{1}{2ab} \cdot \log\left(\frac{a+bx}{a-bx}\right) + c$$

3. $$\int \frac{1}{\sqrt{3x^2-7}} \cdot dx$$

Solution : $$I = \int \frac{1}{\sqrt{3\left(x^2-\frac{7}{3}\right)}} \cdot dx$$

$$= \int \frac{1}{\sqrt{3} \cdot \sqrt{x^2-\left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}} \cdot dx$$

$$= \frac{1}{\sqrt{3}} \cdot \int \frac{1}{\sqrt{x^2-\left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}} \cdot dx$$

$$\because \int \frac{1}{\sqrt{x^2-a^2}} \cdot dx = \log\left|x+\sqrt{x^2-a^2}\right| + c$$

$$I = \frac{1}{\sqrt{3}} \cdot \log\left(x+\sqrt{x^2-\left(\frac{\sqrt{7}}{\sqrt{3}}\right)^2}\right) + c$$

$$= \frac{1}{\sqrt{3}} \cdot \log\left(x+\sqrt{x^2-\frac{7}{3}}\right) + c$$

4. $$\int \frac{1}{x^2+8x+12} \cdot dx$$

Solution : $$I = \int \frac{1}{x^2+8x+16-4} \cdot dx$$

$$= \int \frac{1}{(x+4)^2-(2)^2} \cdot dx$$

$$\because \int \frac{1}{x^2-a^2} \cdot dx = \frac{1}{2a} \log\left(\frac{x-a}{x+a}\right) + c$$

$$I = \frac{1}{2(2)} \cdot \log\left(\frac{(x+4)-2}{(x+4)+2}\right) + c$$

$$= \frac{1}{4} \cdot \log\left(\frac{x+2}{x+6}\right) + c$$

$$\therefore \int \frac{1}{x^2+8x+12} \cdot dx = \frac{1}{4} \cdot \log\left(\frac{x+2}{x+6}\right) + c$$

5. $$\int \frac{1}{\sqrt{3x^2-4x+2}} \cdot dx$$

Solution : $$= \int \frac{1}{\sqrt{3\left(x^2-\frac{4}{3}x+\frac{2}{3}\right)}} \cdot dx$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{1}{2}\left(-\frac{4}{3}\right)\right)^2 = \left(-\frac{2}{3}\right)^2 = \frac{4}{9}\right\}$$

$$= \int \frac{1}{\sqrt{3} \cdot \sqrt{x^2-\frac{4}{3}x+\frac{4}{9}-\frac{4}{9}+\frac{2}{3}}} \cdot dx$$

$$= \frac{1}{\sqrt{3}} \cdot \int \frac{1}{\sqrt{\left(x^2-\frac{4}{3}x+\frac{4}{9}\right)+\left(\frac{2}{3}-\frac{4}{9}\right)}} \cdot dx$$

$$= \frac{1}{\sqrt{3}} \cdot \int \frac{1}{\sqrt{\left(x-\frac{2}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}} \cdot dx$$

$$\because \int \frac{1}{\sqrt{x^2+a^2}} \cdot dx = \log\left|x+\sqrt{x^2+a^2}\right| + c$$

$$= \frac{1}{\sqrt{3}} \cdot \log\left(\left(x-\frac{2}{3}\right)+\sqrt{\left(x-\frac{2}{3}\right)^2+\left(\frac{\sqrt{2}}{3}\right)^2}\right) + c$$

$$= \frac{1}{\sqrt{3}} \cdot \log\left(\left(x-\frac{2}{3}\right)+\sqrt{x^2-\frac{4}{3}x+\frac{2}{3}}\right) + c$$

6. $\displaystyle\int \frac{1}{3-10x-25x^2}\cdot dx$

Solution :

$$I = \int \frac{1}{25\left(\frac{3}{25}-\frac{10}{25}x-x^2\right)}\cdot dx$$

$$= \int \frac{1}{25\left[\frac{3}{25}-\left(x^2+\frac{2}{5}x\right)\right]}\cdot dx$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{1}{2}\left(\frac{2}{5}\right)\right)^2 = \left(\frac{1}{5}\right)^2 = \frac{1}{25}\right\}$$

$$= \frac{1}{25}\cdot\int \frac{1}{\frac{3}{25}-\left(x^2-\frac{2}{5}x+\frac{1}{25}-\frac{1}{25}\right)}\cdot dx$$

$$= \frac{1}{25}\cdot\int \frac{1}{\frac{3}{25}-\left(x^2-\frac{2}{5}x+\frac{1}{25}\right)+\frac{1}{25}}\cdot dx$$

$$= \frac{1}{25}\cdot\int \frac{1}{\frac{4}{25}-\left(x^2-\frac{2}{5}x+\frac{1}{25}\right)}\cdot dx$$

$$= \frac{1}{25}\cdot\int \frac{1}{\left(\frac{2}{5}\right)^2-\left(x-\frac{1}{5}\right)^2}\cdot dx$$

$$\because \int \frac{1}{a^2-x^2}\cdot dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c$$

$$I = \frac{1}{25}\cdot\frac{1}{2\left(\frac{2}{5}\right)}\cdot\log\left(\frac{\frac{2}{5}+\left(x-\frac{1}{5}\right)}{\frac{2}{5}-\left(x-\frac{1}{5}\right)}\right)+c$$

$$= \frac{1}{5}\cdot\log\left(\frac{1+5x}{3-5x}\right)+c$$

Activity :

7. $\displaystyle\int \frac{1}{\sqrt{1+x-x^2}}\cdot dx$

Solution : $$I = \int \frac{1}{\sqrt{1-\left(\ldots\ldots\ldots\ldots\right)}}\cdot dx$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{1}{2}(-1)\right)^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}\right\}$$

$$= \int \frac{1}{\sqrt{1-\left(x^2-x+\frac{1}{4}-\frac{1}{4}\right)}}\cdot dx$$

$$= \int \frac{1}{\sqrt{1-\left(\ldots\ldots\ldots\ldots\ldots\ldots\right)}}\cdot dx$$

$$I = \int \frac{1}{\sqrt{\left(\frac{\sqrt{5}}{2}\right)^2-\left(x-\frac{1}{2}\right)^2}}\cdot dx$$

$$\because \int \frac{1}{\sqrt{a^2-x^2}}\cdot dx = \sin^{-1}\left(\frac{x}{a}\right)+c$$

$$I = \ldots\ldots\ldots\ldots\ldots\ldots$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots$$

8. $$\int \frac{\sin 2x}{3\sin^4 x - 4\sin^2 x + 1} \cdot dx$$

Solution : $$I = \int \frac{\sin 2x}{3(\sin^2 x)^2 - 4(\sin^2 x) + 1} \cdot dx$$

put $\sin^2 x = t$ $\quad\therefore\ 2\sin x \cdot \cos x \cdot dx = 1 \cdot dt$

$\therefore\quad \sin 2x \cdot dx = 1 \cdot dt$

$$= \int \frac{1}{3t^2 - 4t + 1} \cdot dt$$

$$= \int \frac{1}{3\left(t^2 - \frac{4}{3}t + \frac{1}{3}\right)} \cdot dt$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } t\right)^2\right.$$

$$\left. = \left(\frac{1}{2}\left(-\frac{4}{3}\right)\right)^2 = \left(-\frac{2}{3}\right)^2 = \frac{4}{9}\right\}$$

$$I = \frac{1}{3} \cdot \int \frac{1}{t^2 - \frac{4}{3}t + \frac{4}{9} - \frac{4}{9} + \frac{1}{3}} \cdot dt$$

$$= \frac{1}{3} \cdot \int \frac{1}{\left(t^2 - \frac{4}{3}t + \frac{4}{9}\right) - \frac{1}{9}} \cdot dt$$

$$= \frac{1}{3} \cdot \int \frac{1}{\left(t - \frac{2}{3}\right)^2 - \left(\frac{1}{3}\right)^2} \cdot dt$$

$$= \frac{1}{3} \cdot \frac{1}{2\left(\frac{1}{3}\right)} \cdot \log\left(\frac{\left(t - \frac{2}{3}\right) - \frac{1}{3}}{\left(t - \frac{2}{3}\right) + \frac{1}{3}}\right) + c$$

$$= \frac{1}{2} \cdot \log\left(\frac{3t - 3}{3t - 1}\right) + c$$

$$= \frac{1}{2} \cdot \log\left(\frac{3\sin^2 x - 3}{3\sin^2 x - 1}\right) + c$$

$$\therefore \int \frac{\sin 2x}{3\sin^4 x - 4\sin^2 x + 1} \cdot dx$$

$$= \frac{1}{2} \cdot \log\left(\frac{3\sin^2 x - 3}{3\sin^2 x - 1}\right) + c$$

9. $$\int \frac{e^{\frac{x}{2}}}{\sqrt{e^{-x} - e^x}} \cdot dx$$

Solution :

$$I = \int \frac{\sqrt{e^x}}{\sqrt{\frac{1}{e^x} - e^x}} \cdot dx$$

$$= \int \frac{\sqrt{e^x}}{\sqrt{\frac{1 - (e^x)^2}{e^x}}} \cdot dx$$

$$= \int \frac{\sqrt{e^x}}{\frac{\sqrt{1 - (e^x)^2}}{\sqrt{e^x}}} \cdot dx$$

$$= \int \frac{\sqrt{e^x} \cdot \sqrt{e^x}}{\sqrt{1 - (e^x)^2}} \cdot dx$$

$$= \int \frac{e^x}{\sqrt{1 - (e^x)^2}} \cdot dx$$

put $\quad e^x = t$

$\therefore\quad e^x \cdot dx = 1 \cdot dt$

$$I = \int \frac{1}{\sqrt{1 - t^2}} \cdot dt$$

$$= \sin^{-1}(t) + c$$

$$= \sin^{-1}(e^x) + c$$

$$\therefore \int \frac{e^{\frac{x}{2}}}{\sqrt{e^{-x} - e^x}} \cdot dx = \sin^{-1}(e^x) + c$$

10. $\int\left(\sqrt{\tan x}+\sqrt{\cot x}\right)\cdot dx$

Solution : $\text{I} = \int\left(\sqrt{\tan x}+\dfrac{1}{\sqrt{\tan x}}\right)\cdot dx$

$$= \int \frac{\tan x + 1}{\sqrt{\tan x}}\cdot dx$$

put $\sqrt{\tan x} = t \quad \therefore\ \tan x = t^2 \ \therefore\ x = \tan^{-1} t^2$

$\therefore \quad 1\cdot dx = \dfrac{1}{1+(t^2)^2}\cdot 2t\cdot dt$

$\because \quad \sec^2 x\cdot dx = 2t\cdot dt$

$\therefore \quad dx = \dfrac{2t}{\sec^2 x}\cdot dx = \dfrac{2t}{1+\tan^2 x}\cdot dx = \dfrac{2t}{1+t^4}\cdot dt$

$$= \int \frac{t^2+1}{t}\cdot\frac{2t}{1+t^4}\cdot dt = 2\int\frac{t^2+1}{t^4+1}\cdot dt$$

$$= 2\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}\cdot dt = 2\int\frac{\left(1+\frac{1}{t^2}\right)}{\left(t-\frac{1}{t}\right)^2+2}\cdot dt$$

put $t-\dfrac{1}{t} = u \qquad \because\left[\dfrac{d}{dt}\left(t-\dfrac{1}{t}\right) = 1+\dfrac{1}{t^2}\right]$

$\therefore \left(t-\left(-\dfrac{1}{t^2}\right)\right)dt = 1\cdot du$

$\therefore \left(1+\dfrac{1}{t^2}\right)dt = 1\cdot du$

$$\text{I} = 2\int\frac{1}{u^2+2}\cdot du$$

$$= 2\int\frac{1}{u^2+(\sqrt{2})^2}\cdot du$$

$$= 2\cdot\frac{1}{\sqrt{2}}\cdot\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+c$$

$$= \sqrt{2}\cdot\tan^{-1}\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)+c$$

$$= \sqrt{2}\cdot\tan^{-1}\left(\frac{t^2-1}{\sqrt{2}\,t}\right)+c$$

$$= \sqrt{2}\cdot\tan^{-1}\left(\frac{\tan x-1}{\sqrt{2}\cdot\sqrt{\tan x}}\right)+c$$

11. $\displaystyle\int\frac{1}{5-4\cos x}\cdot dx$

Solution : put $\tan\dfrac{x}{2} = t$

$\therefore \quad dx = \dfrac{2}{1+t^2}\cdot dt$ and $\cos x = \dfrac{1-t^2}{1+t^2}$

$$\text{I} = \int\frac{1\left(\frac{2}{1+t^2}\right)}{5-4\left(\frac{1-t^2}{1+t^2}\right)}\cdot dt$$

$$= \int\frac{\frac{2}{1+t^2}}{\frac{5(1+t^2)-4(1-t^2)}{1+t^2}}\cdot dt$$

$$= \int\frac{2}{5-5t^2-4-4t^2}\cdot dt$$

$$= \int\frac{2}{9t^2+1}\cdot dt$$

$$= \int\frac{1}{9\left(t^2+\frac{1}{9}\right)}\cdot dt$$

$$= \frac{2}{9}\cdot\int\frac{1}{t^2+\left(\frac{1}{3}\right)^2}\cdot dt$$

$$= \frac{2}{9}\cdot\frac{1}{\left(\frac{1}{3}\right)}\cdot\tan^{-1}\left(\frac{t}{\left(\frac{1}{3}\right)}\right)+c$$

$$= \frac{2}{3}\cdot\tan^{-1}(2t)+c$$

$$= \frac{2}{3}\cdot\tan^{-1}\left(2\tan\frac{x}{2}\right)+c$$

$$\therefore \int\frac{1}{5-4\cos x}\cdot dx = \frac{2}{3}\cdot\tan^{-1}\left(2\tan\frac{x}{2}\right)+c$$

12. $\displaystyle\int \frac{1}{2-3\sin 2x}\cdot dx$

Solution : put $\tan x = t$

$\therefore \quad dx = \dfrac{1}{1+t^2}\cdot dt$ and $\sin 2x = \dfrac{2}{1+t^2}$

$$I = \int \frac{1\left(\frac{1}{1+t^2}\right)}{2-3\left(\frac{2}{1+t^2}\right)}\cdot dt$$

$$= \int \frac{\frac{1}{1+t^2}}{\frac{2(1+t^2)-3(2t)}{1+t^2}}\cdot dt$$

$$= \int \frac{1}{2+2t^2-6t}\cdot dt = \int \frac{1}{2(t^2-3t+1)}\cdot dt$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } t\right)^2 = \left(\frac{1}{2}(-3)\right)^2 = \left(-\frac{3}{2}\right)^2 = \frac{9}{4}\right\}$$

$$= \frac{1}{2}\cdot\int \frac{1}{t^2-3t+\frac{9}{4}-\frac{9}{4}+1}\cdot dt$$

$$= \frac{1}{2}\cdot\int \frac{1}{\left(t^2-3t+\frac{9}{4}\right)-\frac{5}{4}}\cdot dt$$

$$= \frac{1}{2}\cdot\int \frac{1}{\left(t-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\cdot dt$$

$$= \frac{1}{2}\cdot\frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\cdot\log\left(\frac{\left(t-\frac{3}{2}\right)-\frac{\sqrt{5}}{2}}{\left(t-\frac{3}{2}\right)+\frac{\sqrt{5}}{2}}\right)+c$$

$$= \frac{1}{2\sqrt{5}}\cdot\log\left(\frac{2t-3-\sqrt{5}}{2t-3+\sqrt{5}}\right)+c$$

$$= \frac{1}{2\sqrt{5}}\cdot\log\left(\frac{2\tan x-3-\sqrt{5}}{2\tan x-3+\sqrt{5}}\right)+c$$

13. $\displaystyle\int \frac{1}{3-2\sin x+5\cos x}\cdot dx$

Solution : put $\tan\dfrac{x}{2} = t$

$\therefore \quad dx = \dfrac{2}{1+t^2}$

$\therefore \quad \sin x = \dfrac{2}{1+t^2}\cdot dt$ and $\cos x = \dfrac{1-t^2}{1+t^2}$

$$I = \int \frac{1\left(\frac{2}{1+t^2}\right)}{3-2\left(\frac{2}{1+t^2}\right)+5\left(\frac{1-t^2}{1+t^2}\right)}\cdot dt$$

$$= \int \frac{\frac{2}{1+t^2}}{\frac{3(1+t^2)-2(2t)+5(1-t^2)}{1+t^2}}\cdot dt$$

$$= \int \frac{2}{3+3t^2-4t+5-5t^2}\cdot dt$$

$$= \int \frac{2}{8-4t-2t^2}\cdot dt$$

$$= \int \frac{1}{4-2t-t^2}\cdot dt$$

$$= \int \frac{1}{4-(t^2+2t)}\cdot dt$$

$$= \int \frac{1}{4-(t^2+2t+1-1)}\cdot dt$$

$$= \int \frac{1}{5-(t^2+2t+1)}\cdot dt$$

$$= \int \frac{1}{(\sqrt{5})^2-(t+1)^2}\cdot dt$$

$$= \frac{1}{2(\sqrt{5})}\cdot\log\left(\frac{\sqrt{5}+(t+1)}{\sqrt{5}-(t+1)}\right)+c$$

$$= \frac{1}{2\sqrt{5}}\cdot\log\left(\frac{\sqrt{5}+1+\tan\frac{x}{2}}{\sqrt{5}-1-\tan\frac{x}{2}}\right)+c$$

Activity : 14. $\displaystyle\int \frac{1}{\sin x - \sqrt{3}\cos x}\cdot dx$

Solution : put $\tan\frac{x}{2} = t$ $\quad\therefore dx = \ldots\ldots\ldots$

$\therefore \sin x = \ldots\ldots\ldots$ and $\cos x = \ldots\ldots\ldots$

$$\text{I} = \int \frac{1\left(\dfrac{2}{1+t^2}\right)}{\ldots\ldots\ldots + \sqrt{3}\;\ldots\ldots\ldots}\cdot dt$$

$$= \int \frac{\dfrac{2}{1+t^2}}{\dfrac{\ldots\ldots}{1+t^2}}\cdot dt$$

$$= \int \frac{2}{\ldots\ldots\ldots}\cdot dt$$

$$= \int \frac{2}{\sqrt{3}\left(1-\left(t^2-\dfrac{2}{\sqrt{3}}t\right)\right)}\cdot dt$$

$$= \frac{2}{\sqrt{3}}\cdot\int \frac{1}{1-\left(t^2-\dfrac{2}{\sqrt{3}}t+\dfrac{1}{3}-\dfrac{1}{3}\right)}\cdot dt$$

$$= \frac{2}{\sqrt{3}}\cdot\int \frac{1}{1-\left(t^2-\dfrac{2}{\sqrt{3}}t+\dfrac{1}{3}\right)+\dfrac{1}{3}}\cdot dt$$

$$= \frac{2}{\sqrt{3}}\cdot\int \frac{1}{\left(\ldots\ldots\right)-\left(t-\dfrac{1}{\sqrt{3}}\right)^2}\cdot dt$$

$$= \frac{2}{\sqrt{3}}\cdot\int \frac{1}{(\ldots)^2-(\ldots)^2}\cdot dt$$

$$= \frac{2}{\sqrt{3}}\cdot\frac{1}{2\,(\ldots)}\cdot\log\left(\frac{\quad}{\quad}\right)+c$$

$$= \frac{\quad}{\quad}\cdot\log\left(\frac{\quad}{\quad}\right)+c$$

$$= \frac{1}{2}\cdot\log\left(\frac{1+\sqrt{3}\tan\dfrac{x}{2}}{3-\sqrt{3}\tan\dfrac{x}{2}}\right)+c$$

Alternative method :

14. $\displaystyle\int \frac{1}{\sin x - \sqrt{3}\cos x}\cdot dx$

Solution : For any two positive numbers a and b, we can find an angle θ, such that

$\therefore \sin\theta = \dfrac{a}{\sqrt{a^2-b^2}}$ and $\cos\theta = \dfrac{b}{\sqrt{a^2-b^2}}$

Using this we express $\sin x - \sqrt{3}\cos x$

$= \sqrt{1+3}\,(\cos\theta\cdot\sin x - \sin\theta\cdot\cos x)$

$= 2\cdot\sin(x-\theta)$

$= 2\cdot\sin\left(x-\dfrac{\pi}{3}\right)$

$$\therefore \text{I} = \int \frac{1}{2\cdot\sin\left(x-\dfrac{\pi}{3}\right)}\cdot dx$$

$$= \frac{1}{2}\cdot\int \operatorname{cosec}\left(x-\frac{\pi}{3}\right)\cdot dx$$

$$= \frac{1}{2}\cdot\log\left(\operatorname{cosec}\left(x-\frac{\pi}{3}\right)-\cot\left(x-\frac{\pi}{3}\right)\right)+c$$

$$= \frac{1}{2}\cdot\log\left(\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)\right)+c$$

15. $\displaystyle\int \frac{1}{3+2\sin^2 x+5\cos^2 x}\cdot dx$

Solution : Divide Numerator and Denominator by $\cos^2 x$

$$I = \int \frac{\frac{1}{\cos^2 x}}{\frac{3+2\sin^2 x+5\cos^2 x}{\cos^2 x}}\cdot dx$$

$$= \int \frac{\sec^2 x}{3\sec^2 x+2\tan^2 x+5}\cdot dx$$

$$= \int \frac{\sec^2 x}{3(1+\tan^2 x)+2\tan^2 x+5}\cdot dx$$

$$= \int \frac{\sec^2 x}{5\tan^2 x+8}\cdot dx$$

$$= \frac{1}{5}\int \frac{\sec^2 x}{\tan^2 x+\frac{8}{5}}\cdot dx$$

put $\tan x = t \ \therefore\ \sec^2 x\cdot dx = 1\cdot dt$

$$I = \frac{1}{5}\int \frac{1}{t^2+\frac{8}{5}}\cdot dt$$

$$= \frac{1}{5}\int \frac{1}{t^2+\left(\frac{\sqrt{8}}{\sqrt{5}}\right)^2}\cdot dt$$

$$= \frac{1}{5}\cdot\frac{1}{\frac{\sqrt{8}}{\sqrt{5}}}\cdot\tan^{-1}\left(\frac{t}{\frac{\sqrt{8}}{\sqrt{5}}}\right)+c$$

$$= \frac{1}{\sqrt{5}}\cdot\frac{1}{2\sqrt{2}}\cdot\tan^{-1}\left(\frac{\sqrt{5}\,t}{2\sqrt{2}}\right)+c$$

$$= \frac{1}{2\sqrt{10}}\cdot\tan^{-1}\left(\frac{\sqrt{5}\tan x}{2\sqrt{2}}\right)+c$$

$$\therefore \int \frac{1}{3+2\sin^2 x+5\cos^2 x}\cdot dx = \frac{1}{2\sqrt{10}}\cdot\tan^{-1}\left(\frac{\sqrt{5}\tan x}{2\sqrt{2}}\right)+c$$

16. $\displaystyle\int \frac{\cos\theta}{\cos 3\theta}\cdot d\theta$

Solution : $I = \displaystyle\int \frac{\cos\theta}{4\cos^3\theta-3\cos\theta}\cdot d\theta$

$$= \int \frac{1}{4\cos^2\theta-3}\cdot d\theta$$

Divide Numerator and Denominator by $\cos^2\theta$

$$I = \int \frac{\frac{1}{\cos^2\theta}}{\frac{4\cos^2\theta-3}{\cos^2\theta}}\cdot d\theta$$

$$= \int \frac{\sec^2\theta}{4-3\sec^2\theta}\cdot d\theta$$

$$= \int \frac{\sec^2\theta}{4-3(1+\tan^2\theta)}\cdot d\theta$$

$$= \int \frac{\sec^2\theta}{1-3\tan^2\theta}\cdot d\theta$$

put $\tan\theta = t \ \therefore\ \sec^2\theta\cdot d\theta = 1\cdot dt$

$$I = \int \frac{1}{1-3t^2}\cdot dt$$

$$= \frac{1}{3}\cdot\int \frac{1}{\frac{1}{3}-t^2}\cdot dt$$

$$= \frac{1}{3}\cdot\int \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2-t^2}\cdot dt$$

$$= \frac{1}{3}\cdot\frac{1}{2\left(\frac{1}{\sqrt{3}}\right)}\cdot\log\left(\frac{\frac{1}{\sqrt{3}}+t}{\frac{1}{\sqrt{3}}-t}\right)+c$$

$$= \frac{1}{2\sqrt{3}}\cdot\log\left(\frac{1+\sqrt{3}\,t}{1-\sqrt{3}\,t}\right)+c$$

$$= \frac{1}{2\sqrt{3}}\cdot\log\left(\frac{1+\sqrt{3}\tan\theta}{1-\sqrt{3}\tan\theta}\right)+c$$

$$\therefore \int \frac{\cos\theta}{\cos 3\theta}\cdot d\theta = \frac{1}{2\sqrt{3}}\cdot\log\left(\frac{1+\sqrt{3}\tan\theta}{1-\sqrt{3}\tan\theta}\right)+c$$

EXERCISE 3.2 (B)

I. Evaluate the following :

1. $\int \frac{1}{4x^2-3} \cdot dx$
2. $\int \frac{1}{25-9x^2} \cdot dx$
3. $\int \frac{1}{7+2x^2} \cdot dx$
4. $\int \frac{1}{\sqrt{3x^2+8}} \cdot dx$
5. $\int \frac{1}{\sqrt{11-4x^2}} \cdot dx$
6. $\int \frac{1}{\sqrt{2x^2-5}} \cdot dx$
7. $\int \sqrt{\frac{9+x}{9-x}} \cdot dx$
8. $\int \sqrt{\frac{2+x}{2-x}} \cdot dx$
9. $\int \sqrt{\frac{10+x}{10-x}} \cdot dx$
10. $\int \frac{1}{x^2+8x+12} \cdot dx$
11. $\int \frac{1}{1+x-x^2} \cdot dx$
12. $\int \frac{1}{4x^2-20x+17} \cdot dx$
13. $\int \frac{1}{5-4x-3x^2} \cdot dx$
14. $\int \frac{1}{\sqrt{3x^2+5x+7}} \cdot dx$
15. $\int \frac{1}{\sqrt{x^2+8x-20}} \cdot dx$
16. $\int \frac{1}{\sqrt{8-3x+2x^2}} \cdot dx$
17. $\int \frac{1}{\sqrt{(x-3)(x+2)}} \cdot dx$
18. $\int \frac{1}{4+3\cos^2 x} \cdot dx$
19. $\int \frac{1}{\cos 2x+3\sin^2 x} \cdot dx$
20. $\int \frac{\sin x}{\sin 3x} \cdot dx$

II. Integrate the following functions *w. r. t. x* :

1. $\int \frac{1}{3+2\sin x} \cdot dx$
2. $\int \frac{1}{4-5\cos x} \cdot dx$
3. $\int \frac{1}{2+\cos x-\sin x} \cdot dx$
4. $\int \frac{1}{3+2\sin x-\cos x} \cdot dx$
5. $\int \frac{1}{3-2\cos 2x} \cdot dx$
6. $\int \frac{1}{2\sin 2x-3} \cdot dx$
7. $\int \frac{1}{3+2\sin 2x+4\cos 2x} \cdot dx$
8. $\int \frac{1}{\cos x-\sin x} \cdot dx$
9. $\int \frac{1}{\cos x-\sqrt{3}\sin x} \cdot dx$

3.2.6 Integral of the form $\int \frac{px+q}{ax^2+bx+c} \cdot dx$ and $\int \frac{px+q}{\sqrt{ax^2+bx+c}} \cdot dx$

The integral of the form $\int \frac{px+q}{ax^2+bx+c} \cdot dx$ is evaluated by expressing the integral in the form

$$\int \frac{A \cdot \frac{d}{dx}(ax^2+bx+c)}{ax^2+bx+c} \cdot dx + \int \frac{B}{ax^2+bx+c} \cdot dx$$ for some constants A and B.

The numerator, $px+q = A \cdot \frac{d}{dx}(ax^2+bx+c) + B$

i.e. $\quad \mathrm{Nr} = A \cdot \frac{d}{dx}\mathrm{Dr} + B$

The first integral is evaluated by putting $ax^2+bx+c = t$

The Second integral is evaluated by expressing the integrand in the form either

$\frac{1}{A^2+t^2}$ or $\frac{1}{t^2-A^2}$ or $\frac{1}{A^2-t^2}$ and applying the methods discussed previously.

The integral of the form $\int \frac{px+q}{\sqrt{ax^2+bx+c}} \cdot dx$ is evaluated by expressing the integral in the form

$$\int \frac{A \cdot \frac{d}{dx}(ax^2+bx+c)}{\sqrt{ax^2+bx+c}} \cdot dx + \int \frac{B}{\sqrt{ax^2+bx+c}} \cdot dx$$ for constants A and B.

The numerator, $px+q = A \cdot \frac{d}{dx}(ax^2+bx+c) + B$

The first integral is evaluated by putting $ax^2+bx+c = t$

The second integral is evaluated by expressing the integrand in the form either

$\frac{1}{\sqrt{A^2+t^2}}$ or $\frac{1}{\sqrt{t^2-A^2}}$ or $\frac{1}{\sqrt{A^2-t^2}}$ and applying the methods which discussed previously.

SOLVED EXAMPLES

1. $\displaystyle\int \frac{2x-3}{3x^2+4x+5}\cdot dx$

Solution : $2x-3 = A\cdot\dfrac{d}{dx}\left(3x^2+4x+5\right)+B$

$2x-3 = A(6x+4)+B$

$= (6A)\,x + (4A+B)$

compairing the sides/ the co−efficients of like variables and constants

$6A = 2$ and $4A+B = -3$

$\Rightarrow A = \dfrac{1}{3}$ and $B = -\dfrac{13}{3}$

$$= \int \frac{\frac{1}{3}\cdot\frac{d}{dx}(3x^2+4x+5)+\left(-\frac{13}{3}\right)}{3x^2+4x+5}\cdot dx$$

$$= \frac{1}{3}\cdot\int \frac{\frac{d}{dx}(3x^2+4x+5)}{3x^2+4x+5}\cdot dx - \frac{13}{3}\int \frac{1}{3x^2+4x+5}\cdot dx$$

$$= \frac{1}{3}\cdot\int \frac{6x+4}{3x^2+4x+5}\cdot dx - \frac{13}{3}\int \frac{1}{3x^2+4x+5}\cdot dx$$

$= I_1 - I_2$ ..... (i)

$\therefore\ I_1 = \dfrac{1}{3}\cdot\displaystyle\int \frac{6x+4}{3x^2+4x+5}\cdot dx$

put $3x^2+4x+5 = t$

$\therefore\ (6x+4)\cdot dx = 1\cdot dt$

$I_1 = \dfrac{1}{3}\cdot\displaystyle\int \frac{1}{t}\cdot dt$

$= \dfrac{1}{3}\cdot\log(t) + c_1$

$= \dfrac{1}{3}\cdot\log\left(3x^2+4x+5\right) + c_1$ ..... (ii)

$\therefore\ I_2 = \dfrac{13}{3}\cdot\displaystyle\int \frac{1}{3x^2+4x+5}\cdot dx$

$= \dfrac{13}{3}\cdot\dfrac{1}{3}\cdot\displaystyle\int \frac{1}{x^2+\frac{4}{3}x+\frac{5}{3}}\cdot dx$

$\because \left\{\left(\frac{1}{2}\text{ coefficient of } t\right)^2 = \left(\frac{1}{2}\left(\frac{4}{3}\right)\right)^2 = \left(\frac{2}{3}\right)^2 = \frac{4}{9}\right\}$

$I_2 = \dfrac{13}{9}\cdot\displaystyle\int \frac{1}{x^2+\frac{4}{3}x+\frac{4}{9}-\frac{4}{9}+\frac{5}{3}}\cdot dx$

$= \dfrac{13}{9}\cdot\displaystyle\int \frac{1}{x^2+\frac{4}{3}x+\frac{4}{9}+\frac{11}{9}}\cdot dx$

$= \dfrac{13}{9}\cdot\displaystyle\int \frac{1}{\left(x+\frac{2}{3}\right)^2+\left(\frac{\sqrt{11}}{3}\right)^2}\cdot dx$

$\because \displaystyle\int \frac{1}{X^2+A^2}\cdot dx = \frac{1}{A}\tan^{-1}\left(\frac{X}{A}\right)+c$

$= \dfrac{13}{9}\cdot\dfrac{1}{\frac{\sqrt{11}}{3}}\cdot\tan^{-1}\left(\dfrac{x+\frac{2}{3}}{\frac{\sqrt{11}}{3}}\right) + c_1$

$I_2 = \dfrac{13}{3\sqrt{11}}\cdot\tan^{-1}\left(\dfrac{3x+2}{\sqrt{11}}\right) + c_2$ ..... (iii)

thus, from (i), (ii) and (iii)

$\therefore\ \displaystyle\int \frac{2x-3}{3x^2+4x+5}\cdot dx$

$= \dfrac{1}{3}\cdot\log\left(3x^2+4x+5\right) - \dfrac{13}{3\sqrt{11}}\cdot\tan^{-1}\left(\dfrac{3x+2}{\sqrt{11}}\right) + c$

$(\because c_1 + c_2 = c)$

2. $\displaystyle\int\sqrt{\frac{x-5}{x-7}}\cdot dx$

Solution : I $= \displaystyle\int\sqrt{\frac{(x-5)\cdot(x-5)}{(x-7)\cdot(x-5)}}\cdot dx = \int\sqrt{\frac{(x-5)^2}{x^2-12x+35}}\cdot dx$

$\because \quad x-5 = A\cdot\dfrac{d}{dx}\left(x^2-12x+35\right)+B$

$x-5 = A\,(2x-12)+B$

$= (2A)\,x+(-12A+B)$

compairing, the co–efficients of like variables and constants

$2A = 1$ and $-12A+B=-5$

$\Rightarrow A=\dfrac{1}{2}$ and $B=1$

$$I = \int\frac{\frac{1}{2}\cdot\frac{d}{dx}(x^2-12x+35)+(1)}{\sqrt{x^2-12x+35}}\cdot dx$$

$$= \frac{1}{2}\cdot\int\frac{\frac{d}{dx}(x^2-12x+35)}{\sqrt{x^2-12x+35}}\cdot dx+\int\frac{1}{\sqrt{x^2-12x+35}}\cdot dx$$

$= I_1+I_2$ ..... (i)

$\therefore \quad I_1 = \dfrac{1}{2}\cdot\displaystyle\int\frac{2x-12}{\sqrt{x^2-12x+35}}\cdot dx$

put $x^2-12x+35=t$

$\therefore \quad (2x-12)\cdot dx = 1\cdot dt$

$I_1 = \dfrac{1}{2}\cdot\displaystyle\int\frac{1}{\sqrt{t}}\cdot dt$

$= \displaystyle\int\frac{1}{2\sqrt{t}}\cdot dt$

$= \sqrt{t}+c_1$

$= \sqrt{x^2-12x+35}+c_1$ ..... (ii)

$\therefore \quad I_2 = \displaystyle\int\frac{1}{\sqrt{x^2-12x+35}}\cdot dx$

$= \displaystyle\int\frac{1}{\sqrt{x^2-12x+36-1}}\cdot dx$

$= \displaystyle\int\frac{1}{\sqrt{(x-6)^2-(1)^2}}\cdot dx$

$\because \quad \displaystyle\int\frac{1}{\sqrt{X^2-A^2}}\cdot dx = \log\left(X+\sqrt{X^2-A^2}\right)+c$

$I_2 = \log\left((x-6)+\sqrt{(x-6)^2-1}\right)+c_2$

$= \log\left((x-6)+\sqrt{x^2-12x+35}\right)+c_2$ ..... (iii)

Thus, from (i), (ii) and (iii)

$\displaystyle\int\sqrt{\frac{x-5}{x-7}}\cdot dx$

$= \sqrt{x^2-12x+35}+\log\left((x-6)+\sqrt{x^2-12x+35}\right)+c$

$(c_1+c_2=c)$

Activity :

3. $\displaystyle\int \sqrt{\frac{8-x}{x}}\cdot dx$

Solution : $\displaystyle = \int \sqrt{\frac{(8-x)}{x} \times \frac{(\quad)}{(\quad)}} \cdot dx \qquad = \int \sqrt{\frac{(\qquad)^2}{8x-x^2}} \cdot dx \qquad = \int \frac{(8-x)}{\sqrt{8x-x^2}} \cdot dx$

$\because \qquad 8-x = A \cdot \frac{d}{dx}\left(8x - x^2\right) + B$

$8-x = A\,(\ldots\ldots\ldots\ldots) + B$

$= (8A + B) - 2Ax$

compairing, the co−efficients of like variables and constants

$8A + B = \ldots\ldots \qquad$ and $\; -2A = -1$

$\Rightarrow \; A = \frac{\cdots}{\cdots}$ and $\quad B = \ldots\ldots$

$$= \int \frac{\frac{1}{2}\cdot\frac{d}{dx}(8x-x^2) + (4)}{\sqrt{8x-x^2}}\cdot dx$$

$$= \frac{1}{2}\cdot\int \frac{\frac{d}{dx}(8x-x^2)}{\sqrt{8x-x^2}}\cdot dx + 4\cdot\int \frac{1}{\sqrt{8x-x^2}}\cdot dx$$

$$= \frac{1}{2}\cdot\int \frac{8-2x}{\sqrt{8x-x^2}}\cdot dx + 4\cdot\int \frac{1}{\sqrt{8x-x^2}}\cdot dx$$

$= \mathrm{I}_1 + \mathrm{I}_2 \qquad \ldots\ldots \text{(i)}$

$\therefore \quad \mathrm{I}_1 = \frac{1}{2}\cdot\int \frac{8-2x}{\sqrt{8x-x^2}}\cdot dx$

put $\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots = t$

$\therefore \quad (\ldots\ldots\ldots\ldots)\cdot dx = 1\cdot dt$

$$= \frac{1}{2}\cdot\int \frac{1}{\sqrt{t}}\cdot dt$$

$$= \int \frac{1}{2\sqrt{t}}\cdot dt$$

$= \sqrt{t} + c_1$

$= \sqrt{8x-x^2} + c_1 \quad \ldots\ldots \text{(ii)}$

$$\therefore \quad \mathrm{I}_2 = 4\cdot\int \frac{1}{\sqrt{8x-x^2}}\cdot dx$$

$$= 4\cdot\int \frac{1}{\sqrt{-\,(\ldots\ldots\ldots\ldots\ldots\ldots\ldots)}}\cdot dx$$

$$= 4\cdot\int \frac{1}{\sqrt{\ldots\ldots\ldots - (\ldots\ldots\ldots)}}\cdot dx$$

$$= 4\cdot\int \frac{1}{\sqrt{\ldots\ldots\ldots - (x-4)^2}}\cdot dx$$

$$\because \quad \int \frac{1}{\sqrt{A^2 - X^2}}\cdot dx = \sin^{-1}\left(\frac{X}{A}\right) + c$$

$\mathrm{I}_2 = 4\cdot\sin^{-1}\left(\frac{x-4}{4}\right) + c_2 \qquad \ldots\ldots \text{(iii)}$

thus, from (i), (ii) and (iii)

$$\therefore \quad \int \sqrt{\frac{8-x}{x}}\cdot dx$$

$$= \sqrt{8x-x^2} + 4\cdot\sin^{-1}\left(\frac{x-4}{4}\right) + c$$

$(\because c_1 + c_2 = c)$

EXERCISE 3.2 (C)

I. Evaluate :

1. $\int \frac{3x+4}{x^2+6x+5} \cdot dx$
2. $\int \frac{2x+1}{x^2+4x-5} \cdot dx$
3. $\int \frac{2x+3}{2x^2+3x-1} \cdot dx$
4. $\int \frac{3x+4}{\sqrt{2x^2+2x+1}} \cdot dx$
5. $\int \frac{7x+3}{\sqrt{3+2x-x^2}} \cdot dx$
6. $\int \sqrt{\frac{x-7}{x-9}} \cdot dx$
7. $\int \sqrt{\frac{9-x}{x}} \cdot dx$
8. $\int \frac{3\cos x}{4\sin^2 x + 4\sin x - 1} \cdot dx$
9. $\int \sqrt{\frac{e^{3x}-e^{2x}}{e^x+1}} \cdot dx$

3.3 Integration by parts :

This method is useful when the integrand is expressed as a product of two different types of functions; one of which can be differentiated and the other can be integrated conveniently.

The following theorem gives the rule of integration by parts.

3.3.1 Theorem : If u and v are two differentiable functions of x then

$$\int u \cdot v \cdot dx = u \cdot \int v \cdot dx - \int \left(\frac{d}{dx} \cdot u\right) \left(\int v \cdot dx\right) \cdot dx$$

Proof : Let $\int v \cdot dx = w$ ... (i) $\Rightarrow$ $v = \frac{dw}{dx}$ ... (ii)

Consider, $\frac{d}{dx}(u \cdot w) = u \cdot \frac{d}{dx} w + w \cdot \frac{d}{dx} u$

$$= u \cdot v + w \cdot \frac{du}{dx}$$

By definition of integration

$$u \cdot w = \int \left[u \cdot v + w \cdot \frac{du}{dx}\right] \cdot dx$$

$$= \int u \cdot v \cdot dx + \int w \cdot \frac{du}{dx} \cdot dx$$

$$= \int u \cdot v \cdot dx + \int \frac{du}{dx} \cdot w \cdot dx$$

$$\therefore \quad u \cdot \int v \cdot dx = \int u \cdot v \cdot dx + \int \frac{du}{dx} \cdot \int v \cdot dx \cdot dx$$

$$\therefore \quad \int u \cdot v \cdot dx = u \cdot \int v \cdot dx - \int \left(\frac{d}{dx} \cdot u\right) \left(\int v \cdot dx\right) \cdot dx$$

In short, $\int u \cdot v = u \cdot \int v - \int (u' \int v)$

For example :
$$\begin{aligned}\int x\cdot e^x\cdot dx &= x\int e^x\cdot dx - \int\left(\frac{d(x)}{dx}\cdot\int e^x\cdot dx\right)\cdot dx\\ &= x\cdot e^x - \int(1)\cdot e^x\cdot dx\\ &= x\cdot e^x - \int e^x\cdot dx\\ &= x\cdot e^x - e^x + c\end{aligned}$$

now let us reverse the choise of u and v

$$\begin{aligned}\therefore\quad \int e^x\cdot x\cdot dx &= e^x\cdot\int x^1\cdot dx - \int\frac{d}{dx}\cdot e^x\int x\cdot dx\cdot dx\\ &= e^x\cdot\frac{x^2}{2} - \int e^x\cdot\frac{x^2}{2}\cdot dx\\ &= \frac{1}{2}\cdot e^x\cdot x^2 - \frac{1}{2}\cdot\int e^x\cdot x^2\cdot dx\end{aligned}$$

We arrive at an integral $\int e^x\cdot x^2\cdot dx$ which is more difficult, but it helps to get $\int e^x\cdot x^2\cdot dx$

Thus it is essential to make a proper choise of the first function and the second function. The first function to be selected will be the one, which comes first in the order of **L I A T E**.

L	**Logarithmic function.**
I	**Inverse trigonometric function.**
A	**Algebric function.**
T	**Trigonometric function.**
E	**Exponential function.**

For example : $\int \sin x\cdot x\cdot dx$

$= \int x\cdot\sin x\cdot dx$ . . . . . by LIATE

$= x\cdot\int\sin x\cdot dx - \int\frac{d}{dx}\cdot x\cdot\int\sin x\cdot dx\cdot dx$

$= x\cdot(-\cos x) - \int(1)\,(-\cos x)\cdot dx$

$= -\,x\cdot\cos x + \int\cos x\cdot dx$

$= -\,x\cdot\cos x + \sin x + c$

SOLVED EXAMPLES

1. $\int x^2 \cdot 5^x \cdot dx$

Solution : I $= x^2 \cdot \int 5^x \cdot dx - \int \frac{d}{dx} \cdot x^2 \cdot \int 5^x \cdot dx \cdot dx$

$= x^2 \cdot 5^x \cdot \frac{1}{\log 5} - \int 2x \cdot 5^x \cdot \frac{1}{\log 5} \cdot dx$

$= \frac{1}{\log 5} \cdot x^2 \cdot 5^x - \frac{2}{\log 5} \left\{ x \cdot \int 5^x \cdot dx - \int \frac{d}{dx} \cdot x \cdot \int 5^x \cdot dx \cdot dx \right\}$

$= \frac{1}{\log 5} \cdot x^2 \cdot 5^x - \frac{2}{\log 5} \left\{ x \cdot 5^x \cdot \frac{1}{\log 5} - \int (1) \left(5^x \cdot \frac{1}{\log 5} \right) \cdot dx \right\}$

$= \frac{1}{\log 5} \cdot x^2 \cdot 5^x - \frac{2}{\log 5} \left\{ \frac{1}{\log 5} \cdot x \cdot 5^x \cdot - \int \frac{1}{\log 5} \cdot 5^x \cdot dx \right\}$

$= \frac{1}{\log 5} \cdot x^2 \cdot 5^x - \frac{2}{\log 5} \left\{ \frac{1}{\log 5} \cdot x \cdot 5^x \cdot - \frac{1}{\log 5} \cdot 5^x \cdot \frac{1}{\log 5} \right\} + c$

$= \frac{1}{\log 5} \cdot x^2 \cdot 5^x - \frac{2}{(\log 5)^2} \cdot x \cdot 5^x \cdot + \frac{2}{(\log 5)^3} \cdot 5^x + c$

$\therefore \int x^2 \cdot 5^x \cdot dx = \frac{5^x}{\log 5} \cdot \left\{ x^2 - \frac{2x}{\log 5} + \frac{2}{(\log 5)^2} \right\} + c$

2. $\int x \cdot \tan^{-1} x \cdot dx$

Solution : I $= \int (\tan^{-1} x \cdot) x \cdot dx$ . . . . . by LIATE

$= \tan^{-1} x \cdot \int x \cdot dx - \int \frac{d}{dx} \cdot \tan^{-1} x \cdot \int x \cdot dx \cdot dx$

$= \tan^{-1} x \cdot \frac{x^2}{2} - \int \frac{1}{1 + x^2} \cdot \frac{x^2}{2} \cdot dx$

$= \frac{1}{2} x^2 \cdot \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1 + x^2} \cdot dx$

$= \frac{1}{2} x^2 \cdot \tan^{-1} x - \frac{1}{2} \int \frac{1 + x^2 - 1}{1 + x^2} \cdot dx$

$= \frac{1}{2} x^2 \cdot \tan^{-1} x - \frac{1}{2} \int \left[1 - \frac{1}{1 + x^2} \right] \cdot dx$

$= \frac{1}{2} x^2 \cdot \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] + c$

$\therefore \int x \cdot \tan^{-1} x \cdot dx = \frac{1}{2} x^2 \cdot \tan^{-1} x - \frac{1}{2} x + \frac{1}{2} \tan^{-1} x + c$

3. $\displaystyle\int \frac{x}{1-\sin x}\cdot dx$

Solution : $\displaystyle \text{I} = \int \frac{x}{1-\sin x}\cdot\frac{(1+\sin x)}{(1+\sin x)}\cdot dx$

$$= \int \frac{x\,(1+\sin x)}{1-\sin^2 x}\cdot dx = \int \frac{x\,(1+\sin x)}{\cos^2 x}\cdot dx = \int x\cdot\left(\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right)\cdot dx$$

$$= \int x\cdot(\sec^2 x + \sec x\cdot\tan x)\cdot dx$$

$$= \int x\cdot\sec^2 x\cdot dx + \int x\cdot\sec x\cdot\tan x\cdot dx$$

$$= \left(x\cdot\int\sec^2 x\cdot dx - \int\frac{d}{dx}\,x\cdot\int\sec^2 x\cdot dx\cdot dx\right) + \left(x\cdot\int\sec x\cdot\tan x\cdot dx - \int\frac{d}{dx}\cdot x\cdot\int\sec x\cdot\tan x\cdot dx\cdot dx\right)$$

$$= x\cdot\tan x - \int(1)\cdot\tan x\cdot dx + x\cdot\sec x - \int(1)\cdot\sec x\cdot dx$$

$$= x\cdot\tan x - \log\,(\sec x) + x\cdot\sec x - \log\,(\sec x+\tan x) + c$$

$$= x\cdot(\sec x+\tan x) - \log\,(\sec x) - \log\,(\sec x+\tan x) + c$$

$$\therefore\quad \int \frac{x}{1-\sin x}\cdot dx = x\cdot(\sec x+\tan x) - \log\,[(\sec x)\,(\sec x+\tan x)] + c$$

4. $\int e^{2x}\cdot\,\sin 3x\cdot dx$

Solution : $\text{I} = \int e^{2x}\cdot\,\sin 3x\cdot dx$

Here we use repeated integration by parts.

To evaluate $\int e^{ax}\cdot\,\sin\,(bx+c)\cdot dx\,;\ \int e^{ax}\cdot\,\cos\,(bx+c)\cdot dx$ any function can be taken as a first function.

$$\text{I} = e^{2x}\cdot\int\sin 3x\cdot dx - \int\frac{d}{dx}\cdot e^{2x}\cdot\int\sin 3x\cdot dx\cdot dx$$

$$= e^{2x}\cdot\left(-\cos 3x\cdot\frac{1}{3}\right) - \int e^{2x}\cdot 2\left(-\cos 3x\cdot\frac{1}{3}\right)\cdot dx$$

$$= -\frac{1}{3}\cdot e^{2x}\cdot\cos 3x + \frac{2}{3}\int e^{2x}\cdot\cos 3x\cdot dx$$

$$= -\frac{1}{3}\cdot e^{2x}\cdot\cos 3x + \frac{2}{3}\left(e^{2x}\cdot\int\cos 3x\cdot dx - \int\frac{d}{dx}\cdot e^{2x}\cdot\int\cos 3x\cdot dx\cdot dx\right)$$

$$= -\frac{1}{3}\cdot e^{2x}\cdot\cos 3x + \frac{2}{3}\left[e^{2x}\cdot\left(\sin 3x\cdot\frac{1}{3}\right) - \int e^{2x}\cdot 2\cdot\left(\sin 3x\cdot\frac{1}{3}\right)\cdot dx\right]$$

$$= -\frac{1}{3}\cdot e^{2x}\cdot\cos 3x + \frac{2}{9}\cdot e^{2x}\cdot\sin 3x - \frac{4}{9}\cdot\int e^{2x}\cdot\sin 3x\cdot dx$$

$$\text{I} = -\frac{1}{3}\cdot e^{2x}\cdot\cos 3x + \frac{2}{9}\cdot e^{2x}\cdot\sin 3x - \frac{4}{9}\cdot\text{I}$$

$$\text{I}+\frac{4}{9}\cdot\text{I} = \frac{e^{2x}}{9}[-\,3\cos 3x + 2\sin 3x] + c$$

$$\frac{13}{9}\cdot\text{I} = \frac{e^{2x}}{9}[2\sin 3x - 3\cos 3x\,] + c$$

$$= \frac{e^{2x}}{13}[2\sin 3x - 3\cos 3x\,] + c$$

$$\therefore\ \int e^{2x}\cdot\,\sin 3x\cdot dx = \frac{e^{2x}}{13}[2\sin 3x - 3\cos 3x\,] + c$$

Activity :

Prove the following results.

(i) $\int e^{ax} \cdot \sin(bx+c) \cdot dx = \dfrac{e^{ax}}{a^2+b^2} \cdot [a \sin(bx+c) + b\cos(bx+c)] + c$

(ii) $\int e^{ax} \cdot \cos(bx+c) \cdot dx = \dfrac{e^{ax}}{a^2+b^2} \cdot [a \sin(bx+c) - b\cos(bx+c)] + c$

5. $\displaystyle\int \left[\log(\log x) + \frac{1}{(\log x)^2} \right] \cdot dx$

Solution :

$$\begin{aligned}
\mathrm{I} &= \int \log(\log x) \cdot 1 \cdot dx + \int \frac{1}{(\log x)^2} \cdot dx \\
&= \log(\log x) \cdot \int 1 \cdot dx - \int \frac{d}{dx} \cdot \log(\log x) \int 1 \cdot dx + \int \frac{1}{(\log x)^2} \cdot dx \\
&= \log(\log x) \cdot x - \int \frac{1}{\log x} \cdot \frac{1}{x} \cdot (x) \cdot dx + \int \frac{1}{(\log x)^2} \cdot dx \\
&= \log(\log x) \cdot x - \int \frac{1}{\log x} \cdot dx + \int \frac{1}{(\log x)^2} \cdot dx \\
&= \log(\log x) \cdot x - \int (\log x)^{-1} \cdot 1 \cdot dx + \int \frac{1}{(\log x)^2} \cdot dx \\
&= \log(\log x) \cdot x - \left\{ (\log x)^{-1} \cdot \int 1 \cdot dx + \int \frac{d}{dx} \cdot (\log x)^{-1} \cdot \int 1 \cdot dx \cdot dx \right\} + \int \frac{1}{(\log x)^2} \cdot dx \\
&= \log(\log x) \cdot x - \left\{ (\log x)^{-1} \cdot x - \int -1 (\log x)^{-2} \cdot \frac{1}{x} \cdot x \cdot dx \right\} + \int \frac{1}{(\log x)^2} \cdot dx \\
&= \log(\log x) \cdot x - (\log x)^{-1} \cdot x - \int (\log x)^{-2} \cdot dx + \int \frac{1}{(\log x)^2} \cdot dx \\
&= x \cdot \log(\log x) - \frac{x}{\log x} - \int \frac{1}{(\log x)^2} \cdot dx + \int \frac{1}{(\log x)^2} \cdot dx
\end{aligned}$$

$$\therefore \int \left[\log(\log x) + \frac{1}{(\log x)^2} \right] \cdot dx = x \cdot \log(\log x) - \frac{x}{\log x} + c$$

Note that :

To evaluate the integrals of type $\int \sin^{-1} x \cdot dx$; $\int \tan^{-1} x \cdot dx$; $\int \sec^{-1} x \cdot dx$; $\int \log x \cdot dx$, take the second function (v) to be 1 and then apply integration by parts.

$\int \sqrt{a^2 - x^2} \cdot dx$; $\int \sqrt{a^2 + x^2} \cdot dx$; $\int \sqrt{x^2 - a^2} \cdot dx$

6. $\int\sqrt{a^2-x^2}\cdot dx$

Solution : Let $I = \int\sqrt{a^2-x^2}\cdot 1\cdot dx$

$$= \sqrt{a^2-x^2}\cdot\int 1\cdot dx - \int\frac{d}{dx}\cdot\sqrt{a^2-x^2}\cdot\int 1\cdot dx\cdot dx$$

$$= \sqrt{a^2-x^2}\cdot x - \int\frac{1}{2\sqrt{a^2-x^2}}(-2x)\cdot(x)\cdot dx$$

$$= \sqrt{a^2-x^2}\cdot x + \int\frac{x^2}{\sqrt{a^2-x^2}}\cdot dx$$

$$= \sqrt{a^2-x^2}\cdot x + \int\frac{a^2-(a^2-x^2)}{\sqrt{a^2-x^2}}\cdot dx$$

$$= \sqrt{a^2-x^2}\cdot x + \int\left[\frac{a^2}{\sqrt{a^2-x^2}} - \frac{(a^2-x^2)}{\sqrt{a^2-x^2}}\right]\cdot dx$$

$$= x\cdot\sqrt{a^2-x^2} + a^2\int\frac{1}{\sqrt{a^2-x^2}}\cdot dx - \int\sqrt{a^2-x^2}\cdot dx$$

$$I = x\cdot\sqrt{a^2-x^2} + a^2\int\frac{1}{\sqrt{a^2-x^2}}\cdot dx - I$$

$$\therefore \quad I + I = x\cdot\sqrt{a^2-x^2} + a^2\cdot\sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\therefore \quad I = \frac{x}{2}\cdot\sqrt{a^2-x^2} + \frac{a^2}{2}\cdot\sin^{-1}\left(\frac{x}{a}\right) + c$$

$$\therefore \quad \int\sqrt{a^2-x^2}\cdot dx = \frac{x}{2}\cdot\sqrt{a^2-x^2} + \frac{a^2}{2}\cdot\sin^{-1}\left(\frac{x}{a}\right) + c$$

e.g. $\int\sqrt{9-x^2}\cdot dx = \frac{x}{2}\cdot\sqrt{9-x^2} + \frac{9}{2}\cdot\sin^{-1}\left(\frac{x}{3}\right) + c$

with reference to the above example solve these :

7. $\int\sqrt{a^2+x^2}\cdot dx = \frac{x}{2}\cdot\sqrt{x^2+a^2} + \frac{a^2}{2}\cdot\log\left(x+\sqrt{x^2+a^2}\right) + c$

8. $\int\sqrt{x^2-a^2}\cdot dx = \frac{x}{2}\cdot\sqrt{x^2-a^2} - \frac{a^2}{2}\cdot\log\left(x+\sqrt{x^2+a^2}\right) + c$

9. $\int x \cdot \sin^{-1} x \cdot dx$

Solution : I $= \int \sin^{-1} x \cdot x \cdot dx$ . . . . . by LIATE

$$= \sin^{-1} x \cdot \int x \cdot dx - \int \frac{d}{dx} \cdot \sin^{-1} x \cdot \int x \cdot dx \cdot dx$$

$$= \sin^{-1} x \cdot \frac{x^2}{2} - \int \frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} \cdot dx$$

$$= \frac{1}{2} x^2 \cdot \sin^{-1} x - \frac{1}{2} \int \frac{x^2}{\sqrt{1-x^2}} \cdot dx$$

$$= \frac{1}{2} x^2 \cdot \sin^{-1} x - \frac{1}{2} \int \frac{1-(1-x^2)}{\sqrt{1-x^2}} \cdot dx$$

$$= \frac{1}{2} x^2 \cdot \sin^{-1} x - \frac{1}{2} \int \left[\frac{1}{\sqrt{1-x^2}} - \frac{(1-x^2)}{\sqrt{1-x^2}} \right] \cdot dx$$

$$= \frac{1}{2} x^2 \cdot \sin^{-1} x - \frac{1}{2} \int \frac{dx}{\sqrt{1-x^2}} + \frac{1}{2} \int \sqrt{1-x^2} \cdot dx$$

$$= \frac{1}{2} x^2 \cdot \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{x}{2} \sqrt{1-x^2} + \frac{1}{2} \sin^{-1}(x) \right] + c$$

$$= \frac{1}{2} x^2 \cdot \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} - \frac{1}{4} \sin^{-1} x + c$$

$$\therefore \quad \int x \cdot \sin^{-1} x \cdot dx = \frac{1}{2} x^2 \cdot \sin^{-1} x + \frac{1}{4} x \sqrt{1-x^2} - \frac{1}{4} \sin^{-1} x + c$$

Activity :

10. $\int \cos^{-1} \sqrt{x} \cdot dx$

Solution : put $\sqrt{x} = t$

$\therefore \quad x = t^2$

differentiating *w.r.t.* x

$\therefore \quad 1 \cdot dx = 2t \cdot dt$

I $= \int \cos^{-1} t \cdot 2t \cdot dt$

refer previous (example no. 9) example and solve it.

11. $\int \sqrt{4+3x-2x^2}\cdot dx$

Solution : $I = \int \sqrt{4-2x^2+3x}\cdot dx$

$$= \int \sqrt{4-2\left(x^2-\frac{3}{2}x\right)}\cdot dx$$

$$= \int \sqrt{2}\cdot\sqrt{2-\left(x^2-\frac{3}{2}x\right)}\cdot dx$$

$$\because \left\{\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left[\frac{1}{2}\left(-\frac{3}{2}\right)\right]^2 = \left(-\frac{3}{4}\right)^2 = \frac{9}{16}\right\}$$

$$I = \sqrt{2}\cdot\int \sqrt{2-\left(x^2-\frac{3}{2}x+\frac{9}{16}-\frac{9}{16}\right)}\cdot dx$$

$$= \sqrt{2}\cdot\int \sqrt{2-\left(x^2-\frac{3}{2}x+\frac{9}{16}\right)+\frac{9}{16}}\cdot dx$$

$$= \sqrt{2}\cdot\int \sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-\left(x-\frac{3}{4}\right)^2}\cdot dx$$

$$\because \int \sqrt{a^2-x^2}\cdot dx = \frac{x}{2}\cdot\sqrt{a^2-x^2}+\frac{a^2}{2}\cdot\sin^{-1}\left(\frac{x}{a}\right)+c$$

$$= \sqrt{2}\cdot\left[\frac{\left(x-\frac{3}{4}\right)}{2}\cdot\sqrt{\left(\frac{\sqrt{41}}{4}\right)^2-\left(x-\frac{3}{4}\right)^2}+\frac{\left(\frac{\sqrt{41}}{4}\right)^2}{2}\cdot\sin^{-1}\left(\frac{x-\frac{3}{4}}{\frac{\sqrt{41}}{4}}\right)\right]+c$$

$$= \sqrt{2}\left[\frac{4x-3}{8}\cdot\sqrt{2+\frac{3}{2}x-x^2}+\frac{41}{32}\cdot\sin^{-1}\left(\frac{4x-3}{\sqrt{41}}\right)\right]+c$$

$$\therefore \int \sqrt{4+3x-2x^2}\cdot dx = \frac{4x-3}{8}\cdot\sqrt{4+3x-2x^2}+\frac{41}{16\sqrt{2}}\cdot\sin^{-1}\left(\frac{4x-3}{\sqrt{41}}\right)+c$$

Note that :

3.3.2 :

To evaluate the integral of type $\int (px+q)\sqrt{ax^2+bx+c}\cdot dx$

we express the term $px+q = A\cdot\frac{d}{dx}(ax^2+bx+c)+B$ . . . for constants A, B.

Then the integral will be evaluated by the useual known methods.

3.3.3 Integral of the type $\int e^x [f(x) + f'(x)] \cdot dx = e^x \cdot f(x) + c$

Let $e^x \cdot f(x) = t$

Differentiating *w. r. t. x*

$$\left[e^x [f'(x) + f(x)]\right] = \frac{dt}{dx}$$

$$e^x [f(x) + f'(x)] = \frac{dt}{dx}$$

By definition of integration,

$$\therefore \quad \int e^x [f(x) + f'(x)] \cdot dx = t + c$$

$$\therefore \quad \int e^x [f(x) + f'(x)] \cdot dx = e^x \cdot f(x) + c$$

e.g. $\int e^x [\tan x + \sec^2 x] \cdot dx = e^x \cdot \tan x + c$

$$\left(\because \frac{d}{dx} \tan x = \sec^2 x\right)$$

SOLVED EXAMPLES

1. $\displaystyle\int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x}\right) \cdot dx$

Solution :

$$\text{I} = \int e^x \left(\frac{2 + 2\sin x \cdot \cos x}{2 \cdot \cos^2 x}\right) \cdot dx$$

$$= \int e^x \left(\frac{1}{\cos^2 x} + \frac{\sin x \cdot \cos x}{\cos^2 x}\right) \cdot dx$$

$$= \int e^x [\sec^2 x + \tan x] \cdot dx$$

$$= \int e^x [\tan x + \sec^2 x] \cdot dx$$

$$\therefore \quad f(x) = \tan x \Rightarrow \quad f'(x) = \sec^2 x$$

$$\therefore \int e^x [f(x) + f'(x)] \cdot dx = e^x \cdot f(x) + c$$

$$\text{I} = e^x \cdot \tan x + c$$

$$\therefore \int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x}\right) \cdot dx = e^x \cdot \tan x + c$$

2. $\displaystyle\int e^x \left[\frac{x + 2}{(x + 3)^2}\right] \cdot dx$

Solution :

$$\text{I} = \int e^x \left[\frac{x + 3 - 1}{(x + 3)^2}\right] \cdot dx$$

$$= \int e^x \left[\frac{x + 3}{(x + 3)^2} + \frac{-1}{(x + 3)^2}\right] \cdot dx$$

$$= \int e^x \left[\frac{1}{x + 3} + \frac{-1}{(x + 3)^2}\right] \cdot dx$$

$$\therefore \quad f(x) = \frac{1}{x + 3} \Rightarrow \quad f'(x) = \frac{-1}{(x + 3)^2}$$

$$\therefore \int e^x [f(x) + f'(x)] \cdot dx = e^x \cdot f(x) + c$$

$$= e^x \cdot \left(\frac{1}{x + 3}\right) + c$$

$$= \frac{e^x}{x + 3} + c$$

$$\therefore \quad \int e^x \left[\frac{x + 2}{(x + 3)^2}\right] \cdot dx = \frac{e^x}{x + 3} + c$$

3. $\displaystyle\int e^{\tan^{-1}x}\cdot\left(\frac{1+x+x^2}{1+x^2}\right)\cdot dx$

Solution : put $\tan^{-1}x = t$

$\therefore\quad x = \tan t$

differentiating *w. r. t. x*

$\therefore\quad \dfrac{1}{1+x^2}\cdot dx = 1\cdot dt$

$$\begin{aligned}
\text{I} &= \int e^t\cdot[\,1+\tan t+\tan^2 t\,]\cdot dt\\
&= \int e^t\cdot[\,\tan t+(1+\tan^2 t)]\cdot dt\\
&= \int e^t\cdot[\,\tan t+\sec^2 t\,]\cdot dt
\end{aligned}$$

Here $\quad f(t) = \tan t$

$\Rightarrow\quad f'(t) = \sec^2 t$

$$\begin{aligned}
\text{I} &= e^t\cdot f(t)+c\\
&= e^t\cdot\tan t + c\\
&= e^{\tan^{-1}x}\cdot x + c
\end{aligned}$$

$$\therefore \int e^{\tan^{-1}x}\cdot\left(\frac{1+x+x^2}{1+x^2}\right)\cdot dx = e^{\tan^{-1}x}\cdot x + c$$

4. $\displaystyle\int\frac{(x^2+1)\cdot e^x}{(x+1)^2}\cdot dx$

Solution :

$$\begin{aligned}
\text{I} &= \int e^x\left[\frac{x^2+1}{(x+1)^2}\right]\cdot dx\\
&= \int e^x\left[\frac{x^2-1+2}{(x+1)^2}\right]\cdot dx\\
&= \int e^x\left[\frac{x^2-1}{(x+1)^2}+\frac{2}{(x+1)^2}\right]\cdot dx\\
&= \int e^x\left[\frac{x-1}{x+1}+\frac{2}{(x+1)^2}\right]\cdot dx
\end{aligned}$$

Here $\quad f(x) = \dfrac{x-1}{x+1}$

$$\Rightarrow\quad f'(x) = \frac{(x+1)\,(1)-(x-1)(1)}{(x+1)^2} = \frac{2}{(x+1)^2}$$

$\because\quad \int[f(x)+f'(x)]\cdot dx = e^x\cdot f(x)+c$

$$\text{I} = e^x\cdot\left(\frac{x-1}{x+1}\right)+c$$

$$\therefore\quad \int\frac{(x^2+1)\cdot e^x}{(x+1)^2}\cdot dx = e^x\cdot\left(\frac{x-1}{x+1}\right)+c$$

EXERCISE 3.3

I. Evaluate the following :

1. $\int x^2\cdot\log x\cdot dx$
2. $\int x^2\cdot\sin 3x\cdot dx$
3. $\int x\cdot\tan^{-1}x\cdot dx$
4. $\int x^2\cdot\tan^{-1}x\cdot dx$
5. $\int x^3\cdot\tan^{-1}x\cdot dx$
6. $\int(\log x)^2\cdot dx$
7. $\int\sec^3 x\cdot dx$
8. $\int x\cdot\sin^2 x\cdot dx$
9. $\int x^3\cdot\log x\cdot dx$
10. $\int e^{2x}\cdot\cos 3x\cdot dx$
11. $\int x\cdot\sin^{-1}x\cdot dx$
12. $\int x^2\cdot\cos^{-1}x\cdot dx$
13. $\displaystyle\int\frac{\log(\log x)}{x}\cdot dx$
14. $\displaystyle\int\frac{t\cdot\sin^{-1}t}{\sqrt{1-t^2}}\cdot dt$
15. $\int\cos\sqrt{x}\cdot dx$
16. $\int\sin\theta\cdot\log(\cos\theta)\cdot d\theta$
17. $\int x\cdot\cos^3 x\cdot dx$
18. $\displaystyle\int\frac{\sin(\log x)^2}{x}\cdot\log\cdot x\cdot dx$
19. $\displaystyle\int\frac{\log x}{x}\cdot dx$
20. $\int x\cdot\sin 2x\cdot\cos 5x\cdot dx$
21. $\int\cos\left(\sqrt[3]{x}\right)\cdot dx$

II. Integrate the following functions *w. r. t. x* :

1. $e^{2x} \cdot \sin 3x$
2. $e^{-x} \cdot \cos 2x$
3. $\sin (\log x)$
4. $\sqrt{5x^2 + 3}$
5. $x^2 \cdot \sqrt{a^2 - x^6}$
6. $\sqrt{(x-3)(7-x)}$
7. $\sqrt{4^x(4^x + 4)}$
8. $(x+1)\sqrt{2x^2 + 3}$
9. $x\sqrt{5 - 4x - x^2}$
10. $\sec^2 x \cdot \sqrt{\tan^2 x + \tan x - 7}$
11. $\sqrt{x^2 + 2x + 5}$
12. $\sqrt{2x^2 + 3x + 4}$

III. Integrate the following functions *w. r. t. x* :

1. $(2 + \cot x - \operatorname{cosec}^2 x) \cdot e^x$
2. $\left(\dfrac{1 + \sin x}{1 + \cos x}\right) \cdot e^x$
3. $e^x \cdot \left(\dfrac{1}{x} - \dfrac{1}{x^2}\right)$
4. $\left(\dfrac{x}{(x+1)^2}\right) \cdot e^x$
5. $\dfrac{e^x}{x}[x(\log x)^2 + 2(\log x)]$
6. $e^{5x} \cdot \left(\dfrac{5x \cdot \log x + 1}{x}\right)$
7. $e^{\sin^{-1} x} \cdot \left(\dfrac{x + \sqrt{1 - x^2}}{\sqrt{1 - x^2}}\right)$
8. $\log (1 + x)^{(1+x)}$
9. $\operatorname{cosec}(\log x)[1 - \cot(\log x)]$

3.4 Integration by partial fraction :

If $f(x)$ and $g(x)$ are two polynomials then $\dfrac{f(x)}{g(x)}$, $g(x) \neq 0$ is called a rational algebric function.

$\dfrac{f(x)}{g(x)}$ is called a proper rational function provided degree of $f(x) <$ degree of $g(x)$; otherwise it is called **improper rational function**.

If degree of $f(x) \geq$ degree of $g(x)$ i.e. $\dfrac{f(x)}{g(x)}$ is an improper rational function then express it as in the form Quotient $+ \dfrac{\text{Remainder}}{g(x)}$, $g(x) \neq 0$ where $\dfrac{\text{Remainder}}{g(x)}$ is proper rational function.

Lets see the three different types of the proper rational function $\dfrac{f(x)}{g(x)}$, $g(x) \neq 0$ where the denominator $g(x)$ is expressed as

(i) a non-repeated linear factors

(ii) repeated Linear factors and

(iii) product of Linear factor and non-repeated quadratic factor.

No.	Rational form	Partial form
(i)	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-b)}+\dfrac{C}{(x-c)}$
(ii)	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	$\dfrac{A}{(x-a)}+\dfrac{B}{(x-a)^2}+\dfrac{C}{(x-c)}$
(iii)	$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\dfrac{A}{(x-a)}+\dfrac{Bx+C}{x^2+bx+c}$

Type (i) : $\displaystyle\int \frac{px^2+qx+r}{(x-a)(x-b)(x-c)}\cdot dx$ i.e. denominator is expressed as non-repeated Linear factors.

SOLVED EXAMPLES

1. $\displaystyle\int \frac{3x^2+4x-5}{(x^2-1)(x+2)}\cdot dx$

Solution : $\displaystyle I = \int \frac{3x^2+4x-5}{(x-1)(x+1)(x+2)}\cdot dx$

Consider, $\displaystyle \frac{3x^2+4x-5}{(x-1)(x+1)(x+2)} = \frac{A}{(x-1)}+\frac{B}{(x+1)}+\frac{C}{(x+2)}$

$$= \frac{A(x+1)(x+2)+B(x-1)(x+2)+C(x-1)(x+1)}{(x-1)(x+1)(x+2)}$$

$\therefore\quad 3x^2+4x-5 = A(x+1)(x+2)+B(x-1)(x+2)+C(x-1)(x+1)$

at $x=1$, $\quad 3(1)^2+4(1)-5 = A(2)(3)+B(0)+C(0)$

$2=6A \quad\Rightarrow\quad A = \dfrac{1}{3}$

at $x=-1$, $\quad 3(-1)^2+4(-1)-5 = A(0)+B(-2)(1)+C(0)$

$-6=-2B \quad\Rightarrow\quad B = 3$

at $x=-2$, $\quad 3(-2)^2+4(-2)-5 = A(0)+B(0)+C(-3)(-1)$

$-1=3C \quad\Rightarrow\quad C = -\dfrac{1}{3}$

Thus, $\displaystyle \frac{3x^2+4x-5}{(x-1)(x+1)(x+2)} = \frac{\left(\frac{1}{3}\right)}{(x-1)}+\frac{3}{(x+1)}+\frac{\left(-\frac{1}{3}\right)}{(x+2)}$

$$\therefore\ I = \int\left[\frac{\left(\frac{1}{3}\right)}{(x-1)}+\frac{3}{(x+1)}+\frac{\left(-\frac{1}{3}\right)}{(x+2)}\right]\cdot dx \quad = \frac{1}{3}\log(x-1)+3\log(x+1)-\frac{1}{3}\log(x+2)+c$$

$$= \frac{1}{3}\log\left[\frac{(x-1)(x+1)^9}{(x+2)}\right]+c \qquad \therefore \int\frac{3x^2+4x-5}{(x^2-1)(x+2)}\cdot dx = \frac{1}{3}\log\left[\frac{(x-1)(x+1)^9}{(x+2)}\right]+c$$

2. $\displaystyle\int \frac{2x^2 - 3}{(x^2-5)(x^2+4)} \cdot dx$

Solution : Consider, $\dfrac{2x^2 - 3}{(x^2-5)(x^2+4)}$

Let $x^2 = m$

$\therefore \quad = \dfrac{2m-3}{(m-5)(m+4)}$ . . . proper rational function.

Now, $\dfrac{2m-3}{(m-5)(m+4)} = \dfrac{A}{(m-5)} + \dfrac{B}{(m+4)} = \dfrac{A(m+4) + B(m-5)}{(m-5)(m+4)}$

$\therefore \quad 2m - 3 = A(m+4) + B(m-5)$

at $m = 5$, $\quad 2(5) - 3 = A(9) + B(0)$

$7 = 9A \quad \Rightarrow \quad A = \dfrac{7}{9}$

at $m = -4$, $\quad 2(-4) - 3 = A(0) + B(-9)$

$-11 = -9B \quad \Rightarrow \quad B = \dfrac{11}{9}$

Thus, $\dfrac{2m-3}{(m-5)(m+4)} = \dfrac{\left(\frac{7}{9}\right)}{(m-5)} + \dfrac{\left(\frac{11}{9}\right)}{(m+4)}$ i.e. $\dfrac{2x^2-3}{(x^2-5)(x^2+4)} = \dfrac{\left(\frac{7}{9}\right)}{x^2-5} + \dfrac{\left(\frac{11}{9}\right)}{x^2+4}$

$$\therefore \text{I} = \int \left[\frac{\left(\frac{7}{9}\right)}{x^2-5} + \frac{\left(\frac{11}{9}\right)}{x^2+4} \right] \cdot dx$$

$$= \frac{7}{9} \cdot \int \frac{1}{x^2 - (\sqrt{5})^2} \cdot dx + \frac{11}{9} \cdot \int \frac{1}{x^2 + (2)^2} \cdot dx$$

$$= \frac{7}{9} \cdot \frac{1}{2(\sqrt{5})} \cdot \log\left[\frac{x-\sqrt{5}}{x+\sqrt{5}}\right] + \frac{11}{9} \cdot \frac{1}{2} \cdot \tan^{-1}\left(\frac{x}{2}\right) + c$$

$$\therefore \text{I} = \frac{7}{18(\sqrt{5})} \cdot \log\left[\frac{x-\sqrt{5}}{x+\sqrt{5}}\right] + \frac{11}{18} \cdot \tan^{-1}\left(\frac{x}{2}\right) + c$$

$$\therefore \int \frac{2x^2-3}{(x^2-5)(x^2+4)} \cdot dx = \frac{7}{18(\sqrt{5})} \cdot \log\left[\frac{x-\sqrt{5}}{x+\sqrt{5}}\right] + \frac{11}{18} \cdot \tan^{-1}\left(\frac{x}{2}\right) + c$$

3. $\displaystyle\int \frac{1}{(\sin\theta)(3+2\cos\theta)}\cdot d\theta$

Solution : $\displaystyle I = \int \frac{1}{(\sin\theta)(3+2\cos\theta)}\cdot d\theta = \int \frac{\sin\theta}{(1-\cos^2\theta)(3+2\cos\theta)}\cdot d\theta$

$$= \int \frac{\sin\theta}{(1-\cos\theta)(1+\cos\theta)(3+2\cos\theta)}\cdot d\theta$$

put $\cos\theta = t$ $\quad\therefore\quad -\sin\theta\cdot d\theta = 1\cdot dt$

$\therefore\quad \sin\theta\cdot d\theta = -1\cdot dt$

Consider, $\displaystyle \frac{-1}{(1-t)(1+t)(3+2t)} = \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(3+2t)}$

$$= \frac{A(1+t)(3+2t) + B(1-t)(3+2t) + C(1-t)(1+t)}{(1-t)(1+t)(3+2t)}$$

$\therefore\quad -1 = A(1+t)(3+2t) + B(1-t)(3+2t) + C(1-t)(1+t)$

at $t = 1$, $\quad -1 = A(2)(5) + B(0) + C(0)$

$\displaystyle -1 = 10A \quad\Rightarrow\quad A = -\frac{1}{10}$

at $t = -1$, $\quad -1 = A(0) + B(2)(1) + C(0)$

$\displaystyle -1 = 2B \quad\Rightarrow\quad B = -\frac{1}{2}$

at $t = -\frac{3}{2}$, $\quad \displaystyle -1 = A(0) + B(0) + C\left(+\frac{5}{2}\right)\left(-\frac{1}{2}\right)$

$\displaystyle -1 = -\frac{5}{4}C \quad\Rightarrow C = \frac{4}{5}$

Thus, $\displaystyle \frac{-1}{(1-t)(1+t)(3+2t)} = \frac{\left(-\frac{1}{10}\right)}{(1-t)} + \frac{\left(-\frac{1}{2}\right)}{(1+t)} + \frac{\left(\frac{4}{5}\right)}{(3+2t)}$

$$\therefore I = \int \left[\frac{\left(-\frac{1}{10}\right)}{(1-t)} + \frac{\left(-\frac{1}{2}\right)}{(1+t)} + \frac{\left(\frac{4}{5}\right)}{(3+2t)}\right]\cdot dt$$

$$= -\frac{1}{10}\log\left(1-t\right)\cdot\frac{1}{(-1)} - \frac{1}{2}\log\left(1+t\right) + \frac{4}{5}\log\left(3+2t\right)\cdot\frac{1}{2} + c$$

$$= \frac{1}{10}\log\left(1-\cos\theta\right) - \frac{1}{2}\log\left(1+\cos\theta\right) + \frac{4}{10}\log\left(3+2\cos\theta\right) + c$$

$$= \frac{1}{10}\left(\log\frac{(1-\cos\theta)(3+2\cos\theta)^4}{(1+\cos\theta)^5}\right) + c \qquad \because \log a^m = m\cdot\log a$$

4. $\displaystyle\int \frac{1}{2\cos x + \sin 2x}\cdot dx$

Solution : $\displaystyle I = \int \frac{1}{2\cos x + \sin 2x}\cdot dx \quad = \int \frac{1}{2\cos x + 2\sin x\cdot\cos x}\cdot dx \quad = \int \frac{1}{2(\cos x)(1+\sin x)}\cdot dx$

$$= \frac{1}{2}\cdot\int \frac{\cos x}{\cos^2 x\,(1+\sin x)}\cdot dx = \frac{1}{2}\cdot\int \frac{\cos x}{(1-\sin^2 x)(1+\sin x)}\cdot dx$$

put $\sin x = t$ $\quad\therefore\quad \cos x\cdot dx = 1\cdot dt$

$$= \frac{1}{2}\cdot\int \frac{1}{(1-t^2)(1+t)}\cdot dt \quad = \frac{1}{2}\cdot\int \frac{1}{(1-t)(1+t)(1+t)}\cdot dt = \frac{1}{2}\cdot\int \frac{1}{(1-t)(1+t)^2}\cdot dt$$

Consider, $\displaystyle \frac{1}{(1-t)(1+t)^2} = \frac{A}{(1-t)} + \frac{B}{(1+t)} + \frac{C}{(1+t)^2} = \frac{A(1+t)^2 + B(1-t)(1+t) + C(1-t)}{(1-t)(1+t)^2}$

$$\therefore \quad 1 = A(1+t)^2 + B(1-t)(1+t) + C(1-t)$$

at $t = 1$, $\quad 1 = A(2)^2 + B(0) + C(0)$

$1 = 4A \quad \Rightarrow \quad A = \dfrac{1}{4}$

at $t = -1$, $\quad 1 = A(0) + B(0) + C(2)$

$1 = 2C \quad \Rightarrow \quad C = \dfrac{1}{2}$

at $t = 0$, $\quad 1 = A(1)^2 + B(1)(1) + C(1)$

$1 = A + B + C$

$1 = \dfrac{1}{4} + B + \dfrac{1}{2} \quad \Rightarrow \quad B = \dfrac{1}{4}$

Thus, $\displaystyle \frac{1}{(1-t)(1+t)^2} = \frac{\left(\frac{1}{4}\right)}{(1-t)} + \frac{\left(\frac{1}{4}\right)}{(1+t)} + \frac{\left(\frac{1}{2}\right)}{(1+t)^2}$

$$\therefore I = \int \left[\frac{\left(\frac{1}{4}\right)}{(1-t)} + \frac{\left(\frac{1}{4}\right)}{(1+t)} + \frac{\left(\frac{1}{2}\right)}{(1+t)^2}\right]\cdot dt = \frac{1}{2}\left[\frac{1}{4}\log(1-t)\cdot\frac{1}{(-1)} + \frac{1}{4}\log(1+t) + \frac{1}{2}\cdot\frac{(-1)}{(1+t)}\right] + c$$

$$= \frac{1}{2}\left[\frac{1}{4}\log(1-t)\cdot\frac{1}{(-1)} + \frac{1}{4}\log(1+t) + \frac{1}{2}\cdot\frac{-1}{1+t}\right] + c$$

$$= \frac{1}{8}\left[-\log(1-\sin x) + \log(1+\sin x) - \frac{2}{1+\sin x}\right] + c \quad = \frac{1}{8}\left[\log\left(\frac{1+\sin x}{1-\sin x}\right) - \frac{2}{1+\sin x}\right] + c$$

$$\therefore \quad \int \frac{1}{2\cos x + \sin 2x}\cdot dx = \frac{1}{8}\left[\log\left(\frac{1+\sin x}{1-\sin x}\right) - \frac{2}{1+\sin x}\right] + c$$

5. $\displaystyle\int \frac{\tan\theta + \tan^3\theta}{1+\tan^3\theta}\cdot d\theta$

Solution : $\displaystyle I = \int \frac{(\tan\theta)\,(1+\tan^2\theta)}{1+\tan^3\theta}\cdot d\theta \quad = \int \frac{(\tan\theta)\cdot(1+\tan^2\theta)}{1+\tan^3\theta}\cdot d\theta = \int \frac{\tan\theta \cdot \sec^2\theta}{1+\tan^3\theta}\cdot d\theta$

put $\tan\theta = x$ $\quad\therefore\quad$ $\sec^2\theta\cdot d\theta = 1\cdot dx$

$$= \int \frac{x}{1+x^3}\cdot dx = \int \frac{x}{(1+x)\,(1-x+x^2)}\cdot dx$$

Consider, $\displaystyle \frac{x}{(1+x)\,(1-x+x^2)} = \frac{A}{1+x} + \frac{Bx+C}{(1-x+x^2)}$

$$= \frac{A\,(1-x+x^2) + Bx + C\,(1+x)}{(1+x)\,(1-x+x^2)}$$

$$\therefore x \;= A\,(1-x+x^2) + (Bx+C)\,(1+x) = A - Ax + Ax^2 + Bx + Bx^2 + C + Cx$$

$$0\,x^2 + 1\cdot x + 0 = (A+B)\,x^2 + (-A+B+C)\,x + (A+C)$$

compairing the co-efficients of like powers of variables.

$0 = A + B$ $\quad$ **. . . (I)**

$1 = -A + B + C$ $\quad$ **. . . (II)** $\quad$ and

$0 = A + C$ $\quad$ **. . . (III)**

Solving these equations, we get $A = -\dfrac{1}{3}$; $B = \dfrac{1}{3}$ and $C = \dfrac{1}{3}$

Thus, $\displaystyle \frac{x}{(1+x)\,(1-x+x^2)} = \frac{\left(-\frac{1}{3}\right)}{1+x} + \frac{\left(\frac{1}{3}x + \frac{1}{3}\right)}{(1-x+x^2)}$

$$\therefore I = \int \left[\frac{\left(-\frac{1}{3}\right)}{1+x} + \frac{\left(\frac{1}{3}x + \frac{1}{3}\right)}{(1-x+x^2)}\right]\cdot dx \quad = -\frac{1}{3}\cdot\int \frac{1}{1+x}\cdot dx + \frac{1}{3}\cdot\int \frac{x+1}{1-x+x^2}\cdot dx$$

$$= -\frac{1}{3}\cdot\int \frac{1}{1+x}\cdot dx + \frac{1}{3}\cdot\frac{1}{(2)}\int \frac{2x-1+3}{x^2-x+1}\cdot dx \qquad \because \quad \frac{d}{dx}\,x^2 - x + 1 = 2x - 1$$

$$= -\frac{1}{3}\cdot\int \frac{1}{1+x}\cdot dx + \frac{1}{3}\cdot\frac{1}{2}\cdot\int \frac{2x-1+3}{x^2-x+1}\cdot dx$$

$$= -\frac{1}{3}\cdot\int \frac{1}{1+x}\cdot dx + \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\cdot dx + \frac{1}{6}\cdot\int \frac{3}{x^2-x+1}\cdot dx$$

$= \quad I_1 + I_2 + I_3$ $\qquad$ **. . . (IV)**

$$\therefore \quad I_1 = -\frac{1}{3}\cdot\int \frac{1}{1+x}\cdot dx \qquad = -\frac{1}{3}\left[\log(1+x)\right]$$

$$= -\frac{1}{3}\log\left(1+\tan\theta\right) \qquad \ldots \textbf{(V)}$$

$$\therefore \quad I_2 = \frac{1}{6}\cdot\int \frac{2x-1}{x^2-x+1}\cdot dx \quad = \frac{1}{6}\left[\log(x^2-x+1)\right]$$

$$= \frac{1}{6}\log\left(\tan^2\theta-\tan\theta+1\right) \qquad \ldots \textbf{(VI)}$$

$$\therefore \quad I_3 = \frac{1}{6}\cdot\int \frac{3}{x^2-x+1}\cdot dx$$

$$= \frac{1}{2}\cdot\int \frac{1}{x^2-x+\frac{1}{4}-\frac{1}{4}+1}\cdot dx \qquad \because \left\{\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{1}{2}(-1)\right)^2 = \left(-\frac{1}{2}\right)^2 = \frac{1}{4}\right\}$$

$$= \frac{1}{2}\cdot\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\cdot dx$$

$$= \frac{1}{2}\cdot\left[\frac{1}{\left(\frac{\sqrt{3}}{2}\right)}\right]\cdot\tan^{-1}\left[\frac{x-\frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)}\right]+c$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$$

$$\therefore \quad I_3 = \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta-1}{\sqrt{3}}\right)+c \qquad \ldots \textbf{(VII)}$$

$$\therefore \quad \int \frac{\tan\theta+\tan^3\theta}{1+\tan^3\theta}\cdot d\theta = -\frac{1}{3}\log\left(1+\tan\theta\right)+\frac{1}{6}\log\left(\tan^2\theta-\tan\theta+1\right)+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan\theta-1}{\sqrt{3}}\right)+c$$

EXERCISE 3.4

I. Integrate the following *w. r. t. x* :

1. $\dfrac{x^2+2}{(x-1)(x+2)(x+3)}$ 2. $\dfrac{x^2}{(x^2+1)(x^2-2)(x^2+3)}$ 3. $\dfrac{12x+3}{6x^2+13x-63}$

4. $\dfrac{2x}{4-3x-x^2}$

5. $\dfrac{x^2+x-1}{x^2+x-6}$

6. $\dfrac{6x^3+5x^2-7}{3x^2-2x-1}$

7. $\dfrac{12x^2-2x-9}{(4x^2-1)(x+3)}$

8. $\dfrac{1}{x(x^5+1)}$

9. $\dfrac{2x^2-1}{x^4+9x^2+20}$

10. $\dfrac{x^2+3}{(x^2-1)(x^2-2)}$

11. $\dfrac{2x}{(2+x^2)(3+x^2)}$

12. $\dfrac{2^x}{4^x-3\cdot 2^x-4}$

13. $\dfrac{3x-2}{(x+1)^2(x+3)}$

14. $\dfrac{5x^2+20x+6}{x^3+2x^2+x}$

15. $\dfrac{1}{x(1+4x^3+3x^6)}$

16. $\dfrac{1}{x^3-1}$

17. $\dfrac{(3\sin x-2)\cdot\cos x}{5-4\sin x-\cos^2 x}$

18. $\dfrac{1}{\sin x+\sin 2x}$

19. $\dfrac{1}{2\sin x+\sin 2x}$

20. $\dfrac{1}{\sin 2x+\cos x}$

21. $\dfrac{1}{\sin x\cdot(3+2\cos x)}$

22. $\dfrac{5\cdot e^x}{(e^x+1)(e^{2x}+9)}$

23. $\dfrac{2\log x+3}{x(3\log x+2)[(\log x)^2+1]}$

3.5 Something Interesting :

Students/ now familier with the integration by parts.

The result is $\displaystyle \int u\cdot v\cdot dx = u\cdot\int v\cdot dx - \int\left(\frac{d}{dx}\cdot u\right)\left(\int v\cdot dx\right)\cdot dx$,

u and v are differentiable functions of x and $u\cdot v$ follows L I A T E order.

This result can be extended to the generalisation as -

$\int u\cdot v\cdot dx = u\cdot v_1 - u'\cdot v_2 + u''\cdot v_3 - u'''\cdot v_4 + \ldots$

($'$) dash indicates the derivative.

($_1$) subscript indicates the integration.

This result is more useful where the first function (u) is a polynomial, because $\dfrac{d^n u}{dx^n} = 0$ for some n.

For example : $\int x^2\cdot\cos 3x\cdot dx$

$$= x^2\cdot\left(\sin 3x\cdot\frac{1}{3}\right) - (2x)\left(-\cos 3x\cdot\frac{1}{3}\cdot\frac{1}{3}\right) + (2)\left(-\sin 3x\cdot\frac{1}{3}\cdot\frac{1}{9}\right) - (0)$$

$$= \frac{1}{3}x^2\cdot\sin 3x + \frac{2}{9}x\cdot\cos 3x - \frac{2}{27}\sin 3x + c$$

verify this example with usual rule of integration by parts.

Let us Remember

- We can always add arbitarary constant c to the integration obtained :

(I) i.e. $\frac{d}{dx} \cdot g(x) = f(x) \quad \Rightarrow \quad \int f(x) \cdot dx = g(x) + c$

$f(x)$ is integrand, $g(x)$ is integral of $f(x)$ with respect to x, c is arbitarary constant.

(II) $\int f(ax+b) \cdot dx = g(ax+b) \cdot \frac{1}{a} + c$

(III) (1) $\int [f(x)]^n \cdot f'(x) \cdot dx = \frac{[f(x)]^{n+1}}{n+1} + c$ (2) $\int \frac{f'(x)}{f(x)} \cdot dx = \log(f(x)) + c$

(3) $\int \frac{f'(x)}{\sqrt{f(x)}} \cdot dx = 2\sqrt{f(x)} + c$

(IV) (1) $\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + c$ (2) $\int \frac{1}{\sqrt{x}} \cdot dx = 2\sqrt{x} + c$

(3) $\int \text{constant}(k) \cdot dx = kx + c$ (4) $\int a^x \cdot dx = \frac{a^x}{\log a} + c$

(5) $\int e^x \cdot dx = e^x + c$ (6) $\int \frac{1}{x} \cdot dx = \log(x) + c$

(7) $\int \sin x \cdot dx = -\cos x + c$ (8) $\int \cos x \cdot dx = \sin x + c$

(9) $\int \tan x \cdot dx = \log(\sec x) + c$ (10) $\int \cot x \cdot dx = \log(\sin x) + c$

(11) $\int \sec x \cdot dx = \log(\sec x + \tan x) + c = \log\left[\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right] + c$

(12) $\int \operatorname{cosec} x \cdot dx = \log(\operatorname{cosec} x - \cot x) + c = \log\left[\tan\left(\frac{x}{2}\right)\right] + c$

(13) $\int \sec^2 x \cdot dx = \tan x + c$ (14) $\int \operatorname{cosec}^2 x \cdot dx = -\cot x + c$

(15) $\int \sec x \cdot \tan x \cdot dx = \sec x + c$ (16) $\int \operatorname{cosec} x \cdot \cot x \cdot dx = -\operatorname{cosec} x + c$

(17) $\int \frac{1}{\sqrt{1-x^2}} \cdot dx = \sin^{-1} x + c$ (18) $\int \frac{-1}{\sqrt{1-x^2}} \cdot dx = \cos^{-1} x + c$

(19) $\int \frac{1}{1+x^2} \cdot dx = \tan^{-1} x + c$ (20) $\int \frac{-1}{1+x^2} \cdot dx = \cot^{-1} x + c$

(21) $\int \frac{1}{x \cdot \sqrt{x^2-1}} \cdot dx = \sec^{-1} x + c$ (22) $\int \frac{-1}{x \cdot \sqrt{x^2-1}} \cdot dx = \operatorname{cosec}^{-1} x + c$

(23) $\displaystyle\int \frac{1}{x^2+a^2}\cdot dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$

(24) $\displaystyle\int \frac{1}{x^2-a^2}\cdot dx = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)+c$

(25) $\displaystyle\int \frac{1}{a^2-x^2}\cdot dx = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)+c$

(26) $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\cdot dx = \sin^{-1}\left(\frac{x}{a}\right)+c$

(27) $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\cdot dx = \log\left|\, x+\sqrt{x^2-a^2}\right|+c$

(28) $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\cdot dx = \log\left|x+\sqrt{x^2+a^2}\right|+c$

(29) $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\cdot dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+c$

(30) $\displaystyle\int \sqrt{a^2-x^2}\cdot dx = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

(31) $\displaystyle\int \sqrt{a^2+x^2}\cdot dx = \frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)+c$

(32) $\displaystyle\int \sqrt{x^2-a^2}\cdot dx = \frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)+c$

(V) If u and v are differentiable functions of x then $\int u\cdot v\cdot dx = u\cdot\int v\cdot dx - \int\left(\frac{d}{dx}\cdot u\right)\left(\int v\cdot dx\right)\cdot dx$ where $u\cdot v$ follows the L I A T E order.

(VI) $\int e^x\,[f(x)+f'(x)]\cdot dx = e^x\cdot f(x)+c$

(VII) For the integration of type $\displaystyle\int\frac{f(x)}{g(x)}\cdot dx,\ g(x)\neq 0$ where $\dfrac{f(x)}{g(x)}$ proper rational function.

(i) non-repeated linear factors

(ii) repeated Linear factors and

(iii) product of Linear factor and non-repeated quadratic factor.

(VIII) $\displaystyle\int\frac{1}{x^2+a^2}\cdot dx$, $\displaystyle\int\frac{1}{x^2-a^2}\cdot dx$, $\displaystyle\int\frac{1}{a^2-x^2}\cdot dx$

$\displaystyle\int\frac{1}{ax^2+bx+c}\cdot dx$

Method of completing square

$\displaystyle\int\frac{1}{a\sin^2 x+b\cos^2 x+c}\cdot dx$

Divide Nr and Dr by $\cos^2 x$

$\displaystyle\int\frac{1}{a\sin x+b\cos x+c}\cdot dx$

put $\tan\left(\frac{x}{2}\right)=t$

$\displaystyle\int\frac{px+q}{ax^2+bx+c}\cdot dx$

$px+q = A\dfrac{d}{dx}(ax^2+bx+c)+B$

(VIII) $\displaystyle\int\frac{1}{\sqrt{x^2+a^2}}\cdot dx$, $\displaystyle\int\frac{1}{\sqrt{x^2-a^2}}\cdot dx$, $\displaystyle\int\frac{1}{\sqrt{a^2-x^2}}\cdot dx$

$\displaystyle\int\frac{1}{\sqrt{ax^2+bx+c}}\cdot dx$

$\displaystyle\int\frac{px+q}{\sqrt{ax^2+bx+c}}\cdot dx$

MISCELLANEOUS EXERCISE 3

(I) Choose the correct option from the given alternatives :

(1) $\int \dfrac{1+x+\sqrt{x+x^2}}{\sqrt{x}+\sqrt{1+x}} \cdot dx =$

(A) $\dfrac{1}{2}\sqrt{x+1}+c$ (B) $\dfrac{2}{3}(x+1)^{\frac{3}{2}}+c$ (C) $\sqrt{x+1}+c$ (D) $2(x+1)^{\frac{3}{2}}+c$

(2) $\int \dfrac{1}{x+x^5} \cdot dx = f(x)+c$, then $\int \dfrac{x^4}{x+x^5} \cdot dx =$

(A) $\log x - f(x)+c$ (B) $f(x)+\log x+c$ (C) $f(x)-\log x+c$ (D) $\dfrac{1}{5}x^5 f(x)+c$

(3) $\int \dfrac{\log(3x)}{x\log(9x)} \cdot dx =$

(A) $\log(3x) - \log(9x) + c$ (B) $\log(x) - (\log 3) \cdot \log(\log 9x) + c$

(C) $\log 9 - (\log x) \cdot \log(\log 3x) + c$ (D) $\log(x) + (\log 3) \cdot \log(\log 9x) + c$

(4) $\int \dfrac{\sin^m x}{\cos^{m+2} x} \cdot dx =$

(A) $\dfrac{\tan^{m+1}x}{m+1}+c$ (B) $(m+2)\tan^{m+1}x+c$ (C) $\dfrac{\tan^m x}{m}+c$ (D) $(m+1)\tan^{m+1}x+c$

(5) $\int \tan(\sin^{-1}x) \cdot dx =$

(A) $(1-x^2)^{-\frac{1}{2}}+c$ (B) $(1-x^2)^{\frac{1}{2}}+c$ (C) $\dfrac{\tan^m x}{\sqrt{1-x^2}}+c$ (D) $-\sqrt{1-x^2}+c$

(6) $\int \dfrac{x-\sin x}{1-\cos x} \cdot dx =$

(A) $x\cot\left(\dfrac{x}{2}\right)+c$ (B) $-x\cot\left(\dfrac{x}{2}\right)+c$ (C) $\cot\left(\dfrac{x}{2}\right)+c$ (D) $x\tan\left(\dfrac{x}{2}\right)+c$

(7) If $f(x) = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, $g(x) = e^{\sin^{-1}x}$, then $\int f(x) \cdot g(x) \cdot dx =$

(A) $e^{\sin^{-1}x} \cdot (\sin^{-1}x - 1) + c$ (B) $e^{\sin^{-1}x} \cdot (1 - \sin^{-1}x) + c$

(C) $e^{\sin^{-1}x} \cdot \overline{(\sin^{-1}x + 1)} + c$ (D) $e^{\sin^{-1}x} \cdot (\sin^{-1}x - 1) + c$

(8) If $\int \tan^3 x \cdot \sec^3 x \cdot dx = \left(\dfrac{1}{m}\right)\sec^m x - \left(\dfrac{1}{n}\right)\sec^n x + c$, then $(m, n) =$

(A) $(5, 3)$ (B) $(3, 5)$ (C) $\left(\dfrac{1}{5}, \dfrac{1}{3}\right)$ (D) $(4, 4)$

(9) $\int \frac{1}{\cos x - \cos^2 x} \cdot dx =$

(A) $\log(\operatorname{cosec} x - \cot x) + \tan\left(\frac{x}{2}\right) + c$ (B) $\sin 2x - \cos x + c$

(C) $\log(\sec x + \tan x) - \cot\left(\frac{x}{2}\right) + c$ (D) $\cos 2x - \sin x + c$

(10) $\int \frac{\sqrt{\cot x}}{\sin x \cdot \cos x} \cdot dx =$

(A) $2\sqrt{\cot x} + c$ (B) $-2\sqrt{\cot x} + c$ (C) $\frac{1}{2}\sqrt{\cot x} + c$ (D) $\sqrt{\cot x} + c$

(11) $\int \frac{e^x (x-1)}{x^2} \cdot dx =$

(A) $\frac{e^x}{x} + c$ (B) $\frac{e^x}{x^2} + c$ (C) $\left(x - \frac{1}{x}\right) e^x + c$ (D) $xe^{-x} + c$

(12) $\int \sin(\log x) \cdot dx =$

(A) $\frac{x}{2}[\sin(\log x) - \cos(\log x)] + c$ (B) $\frac{x}{2}[\sin(\log x) + \cos(\log x)] + c$

(C) $\frac{x}{2}[\cos(\log x) - \sin(\log x)] + c$ (D) $\frac{x}{4}[\cos(\log x) - \sin(\log x)] + c$

(13) $\int x^x (1 + \log x) \cdot dx =$

(A) $\frac{1}{2}(1 + \log x)^2 + c$ (B) $x^{2x} + c$ (C) $x^x \log x + c$ (D) $x^x + c$

(14) $\int \cos^{-\frac{3}{7}} x \cdot \sin^{-\frac{11}{7}} x \cdot dx =$

(A) $\log\left(\sin^{-\frac{4}{7}} x\right) + c$ (B) $\frac{4}{7}\tan^{\frac{4}{7}} x + c$

(C) $-\frac{7}{4}\tan^{-\frac{4}{7}} x + c$ (D) $\log\left(\cos^{\frac{3}{7}} x\right) + c$

(15) $2\int \frac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin^2 x} \cdot dx =$

(A) $\sin 2x + c$ (B) $\cos 2x + c$ (C) $\tan 2x + c$ (D) $2\sin 2x + c$

(16) $\int \frac{dx}{\cos x\sqrt{\sin^2 x - \cos^2 x}} \cdot dx =$

(A) $\log\left(\tan x - \sqrt{\tan^2 x - 1}\right) + c$ (B) $\sin^{-1}(\tan x) + c$

(C) $1 + \sin^{-1}(\cot x) + c$ (D) $\log\left(\tan x + \sqrt{\tan^2 x - 1}\right) + c$

(17) $\int \frac{\log x}{(\log ex)^2} \cdot dx =$

(A) $\frac{x}{1 + \log x} + c$ (B) $x(1 + \log x) + c$ (C) $\frac{1}{1 + \log x} + c$ (D) $\frac{1}{1 - \log x} + c$

(18) $\int [\sin(\log x) + \cos(\log x)] \cdot dx =$

(A) $x \cos(\log x) + c$ (B) $\sin(\log x) + c$ (C) $\cos(\log x) + c$ (D) $x \sin(\log x) + c$

(19) $\int \frac{\cos 2x - 1}{\cos 2x + 1} \cdot dx =$

(A) $\tan x - x + c$ (B) $x + \tan x + c$ (C) $x - \tan x + c$ (D) $-x - \cot x + c$

(20) $\int \frac{e^{2x} + e^{-2x}}{e^x} \cdot dx =$

(A) $e^x - \frac{1}{3e^{3x}} + c$ (B) $e^x + \frac{1}{3e^{3x}} + c$ (C) $e^{-x} + \frac{1}{3e^{3x}} + c$ (D) $e^{-x} - \frac{1}{3e^{3x}} + c$

(II) Integrate the following with respect to the respective variable :

(1) $(x - 2)^2 \sqrt{x}$ (2) $\frac{x^7}{x + 1}$ (3) $(6x + 5)^{\frac{3}{2}}$

(4) $\frac{t^3}{(t + 1)^2}$ (5) $\frac{3 - 2 \sin x}{\cos^2 x}$ (6) $\frac{\sin^6 \theta + \cos^6 \theta}{\sin^2 \theta \cdot \cos^2 \theta}$

(7) $\cos 3x \cdot \cos 2x \cdot \cos x$ (8) $\frac{\cos 7x - \cos 8x}{1 + 2 \cos 5x}$ (9) $\cot^{-1}\left(\frac{1 + \sin x}{\cos x}\right)$

(III) Integrate the following :

(1) $\frac{(1 + \log x)^3}{x}$ (2) $\cot^{-1}(1 - x + x^2)$ (3) $\frac{1}{x \cdot \sin^2(\log x)}$

(4) $\sqrt{x} \sec(x^{\frac{3}{2}}) \cdot \tan(x^{\frac{3}{2}})$ (5) $\log(1 + \cos x) - x \cdot \tan\left(\frac{x}{2}\right)$ (6) $\frac{x^2}{\sqrt{1 - x^6}}$

(7) $\frac{1}{(1 - \cos 4x)(3 - \cot 2x)}$ (8) $\log(\log x) + (\log x)^{-2}$ (9) $\frac{1}{2 \cos x + 3 \sin x}$

(10) $\frac{1}{x^3 \cdot \sqrt{x^2 - 1}}$ (11) $\frac{3x + 1}{\sqrt{-2x^2 + x + 3}}$ (12) $\log(x^2 + 1)$

(13) $e^{2x} \cdot \sin x \cdot \cos x$ (14) $\frac{x^2}{(x - 1)(3x - 1)(3x - 2)}$ (15) $\frac{1}{\sin x + \sin 2x}$

(16) $\sec^2 x \cdot \sqrt{7 + 2 \tan x - \tan^2 x}$ (17) $\frac{x + 5}{x^3 + 3x^2 - x - 3}$ (18) $\frac{1}{x \cdot (x^5 + 1)}$

(19) $\frac{\sqrt{\tan x}}{\sin x \cdot \cos x}$ (20) $\sec^4 x \cdot \mathrm{cosec}^2 x$

4. DEFINITE INTEGRATION

Let us Study

- Definite integral as limit of sum.
- Fundamental theorem of integral calculus.
- Methods of evaluation and properties of definite integral.

4. 1 Definite integral as limit of sum :

In the last chapter, we studied various methods of finding the primitives or indefinite integrals of given function. We shall now interprete the definite integrals denoted by $\int_a^b f(x)\, dx$, read as the integral from a to b of the function $f(x)$ with respect to x. Here $a < b$, are real numbers and $f(x)$ is definited on $[a, b]$. At present, we assume that $f(x) \geq 0$ on $[a, b]$ and $f(x)$ is continuous.

$\int_a^b f(x)\, dx$ is defined as the area of the region bounded by $y = f(x)$, X-axis and the ordinates $x = a$ and $x = b$. If $g(x)$ is the primitive of $f(x)$ then the area is $g(b) - g(a)$.

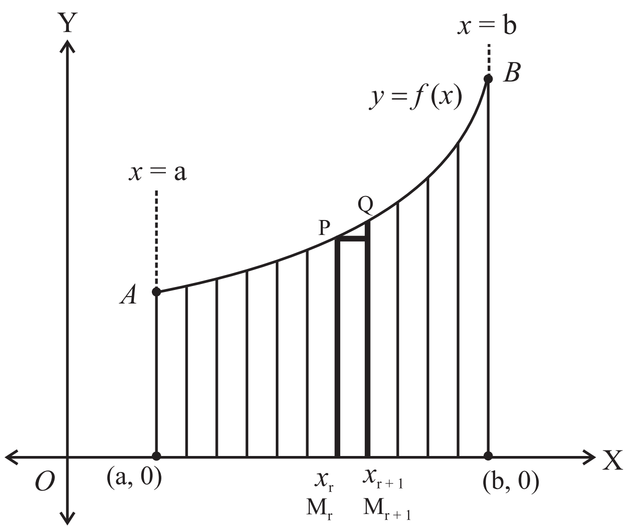

Fig. 4.1

The reason of the above definition will be clear from the figure 4.1. and the discussion that follows here. We are using the mean value theorem learnt earlier. Divide the interval $[a, b]$ into a equal parts by

$a = x_0 < x_1 < x_2 < . . . < x_{n-1} < x_n = b.$

Draw the curve $y = f(x)$ in $[a, b]$ and divide the interval $[a, b]$ into n equal parts by

$a = x_0 < x_1 < x_2 < . . . < x_{n-1} < x_n = b.$

Divide the region whose area is measured into their strips as above.

Note that, the area of each strip can be approximated by the area of a rectangle $M_r\, M_{r+1}$ QP as shown in the figure 4.1, which is $(x_{r-1} - x_r) \times f(\text{T})$ where T is a point on the curve $y = f(x)$ between P and Q.

The mean value theorem states that if $g(x)$ is the primitive of $f(x)$,

$g(x_{r+1}) - g(x_r) = (x_{r+1} - x_r) \cdot f(t_r)$ where $x_r < t_r < x_{r+1}$.

Now we can replace $f(\text{T})$ by $f(t_r)$ given here and express the approximation of the area of the shaded region as $\sum_{r=0}^{n=1} (x_{r+1} - x_r) \cdot f(t_r)$ where $x_r < t_r < x_{r+1}$.

Now we can replace $f(\text{T})$ by $f(t_r)$ given here and express the approximation of the area of the shaed region as

$$\sum_{r=0}^{n=1} (x_{r+1} - x_r) \cdot f(t_r) = \sum_{r=0}^{n=1} g(x_{r+1}) - g(x_r) = g(b) - g(a)$$

Thus taking limit as $n \to \infty$

$$\begin{aligned} g(b) - g(a) &= \lim_{n\to\infty} \sum (x_{r+1} - x_r) \cdot f(t_r) \\ &= \lim_{n\to\infty} \text{S}_n \\ &= \int_a^b f(x)\, dx \end{aligned}$$

The word 'to integrate' means 'to find the sum of'. The technique of integration is very useful in finding plane areas, length of arcs, volume of solid revolution etc...

SOLVED EXAMPLES

Ex. 1 : $\int_1^2 (2x + 5)\, dx$

Solution : Given, $\int_1^2 (2x + 5)\, dx = \int_a^b f(x)\, dx$

$f(x) = 2x + 5 \qquad a = 1 \,;\, b = 2$

$$\begin{aligned} \Rightarrow \quad f(a + rh) &= f(1 + rh) \\ &= 2(1 + rh) + 5 \\ &= 2 + 2rh + 5 \\ &= 7 + 2rh \end{aligned}$$

and $h = \dfrac{b - a}{n}$

$h = \dfrac{2 - 1}{n}$

$\therefore \quad nh = 1$

We know $\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot f(a + rh)$

$$\therefore \quad \int_1^2 (2x+5)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot (7+2rh)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} (7h + 2rh^2)$$

$$= \lim_{n\to\infty} \left(7h\sum_{r=1}^{n} 1 + 2h^2 \sum_{r=1}^{n} r\right)$$

$$= \lim_{n\to\infty} \left[7h\cdot(n) + 2h^2\left(\frac{n\,(n+1)}{2}\right)\right]$$

$$= \lim_{n\to\infty} \left[7nh + h^2n^2\left(1+\frac{1}{n}\right)\right]$$

$$= \lim_{n\to\infty} \left[7\,(1) + (1)^2\left(1+\frac{1}{n}\right)\right]$$

$$= 7 + 1\,(1+0) = 8$$

Ex. 2 : $\int_2^3 7^x \cdot dx$

Solution : Given, $\int_2^3 7^x \cdot dx = \int_a^b f(x)\,dx$

$f(x) = 7^x \qquad a = 2\ ;\ b = 3$

$\Rightarrow \quad f(a+rh) = f(1+rh)$ and $h = \dfrac{b-a}{n}$

$= 7^{2+rh}$ $\qquad h = \dfrac{3-2}{n} \qquad \therefore \quad nh = 1$

$= 7^2 \cdot 7^{rh}$

We know $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot f(a+rh)$

$$\therefore \quad \int_1^3 7^x \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot \left(7^2 \cdot 7^{r\cdot h}\right)$$

$$= \lim_{n\to\infty} 7^2 \cdot \sum_{r=1}^{n} h \cdot 7^{r\cdot h}$$

$$= \lim_{n\to\infty} 7^2 \cdot h \cdot \left[7^h + 7^{2h} + 7^{3h} + 7^{4h} + \ldots + 7^{nh}\right]$$

$$= \lim_{n\to\infty} 7^2\cdot h\cdot\left(\frac{7^h\left[(7^h)^n - 1\right]}{7^h - 1}\right) = \lim_{n\to\infty} 7^2\cdot\left(\frac{7^h\,(7^{nh}-1)}{\frac{7^h-1}{h}}\right)$$

$$= \lim_{n\to\infty} 7^2\cdot\left(\frac{7^h\,(7^{(1)}-1)}{\frac{7^h-1}{h}}\right)$$

$$= \frac{7^2\cdot\, 7^0\cdot\,(7-1)}{\log 7} = \frac{(49)(1)(6)}{\log 7} = \frac{294}{\log 7}$$

Ex. 3 : $\int_0^4 (x - x^2) \cdot dx$

Solution : $\int_0^4 (x - x^2) \cdot dx = \int_a^b f(x)\, dx$

$f(x) = x - x^2$ $\qquad a = 0\ ;\ b = 4$

$\Rightarrow \quad f(a + rh) = f(0 + rh)$ $\qquad$ and $\quad h = \dfrac{b - a}{n}$

$= f(rh)$ $\qquad h = \dfrac{4 - 0}{n}$

$= (rh) - (rh)^2$

$= rh - r^2h^2$ $\qquad \therefore \quad nh = 4$

We know $\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot [f(a + rh)]$

$$\therefore \quad \int_0^4 (x - x^2)\cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot \left(rh - r^2h^2\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \left(rh^2 - r^2h^3\right)$$

$$= \lim_{n\to\infty} \left(h^2 \cdot \sum_{r=1}^{n} r - h^3 \cdot \sum_{r=1}^{n} r^2\right)$$

$$= \lim_{n\to\infty} \left[h^2\left(\frac{n(n+1)}{2}\right) - h^3\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{h^2 \cdot n \cdot n\left(1 + \frac{1}{n}\right)}{2} - \frac{h^3 \cdot n \cdot n\left(1 + \frac{1}{n}\right) n\left(2 + \frac{1}{n}\right)}{6}\right]$$

$$= \lim_{n\to\infty} \left[\frac{(nh)^2\left(1 + \frac{1}{n}\right)}{2} - \frac{(nh)^3\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)}{6}\right]$$

$$= \lim_{n\to\infty} \left[\frac{(4)^2\left(1 + \frac{1}{n}\right)}{2} - \frac{(4)^3\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)}{6}\right]$$

$$= \frac{(4)^2 \cdot (1 + 0)}{2} - \frac{(4)^3(1 + 0)(2 + 0)}{6}$$

$$= 8 - \frac{(64)(2)}{6}$$

$$= -\frac{40}{3}$$

Ex. 4 : $\int_0^{\pi/2} \sin x \cdot dx$

Solution : $\int_0^{\pi/2} \sin x \cdot dx = \int_0^{\pi/2} f(x)\, dx$

$f(x) = \sin x \qquad a = 0\ ;\ b = \dfrac{\pi}{2}$

$$\Rightarrow \quad f(a + rh) = \sin(a + rh) = \sin(0 + rh) = \sin rh$$

and $h = \dfrac{b-a}{n} = \dfrac{\frac{\pi}{2} - 0}{n}$

$\therefore \quad nh = \dfrac{\pi}{2}$

We know $\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot [f(a + rh)]$

$$\therefore \quad \int_0^{\pi/2} \sin x \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot \sin rh$$

$$= \lim_{n\to\infty} h \cdot \sum_{r=1}^{n} \sin rh$$

$$= \lim_{n\to\infty} h \cdot [\sin h + \sin 2h + \sin 3h + \ldots + \sin nh] \qquad \ldots \text{(I)}$$

Consider,

$$\sum_{r=1}^{n} \sin rh = \sin h + \sin 2h + \sin 3h + \ldots + \sin nh$$

$$= 2\sin\frac{h}{2} \cdot \sin h + 2\sin\frac{h}{2} \cdot \sin 2h + 2\sin\frac{h}{2} \cdot \sin 3h + \ldots + 2\sin\frac{h}{2} \cdot \sin nh$$

$\because \quad 2\sin A \cdot \sin B = \cos(A - B) - \cos(A + B)$

$$2\sin\frac{h}{2} \cdot \sum_{r=1}^{n} \sin rh = \left[\left(\cos\frac{h}{2} - \cos\frac{3h}{2}\right) + \left(\cos\frac{3h}{2} - \cos\frac{5h}{2}\right) + \left(\cos\frac{5h}{2} - \cos\frac{7h}{2}\right) + \ldots\right.$$

$$\left. + \ldots + \left(\cos\left(\frac{2n-1}{2}\right)h - \left(\cos\left(\frac{2n+1}{2}\right)h\right.\right.\right]$$

$$= \left[\cos\frac{h}{2} - \cos\left(\frac{2n+1}{2}\right)h\right]$$

$$= \left[\cos\frac{h}{2} - \cos\left(\frac{2nh}{2} + \frac{h}{2}\right)\right]$$

$$= \left[\cos\frac{h}{2} - \cos\left(\frac{\pi}{2} + \frac{h}{2}\right)\right] \qquad \because \quad nh = \frac{\pi}{2}$$

$$= \left(\cos\frac{h}{2} + \sin\frac{h}{2}\right)$$

$$\therefore \qquad \sum_{r=1}^{n} \sin rh = \frac{\cos\frac{h}{2} + \sin\frac{h}{2}}{2\sin\frac{h}{2}}$$

Now from I,

$$\int_0^{\pi/2} \sin x \cdot dx = \lim_{n\to\infty} \sum_{r=1}^{n} h \cdot \sin rh$$

$$= \lim_{n\to\infty} h \cdot \left[\frac{\cos\frac{h}{2} + \sin\frac{h}{2}}{2\sin\frac{h}{2}}\right]$$

$$\because \qquad nh = \frac{\pi}{4} \text{ as } n\to\infty \Rightarrow h\to 0 \left(\frac{1}{n} \to 0\right)$$

$$= \lim_{\substack{n\to\infty \\ h\to 0}} \left[\frac{\cos\frac{h}{2} + \sin\frac{h}{2}}{\frac{2\cdot\sin\frac{h}{2}}{h}}\right]$$

$$= \frac{\cos 0 + \sin 0}{\left(\frac{1}{2}\right)}$$

$$= \frac{1+0}{2\cdot\frac{1}{2}} = 1$$

$$\therefore \qquad \int_0^{\pi/2} \sin x \cdot dx = 1$$

EXERCISE 4.1

I. Evaluate the following integrals as limit of sum.

(1) $\int_1^3 (3x - 4)\cdot dx$ (2) $\int_0^4 x^2\cdot dx$ (3) $\int_0^2 e^x\cdot dx$

(4) $\int_0^2 (3x^2 - 1)\cdot dx$ (5) $\int_1^3 x^3\cdot dx$

4.2 Fundamental theorem of integral calculus :

Let f be the continuous function defined on $[a, b]$ and if $\int f(x)\,dx = g(x) + c$

then
$$\int_a^b f(x)\,dx = \Big[g(x) + c\Big]_a^b$$
$$= [(g(b) + c) - (g(a) + c)]$$
$$= g(b) + c - g(a) - c$$
$$= g(b) - g(a)$$

Thus $\displaystyle\int_a^b f(x)\,dx = g(b) - g(a)$

Ex. :
$$\int_2^5 (x^2 - x)\,dx = \left[\left(\frac{x^3}{3} - \frac{x^2}{2}\right)\right]_2^5$$
$$= \left[\left(\frac{5^3}{3} - \frac{5^2}{2}\right) - \left(\frac{2^3}{3} - \frac{2^2}{2}\right)\right]$$
$$= \frac{125}{3} - \frac{25}{2} - \frac{8}{3} + \frac{4}{2}$$
$$= \frac{117}{3} - \frac{21}{2} = \frac{234 - 83}{6}$$
$$\therefore \int_2^5 (x^2 - x)\,dx = \frac{151}{3}$$

In $\displaystyle\int_a^b f(x)\,dx$ a is called as a lower limit and b is called as an upper limit.

Now let us discuss some fundamental properties of definite integration.

These properties are very useful in evaluation of the definite integral.

4.2.1

Property I : $\displaystyle\int_a^a f(x)\,dx = 0$

Let $\displaystyle\int f(x)\,dx = g(x) + c$

$$\therefore \int_a^a f(x)\,dx = \Big[g(x) + c\Big]_a^a$$
$$= [(g(a) + c) - (g(a) + c)]$$
$$= 0$$

Property II : $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Let $\displaystyle\int f(x)\,dx = g(x) + c$

$$\therefore \int_a^b f(x)\,dx = \Big[g(x) + c\Big]_a^b$$
$$= [(g(b) + c) - (g(a) + c)]$$
$$= g(b) - g(a)$$
$$= -[g(a) - g(b)]$$
$$= -\int_b^a f(x)\,dx$$

Thus $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Ex.
$$\int_1^3 x\,dx = \left[\frac{x^2}{2}\right]_1^3$$
$$= \frac{3^2}{2} - \frac{1^2}{2} = \frac{9}{2} - \frac{1}{2} = 4$$

Ex.
$$\int_3^1 x\,dx = \left[\frac{x^2}{2}\right]_3^1$$
$$= \frac{1^2}{2} - \frac{3^2}{2} = \frac{1}{2} - \frac{9}{2} = -4$$

Property III : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

Let $\int f(x)\,dx = g(x) + c$

L.H.S. : $\int_a^b f(x)\,dx = \left[g(x)+c\right]_a^b$

$= [(g(b)+c) - (g(a)+c)]$

$= g(b) - g(a)$ . . . . . (i)

R.H.S. : $\int_a^b f(t)\,dt = \left[g(t)+c\right]_a^b$

$= [(g(b)+c) - (g(a)+c)]$

$= g(b) - g(a)$ . . . . . (ii)

from (i) and (ii)

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$

i.e. definite integration is independent of the variable.

Ex. $\int_{\pi/6}^{\pi/3} \cos x \cdot dx = \left[\sin x\right]_{\pi/6}^{\pi/3}$

$= \sin\frac{\pi}{3} - \sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2} - \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2}$

Ex. $\int_{\pi/6}^{\pi/3} \cos t \cdot dt = \left[\sin t\right]_{\pi/6}^{\pi/3}$

$= \sin\frac{\pi}{3} - \sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2} - \frac{1}{2}$

$= \frac{\sqrt{3}-1}{2}$

Property IV : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$ i.e. $c \in [a, b]$

Let $\int f(x)\,dx = g(x) + c$

Consider R.H.S. : $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$= \left[g(x)+c\right]_a^c + \left[g(x)+c\right]_c^b$

$= [(g(c)+c) - (g(a)+c)] + [(g(b)+c) - (g(c)+c)]$

$= g(c) + c - g(a) - c + g(b) + c - g(c) - c$

$= g(b) - g(a)$

$= \left[g(x)+c\right]_a^b$

$= \int_a^b f(x)\,dx$: L.H.S.

Thus $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$

Ex. : $\int_{-1}^{5}(2x+3)\cdot dx = \int_{-1}^{3}(2x+3)\cdot dx + \int_{3}^{5}(2x+3)\cdot dx$

L.H.S. : $\int_{-1}^{5}(2x+3)\cdot dx$

$$= \left[2\frac{x^2}{2}+3x\right]_{-1}^{5}$$

$$= \left[x^2+3x\right]_{-1}^{5}$$

$$= [(5)^2+3\,(5)]-[(-1)^2+3\,(-1)]$$

$$= (25+15)-(1-3)$$

$$= 40+2 = 42$$

R.H.S. : $\int_{-1}^{3}(2x+3)\cdot dx + \int_{3}^{5}(2x+3)\cdot dx$

$$= \left[x^2+3x\right]_{-1}^{3} + \left[x^2+3x\right]_{3}^{5}$$

$$= [((3)^2+3\,(3))-((-1)^2+3\,(-1))] + [((5)^2+3\,(5))-((3)^2+3\,(3))]$$

$$= [(9+9)-(1-3)]+[(25+15)-(9-9)]$$

$$= 18+2+40-18$$

$$= 42$$

Property V : $\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$

Let $\int f(x)\,dx = g(x)+c$

Consider R.H.S. : $\int_{a}^{b} f(a+b-x)\,dx$

put $a+b-x=t$ i.e. $x=a+b-t$

$\therefore \; -dx = dt \Rightarrow dx = -dt$

As $x \to a \Rightarrow t \to b$ and $x \to b \Rightarrow t \to a$

therefore $= \int_{b}^{a} f(t)\,(-dt)$

$$= -\int_{b}^{a} f(t)\,dt$$

$$= \int_{a}^{b} f(t)\,dt \ldots \left(\because \int_{a}^{b} f(x)\,dx = -\int_{b}^{a} f(x)\,dx\right)$$

$= \int_{a}^{b} f(x)\,dx$ . . . as definite integration is independent of the variable.

$=$ L. H. S.

Thus $\int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(a+b-x)\,dx$

Ex. :

$\int_{\pi/6}^{\pi/3} \sin^2 x \cdot dx$

$$\text{I} = \int_{\pi/6}^{\pi/3} \sin^2 x \cdot dx \quad \ldots \text{(i)}$$

$$= \int_{\pi/6}^{\pi/3} \sin^2\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)$$

$$= \int_{\pi/6}^{\pi/3} \sin^2\left(\frac{\pi}{2}-x\right)$$

$$\text{I} = \int_{\pi/6}^{\pi/3} \cos^2 x \cdot dx \quad \ldots \text{(ii)}$$

adding (i) and (ii)

$$2\text{I} = \int_{\pi/6}^{\pi/3} \sin^2 x \cdot dx + \int_{\pi/6}^{\pi/3} \cos^2 x \cdot dx$$

$$2\text{I} = \int_{\pi/6}^{\pi/3} (\sin^2 x + \cos^2 x)\cdot dx$$

$$2\text{I} = \int_{\pi/6}^{\pi/3} 1\cdot dx \quad = \left[x\right]_{\pi/6}^{\pi/3}$$

$$2\text{I} = \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6} \qquad \therefore \quad \text{I} = \frac{\pi}{12}$$

$$\int_{\pi/6}^{\pi/3} \sin^2 x \cdot dx = \frac{\pi}{12}$$

Property VI : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Let $\int f(x)\,dx = g(x) + c$

Consider R.H.S. : $\int_0^a f(a-x)\,dx$

put $a - x = t$ i.e. $x = a - t$

$\therefore$ $-dx = dt \Rightarrow dx = -dt$

As x varies from 0 to a, t varies from a to 0

therefore I $= \int_a^0 f(t)\,(-dt)$

$= -\int_a^0 f(t)\,dt$

$= \int_0^a f(t)\,dt \ldots \left(\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right)$

$= \int_0^a f(x)\,dx$ ... as definite integration is independent of the variable.

$=$ L. H. S.

Thus

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

Ex. : $\int_0^{\pi/4} \log(1+\tan x)\cdot dx$

Let $\int_0^{\pi/4} \log(1+\tan x)\cdot dx$ ... (i)

$$\mathrm{I} = \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-x\right)\right]$$

$$= \int_0^{\pi/4} \log\left[1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\cdot\tan x}\right]\cdot dx$$

$$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan x}{1+\tan x}\right]\cdot dx$$

$$= \int_0^{\pi/4} \log\left[\frac{1+\tan x+1-\tan x}{1+\tan x}\right]\cdot dx$$

$$= \int_0^{\pi/4} \log\left[\frac{2}{1+\tan x}\right]\cdot dx$$

$$= \int_0^{\pi/4} [\log 2 - \log(1+\tan x)]\cdot dx$$

$$= \int_0^{\pi/4} (\log 2)\cdot dx - \int_0^{\pi/4} \log(1+\tan x)\cdot dx$$

$$\mathrm{I} = (\log 2)\int_0^{\pi/4} 1\cdot dx - \mathrm{I} \quad \ldots\text{by eq. (i)}$$

$$\mathrm{I}+\mathrm{I} = (\log 2)\Big[x\Big]_0^{\pi/4}$$

$$2\mathrm{I} = (\log 2)\left[\frac{\pi}{4}-0\right]$$

$$\therefore \quad \mathrm{I} = \frac{\pi}{8}(\log 2)$$

Thus

$$\int_0^{\pi/4} \log(1+\tan x)\cdot dx = \frac{\pi}{8}(\log 2)$$

Property VII :

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

R.H.S. : $\displaystyle\int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

$$= I_1 + I_2 \qquad \ldots \text{(i)}$$

Consider $\displaystyle I_2 = \int_0^a f(2a-x)\,dx$

put $2a - x = t$ i.e. $x = 2a - t$

$\therefore\ -1\,dx = 1\,dt \Rightarrow dx = -\,dt$

As x varies from 0 to $2a$, t varies from $2a$ to 0

$$I = \int_{2a}^{a} f(t)\,(-\,dt)$$

$$= -\int_{2a}^{a} f(t)\,dt$$

$$= \int_0^{2a} f(t)\,dt \ldots \left(\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right)$$

$$= \int_0^{2a} f(x)\,dx \ldots \left(\int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right)$$

$$\therefore\ \int_0^a f(x)\,dx = \int_0^{2a} f(x)\,dx$$

from eq. (i)

$$\int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx = \int_0^a f(x)\,dx + \int_0^{2a} f(x)\,dx$$

$$= \int_0^{2a} f(x)\,dx : \text{L.H.S}$$

Thus,

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

Property VIII :

$$\int_{-a}^{a} f(x)\,dx = 2\cdot\int_0^a f(x)\,dx \text{ , if } f(x) \text{ even function}$$

$$= 0 \text{ , if } f(x) \text{ is odd function}$$

$f(x)$ even function if $f(-x) = f(x)$

and $f(x)$ odd function if $f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx \qquad \ldots \text{(i)}$$

Consider $\displaystyle\int_{-a}^{0} f(x)\,dx$

put $x = -t$ $\therefore$ $dx = -dt$

As x varies from $-a$ to 0, t varies from a to 0

$$I = \int_a^0 f(-t)\,(-dt) = -\int_a^0 f(-t)\,dt$$

$$= \int_0^a f(-t)\,dt \ldots \left(\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right)$$

$$= \int_0^a f(-x)\,dx \ldots \left(\int_a^b f(x)\,dx = \int_a^b f(t)\,dt\right)$$

Equation (i) becomes

$$\int_{-a}^{a} f(x)\,dx = \int_0^a f(-x)\,dx + \int_0^a f(x)\,dx$$

$$= \int_0^a [f(-x) + f(x)]\,dx$$

If $f(x)$ is odd function then $f(-x) = -f(x)$, hence

$$\int_{-a}^{a} f(x)\,dx = 0$$

If $f(x)$ is even function then $f(-x) = f(x)$, hence

$$\int_{-a}^{a} f(x)\,dx = 2\cdot\int_0^a f(x)\,dx$$

Hence :

$$\int_{-a}^{a} f(x)\,dx = 2\cdot\int_0^a f(x)\,dx \text{ , if } f(x) \text{ even function}$$

$$= 0 \text{ , if } f(x) \text{ is odd function}$$

Ex. :

1. $\displaystyle\int_{-\pi/4}^{\pi/4} x^3 \cdot \sin^4 x \cdot dx$

Let $f(x) = x^3 \cdot \sin^4 x$

$f(-x) = (-x)^3 \cdot [\sin(-x)]^4 = -x^3 \cdot [-\sin x]^4 = -x^3 \cdot \sin^4 x$

$= -f(x)$

$f(x)$ is odd function.

$\therefore \displaystyle\int_{-\pi/4}^{\pi/4} x^3 \cdot \sin^4 x \cdot dx = 0$

2. $\displaystyle\int_{-1}^{1} \frac{x^2}{1+x^2} \cdot dx$

Let $f(x) = \dfrac{x^2}{1+x^2}$

$$f(-x) = \frac{(-x)^2}{1+(-x)^2}$$

$$= \frac{x^2}{1+x^2}$$

$$= f(x)$$

$f(x)$ is even function.

$$\int_{-1}^{1} \frac{x^2}{1+x^2} \cdot dx = 2\int_{0}^{1} \frac{x^2}{1+x^2} \cdot dx$$

$$= 2\int_{0}^{1} \frac{1+x^2-1}{1+x^2} \cdot dx$$

$$= 2\int_{0}^{1} \left[1 - \frac{1}{1+x^2}\right] \cdot dx$$

$$= 2\left[x - \tan^{-1} x\right]_0^1$$

$$= 2\left\{(1 - \tan^{-1} x) - (0 - \tan^{-1} x)\right\}$$

$$= 2\left\{1 - \frac{\pi}{4} - 0\right\}$$

$$= 2\left(1 - \frac{\pi}{4}\right) = \left(\frac{4-\pi}{2}\right)$$

$$\therefore \int_{-1}^{1} \frac{x^2}{1+x^2} \cdot dx = \frac{4-\pi}{2}$$

SOLVED EXAMPLES

Ex. 1 : $\displaystyle\int_{1}^{3} \frac{1}{\sqrt{2+x}+\sqrt{x}} \cdot dx$

Solution : $\displaystyle = \int_{1}^{3} \left(\frac{1}{\sqrt{2+x}+\sqrt{x}}\right)\left(\frac{\sqrt{2+x}-\sqrt{x}}{\sqrt{2+x}-\sqrt{x}}\right) \cdot dx$

$$= \int_{1}^{3} \left(\frac{\sqrt{2+x}-\sqrt{x}}{2+x-x}\right) \cdot dx$$

$$= \frac{1}{2} \cdot \int_{1}^{3} (\sqrt{2+x}-\sqrt{x}) \cdot dx$$

$$= \frac{1}{2} \cdot \left[\frac{(2+x)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^3$$

$$= \frac{1}{3} \cdot \left[(2+x)^{\frac{3}{2}} - (x)^{\frac{3}{2}}\right]_1^3$$

$$= \frac{1}{3}\left\{\left[(2+3)^{\frac{3}{2}} - (3)^{\frac{3}{2}}\right] - \left[(2+1)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\right]\right\}$$

$$= \frac{1}{3}\left\{5^{\frac{3}{2}} - 3^{\frac{3}{2}} - 3^{\frac{3}{2}} + 1^{\frac{3}{2}}\right\}$$

$$= \frac{1}{3}\left\{5^{\frac{3}{2}} - 2(3)^{\frac{3}{2}} + 1\right\}$$

$$\therefore \int_{1}^{3} \frac{1}{\sqrt{2+x}+\sqrt{x}}\, dx = \frac{1}{3}\left[5^{\frac{3}{2}} - 2(3)^{\frac{3}{2}} + 1\right]$$

Ex. 2 : $\int_0^{\pi/2} \sqrt{1-\cos 4x}\cdot dx$

Solution : Let $I = \int_0^{\pi/2} \sqrt{1-\cos 4x}\cdot dx$

$$I = \int_0^{\pi/2} \sqrt{2\sin^2 2x}\cdot dx$$

$$\left(\because 1-\cos A = 2\sin^2\frac{A}{2}\right)$$

$$= \sqrt{2}\cdot\int_0^{\pi/2} \sin 2x\cdot dx$$

$$= \sqrt{2}\cdot\left[\frac{-\cos 2x}{2}\right]_0^{\pi/2}$$

$$= \frac{\sqrt{2}}{2}\cdot\left[\cos 2\frac{\pi}{2} - \cos 0\right]$$

$$= -\frac{\sqrt{2}}{2}\cdot[\cos\pi - \cos 0]$$

$$= -\frac{\sqrt{2}}{2}\cdot(-1-1) = \sqrt{2}$$

$$\therefore \quad \int_0^{\pi/2}\sqrt{1-\cos 4x}\cdot dx = \sqrt{2}$$

Ex. 3 : $\int_0^{\pi/2} \cos^3 x\cdot dx$

Solution : Let $I = \int_0^{\pi/2} \cos^3 x\cdot dx$

$$= \int_0^{\pi/2} \frac{1}{4}[\cos 3x + 3\cos x]\cdot dx$$

$$= \frac{1}{4}\left[\sin 3x\cdot\frac{1}{3} + 3\sin x\right]_0^{\pi/2}$$

$$= \frac{1}{4}\left[\left(\frac{1}{3}\sin 3\frac{\pi}{2} + 3\sin\frac{\pi}{2}\right) - \left(\frac{1}{3}\sin 3(0) + 3\sin(0)\right)\right]$$

$$= \frac{1}{4}\left[\frac{1}{3}\sin\frac{3\pi}{2} + 3\sin\frac{\pi}{2} - \frac{1}{3}\sin 0 + 3\sin 0\right]$$

$$= \frac{1}{4}\left[\frac{1}{3}(-1) + 3(1) - 0\right]$$

$$= \frac{1}{4}\left[-\frac{1}{3} + 3\right] = \frac{1}{4}\left[\frac{8}{3}\right] = \frac{2}{3}$$

$$\therefore \quad \int_0^{\pi/2}\cos^3 x\cdot dx = \frac{2}{3}$$

Ex. 4 : $\int_0^{\pi/4} \frac{\sec^2 x}{2\tan^2 x + 5\tan x + 1}\cdot dx$

Solution : Let $I = \int_0^{\pi/4} \frac{\sec^2 x}{2\tan^2 x + 5\tan x + 1}\cdot dx$

put $\tan x = t \quad \therefore \sec^2 x\cdot dx = 1\cdot dt$

As x varies from 0 to $\frac{\pi}{4}$

t varies from 0 to 1

$$= \int_0^1 \frac{1}{2t^2+4t+1}\cdot dt$$

$$= \frac{1}{2}\cdot\int_0^1 \frac{1}{t^2+2t+\frac{1}{2}}\cdot dt$$

$$= \frac{1}{2}\cdot\int_0^1 \frac{1}{t^2+2t+1-1+\frac{1}{2}}\cdot dt$$

$$= \frac{1}{2}\cdot\int_0^1 \frac{1}{(t+1)^2 - \left(\frac{1}{\sqrt{2}}\right)^2}\cdot dt$$

$$= \frac{1}{2}\cdot\frac{1}{2\left(\frac{1}{\sqrt{2}}\right)}\left[\log\left[\frac{(t+1)-\frac{1}{\sqrt{2}}}{(t+1)+\frac{1}{\sqrt{2}}}\right]\right]_0^1$$

$$= \frac{\sqrt{2}}{4}\log\left[\left(\frac{\sqrt{2}\,t+\sqrt{2}-1}{\sqrt{2}\,t+\sqrt{2}+1}\right)\right]_0^1$$

$$= \frac{\sqrt{2}}{4}\left[\log\left(\frac{\sqrt{2}(1)+\sqrt{2}-1}{\sqrt{2}(1)+\sqrt{2}+1}\right) - \log\left(\frac{\sqrt{2}(0)+\sqrt{2}-1}{\sqrt{2}(0)+\sqrt{2}+1}\right)\right]$$

$$= \frac{\sqrt{2}}{4}\left[\log\left(\frac{2\sqrt{2}-1}{2\sqrt{2}+1}\right) - \log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$$

$$= \frac{\sqrt{2}}{4}\log\left[\left(\frac{2\sqrt{2}-1}{2\sqrt{2}+1}\right) \div \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]$$

$$= \frac{\sqrt{2}}{4}\log\left[\frac{3+\sqrt{2}}{3-\sqrt{2}}\right]$$

Ex. 5 : $\int_1^2 \frac{\log x}{x^2} \cdot dx$

Solution : Let $I = \int_1^2 (\log x)\left(\frac{1}{x^2}\right) \cdot dx$

$$= \left[(\log x) \cdot \int \frac{1}{x^2} \cdot dx\right]_1^2 - \int_1^2 \frac{d}{dx} \log x \cdot \int \frac{1}{x^2} \cdot dx \cdot dx$$

$$= \left[(\log x) \cdot \left(-\frac{1}{x}\right)\right]_1^2 - \int_1^2 \frac{1}{x} \cdot \left(-\frac{1}{x}\right) \cdot dx$$

$$= \left[-\frac{1}{x} \log x\right]_1^2 + \int_1^2 \frac{1}{x^2} \cdot dx$$

$$= \left[-\frac{1}{x} \log x\right]_1^2 + \left[-\frac{1}{x}\right]_1^2$$

$$= \left[\left(-\frac{1}{2} \log 2\right) - \left(-\frac{1}{1} \log 1\right)\right] + \left[\left(-\frac{1}{2}\right) - \left(-\frac{1}{1}\right)\right]$$

$$= -\frac{1}{2} \log 2 - 0 - \frac{1}{2} + 1 = \frac{1}{2} - \frac{1}{2} \log 2 \qquad \because \log 1 = 0$$

$$\therefore \int_1^2 \frac{\log x}{x^2} \cdot dx = \frac{1}{2}\left(1 - \log 2\right)$$

Ex. 6 : $\int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x} \cdot dx$

Solution : Let $I = \int_0^{\pi/2} \frac{\cos x}{1 + \cos x + \sin x} \cdot dx$

$$= \int_0^{\pi/2} \frac{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{2}\right)} \cdot dx$$

$$= \int_0^{\pi/2} \frac{\left[\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right]}{2\left[\cos\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right]} \cdot dx$$

$$= \int_0^{\pi/2} \left[\frac{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}\right] \cdot dx = \int_0^{\pi/2} \left[1 - \tan\left(\frac{x}{2}\right)\right] \cdot dx$$

$$= \frac{1}{2} \cdot \left[x - \log\left(\sec\frac{x}{2}\right) \cdot \frac{1}{\frac{1}{2}} \right]_0^{\pi/2}$$

$$= \frac{1}{2} \cdot \left[\frac{\pi}{2} - 2 \cdot \log\left(\sec\frac{\pi}{4}\right) - (0 - 2\log\sec 0) \right]$$

$$= \frac{1}{2} \cdot \left[\frac{\pi}{2} - 2\log\sqrt{2} - 0 + 2(0) \right] \qquad = \frac{1}{2} \cdot \left[\frac{\pi}{2} - 2\log\sqrt{2} \right] \qquad = \frac{\pi}{4} - \log\sqrt{2}$$

$$\therefore \quad \int_0^{\pi/2} \frac{\sec^2 x}{1 + \cos x + \sin x} \cdot dx = \frac{\pi}{4} - \log\sqrt{2}$$

Ex. 7 : $\displaystyle\int_0^{1/2} \frac{1}{(1 - 2x^2)\sqrt{1 - x^2}} \cdot dx$

Solution : Let $\displaystyle I = \int_0^{1/2} \frac{1}{(1 - 2x^2)\sqrt{1 - x^2}} \cdot dx$

put $x = \sin\theta$ $\quad\therefore\quad$ $1 \cdot dx = \cos\theta \cdot d\theta$

As x varies from 0 to $\frac{1}{2}$, θ varies from 0 to $\frac{\pi}{6}$

$$= \int_0^{\pi/6} \frac{\cos\theta}{(1 - 2\sin^2\theta)\sqrt{1 - \sin^2\theta}} \cdot d\theta = \int_0^{\pi/6} \frac{\cos\theta}{(\cos 2\theta)\sqrt{\cos^2\theta}} \cdot d\theta$$

$$= \int_0^{\pi/6} \frac{1}{\cos 2\theta} \cdot d\theta$$

$$= \int_0^{\pi/6} \sec 2\theta \cdot d\theta$$

$$= \left[\log(\sec 2\theta + \tan 2\theta) \cdot \frac{1}{2} \right]_0^{\pi/6}$$

$$= \frac{1}{2} \cdot \left[\log\left(\sec 2\left(\frac{\pi}{6}\right) + \tan 2\left(\frac{\pi}{6}\right) \right) - \log(\sec 0 + \tan 0) \right]$$

$$= \frac{1}{2} \cdot \left[\log\left(\sec\frac{\pi}{3} + \tan\frac{\pi}{3} \right) - \log(1 + 0) \right] \qquad \because \log 1 = 0$$

$$= \frac{1}{2} \cdot [\log(2 + \sqrt{3}) - 0]$$

$$= \frac{1}{2} \log(2 + \sqrt{3})$$

$$\therefore \quad \int_0^{1/2} \frac{1}{(1 - 2x^2)\sqrt{1 - x^2}} \cdot dx = \frac{1}{2} \log(2 + \sqrt{3})$$

Ex. 8 : $\int_0^2 \frac{2^x}{2^x(1+4^x)} \cdot dx$

Solution : Let $I = \int_0^2 \frac{2^x}{2^x(1+4^x)} \cdot dx$

put $2^x = t \quad \therefore \quad 2^x \cdot \log 2 \cdot dx = 1 \cdot dt$

As x varies from 0 to 2, t varies from 1 to 4

$$= \int_1^4 \frac{\frac{1}{\log 2}}{t(1+t^2)} \cdot dt$$

$$= \frac{1}{\log 2} \cdot \int_1^4 \frac{1}{t(1+t^2)} \cdot dt$$

$$= \frac{1}{\log 2} \cdot \int_1^4 \frac{1+t^2-t^2}{t(1+t^2)} \cdot dt$$

may be solved by method of partial fraction

$$= \frac{1}{\log 2} \cdot \int_1^4 \left[\frac{1+t^2}{t(1+t^2)} - \frac{t^2}{t(1+t^2)}\right] \cdot dt$$

$$= \frac{1}{\log 2} \cdot \int_1^4 \left[\frac{1}{t} - \frac{t}{1+t^2}\right] \cdot dt$$

$$= \frac{1}{\log 2} \cdot \left[\int_1^4 \frac{1}{t} \cdot dt - \frac{1}{2}\int_1^4 \frac{2t}{1+t^2} \cdot dt\right]$$

$$= \frac{1}{\log 2} \cdot \left[\log(t) - \frac{1}{2}\log(1+t^2)\right]_1^4$$

$$= \frac{1}{\log 2} \cdot \left[\left(\log 4 - \frac{1}{2}\log 17\right) - \left(\log 1 - \frac{1}{2}\log 2\right)\right]$$

$$= \frac{1}{\log 2} \cdot \left[\log 4 - \frac{1}{2}\log 17 + \frac{1}{2}\log 2\right]$$

$\because \quad \log 1 = 0$

$$= \frac{1}{\log 2} \cdot \left[\log \frac{4\sqrt{2}}{\sqrt{17}}\right]$$

$$\therefore \int_0^2 \frac{2^x}{2^x(1+4^x)} \cdot dx = \frac{1}{(\log 2)} \cdot \left[\log \frac{4\sqrt{2}}{\sqrt{17}}\right]$$

$$= \log_2 \left(\frac{4\sqrt{2}}{\sqrt{17}}\right)$$

Ex. 9 : $\int_{-1}^{1} |5x-3| \cdot dx$

Solution : Let $I = \int_{-1}^{1} |5x-3| \cdot dx$

$$|5x-3| = -(5x-3) \text{ for } (5x-3) < 0 \text{ i.e. } x < \frac{3}{5}$$

$$= (5x-3) \text{ for } (5x-3) > 0 \text{ i.e. } x > \frac{3}{5}$$

$$= \int_{-1}^{3/5} |5x-3| \cdot dx + \int_{3/5}^{1} |5x-3| \cdot dx \quad = \int_{-1}^{3/5} -(5x-3) \cdot dx + \int_{3/5}^{1} (5x-3) \cdot dx$$

$$= \left[-\left(5\frac{x^2}{2} - 3x\right)\right]_{-1}^{3/5} + \left[\left(5\frac{x^2}{2} - 3x\right)\right]_{3/5}^{1} \quad = \left[3x - \frac{5}{2}x^2\right]_{-1}^{3/5} + \left[\frac{5}{2}x^2 - 3x\right]_{3/5}^{1}$$

$$= \left[\left(3\left(\frac{3}{5}\right) - \frac{5}{2}\left(\frac{3}{5}\right)^2\right) - \left(3(-1) - \frac{5}{2}(-1)^2\right)\right] + \left[\left(\frac{5}{2}(1)^2 - 3(1)\right) - \left(\frac{5}{2}\left(\frac{3}{5}\right)^2 - 3\left(\frac{3}{5}\right)\right)\right]$$

$$= \left[\left(\frac{9}{5} - \frac{9}{10}\right) - \left(-3 - \frac{5}{2}\right)\right] + \left[\left(\frac{5}{2} - 3\right) - \left(\frac{9}{10} - \frac{9}{5}\right)\right]$$

$$= \frac{9}{5} - \frac{9}{10} + 3 + \frac{5}{2} + \frac{5}{2} - 3 - \frac{9}{10} + \frac{9}{5} = 2\left(\frac{9}{5} - \frac{9}{10} + \frac{5}{2}\right) = 2\left(\frac{18 - 9 + 25}{5}\right) \quad = \frac{34}{5}$$

$$\therefore \quad \int_{-1}^{1} |\, 5x - 3 \,| \cdot dx = \frac{34}{5}$$

Ex. 10 : $\displaystyle\int_0^{\pi/2} \frac{1}{1 + \sqrt[3]{\tan x}} \cdot dx$

Solution : Let $\displaystyle I = \int_0^{\pi/2} \frac{1}{1 + \sqrt[3]{\tan x}} \cdot dx$

$$= \int_0^{\pi/2} \left[\frac{1}{1 + \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\cos x}}}\right] \cdot dx$$

$$= \int_0^{\pi/2} \frac{\sqrt[3]{\cos x}}{\sqrt[3]{\cos x} + \sqrt[3]{\sin x}} \cdot dx \qquad \ldots \text{(i)}$$

By property $\displaystyle\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$

$$I = \int_0^{\pi/2} \frac{\sqrt[3]{\cos\left(\frac{\pi}{2} - x\right)}}{\sqrt[3]{\cos\left(\frac{\pi}{2} - x\right)} + \sqrt[3]{\sin\left(\frac{\pi}{2} - x\right)}} \cdot dx$$

$$= \int_0^{\pi/2} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x} + \sqrt[3]{\cos x}} \cdot dx \qquad \ldots \text{(ii)}$$

adding (i) and (ii)

$$I + I = \int_0^{\pi/2} \frac{\sqrt[3]{\cos x}}{\sqrt[3]{\cos x} + \sqrt[3]{\sin x}} \cdot dx + \int_0^{\pi/2} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x} + \sqrt[3]{\cos x}} \cdot dx$$

$$2I = \int_0^{\pi/2} \frac{\sqrt[3]{\cos x} + \sqrt[3]{\sin x}}{\sqrt[3]{\cos x} + \sqrt[3]{\sin x}} \cdot dx$$

$$2I = \int_0^{\pi/2} 1 \cdot dx$$

$$I = \frac{1}{2}\Big[\, x \,\Big]_0^{\pi/2} = \frac{1}{2}\left[\frac{\pi}{4} - 0\right] = \frac{\pi}{4}$$

$$\therefore \quad \int_0^{\pi/2} \frac{1}{1 + \sqrt[3]{\tan x}} \cdot dx = \frac{\pi}{4}$$

with the help of the above solved/ illustrative example verify whether the following examples evaluates their definite integrate to be equal to / as $\frac{\pi}{4}$

$\displaystyle\int_0^{\pi/2} \frac{1}{1 + \cot^3 x} \cdot dx$; $\quad \displaystyle\int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} \cdot dx$; $\quad \displaystyle\int_0^{\pi/2} \frac{\sec x}{\sec x + \operatorname{cosec} x} \cdot dx$;

$\displaystyle\int_0^{\pi/2} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} \cdot dx$; $\quad \displaystyle\int_0^{\pi/2} \frac{\operatorname{cosec}^{\frac{5}{2}} x}{\operatorname{cosec}^{\frac{5}{2}} x + \sec^{\frac{5}{2}} x} \cdot dx$

Ex. 11 : $\displaystyle\int_3^8 \frac{(11-x)^2}{x^2+(1-x)^2}\cdot dx$

Solution : Let $\displaystyle I = \int_3^8 \frac{(11-x)^2}{x^2+(1-x)^2}\cdot dx \qquad \ldots \text{(i)}$

By property $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$$I = \int_3^8 \frac{[11-(8+3-x)]^2}{[8+3-x]^2+[11-(8+3-x)]^2}\cdot dx \quad = \int_3^8 \frac{[11-(11-x)]^2}{(11-x)^2+[11-(11-x)]^2}\cdot dx$$

$$= \int_3^8 \frac{x^2}{(11-x)^2+x^2}\cdot dx \qquad \ldots \text{(ii)}$$

adding (i) and (ii)

$$I+I = \int_3^8 \frac{(11-x)^2}{x^2+(1+x)^2}\cdot dx + \int_3^8 \frac{x^2}{(11-x)^2+x^2}\cdot dx$$

$$2I = \int_3^8 \frac{(11-x)^2+x^2}{x^2+(11-x)^2}\cdot dx$$

$$I = \frac{1}{2}\int_3^8 1\cdot dx$$

$$I = \frac{1}{2}\Big[\,x\,\Big]_3^8 = \frac{1}{2}[8-3] = \frac{5}{2}$$

$$\therefore \int_3^8 \frac{(11-x)^2}{x^2+(1+x)^2}\cdot dx = \frac{5}{2}$$

Note that : In general $\displaystyle\int_a^b \frac{f(x)}{f(x)+f(a+b-x)}\cdot dx = \frac{1}{2}(b-a)$

verify the generalisation for the following examples :

$\displaystyle\int_1^2 \frac{\sqrt{x}}{\sqrt{3-x}+\sqrt{x}}\cdot dx$; $\qquad \displaystyle\int_2^7 \frac{x^3}{(9-x)^3+x^3}\cdot dx$;

$\displaystyle\int_4^9 \frac{x^{\frac{1}{4}}}{(13-x)^{\frac{1}{4}}+x^{\frac{1}{4}}}\cdot dx \qquad \displaystyle\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\cot x}}\cdot dx$

$\displaystyle\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt{\operatorname{cosec} x}}\cdot dx$

Ex. 12 : $\int_0^{\pi} x \cdot \sin^2 x \cdot dx$

Solution :

Consider, $\quad \mathrm{I} = \int_0^{\pi} x \cdot \sin^2 x \cdot dx \ldots \text{(i)}$

$$\mathrm{I} = \int_0^{\pi} (\pi - x) \cdot [\sin(\pi - x)]^2 x \cdot dx$$

$$\mathrm{I} = \int_0^{\pi} (\pi - x) \cdot \sin^2 x \cdot dx$$

$$\mathrm{I} = \int_0^{\pi} \pi \cdot \sin^2 x \cdot dx - \int_0^{\pi} x \cdot \sin^2 x \cdot dx$$

$$\mathrm{I} = \pi \cdot \int_0^{\pi} \frac{1}{2}(1 - \cos 2x) \cdot dx - \mathrm{I} \ldots \text{by (i)}$$

$$\mathrm{I} + \mathrm{I} = \frac{\pi}{2} \int_0^{\pi} (1 - \cos 2x) \cdot dx$$

$$2\mathrm{I} = \frac{\pi}{2}\left[x - \sin 2x \cdot \frac{1}{2}\right]_0^{\pi}$$

$$\mathrm{I} = \frac{\pi}{4}\left[\left(\pi - \frac{1}{2}\sin 2\pi\right) - \left(0 - \frac{1}{2}\sin 0\right)\right]$$

$$= \frac{\pi}{4}[\pi] \qquad \because \sin 0 = 0;\ \sin 2\pi = 0$$

$$= \frac{\pi^2}{4}$$

$$\therefore \int_0^{\pi} x^2 \cdot \sin^2 x \cdot dx = \frac{\pi^2}{4}$$

Ex. 13 : Evaluate the integral $\int_0^{\pi} \cos^2 x \cdot dx$ using the result/ property.

Solution :

$$\int_0^{2a} f(x)\ dx = \int_0^{a} f(x)\ dx + \int_0^{a} f(2a - x)\ dx$$

Let, $\mathrm{I} = \int_0^{\pi} \cos^2 x \cdot dx$

$$= \int_0^{2\left(\frac{\pi}{2}\right)} \cos^2 x \cdot dx$$

$$= \int_0^{\pi/2} \cos^2 x \cdot dx + \int_0^{\pi/2} \left[\cos\left(2\frac{\pi}{2} - x\right)\right]^2 \cdot dx$$

$$= \int_0^{\pi/2} \cos^2 x \cdot dx + \int_0^{\pi/2} \cos^2 x \cdot dx$$

$$\because \ \cos(\pi - x) = -\cos x$$

$$= 2 \cdot \int_0^{\pi/2} \cos^2 x \cdot dx$$

$$= \int_0^{\pi/2} (1 + \cos 2x) \cdot dx$$

$$= \left[x + \sin 2x \cdot \frac{1}{2}\right]_0^{\pi/2}$$

$$= \left[\left(\frac{\pi}{2} + \frac{1}{2}\sin 2\frac{\pi}{2}\right) - \left(0 + \frac{1}{2}\sin 2(0)\right)\right]$$

$$= \frac{\pi}{2} + 0 \qquad \because \sin 0 = 0;\ \sin \pi = 0$$

$$= \frac{\pi}{2}$$

$$\therefore \int_0^{\pi} \cos^2 x \cdot dx = \frac{\pi}{2}$$

Ex. 14 : $\int_{-\pi}^{\pi} \frac{x(1+\sin x)}{1+\cos^2 x} \cdot dx$

Solution : Let $I = \int_{-\pi}^{\pi} \frac{x(1+\sin x)}{1+\cos^2 x} \cdot dx$

$$= \left[\left(\int_{-\pi}^{\pi} \frac{x}{1+\cos^2 x} \cdot dx\right) + \left(\int_{-\pi}^{\pi} \frac{x \cdot \sin x}{1+\cos^2 x} \cdot dx\right)\right]$$

The function $\frac{x}{1+\cos^2 x}$ is odd function and the function $\frac{x \cdot \sin x}{1+\cos^2 x}$ is even function.

$$\int_{-a}^{a} f(x)\, dx = 2 \cdot \int_{0}^{a} f(x)\, dx \quad \text{, if } f(x) \text{ even function}$$

$$= 0 \quad \text{, if } f(x) \text{ is odd function}$$

$$\therefore I = 0 + 2 \cdot \int_{0}^{\pi} \frac{x \cdot \sin x}{1+\cos^2 x} \cdot dx$$

$$\therefore I = 2 \cdot \int_{0}^{\pi} \frac{x \cdot \sin x}{1+\cos^2 x} \cdot dx \qquad \text{. . . (i)}$$

$$= 2 \cdot \int_{0}^{\pi} \frac{(\pi - x) \cdot \sin(\pi - x)}{1+[\cos(\pi - x)]^2} \cdot dx$$

$$= 2 \cdot \int_{0}^{\pi} \frac{(\pi - x) \cdot \sin x}{1+(-\cos x)^2} \cdot dx$$

$$= 2\pi \cdot \int_{0}^{\pi} \frac{\pi \cdot \sin x - x \cdot \sin x}{1+\cos^2 x} \cdot dx$$

$$= 2\pi \cdot \int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x} - 2 \cdot \int_{0}^{\pi} \frac{x \cdot \sin x}{1+\cos^2 x} \cdot dx$$

$$I = 2\pi \cdot \int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x} - I \qquad \text{. . . by eq.(i)}$$

$$I + I = 2\pi \cdot \int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x} \qquad \text{. . . (ii)}$$

put $\cos x = t$ $\therefore$ $-\sin x \cdot dx = + dt$

As varies from 0 to π, t varies from 1 to -1

$$2I = 2\pi \cdot \int_{-1}^{1} \frac{-1}{1+t^2} \cdot dt$$

$$I = \pi \cdot 2 \int_{0}^{1} \frac{1}{1+t^2} \cdot dt \qquad \left(\text{where } \frac{1}{1+t^2} \text{ is even function.}\right)$$

$$I = 2\pi \cdot \left[\tan^{-1} t\right]_0^1 \cdot$$

$$= 2\pi \left[\tan^{-1}(1) - \tan^{-1}(0)\right]$$

$$= 2\pi \left(\frac{\pi}{4} - 0\right) = \frac{\pi^2}{2}$$

$$\therefore \int_{-\pi}^{\pi} \frac{x(1+\sin x)}{1+\cos^2 x} \cdot dx = \frac{\pi^2}{2}$$

Ex. 15 : $\int_0^3 x\,[\,x\,] \cdot dx$, where $[\,x\,]$ denote greatest integrate function not greater than x.

Solution : Let $I = \int_0^3 x\,[\,x\,] \cdot dx$

$$I = \int_0^1 x\,[\,x\,] \cdot dx + \int_1^2 x\,[\,x\,] \cdot dx + \int_2^3 x\,[\,x\,] \cdot dx$$

$$= \int_0^1 x\,(0) \cdot dx + \int_1^2 x\,(1) \cdot dx + \int_2^3 x\,(2) \cdot dx$$

$$= 0 + \left[\frac{x^2}{2}\right]_1^2 + \left[x^2\right]_2^3$$

$$= 0 + \left(\frac{4}{2} - \frac{1}{2}\right) + (9-4)$$

$$= \frac{3}{2} + 5 = \frac{13}{2}$$

$$\therefore \int_0^3 x\,[\,x\,] \cdot dx = \frac{13}{2}$$

EXERCISE 4.2

I. Evaluate :

(1) $\int_1^9 \frac{x+1}{\sqrt{x}} \cdot dx$

(2) $\int_2^3 \frac{1}{x^2+5x+6} \cdot dx$

(3) $\int_0^{\pi/4} \cot^2 \cdot dx$

(4) $\int_{-\pi/4}^{\pi/4} \frac{1}{1-\sin x} \cdot dx$

(5) $\int_3^5 \frac{1}{\sqrt{2x+3} - \sqrt{2x-3}} \cdot dx$

(6) $\int_0^1 \frac{x^2-2}{x^2+1} \cdot dx$

(7) $\int_0^{\pi/4} \sin 4x \sin 3x \cdot dx$

(8) $\int_0^{\pi/4} \sqrt{1+\sin 2x} \cdot dx$

(9) $\int_0^{\pi/4} \sin^4 x \cdot dx$

(10) $\int_{-4}^{2} \frac{1}{x^2+4x+13} \cdot dx$

(11) $\int_0^4 \frac{1}{\sqrt{4x-x^2}} \cdot dx$

(12) $\int_0^1 \frac{1}{\sqrt{3+2x-x^2}} \cdot dx$

(13) $\int_0^{\pi/2} x \cdot \sin x \cdot dx$

(14) $\int_0^1 x \cdot \tan^{-1} x \cdot dx$

(15) $\int_0^{\infty} x \cdot e^{-x} \cdot dx$

II. Evaluate :

(1) $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \frac{\sin^{-1} x}{(1-x^2)^{\frac{3}{2}}} \cdot dx$

(2) $\displaystyle\int_0^{\pi/4} \frac{\sec^2 x}{3\tan^2 x + 4\tan x + 1} \cdot dx$

(3) $\displaystyle\int_0^{4\pi} \frac{\sin 2x}{\sin^4 x + \cos^4 x} \cdot dx$

(4) $\displaystyle\int_0^{2\pi} \sqrt{\cos x} \cdot \sin^3 x \cdot dx$

(5) $\displaystyle\int_0^{\pi/2} \frac{1}{5 + 4\cos x} \cdot dx$

(6) $\displaystyle\int_0^{\pi/4} \frac{\cos x}{4 - \sin^2 x} \cdot dx$

(7) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(2+\sin x)} \cdot dx$

(8) $\displaystyle\int_{-1}^{1} \frac{1}{a^2 e^x + b^2 e^{-x}} \cdot dx$

(9) $\displaystyle\int_0^{\pi} \frac{1}{3 + 2\sin x + \cos x} \cdot dx$

(10) $\displaystyle\int_0^{\pi/4} \sec^4 x \cdot dx$

(11) $\displaystyle\int_0^{1} \sqrt{\frac{1-x}{1+x}} \cdot dx$

(12) $\displaystyle\int_0^{\pi} \sin^3 x\,(1 + 2\cos x)\,(1+\cos x)^2 \cdot dx$

(13) $\displaystyle\int_0^{\pi/2} \sin 2x \cdot \tan^{-1}(\sin x) \cdot dx$

(14) $\displaystyle\int_{\frac{1}{\sqrt{2}}}^{1} \frac{(e^{\cos^{-1} x})(\sin^{-1} x)}{\sqrt{1-x^2}} \cdot dx$

(15) $\displaystyle\int_2^{3} \frac{\cos(\log x)}{x} \cdot dx$

III. Evaluate :

(1) $\displaystyle\int_0^{a} \frac{1}{x + \sqrt{a^2 - x^2}} \cdot dx$

(2) $\displaystyle\int_0^{\pi/2} \log \tan x \cdot dx$

(3) $\displaystyle\int_0^{1} \log\left(\frac{1}{x} - 1\right) \cdot dx$

(4) $\displaystyle\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cdot \cos x} \cdot dx$

(5) $\displaystyle\int_0^{3} x^2 (3-x)^{\frac{5}{2}} \cdot dx$

(6) $\displaystyle\int_{-3}^{3} \frac{x^3}{9 - x^2} \cdot dx$

(7) $\displaystyle\int_{-\pi/2}^{\pi/2} \log\left(\frac{2+\sin x}{2-\sin x}\right) \cdot dx$

(8) $\displaystyle\int_{-\pi/4}^{\pi/4} \frac{x + \frac{\pi}{4}}{2 - \cos 2x} \cdot dx$

(9) $\displaystyle\int_{-\pi/4}^{\pi/4} x^3 \cdot \sin^4 x \cdot dx$

(10) $\displaystyle\int_0^{1} \frac{\log(x+1)}{x^2+1} \cdot dx$

(11) $\displaystyle\int_{-1}^{1} \frac{x^3 + 2}{\sqrt{x^2 + 4}} \cdot dx$

(12) $\displaystyle\int_{-a}^{a} \frac{x + x^3}{16 - x^2} \cdot dx$

(13) $\displaystyle\int_0^{1} t^2 \sqrt{1-t} \cdot dx$

(14) $\displaystyle\int_0^{\pi} x \cdot \sin x \cdot \cos^2 x \cdot dx$

(15) $\displaystyle\int_0^{1} \frac{\log x}{\sqrt{1-x^2}} \cdot dx$

Note that :

To evaluate the integrals of the type $\int_0^{\pi/2} \sin^n x \cdot dx$ and $\int_0^{\pi/2} \cos^n x \cdot dx$, the results used are known as **'reduction formulae'** which are stated as follows :

$$\int_0^{\pi/2} \sin^n x \cdot dx = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{4}{5} \frac{2}{3}, \qquad \text{if } n \text{ is odd.}$$

$$= \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{3}{4} \frac{1}{2} \cdot \frac{\pi}{2}, \qquad \text{if } n \text{ is even.}$$

$$\int_0^{\pi/2} \cos^n x \cdot dx = \int_0^{\pi/2} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^n \cdot dx \qquad \text{. . . by property}$$

$$= \int_0^{\pi/2} [\ \sin x\ ]^n \cdot dx$$

$$= \int_0^{\pi/2} \sin^n x \cdot dx$$

$$\int_0^{\pi/2} \sin^7 x \cdot dx = \frac{(7-1)}{7} \cdot \frac{(7-3)}{(7-2)} \cdot \frac{(7-5)}{(7-4)}$$

$$= \frac{(7-1)\cdot(7-3)\cdot(7-5)}{7\cdot(7-2)\cdot(7-4)}$$

$$= \frac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3} = \frac{16}{35}$$

$$\int_0^{\pi/2} \cos^8 x \cdot dx = \frac{(8-1)}{8} \cdot \frac{(8-3)}{(8-2)} \cdot \frac{(8-5)}{(8-4)} \cdot \frac{(8-7)}{(8-6)} \cdot \frac{\pi}{2}$$

$$= \frac{(8-1)\cdot(8-3)\cdot(8-5)\cdot(8-7)}{8\cdot(8-2)\cdot(8-4)\cdot(8-6)} \cdot \frac{\pi}{2}$$

$$= \frac{7 \cdot 5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{35\pi}{256}$$

Let us Remember

- $\sum_{r=0}^{n-1} (x_{r+1} - x_r) \cdot f(t_r) = \sum_{r=0}^{n-1} g(x_{r+1}) - g(x_r) = g(b) - g(a)$

  Thus taking limit as $n \to \infty$

  $$g(b) - g(a) = \lim_{n\to\infty} \sum (x_{r+1} - x_r) \cdot f(t_r) = \lim_{n\to\infty} S_n = \int_a^b f(x)\, dx$$

- **Fundamental theorem of integral calculus :** $\int_a^b f(x)\, dx = g(b) - g(a)$

  Property I : $\int_a^a f(x)\, dx = 0$

  Property II : $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

  Property III : $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$

  Property IV : $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$ where $a < c < b$ i.e. $c \in [a, b]$

  Property V : $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$

  Property VI : $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$

  Property VII : $\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a - x)\, dx$

  Property VIII : $\int_{-a}^a f(x)\, dx = 2 \cdot \int_0^a f(x)\, dx$, if $f(x)$ even function

  $= 0$, if $f(x)$ is odd function

  $f(x)$ even function if $f(-x) = f(x)$ and $f(x)$ odd function if $f(-x) = -f(x)$

- **'Reduction formulae'** which are stated as follows :

  $$\int_0^{\pi/2} \sin^n x \cdot dx = \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{4}{5}\,\frac{2}{3}, \quad \text{if } n \text{ is odd.}$$

  $$= \frac{(n-1)}{n} \cdot \frac{(n-3)}{(n-2)} \cdot \frac{(n-5)}{(n-4)} \cdots \frac{3}{4}\,\frac{1}{2} \cdot \frac{\pi}{2}, \quad \text{if } n \text{ is even.}$$

  $$\int_0^{\pi/2} \cos^n x \cdot dx = \int_0^{\pi/2} \left[\cos\left(\frac{\pi}{2} - 0\right)\right]^n \cdot dx = \int_0^{\pi/2} [\sin x]^n \cdot dx = \int_0^{\pi/2} \sin^n x \cdot dx$$

MISCELLANEOUS EXERCISE 4

(I) Choose the correct option from the given alternatives :

(1) $\int_2^3 \frac{dx}{x(x^3-1)} =$

(A) $\frac{1}{3}\log\left(\frac{208}{189}\right)$ (B) $\frac{1}{3}\log\left(\frac{189}{208}\right)$ (C) $\log\left(\frac{208}{189}\right)$ (D) $\log\left(\frac{189}{208}\right)$

(2) $\int_0^{\pi/2} \frac{\sin^2 x \cdot dx}{(1+\cos x)^2} =$

(A) $\frac{4-\pi}{2}$ (B) $\frac{\pi-4}{2}$ (C) $4-\frac{\pi}{2}$ (D) $\frac{4+\pi}{2}$

(3) $\int_0^{\log 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}\cdot dx =$

(A) $3+2\pi$ (B) $4-\pi$ (C) $2+\pi$ (D) $4+\pi$

(4) $\int_0^{\pi/2} \sin^6 x \cos^2 x \cdot dx =$

(A) $\frac{7\pi}{256}$ (B) $\frac{3\pi}{256}$ (C) $\frac{5\pi}{256}$ (D) $\frac{-5\pi}{256}$

(5) If $\int_0^1 \frac{dx}{\sqrt{1+x}-\sqrt{x}} = \frac{k}{3}$, then k is equal to

(A) $\sqrt{2}\,(2\sqrt{2}-2)$ (B) $\frac{\sqrt{2}}{3}\left(2-2\sqrt{2}\right)$ (C) $\frac{2\sqrt{2}-2}{3}$ (D) $4\sqrt{2}$

(6) $\int_1^2 \frac{1}{x^2} e^{\frac{1}{x}} \cdot dx =$

(A) $\sqrt{e}+1$ (B) $\sqrt{e}-1$ (C) $\sqrt{e}\left(\sqrt{e}-1\right)$ (D) $\frac{\sqrt{e}-1}{e}$

(7) If $\int_2^e \left[\frac{1}{\log x} - \frac{1}{(\log x)^2}\right] \cdot dx = a + \frac{b}{\log 2}$, then

(A) $a = e, b = -2$ (B) $a = e, b = 2$ (C) $a = -e, b = 2$ (D) $a = -e, b = -2$

(8) Let $I_1 = \int_e^{e^2} \frac{dx}{\log x}$ and $I_2 = \int_1^2 \frac{e^x}{x} \cdot dx$, then

(A) $I_1 = \frac{1}{3} I_2$ (B) $I_1 + I_2 = 0$ (C) $I_1 = 2I_2$ (D) $I_1 = I_2$

(9) $\int_0^9 \frac{\sqrt{x}}{\sqrt{x}+\sqrt{9-x}} \cdot dx =$

(A) 9 (B) $\frac{9}{2}$ (C) 0 (D) 1

(10) The value of $\int_{-\pi/4}^{\pi/4} \log\left(\frac{2+\sin\theta}{2-\sin\theta}\right) \cdot d\theta$ is

(A) 0 (B) 1 (C) 2 (D) π

(II) Evaluate the following :

(1) $\int_0^{\pi/2} \frac{\cos x}{3\cdot\cos x+\sin x}\cdot dx$ (2) $\int_{\pi/4}^{\pi/2} \frac{\cos\theta}{\left[\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right]^3}\cdot d\theta$ (3) $\int_0^1 \frac{1}{1+\sqrt{x}}\cdot dx$

(4) $\int_0^{\pi/4} \frac{\tan^3 x}{1+\cos 2x}\cdot dx$ (5) $\int_0^1 t^5\cdot\sqrt{1-t^2}\cdot dt$ (6) $\int_0^1 (\cos^{-1} x)^2\cdot dx$

(7) $\int_{-1}^1 \frac{1+x^3}{9-x^2}\cdot dx$ (8) $\int_0^{\pi} x\cdot\sin x\cdot\cos^4 x\cdot dx$ (9) $\int_0^{\pi} \frac{x}{1+\sin^2 x}\cdot dx$

(10) $\int_1^{\infty} \frac{1}{\sqrt{x}\,(1+x)}\cdot dx$

(III) Evaluate :

(1) $\int_0^1 \left(\frac{1}{1+x^2}\right)\sin^{-1}\left(\frac{2x}{1+x^2}\right)\cdot dx$ (2) $\int_0^{\pi/2} \frac{1}{6-\cos x}\cdot dx$

(3) $\int_0^a \frac{1}{a^2+ax-x^2}\cdot dx$ (4) $\int_{\pi/5}^{3\pi/10} \frac{\sin x}{\sin x+\cos x}\cdot dx$

(5) $\int_0^1 \sin^{-1}\left(\frac{2x}{1+x^2}\right)\cdot dx$ (6) $\int_0^{\pi/4} \frac{\cos 2x}{1+\cos 2x+\sin 2x}\cdot dx$

(7) $\int_0^{\pi/2} (2\cdot\log\sin x-\log\sin 2x)\cdot dx$ (8) $\int_0^{\pi} (\sin^{-1} x+\cos^{-1} x)^3\cdot\sin^3 x\cdot dx$

(9) $\int_0^4 \left[\sqrt{x^2+2x+3}\right]^{-1}\cdot dx$ (10) $\int_{-2}^3 |x-2|\cdot dx$

(IV) Evaluate the following :

(1) If $\int_0^a \sqrt{x} \cdot dx = 2a \cdot \int_0^{\pi/2} \sin^3 x \cdot dx$ then find the value of $\int_a^{a+1} x \cdot dx$.

(2) If $\int_0^k \frac{1}{2 + 8x^2} \cdot dx = \frac{\pi}{16}$. Find k.

(3) If $f(x) = a + bx + cx^2$, show that $\int_0^1 f(x) = \frac{1}{6}\left[f(0) + 4f\left(\frac{1}{2}\right) + f(1)\right]$.

❖ ❖ ❖

5. APPLICATION OF DEFINITE INTEGRATION

Let us Study

- Area under the curve
  - Area bounded by the curve, axis and given lines
  - Area between two curves.

Let us Recall

- In previous chapter, we have studied definition of definite integral as limit of a sum. Geometrically $\int_a^b f(x) \cdot dx$ gives the area A under the curve $y = f(x)$ with $f(x) > 0$ and bounded by the X−axis and the lines $x = a$, $x = b$; and is given by

  $$\int_a^b f(x)\, dx = \phi(b) - \phi(a)$$

  where $\int f(x)\, dx = \phi(x)$

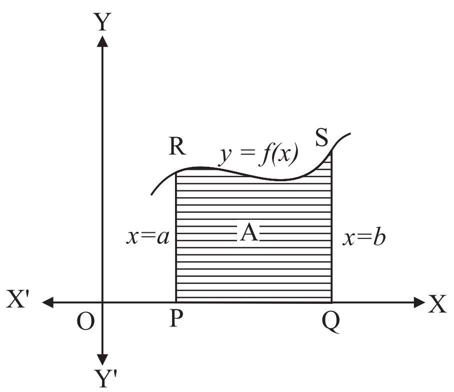

Fig. 5.1

This is also known as fundamental theorem of integral calculus.

We shall find the area under the curve by using definite integral.

5.1 Area under the curve :

For evaluation of area bounded by certain curves, we need to know the nature of the curves and their graphs. We should also be able to draw sketch of the curves.

5.1.1 Area under a curve :

The curve $y = f(x)$ is continuous in $[a, b]$ and $f(x) \geq 0$ in $[a, b]$.

1. The area shaded in figure 5.2 is bounded by the curve $y = f(x)$, X−axis and the lines $x = a$, $x = b$ and is given by the definite integral $\int_{x=a}^{x=b} (y) \cdot dx$

   A = area of the shaded region.

   $$A = \int_a^b f(x) \cdot dx$$

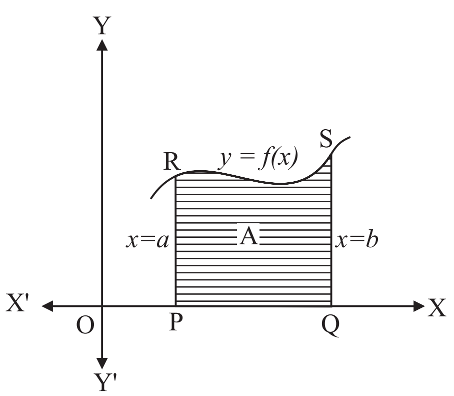

Fig. 5.2

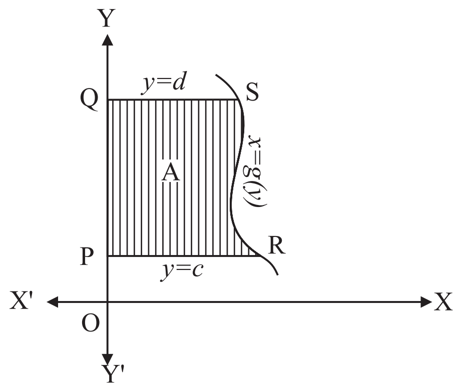

Fig. 5.3

2. The area A, bounded by the curve $x = g(y)$, Y axis and the lines $y = c$ and $y = d$ is given by

$$A = \int_{y=c}^{d} x \cdot dy$$

$$= \int_{y=c}^{y=d} g(y) \cdot dx$$

SOLVED EXAMPLE

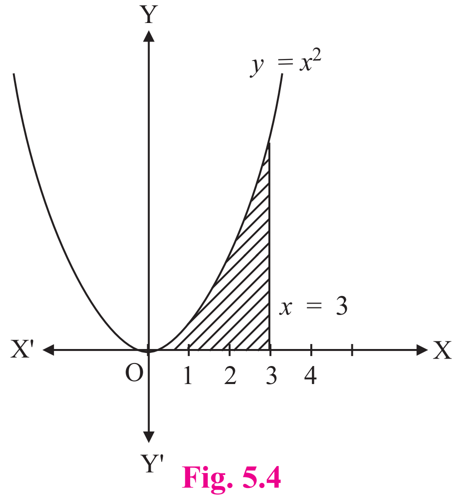

Fig. 5.4

Ex. 1 : Find the area bounded by the curve $y = x^2$, the Y axis the X axis and $x = 3$.

Solution : The required area $A = \int_{x=0}^{3} y \cdot dx$

$$A = \int_0^3 x^2 \cdot dx$$

$$= \left[\frac{x^3}{3}\right]_0^3$$

$$A = 9 - 0$$

$$= 9 \text{ sq.units}$$

5.1.2 Area between two curves :

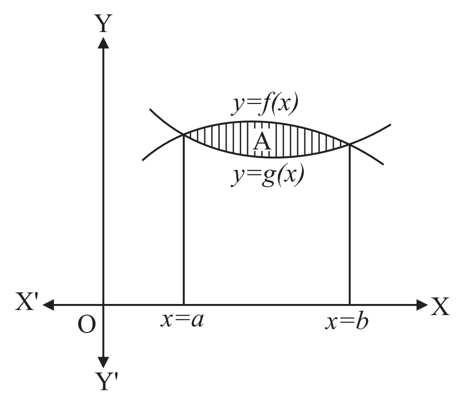

Fig. 5.5

Let $y = f(x)$ and $y = g(x)$ be the equations of the two curves as shown in fig 5.5.

Let A be the area bounded by the curves $y = f(x)$ and $y = g(x)$

$$A = \left| A_1 - A_2 \right| \qquad \text{where}$$

A_1 = Area bounded by the curve $y = f(x)$, X-axis and $x = a$, $x = b$.

A_2 = Area bounded by the curve $y = g(x)$, X-axis and $x = a$, $x = b$.

The point of intersection of the curves $y = f(x)$ and $y = g(x)$ can be obtained by solving their equations simultaneously.

$\therefore$ The required area $\qquad A = \left| \int_a^b f(x)\, dx - \int_a^b g(x)\, dx \right|$

SOLVED EXAMPLES

Ex. 1 : Find the area of the region bounded by the curves $y^2 = 9x$ and $x^2 = 9y$.

Solution : The equations of the curves are

$y^2 = 9x \ldots\ldots$ (I)

and $\quad x^2 = 9y \ldots\ldots$ (II)

Squaring equation (II)

$x^4 = 81y^2$

$x^4 = 81\,(9x) \quad \ldots$ by (1)

$x^4 = 729\,x$

$\therefore \quad x\,(x^3 - 9^3) = 0$

i.e. $\quad x\,(x^3 - 9^3) = 0$

$\Rightarrow \quad x = 0 \quad$ or $\quad x = 9$

From equation (II), $\quad y = 0 \quad$ or $\quad y = 9$

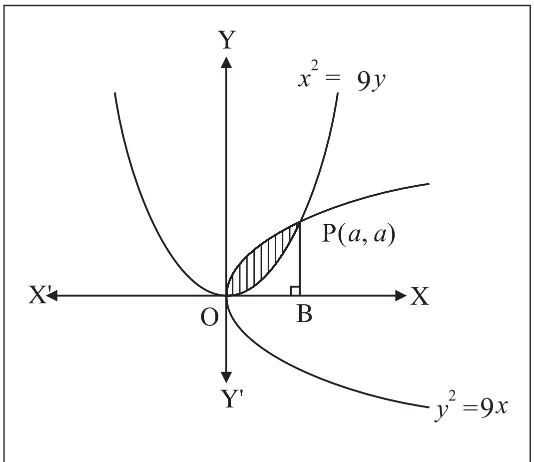

Fig. 5.6

$\therefore$ The points of intersection of the curves are (0, 0), (9, 9).

$\therefore$ Required area $\quad A = \int_0^9 \sqrt{9x}\, dx - \int_0^9 \frac{x^2}{9}\, dx$

$$= \left[3 \cdot \frac{2}{3} \cdot x^{\frac{3}{2}} \right]_0^9 - \left[\frac{1}{9} \cdot \frac{x^3}{3} \right]_0^9$$

$$= 2 \cdot 9^{\frac{3}{2}} - 27$$

$$A = 54 - 27$$

$$= 27 \text{ sq.units}$$

Now, we will see how to find the area bounded by the curve $y = f(x)$, X-axis and lines $x = a$, $x = b$ if $f(x)$ is negative i.e. $f(x) \leq 0$ in $[a, b]$.

Ex. 2 : Find the area bounded by the curve $y = -x^2$, X−axis and lines $x = 1$ and $x = 4$.

Solution : Let A be the area bounded by the curve $y = -x^2$, X−axis and $1 \le x \le 4$.

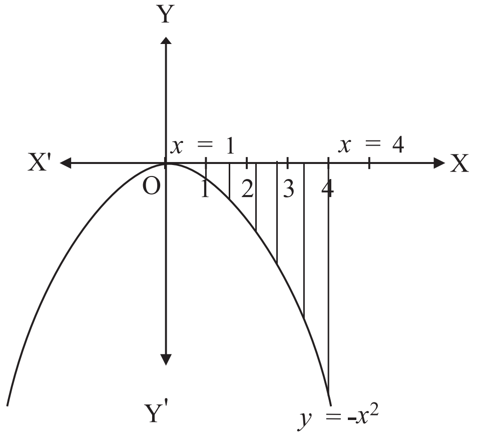

Fig. 5.7

$$\text{The required area} \quad A = \int_1^4 y\,dx$$

$$= \int_1^4 -x^2\ dx$$

$$= \left[-\frac{x^3}{3}\right]_1^4$$

$$= -\frac{64}{3} + \frac{1}{3}$$

$$A = -21,$$

But we consider the area to be positive.

$\therefore$ A $= \left| -21 \right|$sq.units = 21 square units.

Thus, if $f(x) \le 0$ or $f(x) \ge 0$ in $[\ a, b\ ]$ then the area enclosed between $y = f(x)$, X−axis and $x = a, x = b$ is $\left| \int_a^b f(x)\cdot dx \right|$.

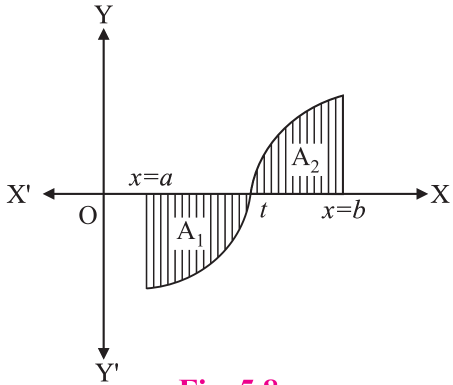

Fig. 5.8

If the area A is divided into two parts A_1 and A_2 such that

A_1 is the part of $a \le x \le t$ where $f(x) \le 0$ and

A_2 is the part of $a \le x \le t$ where $f(x) \ge 0$

then in A_1, the required area is below the X−axis

and in A_2, the required area is above the X-axis.

$$\text{Now the total area} \quad A = A_1 + A_2$$

$$= \left| \int_a^t f(x)\,dx \right| + \left| \int_t^b f(x)\,dx \right|$$

Ex. 3 : Find the area bounded by the line $y = x$, X axis and the lines $x = -1$ and $x = 4$.

Solution : Consider the area A, bounded by straight line $y = x$, X axis and $x = -1, x = 4$.

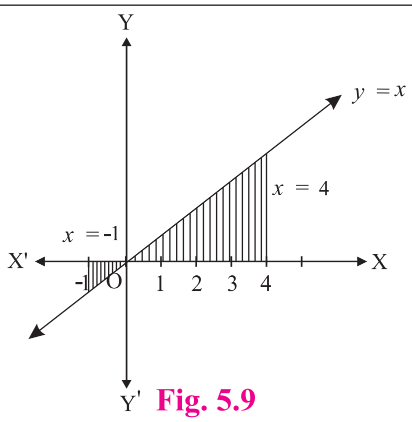

Fig. 5.9

From figure 5.9, A is divided into A_1 and A_2

$$\text{The required area} \quad A_1 = \int_{-1}^0 y\,dx = \int_{-1}^0 x\ dx$$

$$= \left[\frac{x^2}{2}\right]_{-1}^0$$

$$= 0 - \frac{1}{2}$$

$$A_1 = -\frac{1}{2} \text{ square units.}$$

But area is always positive.

$\therefore \quad A_1 = \left| -\dfrac{1}{2} \right| \text{sq.units} = \dfrac{1}{2}$ square units.

$A_2 = \int_0^4 y\, dx = \int_0^4 x\, dx = \left[\dfrac{x^2}{2}\right]_0^4 = \dfrac{4^2}{2} = 8$ square units.

$\therefore \quad$ Required area $\quad A = A_1 + A_2 = \dfrac{1}{2} + 8 = \dfrac{17}{2}$ sq.units

Ex. 4 : Find the area enclosed between the X-axis and the curve $y = \sin x$ for values of x between 0 to 2π.

Solution : The area enclosed between the curve and the X-axis consists of equal area lying alternatively above and below X-axis which are respectively positive and negative.

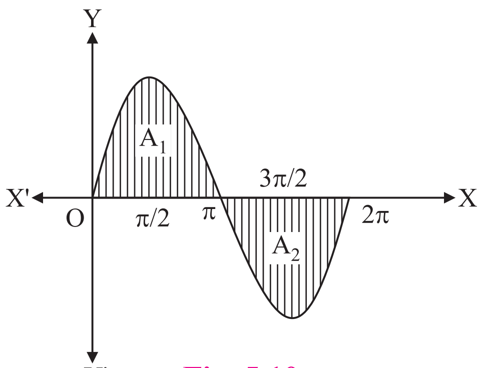

Fig. 5.10

1) Area $\quad A_1 = $ area lying above the X-axis

$$= \int_0^{\pi} \sin x \cdot dx = \left[-\cos x\right]_0^{\pi}$$

$$= -[\cos \pi - \cos 0] = -(-1-1)$$

$$A_1 = 2$$

2) Area $\quad A_2 = $ area lying below the X-axis $= \int_{\pi}^{2\pi} \sin x\, dx = \left[-\cos x\right]_{\pi}^{2\pi} = [-\cos 2\pi - \cos \pi]$

$$= -[-1-(-1)]$$

$$A_2 = -2$$

$\therefore \quad$ Total area $= A_1 + |A_2| = 2 + |(-2)| = 4$ sq.units.

Activity :

Ex. 5 : Find the area enclosed between $y = \sin x$ and X-axis between 0 and 4π.

Ex. 6 : Find the area enclosed between $y = \cos x$ and X-axis between the limits :

(i) $\quad 0 \le x \le \dfrac{\pi}{2}$

(ii) $\quad \dfrac{\pi}{2} \le x \le \pi$

(iii) $\quad 0 \le x \le \pi$

SOLVED EXAMPLES

Ex. 1 : Using integration, find the area of the region bounded by the line $2y + x = 8$, X−axis and the lines $x = 2$ and $x = 4$.

Solution : The required region is bounded by the lines $2y + x = 8$, and $x = 2$, $x = 4$ and X−axis.

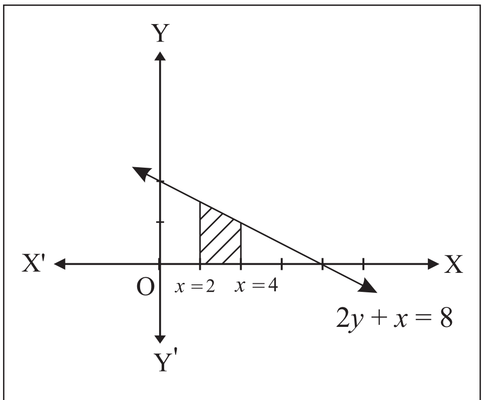

Fig. 5.11

$\therefore \quad y = \dfrac{1}{2}(8 - x)$ and the limits are $x = 2$, $x = 4$.

$$\begin{aligned}
\text{Required area} &= \text{Area of the shaded region} \\
&= \int_{x=2}^{4} y\,dx \\
&= \int_{2}^{4} \frac{1}{2}(8 - x)\,dx \\
&= \frac{1}{2}\left[8x - \frac{x^2}{2}\right]_2^4 \\
&= \frac{1}{2}\left[\left(8\cdot(4) - \frac{4^2}{2}\right) - \left(8\cdot(2) - \frac{2^2}{2}\right)\right] \\
&= 5 \text{ sq. units.}
\end{aligned}$$

Ex. 2 : Find the area of the regions bounded by the following curve, the X−axis and the given lines :

(i) $y = x^2$, $x = 1$, $x = 2$ (ii) $y^2 = 4x$, $x = 1$, $x = 4$, $y \geq 0$

(iii) $y = \sin x$, $x = -\dfrac{\pi}{2}$, $x = \dfrac{\pi}{2}$

Solution : Let A be the required area

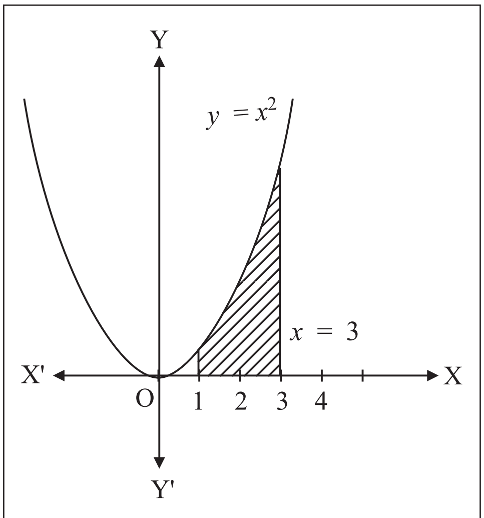

Fig. 5.12

$$\begin{aligned}
\text{(i)} \quad A &= \int_1^3 y\,dx \\
&= \int_1^3 x^2\,dx \\
&= \frac{1}{3}\left[x^3\right]_1^3 \\
&= \frac{1}{3}[27 - 1] \\
A &= \frac{26}{3} \text{ sq. units.}
\end{aligned}$$

(ii) $A = \int_1^4 y\,dx$

$= \int_1^4 2\sqrt{x}\;dx$

$= 2\int_1^4 x^{\frac{1}{2}}\,dx$

$= 2\cdot\frac{2}{3}\left[x^{\frac{3}{2}}\right]_1^4 \qquad = \frac{4}{3}\left[4^{\frac{3}{2}}-1\right]$

$A = \frac{28}{3}$ sq. units.

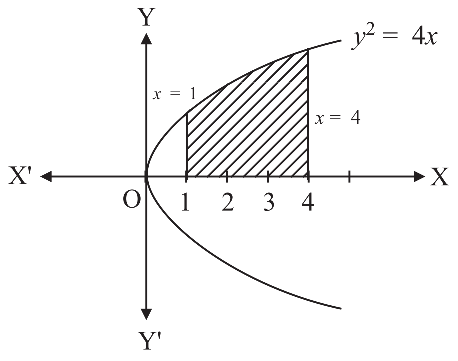

Fig. 5.13

(iii) $A = \int_{-\pi/2}^{\pi/2} y\;dx$

$= \int_{-\pi/2}^{\pi/2} \sin x\,dx$

$= \left|\int_{-\pi/2}^{0} \sin x\cdot dx\right| + \left|\int_{0}^{\pi/2} \sin x\cdot dx\right|$

$= \left|\left[-\cos x\right]_{-\frac{\pi}{2}}^{0}\right| + \left|\left[-\cos x\right]_{0}^{\frac{\pi}{2}}\right|$

$= \left|-\left[\cos 0-\cos\left(\frac{\pi}{2}\right)\right]\right| + \left|\left[-\cos\left(\frac{\pi}{2}\right)+\cos 0\right]\right|$

$= |\,[-1-0] + [\,0+1\,] \;=\; 1+1$

$A = 2$ sq. units.

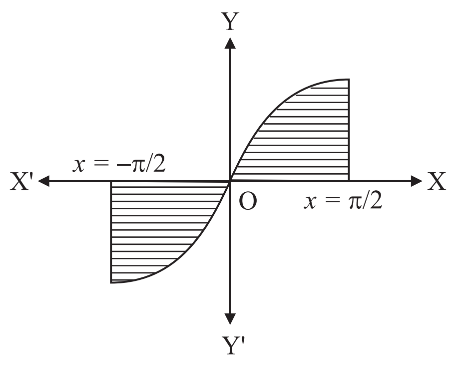

Fig. 5.14

Ex. 3 : Find the area of the region bounded by the parabola $y^2 = 16x$ and the line $x = 4$.

Solution : $y^2 = 16x \qquad \Rightarrow \qquad y = \pm 4\sqrt{x}$

A = Area POCP + Area QOCQ

= 2 (Area POCP)

$= 2\int_0^4 y\cdot dx$

$= 2\int_0^4 4\sqrt{x}\cdot dx$

$A = 8\cdot\frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^4 \qquad = \frac{16}{3}\times 8$

$A = \frac{128}{3}$ sq. units.

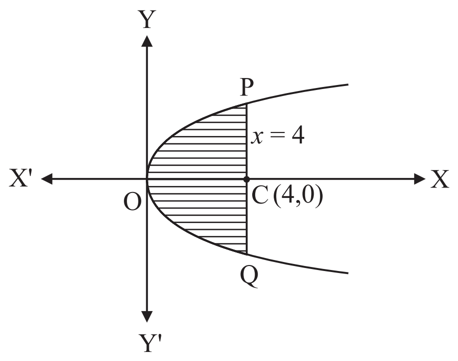

Fig. 5.15

Ex. 4 : Find the area of the region bounded by the curves $x^2 = 16y$, $y = 1$, $y = 4$ and the Y-axis, lying in the first quadrant.

Solution :

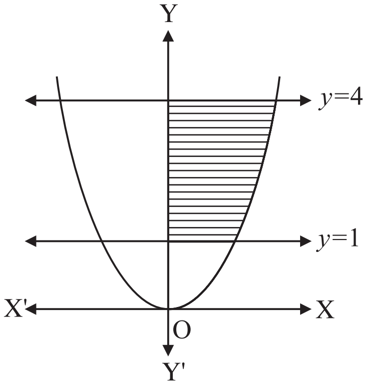

Fig. 5.16

$$\text{Required area} = \int_1^4 x \, dy$$

$$A = \int_1^4 \sqrt{16y} \, dy$$

$$= 4 \int_1^4 \sqrt{y} \cdot dy$$

$$= 4 \cdot \left[\frac{2}{3} \cdot y^{\frac{3}{2}} \right]_1^4$$

$$= \frac{8}{3} \times [8 - 1]$$

$$A = \frac{56}{3} \text{ sq. units.}$$

Ex. 5 : Find the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution : By the symmetry of the ellipse, required area of the ellipse is 4 times the area of the region OPQO. For this region the limit of integration are $x = 0$ and $x = a$.

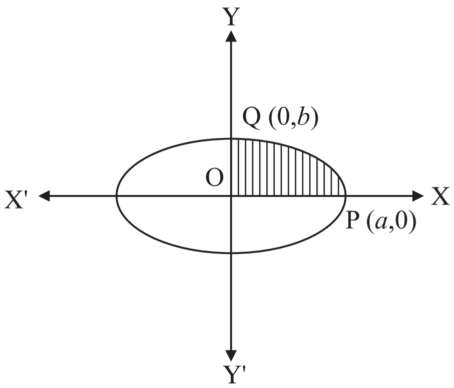

Fig. 5.17

From the equation of ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2 \cdot \left(\frac{a^2 - x^2}{a^2} \right)$$

$$y = \frac{b}{a} \cdot \sqrt{a^2 - x^2} \qquad \text{, In first quadrant, } y > 0$$

$$A = 4 \int_{x=0}^{a} y \, dx$$

$$= \int_0^a \frac{b}{a} \cdot \sqrt{a^2 - x^2} \, dx$$

$$= \frac{4b}{a} \cdot \left[\frac{x}{a} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^a$$

$$= \frac{4b}{a} \cdot \left[\frac{a^2}{2} \cdot \frac{\pi}{2} - 0 \right]_0^a$$

$$A = \pi ab \text{ sq. units}$$

Ex. 6 : Find the area of the region lying between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ where $a > 0$.

Solution : The equations of the parabolas are

$$y^2 = 4ax \quad \ldots \text{(I)}$$

and $$x^2 = 4ay \quad \ldots \text{(II)}$$

From (ii) $y = \dfrac{x^2}{4a}$ substitute in (I)

$$\left(\frac{x^2}{4a}\right)^2 = 4ax$$

$$\Rightarrow x^4 = 64a^3x$$

$\therefore \quad x\,(x^3 - 64a^3) = 0$

$\therefore \quad x\,[x^3 - (4a^3)] = 0$

$\therefore \quad x = 0$ and $x = 4a \qquad \therefore \quad y = 0$ and $y = 4a$

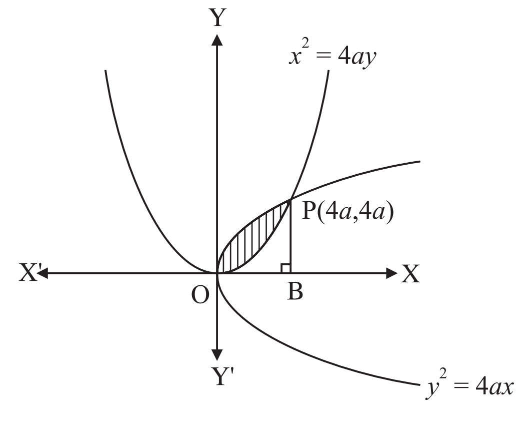

Fig. 5.18

The point of intersection of curves are O (0, 0), P (4a, 4a)

$\therefore$ The required area is in the first quadrant and it is

A = area under the parabola ($y^2 = 4ax$) − area under the parabola ($x^2 = 4ay$)

$$A = \int_0^{4a} \sqrt{4ax}\, dx - \int_0^{4a} \frac{x^2}{4a}\, dx \qquad = \sqrt{4a}\int_0^{4a} x^{\frac{1}{2}}\, dx - \int_0^{4a} \frac{x^2}{4a}\, dx$$

$$= 2\sqrt{a}\cdot\left[\frac{2}{3}\cdot x^{\frac{3}{2}}\right]_0^{4a} - \frac{1}{4a}\cdot\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{a}\cdot\left[4a\,\sqrt{4a} - \frac{1}{4a}\cdot 64a^3\right] = \frac{32}{3}a^2 - \frac{16}{3}a^2 \qquad \therefore A = \frac{16}{3}a^2 \text{ sq. units.}$$

Ex. 7 : Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.

Solution : Required area A = 2 × area of OPQO

$$\therefore \quad A = \int_0^4 x\cdot dy$$

$$A = 2\cdot\int_0^4 \sqrt{y}\cdot dy$$

$$= 2\cdot\left[\frac{2}{3}\cdot y^{\frac{3}{2}}\right]_0^4 \qquad = \left(\frac{4}{3}\times 4^{\frac{3}{2}}\right)$$

$$= \frac{4}{3}\times 8$$

$$A = \frac{32}{3} \text{ sq. units.}$$

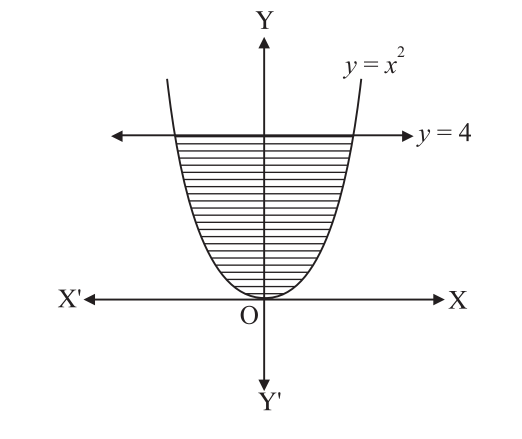

Fig. 5.19

Ex. 8 : Find the area of sector bounded by the circle $x^2 + y^2 = 16$ and the line $y = x$ in the first quadrant.

Solution : Required area A = A (ΔOCB) + A (region ABC)

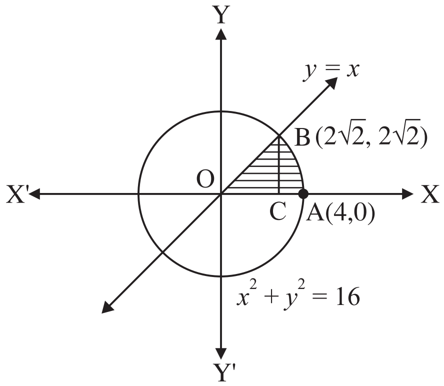

Fig. 5.20

To find,

The point of intersection of $x^2 + y^2 = 16$ . . . (I)

and line $y = x$ . . . (II)

Substitute (II) in (I)

$$\begin{aligned} x^2 + x^2 &= 16 \\ 2x^2 &= 16 \\ x^2 &= 8 \\ x &= \pm 2\sqrt{2}, \qquad y = \pm 2\sqrt{2} \end{aligned}$$

The point of intersection is B $(2\sqrt{2}, 2\sqrt{2})$

$$A = \int_0^{2\sqrt{2}} x \cdot dx + \int_{2\sqrt{2}}^{0} \sqrt{16 - x^2} \cdot dx = \frac{1}{2}\left[x^2\right]_0^{2\sqrt{2}} + \left[\frac{x}{2}\sqrt{16 - x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_{2\sqrt{2}}^{4}$$

$$= \frac{1}{2} \cdot (2\sqrt{2})^2 + \left[8 \sin^{-1} 1 - \left(\frac{2\sqrt{2}}{2}\sqrt{8} + 8 \sin^{-1}\frac{1}{2}\right)\right]$$

$$= 4 + 8 \cdot \frac{\pi}{2} - 4 - 8 \cdot \frac{\pi}{4} \qquad \therefore \quad A = 2\pi \text{ sq. units.}$$

Note that, the required area is $\frac{1}{8}$ times the area of the circle given.

EXERCISE 5.1

(1) Find the area of the region bounded by the following curves, X- axis and the given lines:

(i) $y = 2x, x = 0, x = 5$

(ii) $x = 2y, y = 0, y = 4$

(iii) $x = 0, \ x = 5, y = 0, y = 4$

(iv) $y = \sin x, \ x = 0, x = \frac{\pi}{2}$

(v) $xy = 2, \ x = 1, x = 4$

(vi) $y^2 = x, x = 0, x = 4$

(vii) $y^2 = 16x$ and $x = 0, x = 4$

(2) Find the area of the region bounded by the parabola :

(i) $y^2 = 16x$ and its latus rectum.

(ii) $y = 4 - x^2$ and the X-axis

(3) Find the area of the region included between:

(i) $y^2 = 2x$, line $y = 2x$

(ii) $y^2 = 4x$, line $y = x$

(iii) $y = x^2$ and the line $y = 4x$

(iv) $y^2 = 4ax$ and the line $y = x$

(v) $y = x^2 + 3$ and the line $y = x + 3$

Let us Remember

- The area A, bounded by the curve $y = f(x)$, X-axis and the lines $x = a$ and $x = b$ is given by

  $$A = \int_a^b f(x)\cdot dx = \int_{x=a}^{x=b} f(x)\cdot dx$$

  If the area A lies below the X-axis, then A is negative and in this case we take $|A|$.

- The area A of the region bounded by the curve $x = g(y)$, the Y axis, and the lines $y = c$ and $y = d$ is given by

  $$A = \int_{y=c}^{d} f(x)\cdot dx = \int_{y=c}^{y=d} g(y)\cdot dx$$

- **Tracing of curve :**

  (i) X-axis is an axis of symmetry for a curve C, if $(x, y) \in C \Leftrightarrow (x, -y) \in C$.

  (ii) Y-axis is an axis of symmetry for a curve C, if $(x, y) \in C \Leftrightarrow (-x, y) \in C$.

  (iii) If replacing x and y by $-x$ and $-y$ respectively, the equation of the curve is unchanged then the curve is symmetric about X-axis and Y-axis.

MISCELLANEOUS EXERCISE 5

(I) Choose the correct option from the given alternatives :

(1) The area bounded by the region $1 \le x \le 5$ and $2 \le y \le 5$ is given by

(A) 12 sq. units (B) 8 sq. units (C) 25 sq. units (D) 32 sq. units

(2) The area of the region enclosed by the curve $y = \frac{1}{x}$, and the lines $x = e$, $x = e^2$ is given by

(A) 1 sq. unit (B) $\frac{1}{2}$ sq. unit (C) $\frac{3}{2}$ sq. units (D) $\frac{5}{2}$ sq. units

(3) The area bounded by the curve $y = x^3$, the X-axis and the lines $x = -2$ and $x = 1$ is

(A) -9 sq. units (B) $-\frac{15}{4}$ sq. units (C) $\frac{15}{4}$ sq. units (D) $\frac{17}{4}$ sq. units

(4) The area enclosed between the parabola $y^2 = 4x$ and line $y = 2x$ is

(A) $\frac{2}{3}$ sq. units (B) $\frac{1}{3}$ sq. units (C) $\frac{1}{4}$ sq. units (D) $\frac{3}{4}$ sq. units

(5) The area of the region bounded between the line $x = 4$ and the parabola $y^2 = 16x$ is

(A) $\frac{128}{3}$ sq. units (B) $\frac{108}{3}$ sq. units (C) $\frac{118}{3}$ sq. units (D) $\frac{218}{3}$ sq. units

(6) The area of the region bounded by $y = \cos x$, Y-axis and the lines $x = 0$, $x = 2\pi$ is

(A) 1 sq. unit (B) 2 sq. units (C) 3 sq. units (D) 4 sq. units

(7) The area bounded by the parabola $y^2 = 8x$ the X-axis and the latus rectum is

(A) $\frac{31}{3}$ sq. units (B) $\frac{32}{3}$ sq. units (C) $\frac{32\sqrt{2}}{3}$ sq. units (D) $\frac{16}{3}$ sq. units

(8) The area under the curve $y = 2\sqrt{x}$, enclosed between the lines $x = 0$ and $x = 1$ is

(A) 4 sq. units (B) $\frac{3}{4}$ sq. units (C) $\frac{2}{3}$ sq. units (D) $\frac{4}{3}$ sq. units

(9) The area of the circle $x^2 + y^2 = 25$ in first quadrant is

(A) $\frac{25\pi}{3}$ sq. units (B) 5π sq. units (C) 5 sq. units (D) 3 sq. units

(10) The area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

(A) ab sq. units (B) πab sq. units (C) $\frac{\pi}{ab}$ sq. units (D) πa^2 sq. units

(11) The area bounded by the parabola $y^2 = x$ and the line $2y = x$ is

(A) $\frac{4}{3}$ sq. units (B) 1 sq. units (C) $\frac{2}{3}$ sq. units (D) $\frac{1}{3}$ sq. units

(12) The area enclosed between the curve $y = \cos 3x$, $0 \le x \le \frac{\pi}{6}$ and the X-axis is

(A) $\frac{1}{2}$ sq. units (B) 1 sq. units (C) $\frac{2}{3}$ sq. units (D) $\frac{1}{3}$ sq. units

(13) The area bounded by $y = \sqrt{x}$ and line $x = 2y + 3$, X-axis in first quadrant is

(A) $2\sqrt{3}$ sq. units (B) 9 sq. units (C) $\frac{34}{3}$ sq. units (D) 18 sq. units

(14) The area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the line $\frac{x}{a} + \frac{y}{b} = 1$ is

(A) $\pi\, ab - 2\, ab$ (B) $\frac{\pi ab}{4} - \frac{ab}{2}$ (C) $\pi\, ab - ab$ (D) πab

(15) The area bounded by the parabola $y = x^2$ and the line $y = x$ is

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{6}$ (D) $\frac{1}{12}$

(16) The area enclosed between the two parabolas $y^2 = 4x$ and $y = x$ is

(A) $\frac{8}{3}$ (B) $\frac{32}{3}$ (C) $\frac{16}{3}$ (D) $\frac{4}{3}$

(17) The area bounded by the curve $y = \tan x$, X-axis and the line $x = \frac{\pi}{4}$ is

(A) $\frac{1}{3}\log 2$ (B) $\log 2$ (C) $2\log 2$ (D) $3 \cdot \log 2$

(18) The area of the region bounded by $x^2 = 16y$, $y = 1$, $y = 4$ and $x = 0$ in the first quadrant, is

(A) $\frac{7}{3}$ (B) $\frac{8}{3}$ (C) $\frac{64}{3}$ (D) $\frac{56}{3}$

(19) The area of the region included between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, $(a > 0)$ is given by

(A) $\frac{16\,a^2}{3}$ (B) $\frac{8\,a^2}{3}$ (C) $\frac{4\,a^2}{3}$ (D) $\frac{32\,a^2}{3}$

(20) The area of the region included between the line $x + y = 1$ and the circle $x^2 + y^2 = 1$ is

(A) $\frac{\pi}{2} - 1$ (B) $\pi - 2$ (C) $\frac{\pi}{4} - \frac{1}{2}$ (D) $\pi - \frac{1}{2}$

(II) Solve the following :

(1) Find the area of the region bounded by the following curve, the X-axis and the given lines

(i) $0 \leq x \leq 5$, $0 \leq y \leq 2$ (ii) $y = \sin x$, $x = 0$, $x = \pi$ (iii) $y = \sin x$, $x = 0$, $x = \frac{\pi}{3}$

(2) Find the area of the circle $x^2 + y^2 = 9$, using integration.

(3) Find the area of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ using integration.

(4) Find the area of the region lying between the parabolas.

(i) $y^2 = 4x$ and $x^2 = 4y$ (ii) $4y^2 = 9x$ and $3x^2 = 16y$ (iii) $y^2 = x$ and $x^2 = y$

(5) Find the area of the region in first quadrant bounded by the circle $x^2 + y^2 = 4$ and the x axis and the line $x = y\sqrt{3}$.

(6) Find the area of the region bounded by the parabola $y^2 = x$ and the line $y = x$ in the first quadrant.

(7) Find the area enclosed between the circle $x^2 + y^2 = 1$ and the line $x + y = 1$, lying in the first quadrant.

(8) Find the area of the region bounded by the curve $(y - 1)^2 = 4(x + 1)$ and the line $y = (x - 1)$.

(9) Find the area of the region bounded by the straight line $2y = 5x + 7$, X−axis and $x = 2$, $x = 5$.

(10) Find the area of the region bounded by the curve $y = 4x^2$, Y-axis and the lines $y = 1$, $y = 4$.

6. DIFFERENTIAL EQUATIONS

Let us Study

- Differential Equation
- Formation of differential equation
- Types of differential equation.
- Order and degree of differential equation
- Solution of differential equation
- Application of differential equation.

Let us Recall

- The differentiation and integration of functions and the properties of differentiation and integration.

Let us Learn

6.1.1 Introduction :

In physics, chemistry and other sciences we often have to build mathematical models which involves differential equations. We need to find functions which satisfy those differential equations.

6.1.2 Differential Equation :

Equation which contains the derivative of a function is called a **differential euqation**. The following are differential equations.

(i) $\frac{dy}{dx} = \cos x$ (ii) $\frac{d^2y}{dx^2} + k^2y = 0$ (iii) $\left(\frac{d^2w}{dx^2}\right) - x^2\frac{dw}{dx} + w = 0$

(iv) $\frac{d^2y}{dt^2} + \frac{d^2x}{dt^2} = x$, here x and y are functions of 't'.

(v) $\frac{d^3y}{dx^3} + x\frac{dy}{dx} - 4xy = 0$, here x is a function of y. (vi) $r\frac{dr}{d\theta} + \cos\theta = 5$

6.2 Order and degree of the differential equation :

The order of a differential equation is the highest order of the derivative appearing in the equation.

The degree of differential equation is the power of the highest ordered derivative present in the equation. To find the degree of the differential equation, we need to have a positive integer as the index of each derivative.

SOLVED EXAMPLES

Ex. 1 : Find order and degree of the following differential equations.

(i) $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+4y=0$

Solution : It's order is 2 and degree is 1.

(ii) $\left(\frac{d^3y}{dx^3}\right)^2+xy\frac{dy}{dx}-2x+3y+7=0$

Solution : It's order is 3 and degree is 2.

(iii) $r\frac{dr}{d\theta}+\cos\theta=5$

Solution : It's order is 1 and degree is 1.

(iv) $\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^2=e^x$

Solution : It's order is 2 and degree is 2.

(v) $\frac{dy}{dx}+\frac{3xy}{\frac{dy}{dx}}=\cos x$

Solution : This equation expressed as

$$\left(\frac{dy}{dx}\right)^2+3xy=\cos x\left(\frac{dy}{dx}\right)$$

It's order is 1 and degree is 2.

(vi) $\sqrt{1+\frac{1}{\left(\frac{dy}{dx}\right)^2}}=\left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}}$

Solution : This equation can be expressed as

$$1+\frac{1}{\left(\frac{dy}{dx}\right)^2}=\left(\frac{d^2y}{dx^2}\right)^3$$

$$\therefore\quad \left(\frac{dy}{dx}\right)^2+1=\left(\frac{d^2y}{dx^2}\right)^3\left(\frac{dy}{dx}\right)^2$$

It's order is 2 and degree is 3.

(vii) $\frac{d^4y}{dx^4}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^3$

Solution : It's order is 4 and degree is 1.

(viii) $e^{\frac{dy}{dx}}+\frac{dy}{dx}=x$

Solution : It's order is 1, but equation can not be expressed as a polynomial differential equation.

$\therefore$ The degree is not defined.

(ix) $$\begin{vmatrix} x^3 & y^3 & 3 \\ 2x^2 & 3y\frac{dy}{dx} & 0 \\ 5x & 2\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right] & 0 \end{vmatrix}=0$$

Solution : $\therefore\quad x^3[0-0]-y^2[0-0]+3\left\{4x^2\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-15xy\frac{dy}{dx}\right\}=0$

$\therefore\quad 4x^2y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2-15xy\frac{dy}{dx}=0$ $\qquad\therefore$ Its order is 2 and degree is 1.

Notes : (1) $\frac{dy}{dx}$ is also denoted by y', $\frac{d^2y}{dx^2}$ is also denoted by y'', $\frac{d^3y}{dx^3}$ is also by y''' and so-on.

(2) The order and degree of a differential equation are always positive integers.

EXERCISE 6.1

(1) Determine the order and degree of each of the following differential equations.

(i) $\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right) + y = 2\sin x$

(ii) $\sqrt[3]{1 + \left(\dfrac{dy}{dx}\right)^2} = \dfrac{d^2y}{dx^2}$

(iii) $\dfrac{dy}{dx} = \dfrac{2\sin x + 3}{\dfrac{dy}{dx}}$

(iv) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + x = \sqrt{1 + \dfrac{d^3y}{dx^3}}$

(v) $\dfrac{d^2y}{dt^2} + \left(\dfrac{dy}{dt}\right)^2 + 7x + 5 = 0$

(vi) $(y''')^2 + 3y'' + 3xy' + 5y = 0$

(vii) $\left(\dfrac{d^2y}{dx^2}\right)^2 + \cos\left(\dfrac{dy}{dx}\right) = 0$

(viii) $\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = 8\dfrac{d^2y}{dx^2}$

(ix) $\left(\dfrac{d^3y}{dx^3}\right)^{\frac{1}{2}} - \left(\dfrac{dy}{dx}\right)^{\frac{1}{3}} = 20$

(x) $x + \dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{d^2y}{dx^2}\right)^2}$

6.3 Formation of Differential Equation :

From the given information, we can form the differential equation. Sometimes we need to eliminate the arbitrary constants from a given relation. It may be done by differentiation.

SOLVED EXAMPLES

Ex. 1 : Obtain the differential euqation by eliminating the arbitrary constants from the following :

(i) $y^2 = 4ax$ (ii) $y = Ae^{3x} + Be^{-3x}$ (iii) $y = (c_1 + c_2x)\,e^x$

(iv) $y = c^2 + \dfrac{c}{x}$ (v) $y = c_1e^{3x} + c_2e^{2x}$

Solution :

(i) $y^2 = 4ax \ldots (1)$

Here a is the arbitraty constant, we differentiate *w. r. t. x*,

$\therefore \dfrac{dy}{dx} = 4a$

then eq. (1) gives

$y^2 = \left(\dfrac{dy}{dx}\right)x$ is required differential equation.

(ii) $y = Ae^{3x} + Be^{-3x} \ \ldots (1)$

Here A and B are arbitrary constants.

Differentiate *w. r. t. x*, we get

$\therefore \dfrac{dy}{dx} = 3Ae^{3x} - 3Be^{-3x}$

again Differentiate *w. r. t. x*, we get

$\dfrac{d^2y}{dx^2} = 3 \times 3Ae^{3x} - 3 \times 3Be^{-3x}$

$= 9\left(Ae^{3x} + Be^{-3x}\right) = 9y \quad \ldots$ from eq.(1)

$\therefore \dfrac{d^2y}{dx^2} = 9y$

(iii) $y = (c_1 + c_2 x)\, e^x \ldots (1)$

Here c_1 and c_2 are arbitrary constants.

Differentiate $w.\ r.\ t.\ x$, we get

$$\therefore \quad \frac{dy}{dx} = (c_1 + c_2 x)\, e^x + c_2 e^x$$

$$\therefore \quad \frac{dy}{dx} = y + c_2 e^x \quad \ldots (2) \qquad \ldots \text{from eq.(1)}$$

Again differentiate $w.\ r.\ t.\ x$, we get

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + c_2 e^x$$

$$\therefore \quad c_2 e^x = \frac{d^2y}{dx^2} - \frac{dy}{dx}$$

put in eq.(2)

$$\frac{dy}{dx} = y + \frac{d^2y}{dx^2} - \frac{dy}{dx}$$

$$\therefore \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

(iv) $y = c^2 + \dfrac{c}{x} \qquad \ldots (1)$

Differentiate $w.\ r.\ t.\ x$, we get

$$\therefore \quad \frac{dy}{dx} = -\frac{c}{x^2}$$

$$\therefore \quad c = -x^2 \frac{dy}{dx}$$

then eq.(1) gives

$$y = \left[-x^2\left(\frac{dy}{dx}\right)^2\right]^2 - x^2\frac{dy}{dx} \times \frac{1}{x}$$

$$\therefore \quad y = x^4\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx}$$

$$\therefore \quad x^4\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} - y = 0$$

(v) $y = c_1 e^{3x} + c_2 e^{2x} \qquad \ldots (\mathrm{I})$

Differentiate $w.\ r.\ t.\ x$, we get

$$\therefore \quad \frac{dy}{dx} = 3c_1 e^{3x} + 2c_2 e^{2x} \qquad \ldots (\mathrm{II})$$

Again differentiate $w.\ r.\ t.\ x$, we get

$$\frac{d^2y}{dx^2} = 9c_1 e^{3x} + 4c_2 e^{2x} \qquad \ldots (\mathrm{III})$$

As equations (I), (II) and (III) in $c_1 e^{3x}$ and $c_2 e^{2x}$ are consistent

$$\therefore \quad \begin{vmatrix} y & 1 & 1 \\ \dfrac{dy}{dx} & 3 & 2 \\ \dfrac{d^2y}{dx^2} & 9 & 4 \end{vmatrix} = 0$$

$$\therefore \quad y(12 - 18) - 1\left(4\frac{dy}{dx} - 2\frac{d^2y}{dx^2}\right) + 1\left(9\frac{dy}{dx} - 3\frac{d^2y}{dx^2}\right) = 0$$

$$\therefore \quad -6y - 4\frac{dy}{dx} + 2\frac{d^2y}{dx^2} + 9\frac{dy}{dx} - 3\frac{d^2y}{dx^2} = 0$$

$$\therefore \quad -\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 6y = 0 \qquad \therefore \quad \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$$

Ex. 2 : The rate of decay of the mass of a radioactive substance any time is k times its mass at that time, form the differential equation satisafied by the mass of the substance.

Solution : Let m be the mass of a radioactive substance time 't'

$\therefore$ The rate of of decay of mass is $\dfrac{dm}{dt}$

Here $\dfrac{dm}{dt} \propto m$

$\therefore$ $\dfrac{dm}{dt} = mk$, where $k < 0$

is the required differential equation.

Ex. 3 : Form the differential equation of family of circles above the X-axis and touching the X-axis at the origin.

Solution : Let c (a, b) be the centre of the circle touching X-axis at the origin $(b < 0)$.

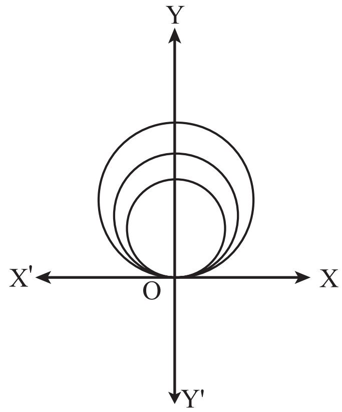

Fig. 6.1

The radius of the circle of b.

The equation of the circle is

$(x - 0)^2 + (y - b)^2 = b^2$

$\therefore$ $x^2 + y^2 - 2by + b^2 = b^2$

$\therefore$ $x^2 + y^2 - 2by = 0$ $\quad \ldots$ (I)

Differentiate $w.\ r.\ t.\ x$, we get

$$2x + 2y\left(\frac{dy}{dx}\right) - 2b\left(\frac{dy}{dx}\right) = 0$$

$$\therefore \quad x + (y - b)\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{x}{\left(\dfrac{dy}{dx}\right)} + (y - b) = 0 \qquad \therefore \quad b = y + \frac{x}{\left(\dfrac{dy}{dx}\right)} \quad \ldots \text{(II)}$$

From eq. (I) and eq. (II)

$$\therefore \quad x^2 + y^2 - 2\left[y + \frac{x}{\left(\dfrac{dy}{dx}\right)}\right]y = 0$$

$$\therefore \quad x^2 + y^2 - 2y^2 - \frac{2xy}{\left(\dfrac{dy}{dx}\right)} = 0$$

$$\therefore \quad x^2 - y^2 = \frac{2xy}{\left(\dfrac{dy}{dx}\right)}$$

$$\therefore \quad (x^2 - y^2)\frac{dy}{dx} = 2xy$$

is the required differential equation.

Activity : Form the differential equation of family of circles touching Y-axis at the origin and having their centres on the X-axis.

Ex. 4 : A particle is moving along the X-axis. Its acceleration at time t is proportional to its velocity at that time. Find the differential equation of the motion of the particle.

Solution : Let s be the displacement of the particle at time 't'.

Its velocity and acceleration are $\dfrac{ds}{dt}$ and $\dfrac{d^2s}{dt^2}$ respectively.

Here $\dfrac{d^2s}{dt^2} \propto \dfrac{ds}{dt}$

$$\therefore \quad \frac{d^2s}{dt^2} = k\frac{ds}{dt}, \qquad \text{(where } k \text{ is constant} \neq 0)$$

is the required differential equation.

EXERCISE 6.2

(1) Obtain the differential equations by elliminating arbitrary constants c_1 and c_2.

(i) $x^3 + y^3 = 4ax$

(ii) $Ax^2 + By^2 = 1$

(iii) $y = A\cos(\log x) + B\sin(\log x)$

(iv) $y^2 = (x + c)^3$

(v) $y = Ae^{5x} + Be^{-5x}$

(vi) $(y - a)^2 = 4(x - b)$

(vii) $y = a + \dfrac{a}{x}$

(viii) $y = c_1 e^{2x} + c_2 e^{5x}$

(ix) $c_1 x^3 + c_2 y^2 = 5$

(x) $y = e^{-2x}(A\cos x + B\sin x)$

(2) Form the differential equation of family of lines having intercepts a and b on the co-ordicate ares respectively.

(3) Find the differential equation of all parabolas having length of latus rectum $4a$ and axis is parallel to the X-axis.

(4) Find the differential euqation of an ellipse whose major axis is twise its minor axis.

(5) Form the differential equation of family of lines parallel to the line $2x + 3y + 4 = 0$

(6) Find the differential equations of all circles having radius 9 and centre at point $A(h, k)$.

(7) Form the differential equation of all parabolas whose axis is the X-axis.

6.4 Solution of differential equation :

Verify that

$$y = a \sin x \text{ and } y = b \cos x$$

are solutions of the differential equation, where a and b are any constants.

Also $y = a \sin x + b \cos x$ is a solution of the equation.

Here $\sin x$ and $\cos x$ are particular solutions where as $a \sin x + b \cos x$ is the general solution which describes all possible solutions.

A solution which can be otained from the general solution by giving particular values to the arbitarary constants is called a **particular solution**.

Therefore the differential equation has infinitely many solutions.

SOLVED EXAMPLES

Ex. 1 : Verify that : $y \sec x = \tan x + c$ is a solution of the differential equation $\frac{dy}{dx} + y \tan x = \sec x$.

Solution : Here $y \sec x = \tan x + c$

Differentiate *w. r. t. x*, we get

$$y \sec x \tan x + \sec x \frac{dy}{dx} = \sec^2 x$$

$$\therefore \quad \frac{dy}{dx} + y \tan x = \sec x$$

Hence $y \sec x = \tan x + c$

is a solution of the differential equation

$$\frac{dy}{dx} + y \tan x = \sec x$$

Ex. 2 : Verify that : $y = \log x + c$ is a solution of the differential equation $x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$.

Solution : Here $y = \log x + c$

Differentiate *w. r. t. x*, we get

$$\frac{dy}{dx} = \frac{1}{x}$$

$$\therefore \quad x \frac{dy}{dx} = 1$$

Differentiate *w. r. t. x*, we get

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} \times 1 = 0$$

$$\therefore \quad x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

$y = \log x + c$ is the solution of

$$x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$$

Consider the example :

$$\frac{dy}{dx} = x^2y + y$$

$$\therefore \qquad \frac{1}{y} \cdot \frac{dy}{dx} = x^2 + 1$$

We can consider x and y both as variables and write this as

$$\frac{dy}{y} = (x^2 + 1) \cdot dx$$

Now we can integrate L.H.S. *w. r. t.* y and R.H.S. *w. r. t.* x, then we get

$$\therefore \qquad \log y = \frac{x^3}{3} + x + c$$

This integration is obtained by separating the variables.

It helps to examine the equation and find out if such a separation is possible.

The above method is known as the method of separation of variables.

In general, if the given differential equation can be written as

$$f(x)\, dx = g(y)\, dy$$

then this method is applicable.

SOLVED EXAMPLES

Ex. 1 : Find the general solution of the following differential equations :

(i) $\dfrac{dy}{dx} = x\sqrt{25 - x^2}$ (ii) $\dfrac{dx}{dt} = \dfrac{x \log x}{t}$

Solution :

(i) $\dfrac{dy}{dx} = x\sqrt{25 - x^2}$

$\therefore \quad dy = x\sqrt{25 - x^2} \cdot dx$

Integrating both sides, we get

$$\int dy = \int \sqrt{25 - x^2} \cdot x \cdot dx \qquad \ldots \text{(I)}$$

Put $25 - x^2 = t$

$\therefore \quad -2x \cdot dx = dt$

$\therefore \quad x \cdot dx = -\dfrac{dt}{2}$

Eq. (I) becomes, $\int dy = \int \sqrt{t}\left(-\dfrac{dt}{2}\right)$

$\therefore \quad 2\int dy = -\int \sqrt{t} \cdot dt$

$\therefore \quad 2\int dy + \int t^{\frac{1}{2}} \cdot dt = 0$

$\therefore \quad 2y + \dfrac{t^{\frac{3}{2}}}{\frac{3}{2}} = c_1$

$\therefore \quad 2y + \dfrac{2}{3}t^{\frac{3}{2}} = c_1$

$\therefore \quad 6y + 2t^{\frac{3}{2}} = 3c_1$

$\therefore \quad 6y + 2(25 - x^2)^{\frac{3}{2}} = c \qquad \ldots [c = 3c_1]$

(ii) $\dfrac{dx}{dt} = \dfrac{x \log x}{t}$

$\therefore \quad \dfrac{dx}{x \log x} = \dfrac{dt}{t}$

Integrating both sides, we get

$\displaystyle\int \frac{dx}{x \log x} = \int \frac{dt}{t}$

$\therefore \quad \log(\log x) = \log(t) + \log c$

$\therefore \quad \log(\log x) = \log(tc)$

$\therefore \quad \log x = ct$

$\therefore \quad e^{ct} = x$

Ex. 2 : Find the particular solution with given initial conditions :

(i) $\dfrac{dy}{dx} = e^{2y}\cos x$ when $x = \dfrac{\pi}{6}, y = 0$

(ii) $\dfrac{y-1}{y+1} + \dfrac{x-1}{x+1} \cdot \dfrac{dy}{dx} = 0$, when $x = y = 2$

Solution :

(i) $\dfrac{dy}{dx} = e^{2y}\cos x$

$\therefore \quad \dfrac{dy}{e^{2y}} = \cos x \cdot dx$

$\therefore \quad e^{-2y} \cdot dy = \cos x \cdot dx$

Integrating both sides, we get

$\therefore \quad \int e^{-2y} \cdot dy = \int \cos x \cdot dx$

$\therefore \quad \dfrac{e^{-2y}}{-2} = \sin x + c \qquad \ldots \text{(I)}$

When $x = \dfrac{\pi}{6}, y = 0$. So eq. (1), becomes

$\therefore \quad \dfrac{e^{0}}{-2} = \sin\dfrac{\pi}{6} + c \quad \therefore -\dfrac{1}{2} = \dfrac{1}{2} + c$

$\therefore \quad -\dfrac{1}{2} - \dfrac{1}{2} = c \qquad \therefore c = -1$

(Given initial condition determines the value of c)

Put in eq. (1), we get

$\therefore \quad \dfrac{e^{-2y}}{-2} = \sin x - 1$

$\therefore \quad -e^{-2y} = 2\sin x - 2$

$\therefore \quad e^{2y}(2\sin x - 2) + 1 = 0$ is the required particular solution.

(ii) $\dfrac{y-1}{y+1} + \dfrac{x-1}{x+1} \cdot \dfrac{dy}{dx} = 0$

$\therefore \quad \dfrac{x+1}{x-1} \cdot dx + \dfrac{y+1}{y-1} \cdot dy = 0$

$\therefore \quad \dfrac{(x-1)+2}{x-1} \cdot dx + \dfrac{(y-1)+2}{y-1} \cdot dy = 0$

$\therefore \quad \left(1 + \dfrac{2}{x-1}\right) \cdot dx + \left(1 + \dfrac{2}{y-1}\right) \cdot dy = 0$

Integrating , we get

$\displaystyle\int dx + 2\int \frac{dx}{x-1} + \int dy + 2\int \frac{dy}{y-1} = 0$

$\therefore \quad x + 2\log(x-1) + y + 2\log(y-1) = c$

$\therefore \quad x + y + 2\log[(x-1)(y-1)] = c \qquad \ldots \text{(I)}$

When $x = 2, y = 2$. So eq. (I), becomes

$\therefore \quad 2 + 2 + 2\log[(2-1)(2-1)] = c$

$\therefore \quad 4 + 2\log(1 \times 1) = c$

$\therefore \quad 4 + 2\log 1 = c$

$\therefore \quad 4 + 2(0) = c$

$\therefore \quad c = 4$ Put in eq. (I), we get

$\therefore \quad x + y + 2\log[(x-1)(y-1)] = 4$ is required particular solution.

Ex. 3 : Reduce each of the following differential equations to the separted variable form and hence find the general solution.

(i) $1 + \frac{dy}{dx} = \text{cosec}\,(x + y)$

(ii) $\frac{dy}{dx} = (4x + y + 1)^2$

Solution :

(i) $1 + \frac{dy}{dx} = \text{cosec}\,(x + y) \quad \ldots \text{(I)}$

Put $x + y = u$

$\therefore \quad 1 + \frac{dy}{dx} = \frac{du}{dx}$

Given differential equation becomes

$\frac{du}{dx} = \text{cosec}\, u$

$\therefore \quad \frac{du}{\text{cosec}\, u} = dx$

$\therefore \quad \sin u \cdot du = dx$

Integrating both sides, we get

$\therefore \quad \int \sin u \cdot du = \int dx$

$\therefore \quad -\cos u = x + c$

$\therefore \quad x + \cos u + c = 0$

$\therefore \quad x + \cos (x + y) + c = 0 \quad \ldots (\because x + y = u)$

(ii) $\frac{dy}{dx} = (4x + y + 1)^2 \quad \ldots \text{(I)}$

Put $4x + y + 1 = u$

$\therefore \quad 4 + \frac{dy}{dx} = \frac{du}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{du}{dx} - 4$

Given differential equation becomes

$\frac{du}{dx} - 4 = u^2$

$\therefore \quad \frac{du}{dx} = u^2 + 4$

$\therefore \quad \frac{du}{u^2 + 4} = dx$

Integrating both sides, we get

$\therefore \quad \int \frac{du}{u^2 + 4} = \int dx$

$\therefore \quad \frac{1}{2} \tan^{-1}\left(\frac{u}{2}\right) = x + c_1$

$\therefore \quad \tan^{-1}\left(\frac{u}{2}\right) = 2x + 2c_1$

$\therefore \quad \tan^{-1}\left(\frac{4x + y + 1}{2}\right) = 2x + c \quad \ldots [2c_1 = c]$

EXERCISE 6.3

(1) In each of the following examples verity that the given expression is a solution of the corresponding differential equation.

(i) $xy = \log y + c\ ;\ \frac{dy}{dx} = \frac{y^2}{1 - xy}$

(ii) $y = (\sin^{-1} x)^2 + c\ ;\ (1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$

(iii) $y = e^{-x} + Ax + B\ ;\ e^x \frac{d^2y}{dx^2} = 1$

(iv) $y = x^m\ ;\ x^2 \frac{d^2y}{dx^2} - mx\frac{dy}{dx} + my = 0$

(v) $y = a + \frac{b}{x}\ ;\ x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = 0$

(vi) $y = e^{ax}\ ;\ x\frac{dy}{dx} = y \log y$

(2) Solve the following differential equations.

(i) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$

(ii) $\log\left(\dfrac{dy}{dx}\right) = 2x + 3y$

(iii) $y - x\dfrac{dy}{dx} = 0$

(iv) $\sec^2 x \cdot \tan y \cdot dx + \sec^2 y \cdot \tan x \cdot dy = 0$

(v) $\cos x \cdot \cos y \cdot dy - \sin x \cdot \sin y \cdot dx = 0$

(vi) $\dfrac{dy}{dx} = -k$, where k = constant.

(vii) $\dfrac{\cos^2 y \cdot dy}{x} + \dfrac{\cos^2 x \cdot dx}{y} = 0$

(viii) $y^3 - \dfrac{dy}{dx} = x^2\dfrac{dy}{dx}$

(ix) $2e^{x+2y} \cdot dx - 3dy = 0$

(x) $\dfrac{dy}{dx} = e^{x+y} + x^2 e^y$

(3) For each of the following differential equations find the particular solution satisfying the given condition.

(i) $3e^x \tan y \cdot dx + (1 + e^x) \sec^2 y \cdot dy = 0$, when $x = 0$, $y = \pi$.

(ii) $(x - y^2 x) \cdot dx - (y + x^2 y) \cdot dy = 0$, when $x = 2$, $y = 0$.

(iii) $y(1 + \log x)\dfrac{dx}{dy} - x \log x = 0$, $y = e^2$, when $x = e$.

(iv) $(e^y + 1)\cos x + e^y \sin x \dfrac{dy}{dx} = 0$, when $x = \dfrac{\pi}{6}$, $y = 0$.

(v) $(x+1)\dfrac{dy}{dx} - 1 = 2e^{-y}$, $y = 0$, $x = 1$

(vi) $\cos\left(\dfrac{dy}{dx}\right) = a$, $a \in 1^2$, $y(0) = 2$

(4) Reduce each of the following differential to the variable separable form and hence solve.

(i) $\dfrac{dy}{dx} = \cos(x + y)$

(ii) $(x - y)^2\dfrac{dy}{dx} = a^2$

(iii) $x + y\dfrac{dy}{dx} = \sec(x^2 + y^2)$

(iv) $\cos^2(x - 2y) = 1 - 2\dfrac{dy}{dx}$

(v) $(2x - 2y + 3)\,dx - (x - y + 1)\,dy = 0$, when $x = 0$, $y = 1$.

6.4.1 Homogeneous differential :

Recall that the degree of a term is the sum of the degrees in all variables in the equation, eg. : degree of $3x^2y^2z$ is 5. If all terms have the same degree, the equation is called **homogeneous differential equation.**

For example : (i) $x + y\dfrac{dy}{dx} = 0$ is a homogeneous differential equation of degree 1.

(ii) $x^3y + xy^3 + x^2y^2\dfrac{dy}{dx} = 0$ is a homogeneous differential equation of degree 4.

(iii) $x\dfrac{dy}{dx} + x^2y = 0$

(iv) $xy\dfrac{dy}{dx} + y^2 + 2x = 0$

(iii) and (iv) are not homogeneous differential equations.

To solve the homogeneous differential equation, we use the substitution $y = vx$ or $u = vy$.

SOLVED EXAMPLES

Ex. 1 : Solve the following differential equations :

(i) $x^2y\cdot dx - (x^3+y^3)\cdot dy = 0$ (ii) $x\dfrac{dy}{dx} = x\tan\left(\dfrac{y}{x}\right) + y$ (iii) $\dfrac{dy}{dx} = \dfrac{y+\sqrt{x^2+y^2}}{x}$

Solution :

(i) $x^2y\cdot dx - (x^3+y^3)\cdot dy = 0$

$\therefore \quad x^2y - (x^3+y^3)\dfrac{dy}{dx} = 0 \qquad \ldots (I)$

This is homogeneous Differential equation.

Put $y = vx \qquad \ldots (II)$

Differentiate $w.\ r.\ t.\ x$, we get

$\therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx} \qquad \ldots (III)$

Put (II) and (III) in Eq. (I), it becomes,

$x^2\cdot vx - (x^3 + v^3x^3)\left(v + x\dfrac{dv}{dx}\right) = 0$

divide by x^3, we get

$v - (1+v^3)\left(v + x\dfrac{dv}{dx}\right) = 0$

$\therefore \quad \cancel{v} - \cancel{v} - x\dfrac{dv}{dx} - v^4 - v^3x\dfrac{dv}{dx} = 0$

$\therefore \quad -x(1+v^3)\dfrac{dv}{dx} = v^4$

$\therefore \quad \dfrac{1+v^3}{v^4}\cdot dv = -\dfrac{dx}{x}$

$\therefore \quad \dfrac{1+v^3}{v^4}\cdot dv = -\dfrac{dx}{x}$

$\therefore \quad \left(\dfrac{1}{v^4} - \dfrac{v^3}{v^4}\right)dv + \dfrac{dx}{x} = 0$

integrating eq., we get

$\therefore \quad \int v^{-4}\,dv + \int\dfrac{dv}{v} + \int\dfrac{dx}{x} = c_1$

$\therefore \quad \dfrac{v^{-3}}{-3} + \log(v) + \log(x) = c_1$

$\therefore \quad \log(vx) = c_1 + \dfrac{v^{-3}}{3} \qquad \therefore \log(y) = c_1 + \dfrac{1}{3}\cdot\dfrac{v^{-3}}{x^{-3}}$

$\therefore \quad 3\log(y) = 3c_1 + \dfrac{x^3}{y^3}$

$\therefore \quad 3\log y = \dfrac{x^3}{y^3} + c \qquad \ldots$ where $c = 3c_1$

(ii) $x\dfrac{dy}{dx} = x\tan\left(\dfrac{y}{x}\right) + y \qquad \ldots (I)$

This is homogeneous Differential equation.

Put $y = vx \qquad \ldots (II)$

Differentiate $w.\ r.\ t.\ x$, we get

$\therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx} \qquad \ldots (III)$

Put (II) and (III) in Eq. (I), it becomes,

$x\left(v + x\dfrac{dv}{dx}\right) = x\tan\left(\dfrac{vx}{x}\right) + vx$

divide by x, we get

$\therefore \quad v + x\dfrac{dv}{dx} = \tan v + v$

$\therefore \quad x\dfrac{dv}{dx} = \tan v$

$\therefore \quad \dfrac{dv}{\tan v} = \dfrac{dx}{x}$

integrating eq., we get

$\therefore \quad \int\cot v\,dv = \int\dfrac{dx}{x}$

$\therefore \quad \log(\sin v) = \log(x) + \log c$

$\therefore \quad \log(\sin v) = \log(x\times c)$

$\therefore \quad \sin v = cx$

$\therefore \quad \sin\left(\dfrac{y}{x}\right) = cx$ is the solution.

(iii) $\dfrac{dy}{dx} = \dfrac{y + \sqrt{x^2 + y^2}}{x}$ . . . (I)

Solution : It is homogeneous differential equation.

Put $y = vx$ . . . (II)

Differentiate *w. r. t. x*, we get

$\therefore \quad \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ . . . (III)

Put (II) and (III) in Eq. (I), it becomes,

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x}$$

$$\therefore \quad \cancel{v} + x\frac{dv}{dx} = \cancel{v} + \sqrt{1 + v^2}$$

$$\therefore \quad x\frac{dv}{dx} = \sqrt{1 + v^2}$$

$$\therefore \quad \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x} \qquad . . . (IV)$$

integrating eq. (IV), we get

$$\therefore \quad \int \frac{dv}{\sqrt{1 + v^2}} = \int \frac{dx}{x}$$

$$\therefore \quad \log\left(v + \sqrt{1 + v^2}\right) = \log(x) + \log c$$

$$\therefore \quad \log\left(v + \sqrt{1 + v^2}\right) = \log(cx)$$

$$\therefore \quad v + \sqrt{1 + v^2} = cx$$

$$\therefore \quad \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = cx$$

$\therefore \quad y + \sqrt{x^2 + y^2} = cx^2$ is the solution.

EXERCISE 6.4

I. Solve the following differential equations :

(1) $x \sin\left(\dfrac{y}{x}\right) dy = \left[y \sin\left(\dfrac{y}{x}\right) - x\right] dx$

(2) $(x^2 - y^2)\, dx - 2xy \cdot dy = 0$

(3) $\left(1 + 2e^{\frac{x}{y}}\right) + 2e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right)\dfrac{dy}{dx} = 0$

(4) $y^2 \cdot dx + (xy + x^2)\, dy = 0$

(5) $(x^2 - y^2)\, dx + 2xy \cdot dy = 0$

(6) $\dfrac{dy}{dx} + \dfrac{x - 2y}{2x - y} = 0$

(7) $x\dfrac{dy}{dx} - y + x \sin\left(\dfrac{y}{x}\right) = 0$

(8) $\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \dfrac{x}{y}\right) dy = 0$

(9) $y^2 - x^2\dfrac{dy}{dx} = xy\dfrac{dy}{dx}$

(10) $xy\dfrac{dy}{dx} = x^2 + 2y^2,\ y(1)=0$

(11) $x\, dy + 2y \cdot dx = 0$, when $x = 2, y = 1$

(12) $x^2\dfrac{dy}{dx} = x^2 + xy + y^2$

(13) $(9x + 5y)\, dy + (15x + 11y)\, dx = 0$

(14) $(x^2 + 3xy + y^2)\, dx - x^2\, dy = 0$

(15) $(x^2 + y^2)\, dx - 2xy \cdot dy = 0$

6.4.2 Linear Differential Equation :

The differential equation of the type, $\frac{dy}{dx} + Py = Q$ (where P, Q are functions of x.)

is called **linear differential equation.**

To get the solution of equation, multiply the equation by $e^{\int Pdx}$, which is helping factor here.

We get,

$$e^{\int Pdx}\left[\frac{dy}{dx} + Py\right] = Q \cdot e^{\int Pdx}$$

Note that, $\frac{d}{dx}\left[y \cdot e^{\int Pdx}\right] = \left[\frac{dy}{dx} + y \cdot P\right] \cdot e^{\int Pdx}$

$\therefore$ $\frac{d}{dx}\left[y \cdot e^{\int Pdx}\right] = Q \cdot e^{\int Pdx}$

$\therefore$ $\int Q \cdot e^{\int Pdx} \cdot dx = y \cdot e^{\int Pdx}$

Hence, $y \cdot e^{\int Pdx} = \int Q \cdot \left(e^{\int Pdx}\right) dx + c$ is the solution of the given equation

Here $e^{\int Pdx}$ is called the integrating factor. (I.F.)

Note : For the linear differential equation.

$\frac{dx}{dy} + py = Q$ (where P, Q are constants or functions of y) the general solution is

$x \text{ (I.F.)} = \int Q \cdot \text{(I.F.)}\, dy + c$, where I.F. (integrating factor) $= e^{\int Pdy}$

SOLVED EXAMPLES

Ex. 1: Solve the following differential equations :

(i) $\frac{dy}{dx} + y = e^{-x}$ (ii) $x \sin \frac{dy}{dx} + (x \cos x + \sin y) = \sin x$

(iii) $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$

Solution :

(i) $\frac{dy}{dx} + y = e^{-x}$ . . . (I)

This is linear differential equation of the form

$\frac{dy}{dx} + Py = \text{Q}$ where $P = 1$, $Q = e^{-x}$

It's Solution is

$y \text{ (I.F.)} = \int Q \cdot \text{(I.F.)}\, dx + c$ . . . (II)

where I.F. $= e^{\int Pdx} = e^{\int dx} = e^{x}$

eq. (II) becomes,

$y \cdot e^{x} = \int e^{-x} \times e^{x} \cdot dx + c$

$\therefore$ $y \cdot e^{x} = \int e^{-x+x} \cdot dx + c$

$\therefore$ $y \cdot e^{x} = \int e^{0} \cdot dx + c$

$\therefore$ $y \cdot e^{x} = \int dx + c$

$\therefore$ $y \cdot e^{x} = x + c$ is the general solution.

(ii) $x \sin x \dfrac{dy}{dx} + (x \cos x + \sin x)\, y = \sin x$

divide by $x \sin x$, we get

$$\frac{dy}{dx} + \left(\cot x + \frac{1}{x}\right) y = \frac{1}{x} \qquad \ldots \text{(I)}$$

It is the linear differential equation of the type

$$\frac{dy}{dx} + Py = Q \qquad \text{where} \qquad P = \cot x + \frac{1}{x},$$

$$Q = \frac{1}{x}$$

Its solution is

$$y\,(\text{I.F.}) = \int Q\cdot(\text{I.F.})\, dx + c \qquad \ldots \text{(II)}$$

where $\quad \text{I.F.} = e^{\int P dx} = e^{\int \left(\cot x + \frac{1}{x}\right) dx}$

$$\text{I.F.} = e^{\int \cot x\, dx + \int \frac{dx}{x}}$$

$$\text{I.F.} = e^{\log|\sin x| + \log x}$$

$$\text{I.F.} = x \sin x$$

eq. (II) becomes,

$$y\cdot x \sin x = \int \frac{1}{x} \times x \sin x \cdot dx + c$$

$\therefore \quad xy\cdot \sin x = -\cos x + c$

$\therefore \quad xy\cdot \sin x + \cos x = c$ is the general solution.

(iii) $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$

$$\therefore \quad \frac{dx}{dy} = \frac{(\tan^{-1} y - x)}{(1 + y^2)}$$

$$\therefore \quad \frac{dx}{dy} + \left(\frac{1}{1 + y^2}\right) x = \frac{\tan^{-1} y}{1 + y^2}$$

This is linear differential equation of the type

$$\frac{dx}{dy} + Px = Q \text{ where } P = \frac{1}{1 + y^2},\ Q = \frac{\tan^{-1} y}{1 + y^2}$$

Its solution is

$$x\,(\text{I.F.}) = \int Q\cdot(\text{I.F.})\, dy + c \qquad \ldots \text{(II)}$$

where $\quad \text{I.F.} = e^{\int P dy} = e^{\int \frac{1}{1 + y^2} dy}$

$$\text{I.F.} = e^{\tan^{-1} y}$$

eq. (II) becomes,

$$x\cdot e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1 + y^2}\cdot e^{\tan^{-1} y}\cdot dy + c \ldots \text{(III)}$$

in R.H.S. Put $\quad \tan^{-1} y = t$

differentiate *w. r. t. x*, we get

$$\therefore \qquad \frac{dy}{1 + y^2} = dt$$

eq. (III) becomes

$$x\cdot e^{\tan^{-1} y} = \int t\cdot e^t \cdot dt + c$$

$$= t\int e^t\cdot dt - \int [1 \times e^t]\, dt + c$$

$$= t\cdot e^t - \int e^t \cdot dt + c$$

$$= t\cdot e^t - e^t + c$$

$$x\cdot e^{\tan^{-1} y} = \tan^{-1} y\cdot e^{\tan^{-1} y} - e^{\tan^{-1} y} + \text{c}$$

$$\therefore \quad x = \tan^{-1} y - 1 + \frac{c}{e^{\tan^{-1} y}}$$

$\therefore \quad x + 1 - \tan^{-1} y = c\cdot e^{-\tan^{-1} y}$ is the solution.

Ex. 2: The slope of the targent to the curve at any point is equal to $y + 2x$. Find the equation of the curve passing through the origin.

Solution : Let P (x, y) be any point on the curve $y = f(x)$

The slope of the tangent at point P (x, y) is $\frac{dy}{dx}$.

$\therefore \quad \frac{dy}{dx} = y + 2x \qquad \therefore \quad \frac{dy}{dx} - y = 2x$

This is linear differential equation of the type

$\frac{dy}{dx} + Py = Q$ where $P = -1$, $Q = 2x$

Its solution is

$y\,(\text{I.F.}) = \int Q\cdot(\text{I.F.})\,dx + c \qquad \ldots \text{(I)}$

where $\quad \text{I.F.} = e^{\int P dx} = e^{\int -dx}$

$\text{I.F.} = e^{-x}$

eq. (I) becomes,

$y\cdot e^{-x} = \int 2x \times e^{-x}\cdot dx + c$

$y\cdot e^{-x} = 2\int x\cdot e^{-x}\cdot dx + c \qquad \ldots \text{(II)}$

Consider, $\int x\cdot e^{-x}\cdot dx$

$= x\int e^{-x}\cdot dx - \int \left(1 \times \frac{e^{-x}}{-1}\right) dx$

$= \frac{x\cdot e^{-x}}{-1} + \int e^{-x}\cdot dx$

$= -xe^{-x}\cdot dx + \int e^{-x}\cdot dx$

$= -xe^{-x} - e^{-x}$

(II) becomes

$y\cdot e^{-x} = 2\,[-xe^{-x} - e^{-x}] + c$

$\therefore \quad y = -2x - 2 + ce^{x} \qquad \ldots \text{(III)}$

The curve passes through the origin (0, 0)

$\therefore \quad 0 = -2\,(0) - 2 + ce^{-0}$

$\therefore \quad 0 = -2 + c$

$\therefore \quad 2 = c \quad$ Put in (III)

$\therefore \quad y = -2x - 2 + 2e^{x}$

$\therefore \quad 2x + y + 2 = 2e^{x}$

is the equation of the curve.

EXERCISE 6.5

(1) Solve the following differential equations :

(i) $\frac{dy}{dx} + \frac{y}{x} = x^3 - 3$

(ii) $\cos^2 x \frac{dy}{dx} + y = \tan x$

(iii) $(x + 2y^3)\frac{dy}{dx} = y$

(iv) $\frac{dy}{dx} + y \sec x = \tan x$

(v) $x\frac{dy}{dx} + 2y = x^2 \log x$

(vi) $(x + y)\frac{dy}{dx} = 1$

(vii) $(x + a)\frac{dy}{dx} - 3y = (x + a)^5$

(viii) $dr + (2r \cot\theta + \sin 2\theta)\, d\theta = 0$

(ix) $ydx + (x - y^2)\, dy = 0$ (x) $(1 - x^2)\dfrac{dy}{dx} + 2xy = x\,(1 - x^2)^{\frac{1}{2}}$

(xi) $(1 + x^2)\dfrac{dy}{dx} + y = e^{\tan^{-1} x}$

(2) Find the equation of the curve which passes through the origin and has slope $x + 3y - 1$ at any point (x, y) on it.

(3) Find the equation of the curve passing through the point $\left(\dfrac{3}{\sqrt{2}}, \sqrt{2}\right)$ having slope of the tangent to the curve at any point (x, y) is $-\dfrac{4x}{9y}$.

(4) The curve passes through the point (0, 2). The sum of the co-ordinates of any point on the curve exceeds the slope of the tangent to the curve at that point by 5. Find the equation of the curve.

(5) If the slope of the tangent to the curve at each of its point is equal to the sum of abscissa and the product of the abscissa and ordinate of the point. Also the curve passes through the point (0, 1). Find the equation of the curve.

6.5 Application of differential Equations :

There are many situations where the relation in the rate of change of a function is known. This gives a differential equation of the function and we may be able to solve it.

6.5.1 Population Growth and Growth of Bacteria :

It is known that a number of bacteria in a culture increase with time. It means there is growth in the number of bacteria. It the population P increases at time t then the rate of change of P is proportional to the population present at that time.

$$\therefore \quad \frac{dP}{dt} \propto P$$

$$\therefore \quad \frac{dP}{dt} = k \cdot P, \quad (k > 0)$$

$$\therefore \quad \frac{dP}{P} = kdt$$

on integrating

$$\therefore \quad \int \frac{dP}{P} = \int kdt$$

$$\therefore \quad \log P = kt + c_1$$

$$\therefore \quad P = c \cdot e^{kt} \qquad \text{where } c = e^{c_1}$$

which gives the population at any time t.

SOLVED EXAMPLES

Ex. 1 : The population of a town increasing at a rate proportional to the population at that time. If the population increases from 40 thousands to 60 thousands in 40 years, what will be the population in another 20 years.

$\left(\text{Given } \sqrt{\frac{3}{2}} = 1{\cdot}2247\right)$.

Solution : Let P be the population at time t. Since rate of increase of P is a proportional to P itself, we have,

$$\frac{dP}{dt} = k{\cdot}P \qquad \ldots (1)$$

where k is constant of proportionality.

Solving this differential equation, we get

$P = a{\cdot}e^{kt}$, where $a = e^{c}$ ... (2)

Initially $P = 40{,}000$ when $t = 0$

$\therefore$ From equation (2), we have

$40{,}000 = a{\cdot}1 \qquad \therefore\ a = 40{,}000$

eq. (2) becomes

$\therefore$ $P = 40{,}000{\cdot}e^{kt}$ ... (3)

Again given that $P = 60{,}000$ when $t = 40$

$\therefore$ From equation (3),

$60{,}000 = 40{,}000{\cdot}e^{40k}$

$$e^{40k} = \frac{3}{2} \qquad \ldots (4)$$

Now we have to find P when $t = 40 + 20$

$= 60$ years

$\therefore$ From equation (3), we have

$$P = 40{,}000{\cdot}e^{60k} = 40{,}000\,(e^{40k})^{\frac{3}{2}}$$

$$= 40{,}000\left(\frac{3}{2}\right)^{\frac{3}{2}} = 73482$$

$\therefore$ Required population will be 73482.

Ex. 2 : Bacteria increase at the rate proporational to the number of bacteria present. If the original number N doubles in 3 hours, find in how many hours the number of bacteria will be 4N?

Solution : Let x be the number of bacteria at time t. Since the rate of increase of x is proporational x, the differential equation can be written as :

$$\frac{dx}{dt} = kx$$

where k is constant of proportionality.

Solving this differential equation we have

$x = c_1{\cdot}e^{kt}$, where $c_1 = e^{c}$ ... (1)

Given that $x = \text{N}$ when $t = 0$

$\therefore$ From equation (1) we get

$\text{N} = c_1{\cdot}1$

$\therefore$ $c_1 = \text{N}$

$\therefore$ $x = \text{N}{\cdot}e^{kt}$ ... (2)

Again given that $x = 2\text{N}$ when $t = 3$

$\therefore$ From equation (2), we have

$2\text{N} = \text{N}{\cdot}e^{3k}$ ... (3)

$e^{3k} = 2$

Now we have to find t, when $x = 4$ N

$\therefore$ From equation (2), we have

$4\text{ N} = \text{N}{\cdot}e^{kt}$

i.e. $4 = e^{kt} = (e^{3k})^{\frac{t}{3}}$

$\therefore$ $2^2 = 2^{\frac{t}{3}}$ ... by eq. (3)

$\therefore$ $\frac{t}{3} = 2$

$\therefore$ $t = 6$

Therefore, the number of bacteria will be 4N in 6 hours.

6.5.2 Radio Active Decay :

We know that the radio active substances (elements) like radium, cesium etc. disintegrate with time. It means the mass of the substance decreases with time.

The rate of disintegration of such elements is always proportional to the amount present at that time.

If x is the amount of any material present at time t then

$$\frac{dx}{dt} = -k \cdot x$$

where k is the constant of proportionality and $k > 0$. The negative sign appears because x decreases as t increases.

Solving this differential equation we get

$$x = a \cdot e^{kt} \qquad \text{where } a = e^c \text{ (check!)} \quad \ldots (1)$$

If x_0 is the initial amount of radio active substance at time $t = 0$, then from equation (1)

$$x_0 = a \cdot 1$$

$$\therefore \quad a = x_0$$

$$\therefore \quad x = x_0\, e^{-kt} \qquad \ldots (2)$$

This expression gives the amount of radio active substance at any time t.

Half Life Period :

Half life period of a radio active substance is defined as the time it takes for half the amount/mass of the substance to disintegrate.

Ex. 3 : Bismath has half life of 5 days. A sample originally has a mass of 800 mg. Find the mass remaining after 30 days.

Solution : Let x be the mass of the Bismath present at time t.

Then $\frac{dx}{dt} = -k \cdot x$ where $k > 0$

Solving the differential equation, we get

$x = c \cdot e^{-kt}$ $\ldots (1)$

where c is constant of proporationality.

Given that $x = 800$, when $t = 0$

using these values in euqation (1), we get

$800 = c \cdot 1 = c$

$\therefore$ $x = 800\, e^{-kt}$ $\ldots (2)$

Since half life is 5 days, we have $x = 400$ when $t = 5$,

$\therefore$ From equation (2), we have

$400 = 800\, e^{-5k}$

$\therefore$ $e^{-5k} = \frac{400}{800} = \frac{1}{2}$ $\ldots (3)$

Now we have determine x, when t = 30,

$\therefore$ From equation (2), we have

$x = 800\, e^{-30k} = 12{\cdot}5$ (verify !)

$\therefore$ The mass after 30 days will be $12{\cdot}5$ mg.

Now let us discuss another application of differential equation.

6.5.3 Newton's Law of Cooling :

Newton's law of cooling states that the rate of change of cooling heated body at any time is proporational to the difference between the temperature of a body and that of its surrounding medium.

Let θ be the temperature of a body at time t and θ_0 be the temperature of the medium.

Then $\frac{d\theta}{dt}$ is the rate of change of temperature with respect to time t and $\theta - \theta_0$ is the difference of temperature at time t. According to Newton's law of cooling.

$$\therefore \qquad \frac{d\theta}{dt} \propto (\theta - \theta_0)$$

$$\therefore \qquad \frac{d\theta}{dt} = -k(\theta - \theta_0) \qquad \ldots (1)$$

where k is constant of proportionality and negative sign indicates that difference of temperature is decreasing.

$$\text{Now} \qquad \frac{d\theta}{dt} = -k(\theta - \theta_0)$$

$$\therefore \qquad \frac{d\theta}{(\theta - \theta_0)} = -k\,dt$$

$\therefore$ Integrating and using the initial condition viz.

$$\therefore \qquad \theta = \theta_1 \qquad \text{when } t = 0\text{, we get}$$

$$\therefore \qquad \theta = \theta_0 + (\theta_1 - \theta_0)\,e^{-kt} \text{ (verify)} \qquad \ldots (2)$$

Thus equation (2) gives the temperature of a body at any time t.

Ex. 4 : Water at 100°c cools in 10 minutes to 88°c in a room temperature of 25°c. Find the temperature of water after 20 minutes.

Solution : Let θ be the temperature of water at time t. Room temperature is given to be 25°c. Then according to Newton's law of cooling. we have

$$\frac{d\theta}{dt} \propto (\theta - 25)$$

$$\frac{d\theta}{dt} = -k(\theta - 25), \qquad \text{where } k > 0$$

After integrating and using initial condition.

We get $\theta = 25 + 75\cdot e^{-kt}$ $\qquad \ldots (1)$

But given that $\theta = 88$°c when $t = 10$

$\therefore$ From equation (1) we get

$$88 = 25 + 75\cdot e^{-10k}$$

$$\therefore \quad 63 = 75\cdot e^{-10k} \quad \therefore \quad e^{-10k} = \frac{63}{75} = \frac{21}{25} \qquad \ldots (2)$$

Now we have to find θ, when $t = 20$,

$\therefore$ From equation (1) we have

$$\begin{aligned} \theta &= 25 + 75\cdot e^{-20k} \\ &= 25 + 75\,(e^{-20k})^2 \\ &= 25 + 75\left(\frac{dy}{dx}\right)^2 \qquad \ldots \text{by (2)} \\ &= 25 + \frac{75 \times 21 \times 21}{25 \times 25} \\ &= 25 + \frac{1323}{25} = 77\cdot 92 \end{aligned}$$

Therefore temperature of water after 20 minutes will be 77·92°c.

6.5.4 Surface Area :

Knowledge of a differential equation is also used to solve problems related to the surface area. We consider the following examples :

Ex. 5 : Water is being poured into a vessel in the form of an inverted right circular cone of semi vertical angle 45°c in such a way that the rate of change of volume at any moment is proporational to the area of the curved surfaces which is wet at that moment. Initially, the vessel is full to a height of 2 cms. And after 2 seconds the height becomes 10 cm. Show that after 3.5 seconds from that start, the height of water will be 16 cms.

Solution : Let the height of water at time t seconds be h cms.

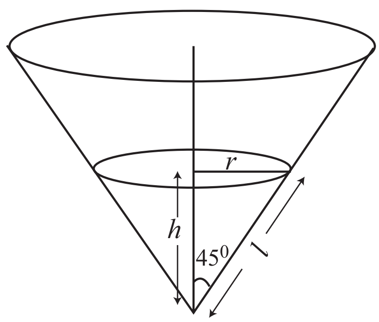

Fig. 6.2

We are given that initial height is 2 cms. and after 2 seconds, the height is 10 cms.

$$\therefore \quad h = 2 \text{ when } t = 0 \qquad \ldots (1)$$

$$\text{and } \ h = 10 \text{ when } t = 2 \qquad \ldots (2)$$

Let v be the volume, r be the radius of the water surface and l be that slant height at time t seconds.

$\therefore$ Area of the curved surface at this moment is πrl.

But the semi vertical angle is 45°.

$$\therefore \quad \tan 45° = \frac{r}{h} = 1$$

$$\therefore \quad r = h$$

$$\text{and} \quad l^2 = r^2 + h^2 = 2h^2$$

$$\therefore \quad l = \sqrt{2}\, h$$

$$\therefore \quad \text{Area of the curved surface} = \pi rl = \pi \cdot h \cdot \sqrt{2}\, h$$

$$= \sqrt{2}\, \pi h^2$$

Since rate of change of volume is proporational to this area, we get

$$\frac{dv}{dt} \propto \sqrt{2}\,\pi h^2$$

$$\therefore \quad \frac{dv}{dt} = c\cdot\sqrt{2}\,\pi h^2$$

where c is constant of proportionality.

Let $\quad c\sqrt{2}\,\pi = k$

$$\therefore \quad \frac{dv}{dt} = kh^2 \qquad \ldots (3)$$

$\therefore$ where k is constant

Now $\quad v = \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3}\pi r^2\cdot h = \frac{1}{3}\pi h^3, \text{ (since } r = h\text{)}$$

Differentiating with respect to t, we get

$$\therefore \quad \frac{dv}{dt} = \pi h^2 \frac{dh}{dt} \qquad \ldots (4)$$

Equating $\frac{dv}{dt}$ from (3) and (4) we get

$$\pi h^2 \frac{dh}{dt} = kh^2$$

$$\therefore \quad \frac{dh}{dt} = \frac{k}{\pi} = a \text{ (say)}$$

where a is constant.

integrating we get

$$h = at + b \qquad \ldots (5)$$

using (1) we have $\quad 2 = a.0 + b \quad \therefore \quad b = 2$

$\therefore$ Equation (5) becomes

$$h = at + 2$$

Now using (2) we get

$$10 = 2a + 2 \quad \therefore \quad a = 4$$

using the values of a and b in equation (5), we have

$$\therefore \quad h = 4t + 2$$

Now put $t = 3\cdot 5$

$$\therefore \quad h = 4 \times 3\cdot 5 + 2$$

$$= 14 + 2 = 16 \text{ cm}$$

Therefore, height of water after 3·5 seconds will be 16 cms.

EXERCISE 6.6

1. In a certain culture of bacteria the rate of increase is proportional to the number present. If it is found that the number doubles in 4 hours, find the number of times the bacteria are increased in 12 hours.

2. If the population of a country doubles in 60 years, in how many years will it be triple (treble) under the assumption that the rate of increase is proporational to the number of inhabitants?
[Given log 2 = 0·6912, log 3 = 1·0986]

3. If a body cools from 80°c to 50°c at room temperature of 25°c in 30 minutes, find the temperature of the body after 1 hour.

4. The rate of growth of bacteria is proportional to the number present. If initially, there were 1000 bacteria and the number double in 1 hour, find the number of bacteria after $2\frac{1}{2}$ hours.
[Take $\sqrt{2} = 1{\cdot}414$]

5. The rate of disintegration of a radio active element at any time t is proportational to its mass at that time. Find the time during which the original mass of 1·5 gm. will disintegrate into its mass of 0·5 gm.

6. The rate of decay of certain substance is directly proportional to the amount present at that instant. Initially, there are 25 gms of certain substance and two hours later it is found that 9 gms are left. Find the amount left after one more hour.

7. Find the population of a city at any time t, given that the rate of increase of population is proporational to the population at the instant and that in a period of 40 years the population increased from 30,000 to 40,000.

8. A body cools according to Newton's law from 100°c to 60°c in 20 minutes. The temperature of the surrounding being 20°c how long will it take to cool down to 30°c?

9. A right circular cone has height 9 cms and radius of the base 5 cms. It is inverted and water is poured into it. If at any instant the water level rises at the rate of $\frac{\pi}{A}$ cms/ sec. where A is the area of water surface at that instant, show that the vessel will the full in 75 seconds.

10. Assume that a spherical raindrop evaporates at a rate proporational to its surface area. If its radius originally is 3mm and 1 hour later has been reduced to 2mm, find an expression for the radius of the raindrop at any time *t*.

11. The rate of growth of the population of a city at any time t is proportional to the size of the population. For a certain city it is found that the constant of proportionality is 0.04. Find the population of the city after 25 years if the initial population is 10,000. [Take $e = 2{\cdot}7182$]

12. Radium decomposes at the rate proportional to the amount present at any time. If p percent of amount disappears in one year, what percent of amount of radium will be left after 2 years ?

Let us Remember

- Equation which contains the derivative of a function is called a **differential equation.**
- The order of a differential equation is the highest order of the derivative appearing in the equation.
- The degree of the differential equation is the power of the highest ordered derivative present in the equation.
- Order and degree of a differential equation are always positive integers.
- Solution of a differential equation in which number of arbitary constants is equal to the order of a differential equation is the **general solution** of the differential equation.
- Solution obtained from the general solution by giving particular values to the arbitrary constants is the particular solution of the differential equation.
- The most general form of a **linear differential equation** of the first order is : $\frac{dy}{dx} + Py = Q$

  where P and Q are functions of x or constant.

  Its solution is given by : $y\,(\text{I.F.}) = \int Q \cdot (\text{I.F.})\, dx + c$, where I.F. (integrating factor) $= e^{\int P dx}$
- Solution of a differential equation $\frac{dx}{dt} = kx$ is in the form $x = a \cdot e^{kt}$ where a is initial value of x.

  Further, $k > 0$ represents growth and $k > 0$, represents decay.
- Newton's law of cooling is $\theta = \theta_0 + (\theta_1 - \theta_0)\, e^{-kt}$.

MISCELLANEOUS EXERCISE 6

(I) Choose the correct option from the given alternatives :

(1) The order and degree of the differential equation $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \left(\frac{d^2y}{dx^2}\right)^{\frac{3}{2}}$ are respectively . . .

(A) 2, 1 (B) 1, 2 (C) 3, 2 (D) 2, 3

(2) The differential equation of $y = c^2 + \frac{c}{x}$ is . . .

(A) $x^4\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} = y$ (B) $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

(C) $x^3\left(\frac{dy}{dx}\right)^2 + x\frac{dy}{dx} = y$ (D) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0$

(3) $x^2 + y^2 = a^2$ is a solution of ...

(A) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - y = 0$

(B) $y = x\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2} + a^2 y$

(C) $y = x\dfrac{dy}{dx} + a\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$

(D) $\dfrac{d^2y}{dx^2} = (x + 1)\dfrac{dy}{dx}$

(4) The differential equation of all circles having their centers on the line $y = 5$ and touching the X-axis is

(A) $y^2\left(1 + \dfrac{dy}{dx}\right) = 25$

(B) $(y - 5)^2\left[1 + \left(\dfrac{dy}{dx}\right)^2\right] = 25$

(C) $(y - 5)^2 + \left[1 + \left(\dfrac{dy}{dx}\right)^2\right] = 25$

(D) $(y - 5)^2\left[1 - \left(\dfrac{dy}{dx}\right)^2\right] = 25$

(5) The differential equation $y\dfrac{dy}{dx} + x = 0$ represents family of . . .

(A) circles (B) parabolas (C) ellipses (D) hyper bolas

(6) The solution of $\dfrac{1}{x}\cdot\dfrac{dy}{dx} = \tan^{-1} x$ is ...

(A) $\dfrac{x^2 \tan^{-1} x}{2} + c = 0$

(B) $x \tan^{-1} x + c = 0$

(C) $x - \tan^{-1} x = c$

(D) $y = \dfrac{x^2 \tan^{-1} x}{2} - \dfrac{1}{2}(x - \tan^{-1} x) + c$

(7) The solution of $(x + y)^2 \dfrac{dy}{dx} = 1$ is ...

(A) $x = \tan^{-1}(x + y) + c$

(B) $y \tan^{-1}\left(\dfrac{x}{y}\right) = c$

(C) $y = \tan^{-1}(x + y) + c$

(D) $y + \tan^{-1}(x + y) = c$

(8) The solution of $\dfrac{dy}{dx} = \dfrac{y + \sqrt{x^2 - y^2}}{2}$ is . . .

(A) $\sin^{-1}\left(\dfrac{y}{x}\right) = 2 \log|x| + c$

(B) $\sin^{-1}\left(\dfrac{y}{x}\right) = \log|x| + c$

(C) $\sin\left(\dfrac{y}{x}\right) = \log|x| + c$

(D) $\sin\left(\dfrac{y}{x}\right) = \log|x| + c$

(9) The solution of $\dfrac{dy}{dx} + y = \cos x - \sin x$ is . . .

(A) $y e^x = \cos x + c$

(B) $y e^x + e^x \cos x = c$

(C) $y e^x = e^x \cos x + c$

(D) $y^2 e^x = e^x \cos x + c$

(10) The integrating factor of linear differential equation $x\dfrac{dy}{dx} + 2y = x^2 \log x$ is . . .

(A) $\dfrac{1}{x}$ (B) k (C) $\dfrac{1}{n^2}$ (D) x^2

(11) The solution of the differential equation $\dfrac{dy}{dx} = \sec x - y \tan x$ is

(A) $y \sec x + \tan x = c$ (B) $y \sec x = \tan x + c$

(C) $\sec x + y \tan x = c$ (D) $\sec x = y \tan x + c$

(12) The particular solution of $\dfrac{dy}{dx} = xe^{y-x}$, when $x = y = 0$ is . . .

(A) $e^{x-y} = x + 1$ (B) $e^{x+y} = x + 1$ (C) $e^{x} + e^{y} = x + 1$ (D) $e^{y-x} = x - 1$

(13) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is a solution of . . .

(A) $\dfrac{d^2y}{dx^2} + yx + \left(\dfrac{dy}{dx}\right)^2 = 0$ (B) $xy\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{dy}{dx} = 0$

(C) $y\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 + y = 0$ (D) $xy\dfrac{dy}{dx} + y\dfrac{d^2y}{dx^2} = 0$

(14) The decay rate of certain substance is directly proporational to the amount present at that instant. Initially there are 27 grams of substance and 3 hours later it is found that 8 grams left. The amount left after one more hour is...

(A) $5\dfrac{2}{3}$ grams (B) $5\dfrac{1}{3}$ grams (C) $5{\cdot}1$ grams (D) 5 grams

(15) If the surrounding air is kept at 20°c and a body cools from 80°c to 70°c in 5 minutes, the temparature of the body after 15 minutes will be...

(A) 51·7°c (B) 54·7°c (C) 52·7°c (D) 50·7°c

(II) Solve the following :

(1) Determine the order and degree of the following differential equations :

(i) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + y = x^3$ (ii) $\left(\dfrac{d^3y}{dx^3}\right)^2 = \sqrt[5]{1 + \dfrac{dy}{dx}}$

(iii) $\sqrt[3]{1 + \left(\dfrac{dy}{dx}\right)^2} = \dfrac{d^2y}{dx^2}$ (iv) $\dfrac{dy}{dx} = 3y + \sqrt[4]{1 + 5\left(\dfrac{dy}{dx}\right)^2}$

(v) $\dfrac{d^4y}{dx^4} + \sin\left(\dfrac{dy}{dx}\right) = 0$

(2) In each of the following examples, verify that the given function is a solution of the differential equation.

(i) $x^2 + y^2 = r^2,\ x\dfrac{dy}{dx} + r\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = y$

(ii) $y = e^{ax}\sin bx\ ,\ \dfrac{d^2y}{dx^2} - 2a\dfrac{dy}{dx} + (a^2+b^2)\,y = 0$

(iii) $y = 3\cos(\log x) + 4\sin(\log x),\ x\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0$

(iv) $y = ae^{x} + be^{-x} + x^2,\ x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + x^3 = xy + 2$

(v) $x^2 = 2y^2\log y,\ x^2 + y^2 = xy\dfrac{dx}{dy}$

(3) Obtain the differential equation by eliminating the arbitrary constants from the following equations.

(i) $y^2 = a\,(b-x)\,(b+x)$

(ii) $y = a\sin(x+b)$

(iii) $(y-a)^2 = b\,(x+4)$

(iv) $y = \sqrt{a\cos(\log x) + b\sin(\log x)}$

(v) $y = Ae^{3x+1} + Be^{-3x+1}$

(4) Form the differential equation of :

(i) all circles which pass through the origin and whose centres lie on X−axis.

(ii) all parabolas which have $4b$ as latus rectum and whose axes is parallel to Y−axis.

(iii) an ellipse whose minor axis is twice its major axis.

(iv) all the lines which are normal to the line $3x - 2y + 7 = 0$.

(v) the hyperbola whose length of transverse and conjugate axes are half of that of the given hyperbola $\dfrac{x^2}{16} - \dfrac{y^2}{36} = k$.

(5) Solve the following differential equations :

(i) $\log\left(\dfrac{dy}{dx}\right) = 2x + 3y$

(ii) $\dfrac{dy}{dx} = x^2y + y$

(iii) $\dfrac{dy}{dx} = \dfrac{2y-x}{2y+x}$

(iv) $x\,dy = (x+y+1)\,dx$

(v) $\dfrac{dy}{dx} + y\cot x = x^2\cot x + 2x$

(vi) $y\log y = (\log y^2 - x)\dfrac{dy}{dx}$

(vii) $4\dfrac{dx}{dy} + 8x = 5e^{-3y}$

(6) Find the particular solution of the following differential equations :

(1) $y(1+\log x) = (\log x^x)\dfrac{dy}{dx}$, when $y(e) = e^2$

(2) $(x+2y^2)\dfrac{dy}{dx} = y$, when $x = 2, y = 1$

(3) $\dfrac{dy}{dx} - 3y\cot x = \sin 2x$, when $y\left(\dfrac{\pi}{2}\right) = 2$

(4) $(x+y)\,dy + (x-y)\,dx = 0$, when $x = 1 = y$

(5) $2e^{\frac{x}{y}}\,dx + \left(y - 2xe^{\frac{x}{y}}\right)dy = 0$, when $y(0) = 1$

(7) Show that the general solution of the differential equation $\dfrac{dy}{dx} = \dfrac{y^2+y+1}{x^2+x+1}$ is given by $(x+y+1) = c(1-x-y-2xy)$

(8) The normal lines to a given curve at each point (x, y) on the curve pass through (2, 0). The curve passes through (2, 3). Find the equation of the curve.

(9) The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of the balloon after t second.

(10) A person's assets start reducing in such a way that the rate of reduction of assets is proportional to the square root of the assets existing at that moment. If the assets at the begining are ₹ 10 lakhs and they dwindle down to ₹ 10,000 after 2 years, show that the person will be bankrupt in $2\dfrac{2}{9}$ years from the start.

❖ ❖ ❖

7. PROBABILITY DISTRIBUTIONS

Let us Learn

- Random variables
- Types of random variables
- Probability distribution of random variable.
    - Discrete random variable
    - Probability mass function
    - Expected values and variance
    - Continuous random variable
    - Probability density function
    - Cumulative distribution function

Let us Recall

- A random experiment and all possible outcomes of an experiment
- The sample space of a random experiment

Let us Study

7.1 Random variables :

We have already studied random experiments and sample spaces corresponding to random experiments. As an example, consider the experiment of tossing two fair coins. The sample space corresponding to this experiment contains four elements, namely {HH, HT,TH,TT}. We have already learnt to construct the sample space of any random experiment. However, the interest is not always in a random experiment and its sample space. We are often not interested in the outcomes of a random experiment, but only in some number obtained from the outcome. For example, in case of the experiment of tossing two fair coins, our interest may be only in the number of heads when two coins are tossed. In general, it is possible to associate a unique real number to every possible outcome of a random experiment. The number obtained from an outcome of a random experiment can take different values for different outcomes. This is why such a number is a variable. The value of this variable depends on the outcome of the random experiment, therefore it is called a random variable.

A random variable is usually denoted by capital letters like X, Y, Z, . . .

Consider the following examples to understand the concept of random variables.

(i) When we throw two dice, there are 36 possible outcomes, but if we are interested in the sum of the numbers on the two dice, then there are only 11 different possible values, from 2 to12.

(ii) If we toss a coin 10 times, then there are $2^{10} = 1024$ possible outcomes, but if we are interested in the number of heads among the 10 tosses of the coin, then there are only 11 different possible values, from 0 to 10.

(iii) In the experiment of randomly selecting four items from a lot of 20 items that contains 6 defective items, the interest is in the number of defective items among the selected four items. In this case, there are only 5 different possible outcomes, from 0 to 4.

In all the above examples, there is a rule to assign a unique value to every possible outcome of the random experiment. Since this number can change from one outcome to another, it is a variable. Also, since this number is obtained from outcomes of a random experiment, it is called a random variable.

A random variable is formally defined as follows.

Definition :

A random variable is a real-valued function defined on the sample space of a random experiment. In other words, the domain of a random variable is the sample space of a random experiment, while its co-domain is the set of real numbers.

Thus $X : S \to R$ is a random variable.

We often use the abbreviation r.v. to denote a random variable. Consider an experiment where three seeds are sown in order to find how many of them germinate. Every seed will either germinate or will not germinate. Let us use the letter Y when a seed germinates and the letter N when a seed does not germinate. The sample space of this experiment can then be written as

$S = \{YYY, YYN, YNY, NYY, YNN, NYN, NNY, NNN\}$, and n (S) = $2^3 = 8$.

None of these outcomes is a number. We shall try to represent every outcome by a number. Consider the number of times the letter Y appears in a possible outcome and denote it by X. Then, we have

$X(YYY) = 3, X(YYN) = X(YNY) = X(NYY) = 2, X(YNN) = X(NYN) = X(NNY) = 1, X(NNN) = 0.$

The variable X has four possible values, namely 0, 1, 2, and 3. The set of possible values of X is called the range of X. Thus, in this example, the range of X is the set $\{0, 1, 2, 3\}$.

A random variable is usually denoted by a capital letter, like X or Y. A particular value taken by the random variable is usually denoted by the small letter x. Note that x is always a real number and the set of all possible outcomes corresponding to a particular value x of X is denoted by the event $[X = x]$.

For example, in the experiment of three seeds, the random variable X takes four possible values, namely 0, 1, 2, 3. The four events are then defined as follows.

$$[X=0]=\{NNN\},$$
$$[X=1]=\{YNN,\ NYN,\ NNY\},$$
$$[X=2]=\{YYN,\ YNY,\ NYY\},$$
$$[X=3]=\{YYY\}.$$

Note that the sample space in this experiment is finite and so is the random variable defined on it.

A sample space need not always be finite. For example, the experiment of tossing a coin until a head is obtained. The sample space for this experiment is $S=\{H,\ TH,\ TTH,\ TTTH,\ \ldots\}$.

Note that S contains an unending sequence of tosses required to get a head. Here, S is countably infinite.The random variable $X: S \to R$, denoting the number of tosses required to get a head, has the range $\{1, 2, 3, \ldots\}$ which is also countably infinite.

7.2 Types of Random Variables :

There are two types of random variables, namely discrete and continuous.

7.2.1 Discrete Random Variables :

Definition : A random variable is said to be a discrete random variable if the number of its possible values is finite or countably infinite.

The values of a discrete random variable are usually denoted by non-negative integers, that is, $\{0, 1, 2, \ldots\}$.

Examples of discrete random variables include the number of children in a family, the number of patients in a hospital ward, the number of cars sold by a dealer, number of stars in the sky and so on.

Note : The values of a discrete random variable are obtained by counting.

7.2.2 Continuous Random Variable :

Definition : A random variable is said to be a continuous random variable if the possible values of this random variable form an interval of real numbers.

A continuous random variable has uncountably infinite possible values and these values form an interval of real numbers.

Examples of continuous random variables include heights of trees in a forest, weights of students in a class, daily temperature of a city, speed of a vehicle, and so on.

The value of a continuous random variable is obtained by measurement. This value can be measured to any degree of accuracy, depending on the unit of measurement. This measurement can be represented by a point in an interval of real numbers.

The purpose of defining a random variable is to study its properties. The most important property of a random variable is its probability distribution. Many other properties of a random variable are obtained with help of its probability distribution. We shall now learn about the probability distribution of a random variable. We shall first learn the probability of a discrete random variable, and then learn the probability distribution of a continuous random variable.

7.3 Probability Distribution of Discrete Random Variables :

Let us consider the experiment of throwing two dice and noting the numbers on the upper-most faces of the two dice. The sample space of this experiment is

$S = \{(1, 1), (1, 2), ...(6, 6)\}$ and $n(S) = 36$.

Let X denote the sum of the two numbers in any single throw.

Then $\{2, 3, \cdots, 12\}$ is the set of possible values of X. Further,

$$[X = 2] = \{(1, 1)\},$$

$$[X = 3] = \{(1, 2), (2, 1)\},$$

$$\cdots$$

$$\cdots$$

$$[X = 12] = \{(6, 6)\}.$$

Next, all of these 36 possible outcomes are equally likely if the two dice are fair, that is, if each of the six faces have the same probability of being uppermost when the die is thrown.

As the result, each of these 36 possible outcomes has the same probability $= \dfrac{1}{36}$

This leads to the following results.

$$P[X = 2] = P\{(1, 1)\} = \frac{1}{36},$$

$$P[X = 3] = P\{(1, 2), (2, 1)\} = \frac{2}{36},$$

$$P[X = 4] = P\{(1, 3), (2, 2), (3, 1)\} = \frac{3}{36},$$

and so on.

The following table shows the probabilities of all possible values of X.

x	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Table 7.1

Such a description giving the values of the random variable X along with the corresponding probabilities is called the probability distribution of the random variable X.

In general, the probability distribution of a discrete random variable X is defined as follows.

Definition : The probability distribution of a discrete random variable X is defined by the following system of numbers. Let the possible values of X be denoted by $x_1, x_2, x_3, \ldots$, and the corresponding probabilities be denoted by $p_1, p_2, p_3, \ldots$, where $p_i = P[X = x_i]$ for $i = 1, 2, 3, \ldots$.

Note : A discrete random variable can have finite or infinite number of possible values, but they are countable.

Sometimes, the probability distribution of a discrete random variable is presented in the form of ordered pairs of the form $(x_1, p_1), (x_2, p_2), (x_3, p_3), \ldots$ A common practice is to present the probability distribution of a discrete random variable in a tabular form as shown below.

x_i	x_1	x_2	x_3	...
$P[X = x_i]$	p_1	p_2	p_3	...

Table 7.2

Note : If x_i is a possible value of X and $p_i = P[X = x_i]$, then there is an event $[E_i]$ in the sample space S such that $p_i = P[E_i]$. Since x_i is a possible value of X, $p_i = P[X = x_i] > 0$. Also, all possible values of X cover all sample points in the sample space S, and hence the sum of their probabilities is 1. That is, $p_i \geq 0$, for all i and $\sum_i p_i = 1$.

7.3.1 Probability Mass Function (p. m. f.) :

Sometimes the probability p_i of X taking the value x_i is a function of x_i for every possible value of X. Such a function is called the **probability mass function (p. m. f.)** of the discrete random variable X.

For example, consider the coin-tossing experiment where the random variable X is defined as the number of tosses required to get a head. Let probability of getting head be 't' and that of not getting head be $1 - t$. The possible values of X are given by the set of natural numbers, 1, 2, 3, . . . and

$$P[X = i] = (1 - t)^{i-1}\, t, \text{ for } i = 1, 2, 3, \ldots$$

This result can be verified by noting that if head is obtained for the first time on the i^{th} toss, then the first $i-1$ tosses have resulted in tail. In other words, $X = i$ represents the event of having $i-1$ tails followed by the first head on the toss.

We now define the probability mass function (p. m. f.) of a discrete random variable.

Definition : Let the possible values of a discrete random variable X be denoted by $x_1, x_2, x_3, \ldots,$ with the corresponding probabilities $p_i = P[X = x_i]$, $i = 1, 2, \ldots$ If there is a function f such that $p_i = P[X = x_i] = f(x_i)$ for all possible values of X, then f is called the probability mass function (p. m. f.) of X.

For example, consider the experiment of tossing a coin 4 times and defining the random variable X as the number of heads in 4 tosses. The possible values of X are 0, 1, 2, 3, 4, and the probability distribution of X is given by the following table.

x	0	1	2	3	4
$P[X = x]$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

Table 7.3

Note that : $$P[X = x] = \left.\begin{matrix}4\\x\end{matrix}\right\}\left(\frac{1}{2}\right)^4, x = 0, 1, 2, 3, 4, \ldots$$

where $\left.\begin{matrix}4\\x\end{matrix}\right\}$ is the number of ways of getting x heads in 4 tosses.

7.3.2 Cumulative Distribution Function (c. d. f.) :

The probability distribution of a discrete random variable can be specified with help of the p. m. f. It is sometimes more convenient to use the cumulative distribution function (c.d.f.) of the random variable. The cumulative distribution function (c. d. f.) of a discrete random variable is defined as follows.

Definition : The cumulative distribution function (c. d. f.) of a discrete random variable X is denoted by F and is defined as follows.

$$F(x) = P[X \le x] = \sum_{x_i < x} P[X = x_i]$$

$$= \sum_{x_i < x} p_i$$

$$= \sum_{x_i < x} f(x_i)$$

where f is the probability mass function (p. m. f.) of the discrete random variable X.

For example, consider the experiment of tossing 4 coins and counting the number of heads.

We can form the next table for the probability distribution of X.

x	0	1	2	3	4
$f(x) = P[X = x]$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$
$F(x) = P[X \leq x]$	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{11}{16}$	$\frac{15}{16}$	1

Table 7.4

For example, consider the experiment of tossing a coin till a head is obtained. The following table shows the p. m. f. and the c. d. f. of the random variable X, defined as the number of tosses required for the first head.

x	1	2	3	4	5	. . .
$f(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	. . .
$F(x)$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	$\frac{31}{32}$	. . .

Table 7.5

It is possible to define several random variables on the same sample space. If two or more random variables are defined on the same sample space, their probability distributions need not be the same.

For example, consider the simple experiment of tossing a coin twice. The sample space of this experiment is $S = \{HH, HT, TH, TT\}$.

Let X denote the number of heads obtained in two tosses. Then X is a discrete random variable and its value for every outcome of the experiment is obtained as follows.

$$X(HH) = 2, X(HT) = X(TH) = 1, X(TT) = 0.$$

Let Y denote the number of heads minus the number of tails in two tosses. Then Y is also a discrete random variable and its value for every outcome of the experiment is obtained as follows.

$$Y(HH) = 2, Y(HT) = Y(TH) = 0, Y(TT) = -2.$$

$$\text{Let} \qquad Z = \frac{\text{Number of heads}}{\text{Number of tails} + 1}$$

Then Z is also a discrete random variable and its values for every outcome of the experiment is obtained as follows. $\quad Z(HH) = 2, Z(HT) = Z(TH) = \frac{1}{2}, Z(TT) = 0.$

These example show that it is possible to define many distinct random variables on the same sample space. Possible values of a discrete random variables can be positive or negative, integer or fraction.

SOLVED EXAMPLES

Ex. 1 : Two persons A and B play a game of tossing a coin thrice. If the result of a toss is head, A gets ₹ 2 from B. If the result of a toss is tail, B gets ₹ 1.5 from A. Let X denote the amount gained or lost by A. Show that X is a discrete random variable and show how it can be defined as a function on the sample space of the experiment.

Solution : X is a number whose value depends on the outcome of a random experiment.

Therefore, X is a random variable. Since the sample space of the experiment has only 8 possible outcomes, X is a discrete random variable. Now, the sample space of the experiment is

$$S = \{\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}\}.$$

The values of X in rupees corresponding to these outcomes of the experiment are as follows.

$$X(\text{HHH}) = 2 \times 3 = ₹\ 6$$

$$X(\text{HHT}) = X(\text{HTH}) = X(\text{THH}) = 2 \times 2 - 1.50 \times 1 = ₹\ 2.50$$

$$X(\text{HTT}) = X(\text{THT}) = X(\text{TTH}) = 2 \times 1 - 1.50 \times 2 = ₹ - 1.00$$

$$X(\text{TTT}) = - 1.50 \times 3 = ₹ - 4.50$$

Here, a negative amount shows a loss to player A. This example shows that X takes a unique value for every element of the sample space and therefore X is a function on the sample space. Further, possible values of X are 4.50, 1, 2.50, 6.

Ex. 2 : A bag contains 1 red and 2 green balls. One ball is drawn from the bag at random, its colour is noted, and then ball is put back in the bag. One more ball is drawn from the bag at random and its colour is also noted. Let X denote the number of red balls drawn from the bag as described above. Derive the probability distribution of X.

Solution : Let the balls in the bag be denoted by r, g_1, g_2. The sample space of the experiment is then given by $S = \{r\,r, r\,g_1, r\,g_2, g_1 r, g_2 r, g_1 g_1, g_1 g_2, g_2 g_1, g_2 g_2\}$.

Since X is defined as the number of red balls, we have

$$X(\{r\,r\}) = 2,$$

$$X(\{r\,g_1\}) = X(r\,g_2) = X(g_1 r) = X(g_2 r) = 1,$$

$$X(\{g_1 g_1\}) = X(g_1 g_2) = X(g_2 g_1) = X(g_2 g_2) = 0.$$

Thus, X is a discrete random variable that can take values 0, 1, and 2.

The probability distribution of X is then obtained as follows :

x	0	1	2
$P[X = x]$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

Ex. 3 : Two cards are randomly drawn, with replacement, from a well shuffled deck of 52 playing cards. Find the probability distribution of the number of aces drawn.

Solution : Let X denote the number of aces among the two cards drawn with replacement.

Clearly, 0, 1, and 2 are the possible values of X. Since the draws are with replacement, the outcomes of the two draws are independent of each other. Also, since there are 4 aces in the deck of 52 cards,

$$\text{P [ace]} = \frac{4}{52} = \frac{1}{13} \text{ and P [non-ace]} = \frac{12}{13}\text{ . Then}$$

$$\text{P }[X = 0] = \text{P [non-ace and non-ace]} = \frac{12}{13} \times \frac{12}{13} = \frac{144}{169},$$

$$\text{P }[X = 1] = \text{P [ace and non-ace] + P [non-ace and ace]}$$

$$= \frac{1}{13} \times \frac{12}{13} + \frac{12}{13} \times \frac{1}{13} = \frac{24}{169},$$

$$\text{and} \quad \text{P }[X = 2] = \text{P [ace and ace]} = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}.$$

The required probability distribution is then as follows.

x	0	1	2
$P[X=x]$	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

Ex. 4 : A fair die is thrown. Let X denote the number of factors of the number on the upper face. Find the probability distribution of X.

Solution : The sample space of the experiment is $S = \{1, 2, 3, 4, 5, 6\}$. The values of X for the possible outcomes of the experiment are as follows.

$X(1) = 1$, $X(2) = 2$, $X(3) = 2$, $X(4) = 3$, $X(5) = 2$, $X(6) = 4$. Therefore,

$$p_1 = \text{P }[X = 1] = \text{P }[\{1\}] = \frac{1}{6}$$

$$p_2 = \text{P }[X = 2] = \text{P }[\{2, 3, 5\}] = \frac{3}{6}$$

$$p_3 = \text{P }[X = 3] = \text{P }[\{4\}] = \frac{1}{6}$$

$$p_4 = \text{P }[X = 4] = \text{P }[\{6\}] = \frac{1}{6}$$

The probability distribution of X is then as follows.

x	1	2	3	4
$P[X=x]$	$\frac{1}{6}$	$\frac{3}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Ex. 5 : Find the probability distribution of the number of doublets in three throws of a pair of dice.

Solution : Let X denote the number of doublets. Possible doublets in a pair of dice are (1, 1), (2, 2), (3, 3), (4, 4), (5, 5,), (6, 6).

Since the dice are thrown thrice, 0, 1, 2, and 3 are possible values of X. Probability of getting a doublet in a single throw of a pair of dice is $p = \frac{1}{6}$ and $q = 1 - \frac{1}{6} = \frac{5}{6}$.

$P[X = 0] = P[\text{ no doublet}] = qqq = \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}$.

$P[X = 1] = P[\text{one doublet}] = pqq + qpq + qqp = 3pq^2 = \frac{75}{216}$.

$P[X = 2] = P[\text{two doublets}] = ppq + pqp + qpp = 3p^2q = \frac{15}{216}$.

$P[X = 3] = \text{P [three doublets]} = ppp = \frac{1}{216}$.

Ex. 6 : The probability distribution of X is as follows :

x	0	1	2	3	4
$P[X = x]$	0·1	k	$2k$	$2k$	k

Find (i) k, (ii) P $[X < 2]$, (iii) P $[X \geq 3]$, (iv) P $[1 \leq X < 4]$, (v) F(2).

Solution : The table gives a probability distribution and therefore

P $[X = 0]$ + P $[X = 1]$ + P $[X = 2]$ + P $[X = 3]$ + P $[X = 4] = 1$.

That is, $0{\cdot}1 + k + 2k + 2k + k = 1$.

That is, $6k = 0{\cdot}9$. Therefore $k = 0{\cdot}15$.

(i) $k = 0{\cdot}15$.

(ii) P $[X < 2]$ = P $[X = 0]$ + P $[X = 1] = 0{\cdot}1 + k = 0{\cdot}1 + 0.15 = 0{\cdot}25$

(iii) P $[X \geq 3]$ = P $[X = 3]$ + P $[X = 4] = 2k + k = 3(0{\cdot}15) = 0{\cdot}45$

(iv) P $[1 \leq X < 4]$ = P $[X = 1]$ + P $[X = 2]$ + P $[X = 3]$ $= k + 2k + 2k = 5k$

$= 5(0{\cdot}15) = 0{\cdot}75$.

(v) F (2) = P $[X \leq 2]$ = P $[X = 0]$ + P $[X = 1]$ + P $[X = 2]$ $= 0{\cdot}1 + k + 2k = 0{\cdot}1 + 3k$

$= 0{\cdot}1 + 3(0{\cdot}15) = 0{\cdot}1 + 0{\cdot}45 = 0{\cdot}55$.

7.3.3 Expected value and Variance of a random variable :

In many problems, it is desirable to describe some feature of the random variable by means of a single number that can be computed from its probability distribution. Few such numbers are mean, median, mode and variance and standard deviation. In this section, we shall discuss mean and variance only. Mean is a measure of location or central tendency in the sense that it roughly locates a **middle** or **average value** of the random variable.

Definition : Let X be a random variable whose possible values $x_1, x_2, x_3, \ldots, x_n$ occur with probabilities $p_1, p_2, p_3, \ldots, p_n$ respectively. The expected value or arithmetic mean of X, denoted by E (X) or μ is defined by

$$E(X) = \mu = \sum_{i=1}^{n} x_i p_i = x_1 p_1 + x_2 p_2 + x_3 p_3 + \ldots + x_n p_n$$

In other words, the mean or expectation of a random variable X is the sum of the products of all possible values of X by their respective probabilities.

Definition : Let X be a random variable whose possible values $x_1, x_2, x_3, \ldots, x_n$ occur with probabilities $p_1, p_2, p_3, \ldots, p_n$ respectively. The variance of X, denoted by $Var\ (X)$ or σ_x^2 is defined as

$$\sigma_x^2 = Var\ (X) = \sum_{i=1}^{n} (x_i - \mu)^2 p_i$$

The non-negative number $\sigma_x = \sqrt{Var\ (X)}$ is called the **standard deviation** of the random variable X.

We can also use the simplified form of

$$Var\ (X) = \left(\sum_{i=1}^{n} x_i^2 p_i\right) - \left(\sum_{i=1}^{n} x_i p_i\right)^2$$

$$Var\ (X) = E(X^2) - [E(X)]^2 \quad \text{where} \sum_{i=1}^{n} x_i^2 p_i = E(X^2)$$

SOLVED EXAMPLES

Ex. 1 : Three coins are tossed simultaneously, X is the number of heads. Find expected value and variance of X.

Solution : $S = \{$HHH, HHT, HTH, THH, HTT, THT, TTH, TTT$\}$ and $X = \{0, 1, 2, 3\}$

$X = x_i$	$P = p_i$	$x_i p_i$	$x_i^2 p_i$
0	$\frac{1}{8}$	0	0
1	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$
2	$\frac{3}{8}$	$\frac{6}{8}$	$\frac{12}{8}$
3	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{9}{8}$
		$\sum_{i=1}^{n} x_i p_i = \frac{12}{8}$	$\sum_{i=1}^{n} x_i^2 p_i = \frac{24}{8}$

Then E (X) $= \sum_{i=1}^{n} x_i p_i = \frac{12}{8} = 1{\cdot}5$

$$Var\ (X) = \left(\sum_{i=1}^{n} x_i^2 p_i\right) - \left(\sum_{i=1}^{n} x_i p_i\right)^2 = \frac{24}{8} - (1{\cdot}5)^2 = 3 - 2{\cdot}25 = 0{\cdot}75$$

Ex. 2 : Let a pair of dice be thrown and the random variable X be the sum of the numbers that appear on the two dice. Find the mean or expectation of X and variance of X.

Solution : The sample space of the experiment consists of 36 elementary events in the form of ordered pairs (x_i, y_i), where $x_i = 1, 2, 3, 4, 5, 6$ and $y_i = 1, 2, 3, 4, 5, 6$.

The random variable X i.e. the sum of the numbers on the two dice takes the values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 or 12.

$X = x_i$	$P = p_i$	$x_i p_i$	$x_i^2 p_i$
2	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{4}{36}$
3	$\frac{2}{36}$	$\frac{6}{36}$	$\frac{18}{36}$
4	$\frac{3}{36}$	$\frac{12}{36}$	$\frac{48}{36}$
5	$\frac{4}{36}$	$\frac{20}{36}$	$\frac{100}{36}$
6	$\frac{5}{36}$	$\frac{30}{36}$	$\frac{180}{36}$
7	$\frac{6}{36}$	$\frac{42}{36}$	$\frac{294}{36}$
8	$\frac{5}{36}$	$\frac{40}{36}$	$\frac{320}{36}$
9	$\frac{4}{36}$	$\frac{36}{36}$	$\frac{324}{36}$
10	$\frac{3}{36}$	$\frac{30}{36}$	$\frac{300}{36}$
11	$\frac{2}{36}$	$\frac{22}{36}$	$\frac{242}{36}$
12	$\frac{1}{36}$	$\frac{12}{36}$	$\frac{144}{36}$
		$\sum_{i=1}^{n} x_i p_i = \frac{252}{36} = 7$	$\sum_{i=1}^{n} x_i^2 p_i = \frac{1974}{36} = 54{\cdot}83$

Then E (X) $= \sum_{i=1}^{n} x_i p_i = 7$

$$Var\,(X) = \left(\sum_{i=1}^{n} x_i^2 p_i\right) - \left(\sum_{i=1}^{n} x_i p_i\right)^2 = 54{\cdot}83 - (7)^2$$

$= 54{\cdot}83 - 49$

$= 5{\cdot}83$

Ex. 3 : Find the mean and variance of the number randomly selected from 1 to 15.

Solution : The sample space of the experiment is $S = \{1, 2, 3, \ldots, 15\}$.

Let X denote the number selected.

Then X is a random variable which can take values 1, 2, 3, …, 15. Each number selected is equiprobable therefore

$$P(1) = P(2) = P(3) = \ldots = P(15) = \frac{1}{15}$$

$$\mu = E(X) = \sum_{i=1}^{n} x_i p_i = 1 \times \frac{1}{15} + 2 \times \frac{1}{15} + 3 \times \frac{1}{15} + \ldots + 15 \times \frac{1}{15}$$

$$= (1 + 2 + 3 + \ldots + 15) \times \frac{1}{15} = \left(\frac{15 \times 16}{2}\right) \times \frac{1}{15} = 8$$

$$Var(X) = \left(\sum_{i=1}^{n} x_i^2 p_i\right) - \left(\sum_{i=1}^{n} x_i p_i\right)^2 = 1^2 \times \frac{1}{15} + 2^2 \times \frac{1}{15} + 3^2 \times \frac{1}{15} + \ldots + 15^2 \times \frac{1}{15} - (8)^2$$

$$= (1^2 + 2^2 + 3^2 + \ldots + 15^2) \times \frac{1}{15} - (8)^2$$

$$= \left(\frac{15 \times 16 \times 31}{6}\right) \times \frac{1}{15} - (8)^2$$

$$= 82 \cdot 67 - 64 = 18 \cdot 67$$

Ex. 4 : Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean, variance and standard deviation of the number of kings drawn.

Solution : Let X denote the number of kings in a draw of two cards. X is a random variable which can assume the values 0, 1 or 2.

Then $P(X = 0) = P(\text{ no card is king}) = \frac{{}^{48}C_2}{{}^{52}C_2} = \frac{48 \times 47}{52 \times 51} = \frac{188}{221}$

Then $P(X = 1) = P(\text{ exactly one card is king}) = \frac{{}^{4}C_1 \times {}^{48}C_1}{{}^{52}C_2} = \frac{4 \times 48 \times 27}{52 \times 51} = \frac{32}{221}$

Then $P(X = 2) = P(\text{ both cards are king}) = \frac{{}^{4}C_2}{{}^{52}C_2} = \frac{4 \times 3}{52 \times 51} = \frac{1}{221}$

$$\mu = E(X) = \sum_{i=1}^{n} x_i p_i = 0 \times \frac{188}{221} + 1 \times \frac{32}{221} + 2 \times \frac{1}{221} = \frac{34}{221}$$

$$Var(X) = \left(\sum_{i=1}^{n} x_i^2 p_i\right) - \left(\sum_{i=1}^{n} x_i p_i\right)^2 = \left(0^2 \times \frac{188}{221} + 1^2 \times \frac{32}{221} + 2^2 \times \frac{1}{221}\right) - \left(\frac{34}{221}\right)^2$$

$$= \frac{36}{221} - \frac{1156}{48841} = \frac{6800}{48841} = 0 \cdot 1392$$

$$\sigma = \sqrt{Var(X)} = \sqrt{0 \cdot 1392}$$

EXERCISE 7.1

1. Let X represent the difference between number of heads and number of tails obtained when a coin is tossed 6 times. What are possible values of X ?

2. An urn contains 5 red and 2 black balls. Two balls are drawn at random. X denotes number of black balls drawn. What are possible values of X ?

3. State which of the following are not the probability mass function of a random variable. Give reasons for your answer.

(i)

X	0	1	2
$P(X)$	0·4	0·4	0·2

(ii)

X	0	1	2	3	4
$P(X)$	0·1	0·5	0·2	–0·1	0·2

(iii)

X	0	1	2
$P(X)$	0·1	0·6	0·3

(iv)

Z	3	2	1	0	–1
$P(Z)$	0·3	0·2	0·4	0	0·05

(v)

Y	–1	0	1
$P(Y)$	0·6	0·1	0·2

(vi)

X	0	–1	–2
$P(X)$	0·3	0·4	0·3

4 Find the probability distribution of (i) number of heads in two tosses of a coin. (ii) Number of tails in the simultaneous tosses of three coins. (iii) Number of heads in four tosses of a coin.

5. Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as number greater than 4 appears on at least one die.

6. From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

7. A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution of number of tails.

8. A random variable X has the following probability distribution :

X	0	1	2	3	4	5	6	7
$P(X)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

Determine : (i) k (ii) $P(X < 3)$ (iii) $P(X > 4)$

9. Find expected value and variance of X for the following p.m.f.

X	–2	–1	0	1	2
$P(X)$	0·2	0·3	0·1	0·15	0·25

10. Find expected value and variance of X ,where X is number obtained on uppermost face when a fair die is thrown.

11. Find the mean number of heads in three tosses of a fair coin.

12. Two dice are thrown simultaneously. If X denotes the number of sixes, find the expectation of X.

13. Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find E (X).

14. Let X denote the sum of the numbers obtained when two fair dice are rolled. Find the standard deviation of X.

15. A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 years. One student is selected in such a manner that each has the same chance of being chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X ? Find mean, variance and standard deviation of X.

16. In a meeting, 70% of the members favour and 30% oppose a certain proposal. A member is selected at random and we take $X = 0$ if he opposed, and $X = 1$ if he is in favour. Find E (X) and *Var* (X).

7.4 Probability Distribution of a Continuous Random Variable :

A continuous random variable differs from a discrete random variable in the sense that the possible values of a continuous random variable form an interval of real numbers. In other words, a continuous random variable has uncountably infinite possible values.

For example, the time an athlete takes to complete a thousand-meter race is a continuous random variable.

We shall extend what we learnt about a discrete random variable to a continuous random variable. More specifically, we shall study the probability distribution of a continuous random variable with help of its probability density function (p. d. f.) and its cumulative distribution function (c. d. f.). If the possible values of a continuous random variable X form the interval $[a, b]$, where a and b are real numbers and $a < b$, then the interval $[a, b]$ is called the support of the continuous random variable X. The support of a continuous random variable is often denoted by S.

In case of a discrete random variable X that takes finite or countably infinite distinct values, the probability $P\ [X = x]$ is determined for every possible value x of the random variable X. The probability distribution of a continuous random variable is not defined in terms of probabilities of possible values of the random variable since the number of possible values are unaccountably infinite. Instead, the probability distribution of a continuous random variable is characterized by probabilities of intervals of the form $[\ c, d\ ]$, where $c < d$. That is, for a continuous random variable, the interest is in probabilities of the form $P\ [c < X < d]$, where $a \leq c < d \leq b$.

This probability is obtained by integrating a function of X over the interval $[c, d]$. Let us first define the probability density function (p.d.f.) of a continuous random variable.

7.4.1 Probability Density Function (p. d. f.) :

Let X be a continuous random variable defined on interval $S = (a\ , b)$. A non-negative integrable function $f\,(x)$ is called the probability density function (p. d. f.) of X if it satisfies the following conditions.

1. $f\,(x)$ is positive or zero every where in S, that is, $f\,(x) \geq 0$, for all $x \in S$.

2. The area under the curve $y = f\,(x)$ over S is 1. That is, $\int_S f\,(x)\,dx = 1$

3. The probability that X takes a value in A, where A is some interval, is given by the integral of $f\,(x)$ over that interval. That is

$$P\,[X \in A] = \int_A f\,(x)\,dx$$

7.4.2 Cumulative Distribution Functions (c. d. f.) :

The cumulative distribution function for continuous random variables is just a straightforward extension of that of the discrete case. All we need to do is replace the summation with an integral.

Definition : The **cumulative distribution function** (c. d. f.) of a continuous random variable X is defined as :

$$F\,(x) = \int_a^x f\,(t)\,dt \qquad \text{for } a < x < b.$$

You might recall, for discrete random variables, that $F\,(x)$ is, in general, a non-decreasing step function. For continuous random variables, $F\,(x)$ is a non-decreasing **continuous function**.

SOLVED EXAMPLES

Ex. 1 : Let X be a continuous random variable whose probability density function is $f\,(x) = 3x^2$, for $0 < x < 1$. note that $f\,(x)$ is not $P\,[X = x]$.

For example, $f\,(0.9) = 3(0.9)^2 = 2{\cdot}43 > 1$, which is clearly not a probability. In the continuous case, $f\,(x)$ is the height of the curve at $X = x$, so that the total area under the curve is 1. Here it is areas under the curve that define the probabilities.

Solution : Now, let's start by verifying that $f\,(x)$ is a valid probability density function.

For this, note the following results.

1. $f\,(x) = 3x^2 \geq 0$ for all $x \in [0, 1]$.

2. $\int_0^1 f\,(x) = \int_0^1 3x^2\,dx = 1$

Therefore, the function $f\,(x) = 3x^2$, for $0 < x < 1$ is a proper probability density function.

Also, for real numbers c and d such that $0 \le c < d \le 1$, note that

$$P[c < X < d] = \int_c^d f(x)\, dx = \int_c^d 3x^2\, dx = \left[x^3\right]_c^d = d^3 - c^3 > 0$$

- What is the probability that X falls between $\frac{1}{2}$ and 1? That is, what is $P\left[\frac{1}{2} < X < 1\right]$?

Substitute $c = \frac{1}{2}$ and $d = 1$ in the above integral to obtain

$$P\left[\frac{1}{2} < X < 1\right] = 1^3 - \left(\frac{1}{2}\right)^3 = 1 - \frac{1}{8} = \frac{7}{8}.$$

- What is $P\left(X = \frac{1}{2}\right)$?

See that the probability is 0 . This is so because

$$\int_c^d f(x)\, dx = \int_{1/2}^{1/2} x^3\, dx = 1 - 1 = 0.$$

$\left[\text{The ordinate AB, with A}\left(\frac{1}{2}, 0\right) \text{and B}\left(\frac{1}{2}, \frac{1}{8}\right) \text{is degenerate case of rectangle and has area 0}\right]$

As a matter of fact, in general, if X is a continuous random variable, then the probability that X takes any specific value x is 0. That is, when X is a continuous random variable, then

$P[X = x] = 0$ for every x in the support.

Ex. 2 : Let X be a continuous random variable whose probability density function is $f(x) = \frac{x^3}{4}$ for an interval $0 < x < c$. What is the value of the constant c that makes $f(x)$ a valid probability density function?

Solution : Note that the integral of the p. d. f. over the support of the random variable must be

That is, $\int_0^c f(x)\, dx = 1$.

That is, $\int_0^c \left(\frac{x^3}{4}\right) dx = 1$. But, $\int_0^c \left(\frac{x^3}{4}\right) dx = \left[\frac{x^4}{16}\right]_0^c = \frac{c^4}{16}$. Since this integral must be equal to 1,

we have $\frac{c^4}{16} = 1$, or equivalently $c^4 = 16$, so that $c = 2$ since c must be positive.

Ex. 3 : Let's return to the example in which X has the following probability density function :

$$f(x) = 3x^2$$

for $0 < x < 1$. What is the cumulative distribution function $F(x)$?

Solution : $F(x) = \int_{-0}^{x} f(x)\, dx = \int_0^x 3x^2\, dx = \left[x^3\right]_0^x = x^3$

Ex. 4 : Let's return to the example in which X has the following probability density function :

$$f(x) = \frac{x^3}{4} \quad \text{for } 0 < x < 4. \text{ What is the cumulative distribution function } X\text{?}$$

Solution : $F(x) = \int_0^x f(x)\,dx = \int_0^x \frac{x^3}{4}\,dx = \frac{1}{4}\left[\frac{x^4}{4}\right]_0^x = \frac{1}{16} = [x^4 - 0] = \frac{x^4}{16}$

Ex. 5 : Suppose the p.d.f. of a continuous random variable X is defined as:

$f(x) = x + 1$, for $-1 < x < 0$, and $f(x) = 1 - x$, for $0 \le x < 1$.

Find the c.d.f. $F(x)$.

Solution : If we look the p.d.f. it is defined in two steps

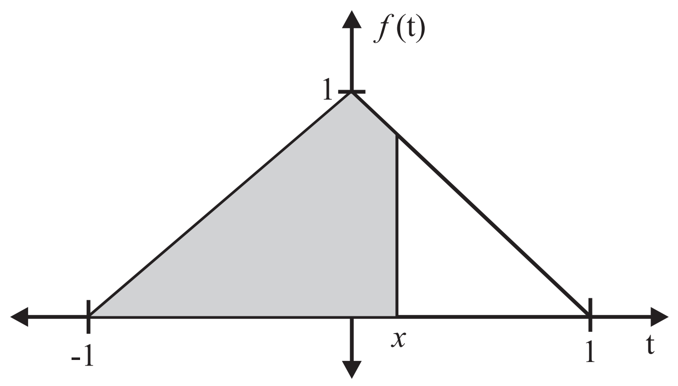

Fig. 7.1

Now for the other two intervals :

For $-1 < x < 0$ and $0 < x < 1$.

$$F(x) = \int_{-1}^{x} (x+1)\,dx$$

$$= \left[\frac{x^2}{2} + x\right]_{-1}^{x} = \left(\frac{x^2}{2} + x\right) - \left(\frac{1}{2} - 1\right)$$

$$= \frac{x^2}{2} + x + \frac{1}{2} = \frac{x^2 + 2x + 2}{2}$$

$$= \frac{(x+1)^2}{2}$$

$$F(0) = \frac{1}{2}$$

For $0 < x < 1$

$$F(x) = P(0 < x < 1) = P(-1 < x < 0) + P(0 < x < 1) = \int_{-1}^{0} (x+1)\,dx + \int_{0}^{x} (1-x)\,dx$$

$$= \frac{1}{2} + \int_0^x (1-x)\,dx$$

$$= \frac{1}{2} + x + \frac{x^2}{2}$$

$F(x) = 0$, for $x \le -1$

$F(x) = \frac{1}{2}(x+1)^2$, for $-1 < x \le 0$

$= \frac{1}{2} + x - \frac{x^2}{2}$, for $0 < x < 1$

If probability function $f(x)$ is defined on (a, b) with $f(x) \geq 0$ and $\int_a^b f(x)\, dx = 1$, then we can extend this function to the whole of 1R as follows.

For $x \leq a$ and $x \geq b$, define $f(x) = 0$.

Then note that $\int_{-\infty}^{t} f(x) = 0$, for $t \leq a$ and for $x \geq b$

$$\int_{-\infty}^{t} f(x)\, dx = \int_{-\infty}^{a} f(x)\, dx + \int_{a}^{b} f(x)\, dx + \int_{b}^{t} f(x)\, dx = 0 + 1 + 0$$

Thus $F(t) = 0$, for $t \leq a$ and $F(t) = 1$, for $t \geq b$

Ex. 6 : Verify if the following functions are p.d.f. of a continuous r.v. X.

(i) $f(x) = e^{-x}$, for $0 < x < \infty$ and $= 0$, otherwise.

(ii) $f(x) = \dfrac{x}{2}$, for $-2 < x < 2$ and $= 0$, otherwise.

Solution : (i) e^{-x} is ≥ 0 for any value of x since $e > 0$,

$\therefore \quad e^{-x} > 0$, for $0 < x < \infty$

$$\int_0^{\infty} f(x)\, dx = \int_0^{\infty} e^{-x}\, dx = \left[-e^{-x}\right]_0^{\infty} = \left[\frac{1}{e^{\infty}} - e^{0}\right] = -(0-1) = 1$$

Both the conditions of p.d.f. are satisfied $f(x)$ is p.d.f. of r.v.

(ii) $f(x) < 0$ i.e. negative for $-2 < x < 0$ therefore f(x) is not p.d.f.

Ex. 7 : Find k if the following function is the p.d.f. of r.v. X.

$f(x) = kx^2(1-x)$, for $0 < x < 1$ and $= 0$, otherwise.

Solution : Since $f(x)$ is the p.d.f. of r.v. X

$$\int_0^1 kx^2(1-x)\, dx = 1$$

$$\therefore \quad \int_0^1 k(x^2 - x^3)\, dx = 1$$

$$\therefore \quad k\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 1$$

$$\therefore \quad k\left\{\left[\frac{1}{3} - \frac{1}{4}\right] - (0)\right\} = 1$$

$$\therefore \quad k \times \frac{1}{12} = 1 \qquad \therefore \quad k = 12$$

Ex. 8 : For each of the following p.d.f. of r.v. X, find (a) $P(X < 1)$ and (b) $P(|X| < 1)$

(i) $f(x) = \dfrac{x^2}{18}$, for $-3 < x < 3$ and $= 0$, otherwise.

(ii) $f(x) = \dfrac{x+2}{18}$, for $-2 < x < 4$ and $= 0$, otherwise.

Solution :

(i) (a) $P(X < 1) = \displaystyle\int_{-3}^{1} \frac{x^2}{18}\,dx = \left[\frac{\frac{1}{18}(x^3)}{3}\right]_{-3}^{1} = \frac{1}{54}[1-(-3)^3] = \frac{1}{54}(1+27) = \frac{28}{54} = \frac{14}{27}$

(b) $P(|X| < 1) = P(-1 < x < 1) = \displaystyle\int_{-1}^{1} \frac{x^2}{18}\,dx = \left[\frac{\frac{1}{18}(x^3)}{3}\right]_{-1}^{1}$

$= \dfrac{1}{54}[1-(-1)^3] = \dfrac{1}{54}(1+1) = \dfrac{2}{54} = \dfrac{1}{27}$

(ii) (a) $P(X < 1) = \displaystyle\int_{-2}^{1} \frac{x+2}{18}\,dx = \frac{1}{18}\left[\frac{x^2}{2} + 2x\right]_{-2}^{1}$

$= \dfrac{1}{18}\left\{\left(\dfrac{1}{2}+2\right) - \left(\dfrac{(-2)^2}{2} + 2(-2)\right)\right\} = \dfrac{1}{18}\left\{\dfrac{5}{2}+2\right\} = \dfrac{1}{18} \times \dfrac{9}{2} = \dfrac{1}{4}$

(b) $P(|X| < 1) = P(-1 < x < 1) = \displaystyle\int_{-1}^{1} \frac{x+2}{18}\,dx = \frac{1}{18}\left[\frac{x^2}{2} + 2x\right]_{-1}^{1}$

$= \dfrac{1}{18}\left\{\left(\dfrac{1}{2}+2\right) - \left(\dfrac{1}{2} - 2\right)\right\} = \dfrac{1}{18}\left\{\dfrac{5}{2}+\dfrac{3}{2}\right\} = \dfrac{1}{18} \times 4 = \dfrac{2}{9}$

Ex. 9 : Find the c.d.f. $F(x)$ associated with p.d.f. $f(x)$ of r.v. X where

$f(x) = 3(1-2x^2)$ for $0 < x < 1$ and $= 0$, otherwise.

Solution : Since $f(x)$ is p.d.f. of r.v. therefore c.d.f. is

$$F(x) = \int_{0}^{x} 3(1-2x^2)\,dx = \left[3\left(x - \frac{2x^3}{3}\right)\right]_{0}^{x} = [\,3x - 2x^3\,] = 3x - 2x^3$$

EXERCISE 7.2

1. Verify which of the following is p.d.f. of r.v. X :

(i) $f(x) = \sin x$, for $0 \le x \le \dfrac{\pi}{2}$ (ii) $f(x) = x$, for $0 \le x \le 1$ and $= 2 - x$ for $1 < x < 2$

(iii) $f(x) = 2$, for $0 \le x \le 1$

2. The following is the p.d.f. of r.v. $X : f(x) = \frac{x}{8}$, for $0 < x < 4$ and $= 0$ otherwise

Find (a) $P(x < 1{\cdot}5)$ (b) $P(1 < x < 2)$ (c) $P(x > 2)$

3. It is known that error in measurement of reaction temperature (in 0° c) in a certain experiment is continuous r.v. given by

$f(x) = \frac{x^2}{3}$, for $-1 < x < 2$ and $= 0$ otherwise

(i) Verify whether $f(x)$ is p.d.f. of r.v. X. (ii) Find $P(0 < x \leq 1)$

(iii) Find probability that X is negative.

4. Find k if the following function represent p.d.f. of r.v. X.

(i) $f(x) = kx$, for $0 < x < 2$ and $= 0$ otherwise, Also find $P\left(\frac{1}{4} < x < \frac{3}{2}\right)$.

(ii) $f(x) = kx(1 - x)$, for $0 < x < 1$ and $= 0$ otherwise, Also find $P\left(\frac{1}{4} < x < \frac{1}{2}\right)$, $P\left(x < \frac{1}{2}\right)$.

5. Let X be amount of time for which a book is taken out of library by randomly selected student and suppose X has p.d.f.

$f(x) = 0{\cdot}5x$, for $0 \leq x \leq 2$ and $= 0$ otherwise.

Calculate: (i) $P(X \leq 1)$ (ii) $P(0{\cdot}5 \leq x \leq 1{\cdot}5)$ (iii) $P(x \geq 1{\cdot}5)$

6. Suppose that X is waiting time in minutes for a bus and its p.d.f. is given by

$f(x) = \frac{1}{5}$, for $0 \leq x \leq 5$ and $= 0$ otherwise.

Find the probability that (i) waiting time is between 1 and 3

(ii) waiting time is more than 4 minutes.

7. Suppose error involved in making a certain measurement is continuous r.v. X with p.d.f.

$f(x) = k(4 - x^2)$, for $-2 \leq x \leq 2$ and $= 0$ otherwise.

Compute : (i) $P(x > 0)$ (ii) $P(-1 < x < 1)$ (iii) $P(-0{\cdot}5 < x$ or $x > 0{\cdot}5)$

8. The following is the p.d.f. of continuous r.v.

$f(x) = \frac{x}{8}$, for $0 < x < 4$ and $= 0$ otherwise.

(i) Find expression for c.d.f. of X (ii) Find $F(x)$ at $x = 0{\cdot}5$, 1.7 and 5.

9. Given the p.d.f. of a continuous r.v. X, $f(x) = \frac{x^2}{3}$, for $-1 < x < 2$ and $= 0$ otherwise

Determine c.d.f. of X hence find $P(x < 1)$, $P(x < -2)$, $P(X > 0)$, $P(1 < x < 2)$

10. If a r.v. X has p.d.f.,

$f(x) = \frac{c}{x}$, for $1 < x < 3$, $c > 0$, Find c, $E(X)$ and $Var(X)$.

Let us Remember

- A random variable (r.v.) is a real-valued function defined on the sample space of a random experiment.

  The domain of a random variable is the sample space of a random experiment, while its co-domain is the real line.

  Thus $X : S \rightarrow R$ is a random variable.

  There are two types of random variables, namely discrete and continuous.

- **Discrete random variable :** Let the possible values of discrete random variable X be denoted by $x_1, x_2, x_3, \ldots,$ and the corresponding probabilities be denoted by $p_1, p_2, p_3, \ldots,$ where $p_i = P[X = x_i]$ for $i = 1, 2, 3, \ldots$. If there is a function f such that $p_i = P[X = x_i] = f(x_i)$ for all possible values of X, then f is called the probability mass function (p. m. f.) of X.

  Note : If x_i is a possible value of X and $p_i = P[X = x_i]$, then there is an event E_i in the sample space S such that $p_i = P[E_i]$. Since x_i is a possible value of X, $p_i = P[X = x_i] > 0$. Also, all possible values of X cover all sample points in the sample space S, and hence the sum of their probabilities is 1. That is, $p_i > 0$ for all i and $\sum p_i = 1$.

- **c.d.f ($F(x)$) :** The cumulative distribution function (c. d. f.) of a discrete random variable X is denoted by F and is defined as follows.

$$
\begin{aligned}
F(x) = P[X \leq x] &= \sum_{x_i < x} P[X = x_i] \\
&= \sum_{x_i < x} p_i \\
&= \sum_{x_i < x} f(x_i)
\end{aligned}
$$

- **Expected Value or Mean of Discrete r. v. :** Let X be a random variable whose possible values $x_1, x_2, x_3, \ldots, x_n$ occur with probabilities $p_1, p_2, p_3, \ldots, p_n$ respectively. The expected value or arithmetic mean of X, denoted by E (X) or μ is defined by

$$
\mathrm{E}(X) = \mu = \sum_{i=1}^{n} x_i p_i = x_1 p_1 + x_2 p_2 + x_3 p_3 + \ldots + x_n p_n
$$

  In other words, the mean or expectation of a random variable X is the sum of the products of all possible values of X by their respective probabilities.

Variance of Discrete r. v. : Let X be a random variable whose possible values $x_1, x_2, x_3, \ldots, x_n$ occur with probabilities $p_1, p_2, p_3, \ldots, p_n$ respectively. The variance of X, denoted by $Var(X)$ or σx^2 is defined as

$$\sigma_x^2 = Var(X) = \sum_{i=1}^{n} (x_i - \mu)^2 p_i$$

The non-negative number $\sigma_x = \sqrt{Var(X)}$

is called the **standard deviation** of the random variable X.

Another formula to find the variance of a random variable. We can also use the simplified form of

$$Var(X) = \left(\sum_{i=1}^{n} x_i^2 p_i\right) - \left(\sum_{i=1}^{n} x_i p_i\right)^2$$

$$Var(X) = E(X^2) - [E(X)]^2 \qquad \text{where } \sum_{i=1}^{n} x_i^2 p_i = E(X^2)$$

Probability Density Function (p. d. f.) : Let X be a continuous random variable defined on interval $S = (a, b)$. A non-negative integrable function $f(x)$ is called the probability density function (p. d. f.) of X if it satisfies the following conditions.

1. $f(x)$ is positive every where in S, that is, $f(x) > 0$, for all $x \in S$.

2. The area under the curve $f(x)$ over S is 1. That is, $\int_S f(x)\, dx = 1$

3. The probability that X takes a value in A, where A is some interval, is given by the integral of $f(x)$ over that interval. That is

$$P[X \in A] = \int_A f(x)\, dx$$

The **cumulative distribution function** (c. d. f.) of a continuous random variable X is defined as :

$$F(x) = \int_a^x f(t)\, dt \qquad \text{for } a < x < b.$$

MISCELLANEOUS EXERCISE 7

(I) Choose the correct option from the given alternatives :

(1) P.d.f. of a.c.r.v X is $f(x) = 6x(1-x)$, for $0 \le x \le 1$ and $= 0$, otherwise (elsewhere)

If $P(X < a) = P(X > a)$, then $a =$

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

(2) If the p.d.f of a.c.r.v. X is $f(x) = 3(1 - 2x^2)$, for $0 < x < 1$ and $= 0$, otherwise (elsewhere) then the c.d.f of X is $F(x) =$

(A) $2x - 3x^2$ (B) $3x - 4x^3$ (C) $3x - 2x^3$ (D) $2x^3 - 3x$

(3) If the p.d.f of a.c.r.v. X is $f(x) = \frac{x^2}{18}$, for $-3 < x < 3$ and $= 0$, otherwise then $P(|X| < 1) =$

(A) $\frac{1}{27}$ (B) $\frac{1}{28}$ (C) $\frac{1}{29}$ (D) $\frac{1}{26}$

(4) If a d.r.v. X takes values 0, 1, 2, 3, . . . which probability $P(X = x) = k(x + 1)\cdot 5^{-x}$, where k is a constant, then $P(X = 0) =$

(A) $\frac{7}{25}$ (B) $\frac{16}{25}$ (C) $\frac{18}{25}$ (D) $\frac{19}{25}$

(5) If p.m.f. of a d.r.v. X is $P(X = x) = \frac{({}^5C_x)}{2^5}$, for $x = 0, 1, 2, 3, 4, 5$ and $= 0$, otherwise

If $a = P(X \leq 2)$ and $b = P(X \geq 3)$, then $E(X) =$

(A) $a < b$ (B) $a > b$ (C) $a = b$ (D) $a + b$

(6) If p.m.f. of a d.r.v. X is $P(X = x) = \frac{x}{n(n+1)}$, for $x = 1, 2, 3, \ldots, n$ and $= 0$, otherwise then $E(X) =$

(A) $\frac{n}{1} + \frac{1}{2}$ (B) $\frac{n}{3} + \frac{1}{6}$ (C) $\frac{n}{2} + \frac{1}{5}$ (D) $\frac{n}{1} + \frac{1}{3}$

(7) If p.m.f. of a d.r.v. X is $P(x) = \frac{c}{x^3}$, for $x = 1, 2, 3$ and $= 0$, otherwise (elsewhere) then $E(X) =$

(A) $\frac{343}{297}$ (B) $\frac{294}{251}$ (C) $\frac{297}{294}$ (D) $\frac{294}{297}$

(8) If the a d.r.v. X has the following probability distribution :

X	-2	-1	0	1	2	3
$P(X = x)$	0.1	k	0.2	$2k$	0.3	k

then $P(X = -1) =$

(A) $\frac{1}{10}$ (B) $\frac{2}{10}$ (C) $\frac{3}{10}$ (D) $\frac{4}{10}$

(9) If the a d.r.v. X has the following probability distribution :

X	1	2	3	4	5	6	7
$P(X = x)$	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

then $k =$

(A) $\frac{1}{7}$ (B) $\frac{1}{8}$ (C) $\frac{1}{9}$ (D) $\frac{1}{10}$

(10) Find expected value of and variance of X for the following p.m.f.

X	-2	-1	0	1	2
$P(X)$	$0{\cdot}3$	$0{\cdot}4$	$0{\cdot}2$	$0{\cdot}15$	$0{\cdot}25$

(A) $0{\cdot}85$ (B) $-0{\cdot}85$ (C) $0{\cdot}15$ (D) $-0{\cdot}15$

(II) Solve the following :

(1) Identify the random variable as either discrete or continuous in each of the following. Write down the range of it.

(i) An economist is interested the number of unemployed graduate in the town of population 1 lakh.

(ii) Amount of syrup prescribed by physician.

(iii) The person on the high protein diet is interested gain of weight in a week.

(iv) 20 white rats are available for an experiment. Twelve rats are male. Scientist randomly selects 5 rats number of female rats selected on a specific day.

(v) A highway safety group is interested in studying the speed (km/hrs) of a car at a check point.

(2) The probability distribution of discrete r.v. X is as follows

$X=x$	1	2	3	4	5	6
$P(X=x)$	k	$2k$	$3k$	$4k$	$5k$	$6k$

(i) Determine the value of k. (ii) Find $P(X \leq 4)$, $P(2 < X < 4)$, $P(X \geq 3)$.

(3) The following probability distribution of r.v. X

$X=x$	-3	-2	-1	0	1	2	3
$P(X=x)$	$0{\cdot}05$	$0{\cdot}1$	$0{\cdot}15$	$0{\cdot}20$	$0{\cdot}25$	$0{\cdot}15$	0.1

Find the probability that

(i) X is positive. (ii) X is non negative. (iii) X is odd. (iv) X is even.

(4) The p.m.f. of a r.v. X is given by $P(X=x) = \dfrac{({}^5C_x)}{2^5}$, for $x = 0, 1, 2, 3, 4, 5$ and $= 0$, otherwise. Then show that $P(X \leq 2) = P(X \geq 3)$.

(5) In the p.m.f. of r.v. X

x	1	2	3	4	5
$P(X)$	$\frac{1}{20}$	$\frac{3}{20}$	a	$2a$	$\frac{1}{20}$

Find a and obtain c.d.f. of X.

(6) A fair coin is tossed 4 times. Let X denotes the number of heads obtained write down the probability distribution of X. Also find the formula for p.m.f. of X.

(7) Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as (i) number greater than 4 (ii) six appears on at least one die.

(8) A random variable X has the following probability distribution.

X	0	1	2	3	4	5	6	7
$P(X)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Determine (i) k (ii) $P(X>6)$ (iii) $P(0<X<3)$

(9) The following is the c.d.f. of r.v. X

X	–3	–2	–1	0	1	2	3	4
$F(X)$	0·1	0·3	0·5	0·65	0·75	0·85	0·9	1

Find p.m.f. of X. (i) $P(-1 \le X \le 2)$ (ii) $P(X \le 3 / X > 0)$

(10) Find the expected value, variance and standard deviation of r.v. X whose p.m.f. are given below.

(i)

$X=x$	1	2	3
$P(X=x)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

(ii)

$X=x$	–1	0	1
$P(X=x)$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{2}{5}$

(iii)

$X=x$	1	2	3	. . .	n
$P(X=x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	. . .	$\frac{1}{n}$

(iv)

$X=x$	0	1	2	3	4	5
$P(X=x)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

(11) A player tosses two coins he wins ₹ 10 if 2 heads appears , ₹ 5 if 1 head appears and ₹ 2 if no head appears. Find the expected winning amount and variance of winning amount.

(12) Let the p.m.f. of r.v. X be $P(x) = \frac{3-x}{10}$, for $x = -1, 0, 1, 2$ and $= 0$, otherwise

Calculate $E(X)$ and $Var(X)$.

(13) Suppose error involved in making a certain measurement is continuous r.v. X with p.d.f.

$f(x) = k(4-x^2)$, for $-2 \le x \le 2$ and $= 0$ otherwise.

Compute (i) $P(X>0)$ (ii) $P(-1<x<1)$ (iii) $P(X<0.5 \text{ or } X>0.5)$

(14) The p.d.f. of r.v. X is given by $f(x) = \frac{1}{2a}$, for $0 < x < 2a$ and $= 0$, otherwise.

Show that $P\left(X < \frac{a}{2}\right) = P\left(X > \frac{3a}{2}\right)$.

(15) The p.d.f. of r.v. of X is given by $f(x) = \frac{k}{\sqrt{x}}$, for $0 < x < 4$ and $= 0$, otherwise.

Determine k . Determine c.d.f. of X and hence $P(X \le 2)$ and $P(X \le 1)$.

8. BINOMIAL DISTRIBUTION

Let us Study

- Bernoulli Trial
- Binomial distribution
- Mean and variance of Binomial Distribution.

Let us Recall

- Many experiments are dichotomous in nature. For example, a tossed coin shows a 'head' or 'tail', A result of student 'pass' or 'fail', a manufactured item can be 'defective' or 'non-defective', the response to a question might be 'yes' or 'no', an egg has 'hatched' or 'not hatched', the decision is 'yes' or 'no' etc. In such cases, it is customary to call one of the outcomes a 'success' and the other 'not success' or 'failure'. For example, in tossing a coin, if the occurrence of the head is considered a success, then occurrence of tail is a failure.

Let us Learn

8.1.1 Bernoulli Trial :

Each time we toss a coin or roll a die or perform any other experiment, we call it a trial. If a coin is tossed, say, 4 times, the number of trials is 4, each having exactly two outcomes, namely, success or failure. The outcome of any trial is independent of the outcome of any other trial. In each of such trials, the probability of success or failure remains constant. Such independent trials which have only two outcomes usually referred to as 'success' or 'failure' are called Bernoulli trials.

Definition:

Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions :

(i) Each trial has exactly two outcomes : success or failure.

(ii) The probability of success remains the same in each trial.

Throwing a die 50 times is a case of 50 Bernoulli trials, in which each trial results in success (say an even number) or failure (an odd number) and the probability of success (p) is same for all 50 throws. Obviously, the successive throws of the die are independent trials. If the die is fair and has six numbers 1 to 6 written on six faces, then

$$p = \frac{1}{2} \text{ and } q = 1 - p \qquad \therefore\ q = \frac{1}{2}$$

For example :

Consider a die to be thrown 20 times. if the result is an even number, consider it a success, else it is a failure. Then $p = \frac{1}{2}$ as there are 3 even numbers in the possible outcomes.

If in the same experiment, we consider the result a success if it is a multiple of 3, then $p = \frac{1}{3}$ as there are 2 multiples of 3 among the six possible outcomes. Both above trials are Bernoulli trials.

SOLVED EXAMPLE

Ex. 1 : Six balls are drawn successively from an urn containing 7 red and 9 black balls. Tell whether or not the trials of drawing balls are Bernoulli trials when after each draw the ball drawn is

(i) replaced (ii) not replaced in the urn.

Solution :

(i) The number of trials is finite. When the drawing is done with replacement, the probability of success (say, red ball) is $p = \frac{7}{16}$ which is same for all six trials (draws). Hence, the drawing of balls with replacements are Bernoulli trials.

(ii) When the drawing is done without replacement, the probability of success (i.e. red ball) in first trial is $\frac{7}{16}$ in second trial is $\frac{6}{15}$ if first ball drawn is red and is $\frac{7}{15}$ if first ball drawn is black and so on. Clearly probability of success is not same for all trials, hence the trials are not Bernoulli trials.

8.2 Binomial distribution:

Consider the experiment of tossing a coin in which each trial results in success (say, heads) or failure (tails). Let S and F denote respectively success and failure in each trial. Suppose we are interested in finding the ways in which we have one success in six trials. Clearly, six different cases are there as listed below:

SFFFFF, FSFFFF, FFSFFF, FFFSFF, FFFFSF, FFFFFS.

Similarly, two successes and four failures can have $\frac{6!}{4\,!\times 2\,!} = 15$ combinations.

But as n grows large, the calculation can be lengthy. To avoid this the number for certain probabilities can be obtained with Bernoullis formula.For this purpose, let us take the experiment made up of three Bernoulli trials with probabilities p and $q = 1 - p$ for success and failure respectively in each trial. The sample space of the experiment is the set

$$S = \{SSS, SSF, SFS, FSS, SFF, FSF, FFS, FFF\}$$

The number of successes is a random variable X and can take values 0, 1, 2, or 3.The probability distribution of the number of successes is as below :

$$\begin{aligned}
P(X=0) &= P\text{ (no success)} \\
&= P(\{\text{FFF}\}) = P(F)\cdot P(F)\cdot P(F)\text{, since trials are independent.} \\
&= q \cdot q \cdot q = q^3 \\
P(X=1) &= P\text{ (one success)} \\
&= P(\{\text{SFF, FSF, FFS}\}) \\
&= P(\{\text{SFF}\}) + P(\{\text{FSF}\}) + P(\{\text{FFS}\}) \\
&= P(S)\cdot P(F)\cdot P(F) + P(F)\cdot P(S)\cdot P(F) + P(F)\cdot P(F)\cdot P(S) \\
&= p\cdot q\cdot q + q\cdot p\cdot q + q\cdot q\cdot p = 3pq^2 \\
P(X=2) &= P\text{ (two success)} \\
&= P(\{\text{SSF, SFS, FSS}\}) \\
&= P(\{\text{SSF}\}) + P(\{\text{SFS}\}) + P(\{\text{FSS}\}) \\
&= P(S)\cdot P(S)\cdot P(F) + P(S)\cdot P(F)\cdot P(S) + P(F)\cdot P(S)\cdot P(S) \\
&= p\cdot p\cdot q + p\cdot q\cdot p + q\cdot p\cdot p = 3p^2q
\end{aligned}$$

and

$$\begin{aligned}
P(X=3) &= P\text{ (three successes)} \\
&= P(\{\text{SSS}\}) \\
&= P(S)\cdot P(S)\cdot P(S) \\
&= p^3
\end{aligned}$$

Thus, the probability distribution of X is

X	0	1	2	3
$P(X)$	q^3	$3q^2p$	$3qp^2$	p^3

Also, the binominal expansion of

$$(q+p)^3 \text{ is } q^3 + 3q^2 p + 3\, qp^2 + p^3$$

Note that the probabilities of 0, 1, 2 or 3 successes are respectively the 1st, 2nd, 3rd and 4th term in the expansion of $(q+p)^3$.

Also, since $q+p=1$, it follows that the sum of these probabilities, as expected, is 1.Thus, we may conclude that in an experiment of n-Bernoulli trials, the probabilities of 0, 1, 2,..., n successes can be obtained as 1^{st}, 2^{nd}, 3^{rd}, . . . , $(n+1)^{th}$ terms in the expansion of $(q+p)^n$. To prove this assertion (result), let us find the probability of x successes in an experiment of n-Bernoulli trials.

Clearly, in case of x successes (S), there will be $(n-x)$ failures (F). Now x successes (S) and $(n-x)$ failures (F) can be obtained in $\dfrac{n!}{x!\,(n-x)!}$ ways.

In each of these ways the probability of x successes and $(n-x)$ failures

$$= P\,(x \text{ successes})\cdot P\big((n-x) \text{ failures}\big)$$

$$= \big(P\,(S)\cdot P\,(S) \ldots P\,(S)\ x \text{ times}\big)\cdot\big(P\,(F)\cdot P(F)\big)\cdot \ldots \cdot\big(P(F)\cdot(n-x) \text{ times}\big)$$

$$= \big(p\cdot p\cdot p \ldots p\ x \text{ times}\big)\ \big(q\cdot q\cdot q \ldots q\ (n-x) \text{ times}\big)$$

$$= p^x\cdot q^{n-x}$$

Thus probability of getting x successes in n-Bernoulli trial is

$$P\,(x \text{ successes out of } n \text{ trials}) = \frac{n!}{x!\,(n-x)!} \times p^x \times q^{n-x} = {}^nC_x\, p^x \times q^{n-x}$$

Clearly, P (x successes), i.e. ${}^nC_x\, p^x\, q^{n-x}$ is the $(x+1)^{th}$ term in the binomial expansion of $(q+p)^n$.

Thus, the probability distribution of number of successes in an experiment consisting of n-Bernoulli trials may be obtained by the binomial expansion of $(q+p)^n$. Hence, this distribution of number of successes X can be written as

X	0	1	2	. . .	x	. . .	n
$P\,(X)$	${}^nC_0\, p^0 \times q^n$	${}^nC_1 p^1 \times q^{n-1}$	${}^nC_2\, p^2 \times q^{n-2}$	. . .	${}^nC_x\, p^x \times q^{n-x}$	. . .	${}^nC_n\, p^n \times q^0$

The above probability distribution is known as binomial distribution with parameters n and p, because for given values of n and p, we can find the complete probability distribution. It is represented $X \sim B\,(n, p)$ as read as X follows binomial distribution with parameters n, p

The probability of x successes $P\,(X = x)$ is also denoted by $P\,(x)$ and is given by

$$P\,(x) = {}^nC_x\cdot q^{n-x} \times p^x,\ x = 0, 1, \ldots, n,\ \ (q = 1-p)$$

This $P\,(x)$ is called the **probability function** of the binomial distribution.

A binomial distribution with n-Bernoulli trials and probability of success in each trial as p, is denoted by $B\,(n, p)$ or $X \sim B\,(n, p)$.

Lets Note : (i) The number of trials should be fixed.

(ii) The trials should be independent.

SOLVED EXAMPLES

Ex. 1 : If a fair coin is tossed 10 times, find the probability of getting

(i) exactly six heads (ii) at least six heads (iii) at most six heads

Solution : The repeated tosses of a coin are Bernoulli trials. Let X denote the number of heads in an experiment of 10 trials.

Clearly, $X \sim B(n, p)$ with $n = 10$ and $p = \frac{1}{2}$, $q = 1 - p = 1 - \frac{1}{2}$ $\therefore$ $q = \frac{1}{2}$

$$P(X = x) = {}^{n}C_{x}\, p^{x} \times q^{n-x}$$

$$= {}^{10}C_{x} \left(\frac{1}{2}\right)^{x} \times \left(\frac{1}{2}\right)^{n-x}$$

(i) Exactly six successes means $x = 6$

$$P(X = 6) = {}^{10}C_{6} \left(\frac{1}{2}\right)^{6} \times \left(\frac{1}{2}\right)^{10-6} = \frac{10\,!}{6\,!\,(10-6)\,!} \times \left(\frac{1}{2}\right)^{6} \times \left(\frac{1}{2}\right)^{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \times \left(\frac{1}{2}\right)^{10}$$

$$= \frac{105}{512}$$

(ii) At least six successes means $x \geq 6$

$$P(X \geq 6) = \left[P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)\right]$$

$$= {}^{10}C_{6}\left(\frac{1}{2}\right)^{6} \times \left(\frac{1}{2}\right)^{4} + {}^{10}C_{7}\left(\frac{1}{2}\right)^{7} \times \left(\frac{1}{2}\right)^{3} + {}^{10}C_{8}\left(\frac{1}{2}\right)^{8} \times \left(\frac{1}{2}\right)^{2} + {}^{10}C_{9}\left(\frac{1}{2}\right)^{9} \times \left(\frac{1}{2}\right)^{1} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10} \times \left(\frac{1}{2}\right)^{0}$$

$$= \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \times \left(\frac{1}{2}\right)^{10} + \frac{10 \times 9 \times 8}{3 \times 2 \times 1} \times \left(\frac{1}{2}\right)^{10} + \frac{10 \times 9}{2 \times 1} \times \left(\frac{1}{2}\right)^{10} + 10\left(\frac{1}{2}\right)^{10} + \left(\frac{1}{2}\right)^{10}$$

$$= (210 + 120 + 45 + 10 + 1) \times \frac{1}{1024}$$

$$= \frac{386}{1024} = \frac{193}{512}$$

(iii) At most six successes means $x \leq 6$

$$P(X \leq 6) = 1 - (\mathrm{P}(X > 6)$$

$$= 1 - \left[P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)\right]$$

$$= 1 - \left[{}^{10}C_{7}\left(\frac{1}{2}\right)^{7} \times \left(\frac{1}{2}\right)^{3} + {}^{10}C_{8}\left(\frac{1}{2}\right)^{8} \times \left(\frac{1}{2}\right)^{2} + {}^{10}C_{9}\left(\frac{1}{2}\right)^{9} \times \left(\frac{1}{2}\right)^{1} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10} \times \left(\frac{1}{2}\right)^{0}\right]$$

$$= 1 - \left[\frac{10 \times 9 \times 8}{3 \times 2 \times 1} \times \left(\frac{1}{2}\right)^{10} + \frac{10 \times 9}{2 \times 1} \times \left(\frac{1}{2}\right)^{10} + 10\left(\frac{1}{2}\right)^{10} \times \left(\frac{1}{2}\right)^{10}\right]$$

$$= 1 - \left[(120 + 45 + 10 + 1) \times \frac{1}{1024}\right] = 1 - \frac{176}{1024} = 1 - \frac{88}{512} = \frac{512 - 88}{512} = \frac{424}{512} = \frac{53}{64}$$

Ex. 2 : Ten eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.

Solution : Let X denote the number of defective eggs in the 10 eggs drawn.

Since the drawing is done with replacement, the trials are Bernoulli trials.

Probability of success $= \frac{1}{10}$

$$p = \frac{1}{10}, \qquad q = 1 - p = 1 - \frac{1}{10} \qquad \therefore \quad q = \frac{9}{10}$$

$$n = 10$$

$$X \sim B\left(10, \frac{1}{10}\right)$$

$$P(X = x) = {}^{10}C_x \left(\frac{1}{10}\right)^x \times \left(\frac{9}{10}\right)^{10-x}$$

Here $X \geq 1$

$$P(X \geq 1) = 1 - {}^{10}C_0 \left(\frac{1}{10}\right)^0 \times \left(\frac{9}{10}\right)^{10}$$

$$= 1 - 1 \times 1 \times \left(\frac{9}{10}\right)^{10}$$

$$= 1 - \left(\frac{9}{10}\right)^{10}$$

8.3 Mean and Variance of Binomial Distribution (Formulae without proof) :

Let $X \sim B(n, p)$ then mean or expected value of r.v. X is denoted by μ or $E(X)$ and given by $\mu = E(X) = np$.

The variance is denoted by $Var(X)$ and given by $Var(X) = npq$.

Standard deviation of X is denoted by $SD(X)$ or σ and given by $SD(X) = \sigma_x = \sqrt{Var(X)}$

For example : If $X \sim B(10, 0{\cdot}4)$ then find $E(X)$ and $Var(X)$.

Solution : Here $n = 10$, $p = 0{\cdot}4$, $\quad q = 1 - p$

$$q = 1 - 0{\cdot}4 = 0{\cdot}6$$

$$E(X) = np$$

$$= 10 \times 0{\cdot}4 = 4$$

$$Var(X) = npq$$

$$= 10 \times 0{\cdot}4 \times 0{\cdot}6$$

$$= 2.4$$

SOLVED EXAMPLES

Ex. 1 : Let the p.m.f. of r.v. X be

$$P(X=x) = {}^4C_x \left(\frac{5}{9}\right)^x \times \left(\frac{4}{9}\right)^{4-x}, \text{ for } x = 0, 1, 2, 3, 4.$$

then find $E(X)$ and $Var(X)$.

Solution : $P(X=x)$ is binomial distribution with $n = 4$, $p = \frac{5}{9}$ and $q = \frac{4}{9}$

$$E(X) = np$$

$$= 4 \times \left(\frac{5}{9}\right) = \frac{20}{9}$$

$$Var(X) = npq$$

$$= 4 \times \left(\frac{5}{9}\right) \times \left(\frac{4}{9}\right) = \frac{80}{81}$$

Ex. 2 : If $E(X) = 6$ and $Var(X) = 4{\cdot}2$, find n and p.

Solution : $E(X) = 6$ therefore $np = 6$ and $Var(X) = 4{\cdot}2$ therefore $npq = 4{\cdot}2$

$$\frac{npq}{np} = \frac{4{\cdot}2}{6} \qquad \therefore \quad q = 0{\cdot}7$$

$$\therefore p = 1 - q = 1 - 0{\cdot}7 \qquad \therefore \quad p = 0{\cdot}3$$

$$np = 6$$

$$\therefore n \times 0{\cdot}3 = 6 \qquad \therefore \quad n = \frac{6}{0{\cdot}3} = 20$$

EXERCISE 8.1

(1) A die is thrown 6 times. If 'getting an odd number' is a success, find the probability of

(i) 5 successes (ii) at least 5 successes (iii) at most 5 successes.

(2) A pair of dice is thrown 4 times. If getting a doublet is considered a success, find the probability of two successes.

(3) There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

(4) Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. find the probability that

(i) all the five cards are spades (ii) only 3 cards are spades (iii) none is a spade.

(5) The probability that a bulb produced by a factory will fuse after 150 days of use is 0·05. Find the probability that out of 5 such bulbs

(i) none (ii) not more than one (iii) more than one (iv) at least one

will fuse after 150 days of use.

(6) A bag consists of 10 balls each marked with one of the digits 0 to 9. If four balls are drawn successively with replacement from the bag, what is the probability that none is marked with the digit 0?

(7) On a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

(8) A person buys a lottery ticket in 50 lotteries, in each of which his chance of winning a prize is 1/100. find the probability that he will win a prize

(i) at least once (ii) exactly once (iii) at least twice

(9) In a box of floppy discs it is known that 95% will work. A sample of three of the discs is selected at random. Find the probability that

(i) none (ii) 1 (iii) 2 (iv) all 3 of the sample will work.

(10) Find the probability of throwing at most 2 sixes in 6 throws of a single die.

(11) It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample of 12 such articles, 9 are defective?

(12) Given that X~ B (n, p)

(i) If $n = 10$ and $p = 0{\cdot}4$, find $E(X)$ and $Var(X)$ (ii) If $p = 0{\cdot}6$ and $E(X) = 6$, find n and $Var(X)$.

(iii) If $n = 25$, $E(X) = 10$ find p and $SD(X)$. (iv) If $n = 10$, $E(X) = 8$, find $Var(X)$.

Let us Remember

- Trials of a random experiment are called **Bernoulli trials**, if they satisfy the following conditions :

  (i) Each trial has exactly two outcomes : success or failure.

  (ii) The probability of success remains the same in each trial.

  Thus probability of getting x successes in n-Bernoulli trial is

  $$P\,(x \text{ successes out of } n \text{ trials}) = \frac{n\,!}{x\,!\,(n-x)\,!} \times p^x \times q^{n-x} = {}^nC_x\, p^x \times q^{n-x}$$

  Clearly, P (x successes), i.e. ${}^nC_x\, p^x\, q^{n-x}$ is the $(x+1)^{\text{th}}$ term in the binomial expansion of $(q+p)^n$.

- Let $X \sim B\,(n, p)$ then mean of expected value of r.v. X is denoted by μ.
  $E\,(X)$ and given by $\mu = E\,(X) = np$.

  The variance is denoted by $Var\,(X)$ and given by $Var\,(X) = npq$.

  Standard deviation of X is denoted by $SD\,(X)$ or σ and given by $SD\,(X) = \sigma_x = \sqrt{Var\,(X)}$

MISCELLANEOUS EXERCISE 8

(I) Choose the correct option from the given alternatives :

(1) A die is thrown 100 times. If getting an even number is considered a sucess, then the standard deviation of the number of successes is

(A) $\sqrt{50}$ (B) 5 (C) 25 (D) 10

(2) The mean and the variance of a binomial distribution are 4 and 2 respectively. Then the probablity of 2 successes is

(A) $\frac{128}{256}$ (B) $\frac{219}{256}$ (C) $\frac{37}{256}$ (D) $\frac{28}{256}$

(3) For a binomial distribution, $n = 5$. If $P(X = 4) = P(X = 3)$ then $p = \ldots$

(A) $\frac{1}{3}$ (B) $\frac{3}{4}$ (C) 1 (D) $\frac{2}{3}$

(4) In a binomial distribution, $n = 4$. If $2P(X = 3) = 3P(X = 2)$ then $p = \ldots$

(A) $\frac{4}{13}$ (B) $\frac{5}{13}$ (C) $\frac{9}{13}$ (D) $\frac{6}{13}$

(5) If $X \sim B(4, p)$ and $P(X = 0) = \frac{16}{81}$, then $P(X = 4) = \ldots$

(A) $\frac{1}{16}$ (B) $\frac{1}{81}$ (C) $\frac{1}{27}$ (D) $\frac{1}{8}$

(6) The probability of a shooter hitting a target is $\frac{3}{4}$.

How many minimum number of times must he fire so that the probability of hitting the target at least once is more than $0{\cdot}99$?

(A) 2 (B) 3 (C) 4 (D) 5

(7) If the mean and variance of a binomial distribution are 18 and 12 respectively, then $n = \ldots$

(A) 36 (B) 54 (C) 18 (D) 27

(II) Solve the following :

(1) Let $X \sim B(10, 0{\cdot}2)$, Find (i) $P(X = 1)$ (ii) $P(X \geq 1)$ (iii) $P(X \leq 8)$.

(2) Let $X \sim B(n, p)$

(i) If $n = 10$, $E(X) = 5$, find p and $Var(X)$.

(ii) If $E(X) = 5$ and $Var(X) = 2{\cdot}5$, find n and p.

(3) If fair coin is tossed 10 times find the probability that it shows heads

(i) 5 times. (ii) in the first four tosses and tail in last six tosses.

(4) Probability that bomb will hit target is $0 \cdot 8$. Find the probability that out of 10 bombs dropped exactly 2 will miss the target.

(5) The probability that a mountain-bike rider travelling along a certain track will have a tyre burst is $0 \cdot 05$. Find the probability that among 17 riders : (i) exactly one has a burst tyre

(ii) at most three have a burst tyre (iii) two or more have burst tyres.

(6) Probability that a lamp in a classroom will burnt out will be $0 \cdot 3$. Six lamps are fitted in the classroom. If it is known that the classroom is unusable if the number of lamps burning in it is less than four, find the probability that classroom can not used at random occasion.

(7) Lot of 100 items contains 10 defective items. Five items are selected at random from the lot and sent to the retail store. What is the probability that the store will receive at most one defective item?

(8) A large chain retailer purchases certain kind of electric device from manufacturer. The manufacturer indicates that the defective rate of the device is 3% . The inspector of the retailer picks 20 items from a shipment. What is the probability that the store will receive at most one defective item?

(9) The probability that the certain kind of component will survive a check test is $0 \cdot 6$. Find the probability that exactly 2 of the next 4 components tested survive.

(10) An examination consists of 10 multiple-choice questions, in each of which a candidate has to deduce which one of five suggested answers is correct. A completely unprepared student guesses each answer completely randomly. What is the probability that this student gets 8 or more questions correct? Draw the appropriate moral !

(11) The probability that a machine will produce all bolts in a production run within specification is $0 \cdot 998$. A sample of 8 machines is taken at random. Calculate the probability that

(i) all 8 machines (ii) 7 or 8 machines

(iii) at least 6 machines will produce all bolts within specification

(12) The probability that a machine develops a fault within the first 3 years of use is $0 \cdot 003$. If 40 machines are selected at random, calculate the probability that 38 or more will not develop any faults within the first 3 years of use.

(13) A computer installation has 10 terminals. Independently, the probability that any one terminal will require attention during a week is $0 \cdot 1$. Find the probabilities that

(i) 0 (ii) 1 (iii) 2

(iv) 3 or more, terminals will require attention during the next week.

(14) In a large school, 80% of the pupils like mathematics. A visitor to the school asks each of 4 pupils, chosen at random, whether they like mathematics.

(i) Calculate the probabilities of obtaining an answer yes from 0, 1, 2, 3, 4 of the pupils

(ii) Find the probability that the visitor obtains the answer yes from at least 2 pupils:

(a) when the number of pupils questioned remains at 4

(b) when the number of pupils questioned is increased to 8.

(15) It is observed that, it rains on 12 days out of 30 days. Find the probability that

(i) it rains exactly 3 days of week. (ii) it will rain on at least 2 days of given week.

(16) If probability of success in a single trial is $0 \cdot 01$. How many trials are required in order to have probability greater than $0 \cdot 5$ of getting at least one success?

(17) In binomial distribution with five Bernoulli's trials, probability of one and two success are $0 \cdot 4096$ and $0 \cdot 2048$ respectively. Find probability of success.

❖ ❖ ❖

ANSWERS

1. DIFFERENTIATION

EXERCISE 1.1

(1) (i) $5\,(3x^2-2)\,(x^3-2x-1)^4$

(ii) $\dfrac{5}{2}\left(3\sqrt{x}-4\sqrt[3]{x}\right)\left(2x^{\frac{3}{2}}-3x^{\frac{4}{3}}-5\right)^{\frac{3}{2}}$

(iii) $\dfrac{x+2}{\sqrt{x^2+4x-7}}$

(iv) $\dfrac{x\,(2\sqrt{x^2+1}+1)}{2\sqrt{x^2+1}\cdot\sqrt{x^2+\sqrt{x^2+1}}}$

(v) $-\dfrac{4x-7}{(2x^2-7x-5)^{\frac{8}{3}}}$

(vi) $\dfrac{15\,(3x-4)}{2\,(3x-5)^{\frac{3}{2}}}\left(\sqrt{3x-5}-\dfrac{1}{\sqrt{3x-5}}\right)^4$

(2) (i) $-\,2x\sin(x^2+a^2)$ (ii) $\dfrac{3e^{(3x+2)}}{2\sqrt{e^{(3x+2)}+5}}$

(iii) $\operatorname{cosec} x$ (iv) $\dfrac{\sec^2\sqrt{x}}{4\sqrt{x}\cdot\sqrt{\tan\sqrt{x}}}$

(v) $\dfrac{-\,9\operatorname{cosec}^2[\log(x^3)]\cdot\cot^2[\log(x^3)]}{x}$

(vi) $3\sin^2 x\cdot\cos x\cdot 5^{\sin^3 x+3}\cdot\log 5$

(vii) $\dfrac{\sin x\operatorname{cosec}\sqrt{\cos x}\cdot\cot(\sqrt{\cos x})}{2\sqrt{\cos x}}$

(viii) $-\,3x^2\tan(x^3-5)$

(ix) $5\sin 2x\cdot e^{3\sin^2 x-2\cos^2 x}$

(x) $\dfrac{-\,2x\cdot\sin[\,2\log(x^2+7)]}{x^2+7}$

(xi) $-\sec^2\left[\cos(\sin x)\right]\cdot\sin(\sin x)\cdot\cos x$

(xii) $4x^3\cdot\sec^2(x^4+4)\cdot\sec\left[\tan(x^4+4)\right]\cdot\tan\left[\tan(x^4+4)\right]$

(xiii) $\dfrac{2\log x}{x}-\dfrac{2}{x}$

(xiv) $\dfrac{\cos\sqrt{\sin\sqrt{x}}\cdot\cos\sqrt{x}}{4\sqrt{x}\cdot\sqrt{\sin\sqrt{x}}}$

(xv) $2x\cdot e^{x^2}\left[\tan(e^{x^2})\right]$ (xvi) $\dfrac{1}{2x\log x}$

(xvii) $\dfrac{2\left[\log[\log(\log x)]\right]}{x\log x\cdot\log(\log x)}$

(xviii) $4x\sin(2x^2)$

(3) (i) $6\,(x+2)(x^2+4x+1)^2+4\,(3x^2-5)\,(x^3-5x-2)^3$

(ii) $8\,(1-2x)\,(1+4x)^5\,(3+x-x^2)^7+20\,(1+4x)^4\,(3+x-x^2)^8$

(iii) $\dfrac{14-3x}{2(7-3x)^{\frac{3}{2}}}$ (iv) $\dfrac{6x^2(x^3+15)\,(x^3-5)^4}{(x^3+3)^6}$

(v) $\sin 2x\,(1+\sin^2 x)\,(1+\cos^2 x)^2\,(1-5\sin^2 x)$

(vi) $-\dfrac{\sin x}{2\sqrt{\cos x}}-\dfrac{\sin\sqrt{x}}{4\sqrt{x}\cdot\sqrt{\cos\sqrt{x}}}$

(vii) $3\sec 3x$ (viii) $\dfrac{\pi\cos x^\circ}{90\,(1-\sin x^\circ)^2}$

(ix) $-\dfrac{\operatorname{cosec}^2\left(\dfrac{\log x}{2}\right)}{2x}+\tan x\cdot\operatorname{cosec}^2 x$

(x) $\dfrac{8\,e^{4x}}{(e^{4x}+1)^2}$ (xi) $-\dfrac{e^{\sqrt{x}}}{\sqrt{x}\,(e^{\sqrt{x}}-1)^2}$

(xii) $6\operatorname{cosec}2x+4\cot x+\dfrac{14x}{x^2+7}$

(xiii) $3\operatorname{cosec}3x$ (xiv) $-\dfrac{5}{2}\operatorname{cosec}\left(\dfrac{5x}{2}\right)$

(xv) $-\sec x$

(xvi) $2\log 4 + \dfrac{3x}{x^2+5} - \dfrac{9x^2}{2(2x^3-4)}$

(xvii) $2x - \dfrac{6}{5-4x} + \dfrac{2}{7-6x}$

(xviii) $-\sin x \log a - \dfrac{6x}{x^2-3} - \dfrac{1}{x\log x}$

(xix) 0 (xx) $\dfrac{x(x^2+2)^3(7x^2+38)}{(x^2+5)^{\frac{3}{2}}}$

(4) (i) -16 (ii) 35 (iii) -20 (iv) 28

(5) -5 (6) $\dfrac{12}{5}$ (7) $x = 0$ or $\dfrac{2\pi}{3}$ or 2π

(8) $e^{2x} + 6e^x + 14$, e^{x^2+5}, $2x$, e^x, $f'[g(x)]\cdot g'(x)$, $2e^{2x} + 6e^x$, 8, $g'[f(x)]\cdot f'(x)$, $2xe^{x^2+5}$, $-2e^6$.

EXERCISE 1.2

(1) (i) $\dfrac{1}{2\sqrt{x}}$ (ii) $-\dfrac{1}{4\sqrt{x}\sqrt{2-\sqrt{x}}}$

(iii) $\dfrac{1}{3\sqrt[3]{(x-2)^2}}$, for $x > 2$ (iv) $\dfrac{2}{2x-1}$

(v) 2 (vi) e^x

(vii) $2e^{2x-3}$ (viii) $\dfrac{1}{x\log 2}$

(2) (i) $\dfrac{1}{x\cdot e^x(x+2)}$ (ii) $\dfrac{1}{\cos x - x\sin x}$

(iii) $\dfrac{1}{7^x(x\log 7+1)}$ (iv) $\dfrac{x}{2x^2+1}$

(v) $\dfrac{1}{1+\log x}$

(3) (i) $\dfrac{1}{14}$ (ii) $\dfrac{1}{4}$ (iii) $\dfrac{1}{12}$ (iv) $\dfrac{1}{5}$

(4) 1

(5) (i) and (ii) derivative proved.

(6) (i) $\dfrac{1}{x[1+(\log x)^2]}$ (ii) $\dfrac{e^x}{\sqrt{1-e^{2x}}}$

(iii) $-\dfrac{3x^2}{1+x^6}$ (iv) $-\dfrac{4^x\log 4}{1+4^{2x}}$

(v) $\dfrac{1}{2\sqrt{x}(1+x)}$ (vi) $\dfrac{x}{\sqrt{1-x^4}}$

(vii) $\dfrac{2}{\sqrt{2-x^2}}$ (viii) $\dfrac{3\sqrt{x}}{2\sqrt{1-x^3}}$

(ix) $9x^8$ (x) $2x$

(7) (i) $2xe^{x^2}$ (ii) $-5^x\log 5$ (iii) $\dfrac{1}{2}$

(iv) $-x$ (v) $-\dfrac{1}{2}$ (vi) -6

(vii) $-\dfrac{1}{6}$ (viii) $-\dfrac{3}{2}$ (ix) $-\dfrac{7}{2}$

(x) $-\dfrac{1}{2}$ (xi) $-\dfrac{1}{2}$ (xii) $\dfrac{2}{3}$

(8) (i) 1 (ii) 1 (iii) $\dfrac{1}{2\sqrt{x}}$

(iv) 3 (v) e^x (vi) $2^x\log 2$

(9) (i) $\dfrac{2}{1+x^2}$ (ii) $\dfrac{2}{1+x^2}$

(iii) $-\dfrac{2}{1+x^2}$ (iv) $\pm\dfrac{2}{\sqrt{1-x^2}}$

(v) $-\dfrac{3}{\sqrt{1-x^2}}$ (vi) $-\dfrac{2e^x}{1+e^{2x}}$

(vii) $\dfrac{2\cdot 3^x\log 3}{1+3^{2x}}$

(viii) $\dfrac{2\cdot 4^x\log 4}{1+4^{2x}}$ or $\left(\dfrac{4^{x+\frac{1}{2}}\log 4}{1+4^{2x}}\right)$

(ix) $-\dfrac{10}{1+25x^2}$ (x) $-\dfrac{3\sqrt{x}}{1+x^3}$

(xi) $\dfrac{5x\sqrt{x}}{1+x^5}$ (xii) $\dfrac{1}{2\sqrt{x}(1+x)}$

(10) (i) $\dfrac{3}{1+9x^2} + \dfrac{5}{1+25x^2}$

(ii) $\dfrac{7}{1+49x^2}-\dfrac{5}{1+25x^2}$

(iii) $\dfrac{1}{2\sqrt{x}}\left(\dfrac{3}{1+9x}-\dfrac{1}{1+x}\right)$

(iv) $2^x\log 2\left(\dfrac{3}{1+9(2^{2x})}+\dfrac{1}{1+2^{2x}}\right)$

(v) $2^x\log 2\left(\dfrac{2}{1+4(2^{2x})}-\dfrac{1}{1+2^{2x}}\right)$

(vi) $\dfrac{3a}{a^2+9x^2}+\dfrac{2a}{a^2+4x^2}$ (vii) 1

(viii) $\dfrac{2}{1+(2x+1)^2}-\dfrac{3}{1+(3x-4)^2}$

(ix) $\dfrac{2}{1+(2x+3)^2}+\dfrac{1}{1+(x-1)^2}$

EXERCISE 1.3

(1) (i) $\dfrac{(x+1)^2}{(x+3)^3\,(x+3)^4}\left[\dfrac{2}{x+1}-\dfrac{3}{x+2}-\dfrac{4}{x+3}\right]$

(ii) $\dfrac{1}{3}\sqrt[3]{\dfrac{4x-1}{(2x+3)(5-2x)^2}}\left(\dfrac{4}{4x-1}-\dfrac{2}{2x+3}+\dfrac{4}{5-2x}\right)$

(iii) $(x^2+3)^{\frac{3}{2}}\cdot\sin^3 2x\cdot 2^{x^2}\left[\dfrac{3x}{x^2+3}+6\cot 2x+2x\log 2\right]$

(iv) $\dfrac{(x^2+2x+2)^{\frac{3}{2}}}{(\sqrt{x}+3)^3\,(\cos x)^x}\left[\dfrac{3(x+1)}{x^2+2x+2}-\dfrac{3}{2\sqrt{x}\,(\sqrt{x}+3)}+x\tan x-\log(\cos x)\right]$

(v) $\dfrac{x^5\cdot\tan^3 4x}{\sin^2 3x}\left[\dfrac{5}{x}+24\operatorname{cosec} 8x-6\cot 3x\right]$

(vi) $x^{\tan^{-1}x}\left[\dfrac{\tan^{-1}x}{x}+\dfrac{\log x}{1+x^2}\right]$

(vii) $\sin^x x\quad[x\cot x+\log(\sin x)]$

(viii) $\cos(x^x)\cdot x^x(1+\log x)$

(2) (i) $ex^{e-1}+e^x+x^x(1+\log x)$

(ii) $x^{x^x}\cdot x^x\cdot\log x\left[1+\log x+\dfrac{1}{x\log x}\right]+e^{x^x}\cdot x^x(1+\log x)$

(iii) $(\log x)^x\left[\dfrac{1}{\log x}+\log(\log x)\right]+(\cos x)^{\cot x}[1+\operatorname{cosec}^2 x\log(\cos x)]$

(iv) $x^{e^x}\cdot e^x\left[\dfrac{1}{x}+\log x\right]+(\log x)^{\sin x}\left[\dfrac{\sin x}{x\log x}+\cos x\log(\log x)\right]$

(v) $\sec^2 x\cdot e^{\tan x}+(\log x)^{\tan x}\left[\dfrac{\tan x}{x\log x}+\sec^2 x\log(\log x)\right]$

(vi) $(\sin x)^{\tan x}[1+\sec^2 x\log(\sin x)]+(\cos x)^{\cot x}[1+\operatorname{cosec}^2 x\log(\cos x)]$

(vii) $10^{x^x}x^x\log 10(1+\log x)+x^{x^{10}}\cdot x^9(1+10\log x)+x^{10^x}\cdot 10^x\left(\dfrac{1}{x}+\log x\cdot\log 10\right)$

(viii) 2

(3) (i) $-\sqrt{\dfrac{y}{x}}$ (ii) $-\sqrt{\dfrac{x}{y}}$

(iii) $-\dfrac{\sqrt{y}\,(2\sqrt{x}+\sqrt{y})}{\sqrt{x}\,(2\sqrt{y}+\sqrt{x})}$ (iv) $-\dfrac{3x^2+2xy+y^2}{x^2+2xy+3y^2}$

(v) $-\dfrac{y}{x}$ (vi) $-\dfrac{e^y+ye^x}{e^x+xe^y}$

(vii) $\dfrac{\sin(x-y)+e^{x+y}}{\sin(x-y)-e^{x+y}}$ (viii) $-\dfrac{1+y\sin(xy)}{1+x\sin(xy)}$

(ix) $\dfrac{y\,(1-xe^{x-y})}{x\,(1-ye^{x-y})}$

(x) $\dfrac{\sin(x-y)-\cos(x+y)-1}{\sin(x-y)+\cos(x+y)-1}$

EXERCISE 1.4

(1) (i) $\dfrac{1}{t}$ (ii) $\dfrac{b}{a}\cos\theta$ (iii) $\dfrac{2}{\sqrt{a^2+m^2}}$

(iv) $\sec^3\theta$ (v) $\dfrac{b}{a}\tan\left(\dfrac{\theta}{2}\right)$

(vi) $\dfrac{y\,(t^2+1)\log a}{axt}$ (vii) $-\dfrac{1}{2}$ (viii) $\dfrac{1}{3}$

(2) (i) $\dfrac{3\sqrt{3}}{2}$ (ii) $-\sqrt{3}$ (iii) $-\dfrac{\pi}{6}$

(iv) $1-\sqrt{2}$ (v) $3+\pi$

(4) (i) $\dfrac{x\cos x+\sin x}{\sec^2 x}$ (ii) 1

(iii) $-\dfrac{1}{2}$ (iv) 2 (v) $-x\,(\log x)^2\cdot 3^x$

(vi) $-\dfrac{x\sqrt{x^2-1}}{2}$

(vii) $\dfrac{(1+\log x)\cdot x^{x+1-\sin x}}{\sin x+x\cos x\cdot\log x}$

(viii) $\dfrac{\sqrt{1-x^2}}{4(1+x^2)}$

EXERCISE 1.5

(1) (i) $40x^3-24x-\dfrac{12}{x^4}$

(ii) $2e^{2x}(1+\tan x)\cdot(2+\tan x+\tan^2 x)$

(iii) $-e^{4x}(9\cos 5x+40\sin 5x)$

(iv) $x(5+6\log x)$ (v) $-\dfrac{1+\log x}{(x\log x)^2}$

(iv) $x^{x-1}+x^x(1+\log x)^2$

(2) (i) $-\dfrac{1}{4a}\operatorname{cosec}^4\left(\dfrac{\theta}{2}\right)$ (ii) $-\dfrac{1}{4at^3}$

(iii) 6 (iv) $-\dfrac{2\sqrt{2}\,b}{a^2}$

(4) (i) $\dfrac{d^ny}{dx^n}=\dfrac{m!\,a^n(ax+b)^{m-n}}{(m-n)!}$ if $m>0, m>n$,

$\dfrac{d^ny}{dx^n}=0$ if $m>0,\ m<n$

$\dfrac{d^ny}{dx^n}=n!\,a^n$ if $m>0,\ m=n$

(ii) $\dfrac{(-1)^n\,n!}{x^{n+1}}$ (iii) a^ne^{ax+b}

(iv) $p^na^{px+q}(\log a)^n$

(v) $\dfrac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$ (vi) $\cos\left(\dfrac{n\pi}{2}+x\right)$

(vii) $a^n\sin\left(\dfrac{n\pi}{2}+ax+b\right)$

(viii) $(-2)^n\cos\left(\dfrac{n\pi}{2}+3-2x\right)$

(ix) $\dfrac{(-1)^{n-1}(n-1)!\,2^n}{(2x+3)^n}$

(x) $\dfrac{(-1)^n\cdot n!\cdot 3^n}{(3x-5)^{n+1}}$

(xi) $e^{ax}(a^2+b^2)^{\frac{n}{2}}\cdot\cos\left[bx+c+n\tan^{-1}\left(\dfrac{b}{a}\right)\right]$

(xii) $e^{8x}\cdot(10)^n\cos\left[6x+7+n\tan^{-1}\left(\dfrac{3}{4}\right)\right]$

MISCELLANEOUS EXERCISE 1

(I)

1	2	3	4	5	6	7	8	9	10	11	12
D	C	C	B	A	C	D	C	B	C	A	B

(II) (1) $\frac{3}{4}$ (ii) Does not exist (iii) -2

(iv) $-\frac{1}{2\cdot\sqrt{1-x^2}}$ [Hint : $x = \cos 2\theta$]

(v) $\frac{3}{1+9x^2}+\frac{5}{1+25x^2}$

(vi) $\frac{1}{2\,(1+x^2)}$ [Hint : $x = \tan\theta$]

(2) (A) 3, (B) 5, (C) 4, (D) 1.

(3) (i) $-\frac{1}{9}$ (ii) $-\frac{40}{3}$ (iii) $-\frac{29}{96}$

(iv) $-\frac{4}{9}$

(4) (i) $-\frac{x}{\sqrt{1-x^2}}$ [Hint : $x = \cos 2\theta$]

(ii) $-\frac{1}{2}$ [Hint : $x = \cos 2\theta$]

(iii) $\frac{3}{2\sqrt{x}\,(1+x)}$ [Hint : $\sqrt{x} = \tan\theta$]

(6) (i) $\frac{\sqrt{1-x^2}}{4\,(1+x^2)}$

(ii) $-\frac{2x}{\sqrt{1+x^2}.\ \sin(\log x)}$ (iii) 1

2. APPLICATIONS OF DERIVATIVES

EXERCISE 2.1

(1) (i) $2x - y + 4 = 0,\ x + 2y - 8 = 0$

(ii) $4x - 5y + 12 = 0,\ 5x + 4y - 26 = 0,$

(iii) $y = 2,\ x = \sqrt{3}$

(iv) $\pi x + 2y - 2\pi = 0,$

$4x - 2\pi y + \pi^2 - 4 = 0$

(v) $2x - y = 0,\ 4x + 8y - 5\pi = 0$

(vi) $4x + 2y - 3 = 0,\ 2x - 4y + 1 = 0$

(vii) $17x - 4y - 20 = 0,\ 8x + 34y - 135 = 0$

(2) (4, 1)

(3) $(2, -2)\left(-\frac{2}{3}, -\frac{14}{27}\right)$ (4) $y = 0$ and $y = 4$

(5) $x + 3y - 8 = 0,\ x + 3y + 8 = 0$

(6) $a = 2,\ b = -7$ (7) (4, 11) and $\left(-4, -\frac{31}{3}\right)$

(8) 0.8 π cm^2/sec. (9) 6 cm^3/ sec.

(10) $\frac{3\sqrt{6}}{2}$ cm^2/ sec. (11) 8 cm^2/ sec

(12) 7.2 cm^3/ sec (13) 3 km/hr

(14) (i) $\left(\frac{3}{8}\right)$ meter/sec. (ii) $\frac{9}{8}$ meter/sec.

(15) 0.9 meter/sec. (16) $\left(\frac{4\pi}{3}\right)$ cm^3/ sec

EXERCISE 2.2

(1) (i) 2.9168 (ii) 3.03704 (iii) 1.9997

(iv) 248.32 (v) 64.48

(2) (i) 0.953 (ii) 0.42423 (iii) 0.4924

(iv) 1.02334

(3) (i) 0.7845 (ii) 0.7859 (iii) 0.7859

(4) (i) 2.70471 (ii) 8.1279 (iii) 9.09887

(5) (i) 4.6152 (ii) 2.1983 (iii) 3.006049

(6) (i) 6.91 (ii) 9.72

EXERCISE 2.3

(1) (i) Valid (ii) Valid

(iii) Invalid (iv) Valid

(v) Invalid (vi) Invalid

(2) $b = 1$

(3) (i) $\frac{\pi}{4}$ or $\frac{5\pi}{4}$ (ii) $c = \pi$ (iii) $c = \frac{5}{2}$

(4) $p = -6, q = 11$ (6) $c = -2$

(7) (i) $e - 1$ (ii) $2 \pm \frac{2}{\sqrt{3}}$ (iii) $\frac{1}{7}$

(iv) $\frac{1}{2}$ (v) $3 + \sqrt{2}$

EXERCISE 2.4

(1) (i) Increasing $\forall\ x \in R$

(ii) Decreasing$\forall\ x \in R$

(iii) Increasing $\forall\ x \in R$

(2) (i) $x < -1$ and $x > 2$ (ii) $R - \{1\}$

(iii) $x < -2$ and $x > 6$

(3) (i) $-1 < x < 2$ (ii) $(-5, 5) - \{0\}$

(iii) $x \in (2, 4)$

(4) (a) $(-\infty, -4] \cup [12, \infty)$

(b) $-4 \leq x \leq 12$ i.e. $[-4, 12]$

(5) (a) $x < -3$ and $x > 8$ (b) $-3 < x < 8$

(6) (a) $-1 < x < 1$ (b) $(-\infty, -1) \cup (1, \infty)$

(9) (i) Max $= \frac{36}{25}$, Min $= -\frac{16}{27}$

(ii) Max $= -3$, Min $= -128$

(iii) Max $= 20$, Min $= 16$ (iv) Min $= 8$

(v) Min $= -\frac{1}{e}$ (vi) Max $= \frac{1}{e}$

(10) 15, 15 (11) 10, 10 (12) 9 (13) 12.8

(14) $l = \sqrt{2}$ and $b = \frac{1}{\sqrt{2}}$

(15) Radius = Height = a (16) 3, 3

(17) Side of square base = 8 cm, Height = 4 cm

(18) $x = 75$, $P = 4000$ (19) 6, 9

(22) $\frac{4\pi r^3}{3\sqrt{3}}$ cm^3

MISCELLANEOUS EXERCISE 2

(I)

1	2	3	4	5	6	7	8	9	10
A	C	B	B	D	C	D	A	D	D

(II) (2) 4

(3) $14x - 13y + 12 = 0,\ 13x + 14y - 41 = 0$

(4) $\dfrac{2}{9\pi}$ ft/sec (5) $\left(\dfrac{16}{3}, 3\right), \left(-\dfrac{16}{3}, -3\right)$

(6) $c = 0$ (7) $c = 2$ (8) 2.025

(9) 1.03565

(10) Decreasing in $\left(0, \dfrac{1}{e}\right]$ and

Increasing in $\left[\dfrac{1}{e}, \infty\right)$

(11) Increasing in $[e, \infty)$, Decreasing in $(1, e]$

(15) $l = \dfrac{60}{\pi + 4},\ b = \dfrac{30}{\pi + 4},\ r = \dfrac{30}{\pi + 4}$

(17) Side $= \dfrac{l}{\pi + 4}$, Radius $= \dfrac{l}{2(\pi + 4)} = \dfrac{x}{2}$

(18) 24, 45 (21) Max $= \dfrac{5}{4}$, Min $= 1$

3. INDEFINITE INTEGRATION

EXERCISE 3.1

(1) (i) $\dfrac{x^4}{4} + \dfrac{x^3}{3} - \dfrac{x^2}{2} + x + c$ (ii) $\dfrac{x^3}{3} - 2x^2 + 4x + c$

(iii) $3 \tan x - 4 \log x - \dfrac{2}{\sqrt{x}} - 7x + c$

(iv) $\dfrac{x^2}{4} - \dfrac{5x^2}{2} + 3 \log x - \dfrac{1}{x^4} + c$

(v) $\dfrac{6}{5} x^2 \sqrt{x} - 4\sqrt{x} - \dfrac{10}{\sqrt{x}} + c$

(2) (i) $\tan x - x + c$ (ii) $-2 \cos x + c$

(iii) $\sec x + c$ (iv) $-\cot x - 2x + c$

(v) $-\cot x - \tan x + x + c$

(vi) $\sec x - \tan x + x + c$

(vii) $\sec x - \tan x + x + c$

(viii) $\sin x - \cos x + c$ (ix) $-\sqrt{2} \cos x + c$

(x) $-\dfrac{1}{14} \cos 7x - \dfrac{1}{2} \cos x + c$

(3) (i) $x - 2 \log (x + 2) + c$

(ii) $2x + \dfrac{1}{2} \log (2x + 1) + c$

(iii) $\dfrac{5}{3} x - \dfrac{26}{9} \log (3x - 4) + c$

(iv) $\dfrac{2 (x + 5)^{\frac{3}{2}}}{3} - 14 \sqrt{x + 5} + c$

(v) $\dfrac{1}{12} (4x - 1)^{\frac{3}{2}} - \dfrac{13}{4} \sqrt{4x - 1} + c$

(vi) $-\cos 2x + c$

(vii) $\dfrac{2}{5} \left(\sin \dfrac{5x}{2} - \cos \dfrac{5x}{2}\right) + c$

(viii) $\dfrac{1}{4} (2x + \sin 2x) + c$

(ix) $-\dfrac{4}{9} \left[x^{\frac{3}{2}} + (x + 3)^{\frac{3}{2}}\right] + c$

(x) $\dfrac{2}{21} \left[(7x - 2)^{\frac{3}{2}} + (7x - 5)^{\frac{3}{2}}\right] + c$

(4) $f(x) = \dfrac{x^2}{2} + \dfrac{3}{2x^2} + \dfrac{7}{2}$

EXERCISE 3.2 (A)

I. 1. $\dfrac{(\log x)^{n+1}}{n + 1} + c$ 2. $\dfrac{2}{5} (\sin^{-1} x)^{\frac{5}{2}} + c$

3. $\log \left(\operatorname{cosec} (x + \log x) - \cot (x + \log x)\right) + c$

4. $\dfrac{-1}{\sqrt{\tan (x^2)}} + c$ 5. $\dfrac{1}{3} (e^{3x} + 1) + c$

6. $\dfrac{1}{\log a} \cdot a^{x+\tan^{-1}x} + c$

7. $\dfrac{1}{2}\left[\log(\sin e^x)\right]^2 + c$

8. $\log(e^x - e^{-x}) + c$

9. $\dfrac{1}{5}\sin^5 x - \dfrac{1}{7}\sin^7 x + c$

10. $\dfrac{1}{48}\log\left(4x^{12}+5\right) + c$

11. $\dfrac{1}{10}\tan x^{10} + c$ 12. $\dfrac{1}{4}\log(x^4+1) + c$

13. $2\sqrt{\tan x} + c$ 14. $\tan^{-1}x + \dfrac{1}{x^2+1} + c$

15. $\log(3\cos^2 x + 4\sin^2 x) + c$

16. $2\tan^{-1}\sqrt{x} + c$ 17. $\log\left(10^x + x^{10}\right) + c$

18. $\dfrac{\sqrt{1+4x^n}}{2n} + c$

19. $\dfrac{4}{5}(x+2)^{\frac{5}{2}} - 2(x+2)^{\frac{3}{2}} + c$

20. $\dfrac{1}{7}(a^2+x^2)^{\frac{7}{2}} - \dfrac{4a^2}{5}(a^2+x^2)^{\frac{5}{2}} + \dfrac{2a^4}{3}(a^2+x^2)^{\frac{3}{2}} + c$

21. $-2\sqrt{2-3x} - \dfrac{2}{9}(2-3x)^{\frac{3}{2}} + c$

22. $\dfrac{5}{12}(2x+3)^{\frac{3}{2}} - \dfrac{11}{2}(2x+3)^{\frac{1}{2}} - \dfrac{49}{4\sqrt{2x+3}} + c$

23. $\dfrac{1}{3}\sin^{-1}\left(\dfrac{x^3}{3}\right) + c$ 25. $\dfrac{1}{3}\log\left(\dfrac{x^3-1}{x^3}\right) + c$

24. $\log(\log(\log x)) + c$

II. 1. $2\cdot\log\left(\sec\dfrac{x}{2}\right) + c$

2. $\cos a\cdot\log(\sin(x-a)) - (\sin a)\,x + c$

3. $\cos(a+b)\cdot\log(\sec(x+b)) - (\sin(a+b))\cdot x + c$

4. $\log(\tan x + 2) + c$

5. $\dfrac{11}{75}x + \dfrac{2}{25}\log(3\sin x + 4\cos x) + c$

6. $\dfrac{2x}{13} + \dfrac{3}{13}\log(2\cos x + 3\sin x) + c$

7. $5x - 3\log|2e^x - 5| + c$

8. $-5x - \log|3e^x - 4| + c$

9. $-x + \dfrac{7}{8}\log|4e^{2x} - 5| + c$

10. $\dfrac{\cos^8 x}{8} + \dfrac{\cos^6 x}{6} + \dfrac{\cos^4 x}{4} + \dfrac{\cos^2 x}{2} + \dfrac{1}{2}\log(\cos^2 x - 1) + c$

11. $\dfrac{\tan^4 x}{4} - \dfrac{\tan^2 x}{2} + \log(\sec x) + c$

12. $\sin x - \sin^3 x + \dfrac{3}{5}\sin^5 x - \dfrac{1}{7}\sin^7 x + c$

13. $\dfrac{1}{6}\log\left[\dfrac{(\sec 3x)^2}{(\sec 2x)^3(\sec x)^6}\right] + c$

14. $\dfrac{1}{6}\cos^{11} x - \dfrac{1}{9}\cos^9 x + \dfrac{1}{13}\cos^{13} x + c$

15. $-\dfrac{1}{\log 3}\cdot 3^{\cos^2 x} + c$

16. $\dfrac{1}{20}\log\left[\dfrac{\sin^5 4x}{\sin^2 10x}\right] + c$

17. $\dfrac{1}{2}\log\left[(1+\cos^2 x) - \cos^2 x\right] + c$

EXERCISE 3.2 (B)

I. 1. $\dfrac{1}{4\sqrt{3}}\log\left(\dfrac{2x-\sqrt{3}}{2x+\sqrt{3}}\right) + c$

2. $\dfrac{1}{30}\log\left(\dfrac{5+3x}{5-3x}\right) + c$

3. $\dfrac{1}{\sqrt{14}}\tan^{-1}\left(\dfrac{\sqrt{2}x}{\sqrt{7}}\right) + c$

4. $\dfrac{1}{\sqrt{3}}\log\left(x + \sqrt{x^2 + \dfrac{8}{3}}\right) + c$

5. $\frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{11}}\right)+c$

6. $\frac{1}{\sqrt{2}}\log\left(x+\sqrt{x^2-\frac{5}{2}}\right)+c$

7. $9\sin^{-1}\left(\frac{x}{9}\right)-\sqrt{9-x^2}+c$

8. $2\sin^{-1}\left(\frac{x}{2}\right)-\sqrt{4-x^2}+c$

9. $2\sin^{-1}\left(\frac{x}{10}\right)-\frac{1}{2}(\sqrt{100-x^2})+c$

10. $\frac{1}{4}\log\left|\frac{x+2}{x+6}\right|+c$

11. $\frac{1}{\sqrt{5}}\log\left(\frac{\sqrt{5}-1+2x}{\sqrt{5}+1-2x}\right)+c$

12. $\frac{1}{8\sqrt{2}}\log\left(\frac{2x-5-2\sqrt{2}}{2x-5+2\sqrt{2}}\right)+c$

13. $\frac{1}{2\sqrt{19}}\log\left(\frac{3x+2+\sqrt{19}}{3x+2-\sqrt{19}}\right)+c$

14. $\frac{1}{\sqrt{3}}\log\left(x+\frac{5}{6}+\sqrt{x^2+\frac{5}{3}x+\frac{7}{3}}\right)+c$

15. $\log\left(x+4+\sqrt{x^2-8x-20}\right)+c$

16. $\frac{1}{\sqrt{2}}\log\left(x-\frac{3}{4}+\sqrt{x^2-\frac{3}{2}x+4}\right)+c$

17. $\log\left(x-\frac{1}{2}+\sqrt{x^2-x-6}\right)+c$

18. $\frac{1}{2\sqrt{7}}\tan^{-1}\left(\frac{2\tan x}{\sqrt{7}}\right)+c$

19. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right)+c$

20. $\frac{1}{2\sqrt{3}}\log\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|+c$

II. 1. $\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{2\tan\frac{x}{2}+2}{\sqrt{5}}\right)+c$

2. $\frac{1}{3}\log\left[\frac{3\tan\left(\frac{x}{2}\right)-1}{3\tan\left(\frac{x}{2}\right)+1}\right]+c$

3. $\sqrt{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}-1}{\sqrt{2}}\right)+c$

4. $\tan^{-1}\left[2\tan\left(\frac{x}{2}\right)+1\right]+c$

5. $\frac{1}{\sqrt{5}}\tan^{-1}\left(\sqrt{5}\tan x\right)+c$

6. $-\frac{1}{\sqrt{5}}\tan^{-1}\left(\frac{3\tan x-2}{\sqrt{5}}\right)+c$

7. $\frac{1}{2\sqrt{11}}\log\left(\frac{\sqrt{11}-2+\tan x}{\sqrt{11}+2-\tan x}\right)+c$

8. $\frac{1}{\sqrt{2}}\log\left[\sec\left(x+\frac{\pi}{4}\right)+\tan\left(x+\frac{\pi}{4}\right)\right]+c$

9. $\frac{1}{2}\log\left[\sec\left(x+\frac{\pi}{4}\right)+\tan\left(x+\frac{\pi}{4}\right)\right]+c$

EXERCISE 3.2 (C)

I. 1. $\frac{3}{2}\log(x^2+6x+5)-\frac{5}{4}\log\left(\frac{x+1}{x+5}\right)+c$

2. $\log(x^2+4x-5)-\frac{1}{2}\log\left(\frac{x-1}{x+5}\right)+c$

3. $\frac{1}{2}\log(2x^2+3x-1)+\frac{3}{2\sqrt{17}}\cdot\log\left(\frac{4x+3-\sqrt{17}}{4x+3+\sqrt{17}}\right)+c$

4. $\frac{3}{2}\sqrt{2x^2+2x+1}+\frac{5}{2\sqrt{2}}\cdot\log\left(x+\frac{1}{2}+\sqrt{x^2+x+\frac{1}{2}}\right)+c$

5. $-7\sqrt{3+2x-x^2}+10\cdot\sin^{-1}\left(\frac{x-1}{2}\right)+c$

6. $\sqrt{x^2 - 16x + 63} + \log\left\{(x-8) + \sqrt{x^2 - 16x + 63}\right\} + c$

7. $\sqrt{9x - x^2} + \dfrac{9}{2}\sin^{-1}\left(\dfrac{2x-9}{9}\right) + c$

8. $\dfrac{3}{4\sqrt{2}}\log\left(\dfrac{2\sqrt{2}\sin x + \sqrt{2} - 2}{2\sqrt{2}\sin x + \sqrt{2} + 2}\right) + c$

9. $\sqrt{e^{2x} - 1} - \log\left(e^x + \sqrt{e^{2x} - 1}\right) + c$

EXERCISE 3.3

I.

1. $\dfrac{x^3}{9}\left(3\cdot\log x - 1\right) + c$

2. $-\dfrac{x^2}{3}\cos 3x + \dfrac{2}{9}x\sin 3x + \dfrac{2}{27}\cos 3x + c$

3. $\dfrac{x^2}{2}\tan^{-1}x - \dfrac{1}{2}\left(x - \tan^{-1}x\right) + c$

4. $\dfrac{x^3}{3}\tan^{-1}x - \dfrac{x^2}{6} + \dfrac{1}{6}\log\left(1 + x^2\right) + c$

5. $\dfrac{1}{4}(\tan^{-1}x)\left(x^4 - 1\right) - \dfrac{x}{12}\left(x^3 - 3x\right) + c$

6. $x\,[(\log x)^2 - 2(\log x) + 2] + c$

7. $\dfrac{1}{2}\log(\sec x + \tan x) + \dfrac{1}{2}\sec x\cdot\tan x + c$

8. $\dfrac{1}{4}\left[x^2 - x\cdot\sin 2x - \dfrac{1}{2}\cos 2x\right] + c$

9. $\dfrac{x^4}{4}\log x - \dfrac{x^4}{16} + c$

10. $\dfrac{e^{2x}}{13}\left[2\cos 3x + 3\sin 3x\right] + c$

11. $\dfrac{x^2}{2}\sin^{-1}x + \dfrac{1}{4}x\sqrt{1 - x^2} - \dfrac{1}{4}\sin^{-1}x + c$

12. $\dfrac{x^3}{3}\cos^{-1}x - \dfrac{1}{3}\sqrt{1 - x^2} + \dfrac{1}{9}(1 - x^2)^{\frac{3}{2}} + c$

13. $(\log x)\,[\log(\log x) - 1] + c$

14. $-(\sin^{-1}t)\sqrt{1 - t^2} + t + c$

15. $2\left[\sqrt{x}\cdot\sin\sqrt{x} + \cos\sqrt{x}\right] + c$

16. $(\cos\theta)\cdot[1 - \log(\cos\theta)] + c$

17. $\dfrac{1}{4}\left[\dfrac{x}{3}\sin 3x + \dfrac{1}{9}\cos 3x + 3x\sin x + 3\cos x\right] + c$

18. $-\dfrac{1}{2}\cos(\log x)^2 + c$

19. $-\dfrac{1}{2}(\log x)^2 + c$

20. $\dfrac{x}{6}\sin 3x + \dfrac{1}{18}\cos x - \dfrac{1}{14}x\sin 7x - \dfrac{1}{98}\cos 7x + c$

21. $\left(3x^{\frac{2}{3}} - 6\right)\sin\sqrt[3]{x} + 6\sqrt[3]{x}\cos\sqrt[3]{x} + c$

II.

1. $\dfrac{e^{2x}}{13}[2\sin 3x - 3\cos 3x] + c$

2. $\dfrac{e^{-x}}{5}[-\cos x + 2\sin 2x] + c$

3. $\dfrac{x}{2}[\sin(\log x) - \cos(\log x)] + c$

4. $\sqrt{5}\left[\dfrac{x}{2}\sqrt{x^2 + \dfrac{3}{5}} + \dfrac{3}{10}\log\left(x + \sqrt{x^2 + \dfrac{3}{5}}\right)\right] + c$

5. $\dfrac{x^3}{6}\cdot\sqrt{a^2 - x^6} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x^3}{a}\right) + c$

6. $\dfrac{x-5}{2}\sqrt{(x-3)(7-x)} + 2\sin^{-1}\left(\dfrac{x-5}{2}\right) + c$

7. $\dfrac{1}{\log 2}\left\{\dfrac{2^x}{2}\sqrt{4^x + 4} + 2\log\left(2^x + \sqrt{4^x + 4}\right)\right\} + c$

8. $\dfrac{1}{6}(2x^2 + 3)^{\frac{3}{2}} + \sqrt{2}\left[\dfrac{x}{2}\sqrt{x^2 + \dfrac{3}{2}} + \dfrac{3}{4}\log\left(x + \sqrt{x^2 + \dfrac{3}{4}}\right)\right] + c$

9. $-\frac{1}{3}(5-4x-x^2)^{\frac{3}{2}} - (x+2)\sqrt{5-4x-x^2} - 9\sin^{-1}\left(\frac{x+2}{3}\right) + c$

10. $\frac{(1+2\tan x)}{4}\sqrt{\tan^2 x + \tan x - 7} - \frac{29}{8}\log\left\{\frac{1}{2} + \tan x + \sqrt{\tan^2 x + \tan x - 7}\right\} + c$

11. $\left(\frac{x+1}{2}\right)\sqrt{x^2+2x+5} + 2\log\left\{x+1+\sqrt{x^2+2x+5}\right\} + c$

12. $\sqrt{2}\left\{\left(\frac{4x+3}{8}\right)\sqrt{x^2+\frac{3}{2}x+2} + \frac{23}{16\sqrt{2}}\log\left[\left(x+\frac{3}{4}\right)+\sqrt{x^2+\frac{3}{2}x+2}\right]\right\} + c$

III. 1. $e^x(2+\cot x) + c$ 2. $e^x \cdot \tan\frac{x}{2} + c$ 3. $e^x \cdot \frac{1}{x} + c$ 4. $e^x \cdot \left(\frac{1}{x+1}\right) + c$

5. $e^x \cdot (\log x)^2 + c$ 6. $e^{5x} \cdot \log x + c$ 7. $e^{\sin^{-1} x} \cdot x + c$

8. $\frac{(1+x)^2}{2}\left(\log(1+x) - \frac{1}{2}\right) + c$ 9. $x \cdot \operatorname{cosec}(\log x) + c$

EXERCISE 3.4

I. 1. $\frac{1}{4}\log(x-1) - 2\log(x+2) + \frac{11}{4}(x+3) + c$

2. $\frac{1}{6}\tan^{-1}x + \frac{1}{15\sqrt{2}}\log\left(\frac{x-\sqrt{2}}{x+\sqrt{2}}\right) - \frac{\sqrt{3}}{10}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$

3. $\frac{51}{41}\log(2x+9) + \frac{31}{41}\log(3x-7) + c$ 4. $-\frac{8}{5}\log(x+4) - \frac{2}{5}\log(x-1) + c$

5. $x - \log(x+3) + \log(x-2) + c$ 6. $x^2 + 3x + \frac{5}{3}\log(3x+1) + \log(x-1) + c$

7. $\frac{1}{2}\log\left|\frac{2x+1}{2x-1}\right| + 3\log(x+3) + c$ 8. $\frac{1}{5}\log\left(\frac{x^5}{x^5+1}\right) + c$ 9. $\frac{11}{\sqrt{5}}\tan^{-1}\left(\frac{x}{2}\right) - \frac{9}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

10. $2\log\left(\frac{x+1}{x-1}\right) + \frac{5}{2\sqrt{2}}\log\left(\frac{x+\sqrt{2}}{x-\sqrt{2}}\right) + c$ 11. $\log\left(\frac{2+x^2}{3+x^2}\right) + c$

12. $\frac{1}{5\cdot\log 2}\log\left(\frac{2^x-4}{2^x+1}\right) + c$ 13. $\frac{5}{2}\left(\frac{1}{x+1}\right) + \frac{11}{4}\log\left(\frac{x+1}{x+3}\right) + c$

14. $6\cdot\log x - \log(x+1) - \frac{9}{x+1} + c$ 15. $\frac{1}{8}\log\left(\frac{x^6(x^3+3)}{(3x^3+1)^3}\right) + c$

16. $\frac{1}{3}\log(x-1) - \frac{1}{6}\log(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

17. $3 \cdot \log(\sin x - 2) - \dfrac{4}{\sin x - 2} + c$

18. $\dfrac{1}{2}\log(\cos x + 1) + \dfrac{1}{6}\log(\cos x - 1) - \dfrac{2}{3}\log(2\cos x + 1) + c$

19. $\dfrac{1}{8}\log\left(\dfrac{\cos x - 1}{\cos x + 1}\right) + \dfrac{1}{4\cdot(\cos x + 1)} + c$

20. $\dfrac{1}{6}\log\left[\dfrac{(1 + 2\sin x)^4}{(1 - \sin x)(1 + \sin x)^3}\right] + c$

21. $\dfrac{1}{10}\log(1 - \cos x) - \dfrac{1}{2}\log(1 + \cos x) + \dfrac{2}{5}\log(3 + 2\cos x) + c$

22. $\dfrac{1}{2}\log\left[\dfrac{e^x + 1}{(e^{2x} + 9)^{\frac{1}{2}}}\right] + \dfrac{1}{6}\tan^{-1}\left(\dfrac{e^x}{3}\right) + c$

23. $\dfrac{5}{26}\log\left[\dfrac{(3\log x + 2)^2}{\sqrt{(\log x)^2 + 1}}\right] + \dfrac{11}{26}\tan^{-1}(\log x) + c$

MISCELLANEOUS EXERCISE 3

(I)

1	2	3	4	5	6	7	8	9	10
B	A	B	A	D	B	A	A	C	B

11	12	13	14	15	16	17	18	19	20
A	A	D	C	A	D	A	D	C	A

(II) (1) $\dfrac{2}{7}x^{\frac{7}{2}} - \dfrac{8}{5}x^{\frac{5}{2}} - \dfrac{8}{3}x^{\frac{3}{2}} + c$

(2) $\dfrac{x^7}{7} - \dfrac{x^6}{6} + \dfrac{x^5}{5} - \dfrac{x^4}{4} + \dfrac{x^3}{3} - \dfrac{x^2}{2} + x - \log(x + 1) + c$

(3) $\dfrac{1}{15}(6x + 5)^{\frac{5}{2}} + c$

(4) $\dfrac{t^2}{2} - 2t + 3\cdot\log(t + 1) + \dfrac{1}{t + 1} + c$

(5) $3\tan x - 2\sec x + c$

(6) $\tan\theta - \cot\theta - 3\theta + c$

(7) $\dfrac{1}{48}(2\sin 6x + 3\sin 4x + 6\sin 2x + 12x) + c$

(8) $\dfrac{1}{2}\sin 2x - \dfrac{1}{3}\sin 3x + c$

(9) $\dfrac{\pi}{4}x - \dfrac{1}{4}x^2 + c$

(III) (1) $\dfrac{1}{4}(1 + \log x)^4 + c$

(2) $(\tan^{-1}x)\,x - \dfrac{1}{2}\log(1 + x^2) - (1 - x)\tan^{-1}(1 - x) + \dfrac{1}{2}\log(x^2 - 2x + 2) + c$

(3) $-\cot(\log x) + c$

(4) $\dfrac{2}{3}\sec x^{\frac{3}{2}} + c$

(5) $x \log(1+\cos x) + c$

(6) $\frac{1}{3}\sin^{-1}(x^3) + c$

(7) $\frac{1}{4}\log(3-2\cot x) + c$

(8) $x\cdot\left(\log(\log x) - \frac{1}{\log x}\right) + c$

(9) $\frac{2}{\sqrt{13}}\tan^{-1}\left(\frac{2\tan\left(\frac{x}{2}\right)-3}{\sqrt{13}}\right) + c$

(10) $\frac{1}{4}\left(2\sec^{-1}x + \frac{2\sqrt{x^2-1}}{x^2}\right) + c$

(11) $-\frac{3}{2}\sqrt{-2x^2+x+3} + \frac{7}{4\sqrt{2}}\sin^{-1}\left(\frac{2x-1}{\sqrt{7}}\right) + c$

(12) $x\cdot\log(x^2+1) - 2\left[x - \tan^{-1}x\right] + c$

(13) $\frac{1}{4}e^{2x}\cdot\left[\sin 2x - \cos 2x\right] + c$

(14) $\frac{1}{18}\log(3x-1) + \frac{1}{2}\log(x-1) - \frac{4}{9}\log(3x-2) + c$

(15) $\frac{1}{6}\log\left\{\frac{(\cos x-1)(\cos x+1)^3}{(2\cos x+1)^4}\right\} + c$

(16) $\left(\frac{\tan x-1}{2}\right)\sqrt{7+2\tan x-\tan^2 x} + 4\sin^{-1}\left(\frac{\tan x-1}{2\sqrt{2}}\right) + c$

(17) $\frac{1}{4}\log\left\{\frac{(x-1)^3(x+3)}{(x+1)^4}\right\} + c$

(18) $\frac{1}{5}\log\left(\frac{x^5}{x^5+1}\right) + c$

(19) $2\sqrt{\tan x} + c$

(20) $\frac{1}{3\cot^3 x} + \frac{2}{\cot x} - \cot x + c$

4. DEFINITE INTEGRATION

EXERCISE 4.1

I. (1) 4 (2) $\frac{64}{3}$ (3) $e^2 - 1$ (4) 6 (5) 20

EXERCISE 4.2

I. (1) $\frac{64}{3}$ (2) $\log\left(\frac{25}{24}\right)$ (3) $-\left(1+\frac{\pi}{4}\right)$ (4) 2

(5) $\frac{1}{18}\left[13\sqrt{13}+7\sqrt{7}-3\sqrt{3}-27\right]$ (6) $1-\frac{3\pi}{4}$ (7) $\frac{4}{7\sqrt{2}}$ (8) 1 (9) $\frac{3\pi}{16}$

(10) $\frac{1}{3}\left(\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{2}{3}\right)$ (11) π (12) $\frac{\pi}{6}$ (13) 1 (14) $\frac{\pi}{4} - \frac{1}{2}$ (15) 1

II. (1) $\frac{\pi}{4} - \frac{1}{2}\log 2$ (2) $\frac{1}{2}\log 2$ (3) $\frac{\pi}{4}$

(4) 0 (5) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)$ (6) $\frac{1}{4}\log\left(\frac{2\sqrt{2}+1}{2\sqrt{2}-1}\right)$

(7) $\log\left(\frac{4}{3}\right)$ (8) $\frac{1}{ab}\left[\tan^{-1}\left(\frac{ae}{b}\right) - \tan^{-1}\left(\frac{a}{be}\right)\right]$

(9) $\frac{\pi}{4}$ (10) $\frac{4}{3}$ (11) $\frac{\pi}{2} - 1$

(12) $\frac{8}{3}$ (13) $\frac{\pi}{2} - 1$

(14) $e^{\frac{\pi}{4}}\left[\frac{\pi}{4}+1\right] - \left[\frac{\pi}{2}+1\right]$ (15) $\sin(\log 3)$

III. (1) $\frac{\pi}{4}$ (2) 0 (3) 0 (4) 0 (5) $\frac{16}{77}(3)^{\frac{7}{2}}$

(6) 0 (7) 0 (8) $\frac{\pi^2}{6\sqrt{3}}$ (9) 0 (10) 0

(11) $4\log\left(\frac{1+\sqrt{5}}{2}\right)$ (12) 0 (13) $\frac{16}{105}$ (14) $\frac{\pi}{3}$ (15) $\frac{\pi}{2}\log\left(\frac{1}{2}\right)$

MISCELLANEOUS EXERCISE 4

(I)

1	2	3	4	5	6	7	8	9	10
A	A	C	C	D	C	A	D	B	A

(II) (1) $\frac{1}{10}(3 - \log 3)$ (2) $2 - \sqrt{2}$ (3) $6 - 4\log 2$ (4) $\frac{1}{8}$ (5) $\frac{1}{21}$

(6) $\pi - 2$ (7) $\frac{1}{3}\log 2$ (8) $\frac{\pi}{5}$ (9) 0 (10) $\frac{\pi}{2}$

(III) (1) $\frac{\pi^2}{16}$ (2) $\frac{2}{\sqrt{35}}\tan^{-1}\sqrt{\frac{7}{5}}$ (3) $\frac{1}{\sqrt{5}\,a}\log\left(\frac{7+3\sqrt{5}}{2}\right)$ (4) $\frac{\pi}{20}$

(5) $\frac{\pi}{2} - \log 2$ (6) $\frac{1}{2}\left(\frac{\pi}{4} - \log\sqrt{2}\right)$ (7) $-\frac{\pi}{2}\log 2$ (8) $\frac{\pi^3}{6}$

(9) $\log\left(\frac{5+3\sqrt{3}}{1+\sqrt{3}}\right)$ (10) $\frac{17}{2}$

(IV) (1) $\frac{1}{2}$ when $a = 0$; $\frac{9}{2}$ when $a = 4$ (2) $k = \frac{1}{2}$

5. APPLICATION OF DEFINITE INTEGRAL

EXERCISE 5.1

(1) (i) 25 (ii) 16 (iii) 20

(iv) 1 (v) 2 log 4 (vi) $\frac{32}{3}$

(vii) $\frac{128}{3}$ sq. units

(2) (i) $\frac{128}{3}$ (ii) $\frac{16}{3}$

(3) (i) $\frac{1}{12}$ (ii) $\frac{8}{3}$ (iii) $\frac{32}{3}$

(iv) $8\frac{a^2}{3}$ (v) $\frac{1}{6}$

MISCELLANEOUS EXERCISE 5

(I)

1	2	3	4	5	6	7	8	9	10
A	A	C	B	A	D	B	D	A	B

11	12	13	14	15	16	17	18	19	20
A	D	B	B	C	C	A	D	A	C

(II)

1. (i) 10 (ii) 2 (iii) $\frac{1}{2}$

2. 9π

3. 20π

4. (i) $\frac{16}{3}$ (ii) $\frac{8}{3}$ (iii) $\frac{1}{3}$

5. $\frac{\pi}{3}$

6. $\frac{1}{6}$

7. $\frac{\pi}{4}-\frac{1}{2}$

8. $\frac{56}{3}$

9. $36\frac{3}{4}$

10. $\frac{7}{3}$

6. DIFFERENTIAL EQUATIONS

EXERCISE 6.1

(1) (i) 2, 1 (ii) 2, 3 (iii) 1, 2

(iv) 3, 1 (v) 2, 1 (vi) 3, 2

(vii) 2, not definded (viii) 2, 2

(ix) 3, 3 (x) 2, 1

EXERCISE 6.2

(1) (i) $2x^3 + 3xy^2\frac{dy}{dx} - y^3 = 0$

(ii) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

(iii) $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ (iv) $8\left(\frac{dy}{dx}\right)^3 - 27y = 0$

(v) $\frac{d^2y}{dx^2} - 25y = 0$ (vi) $2\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$

(vii) $(x^2+xy)\frac{dy}{dx} + y = 0$ (viii) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 10y = 0$

(ix) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} = 0$

(x) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$

(2) $\dfrac{d^2y}{dx^2} = 0$ (3) $2a\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^3 = 0$

(4) $x + 4y\dfrac{dy}{dx} = 0$ (5) $3\dfrac{dy}{dx} + 2 = 0$

(6) $81\left(\dfrac{d^2y}{dx^2}\right)^2 = \left[\left(\dfrac{dy}{dx}\right)^2 + 1\right]^3$

(7) $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 0$

EXERCISE 6.3

(2) (i) $\tan^{-1} y = \tan^{-1} x + c$

(ii) $2e^{-3y} + 3e^{2x} = c$ (iii) $x = cy$

(iv) $\tan x \cdot \tan y = c$ (v) $\sin y \cdot \cos x = c$

(vi) $y = -kx + c$

(vii) $2(x^2 + y^2) + 2(x \sin 2x + y \sin 2y) + \cos 2y + \cos 2x + c = 0$

(viii) $2y^2 \tan^{-1} x + 1 = cy^2$

(ix) $4e^x + 3e^{-2y} = c$

(x) $3e^x + 3e^{-y} + x^3 = c$

(3) (i) $(1 + e^x)^3 \tan y = 0$

(ii) $(1 + x^2)(1 - y^2) = 5$

(iii) $y = ex \log x$ (iv) $(\sin x)(e^y + 1) = \sqrt{2}$

(v) $2(2 + e^y) = 3(x + 1)$

(vi) $\cos\left(\dfrac{y-2}{x}\right) = a$

(4) (i) $\tan\left(\dfrac{x+y}{2}\right) = x + c$

(ii) $c + 2y = a \log\left(\dfrac{x-y-a}{x-y+a}\right)$

(iii) $\sin(x^2 + y^2) + 2x = c$

(iv) $x = \tan(x - 2y) + c$

(v) $(2x - y) - \log(x - y + 2) + 1 = 0$

EXERCISE 6.4

(1) $\cos\left(\dfrac{y}{x}\right) dy = \log(x) + c$

(2) $x^2 - y^2 = cx$ (3) $x + 2ye^{\frac{x}{y}} = c$

(4) $xy^2 = c^2(x + 2y)$ (5) $x^2 + y^2 = cx$

(6) $y = c(x + y)^3 + x$

(7) $x\left[1 - \cos\left(\dfrac{y}{x}\right)\right] = \sin\left(\dfrac{y}{x}\right)$

(8) $x + ye^{\frac{x}{y}} = c$ (9) $\log(y) + \dfrac{y}{x} = c$

(10) $x^2y = 4$ (11) $x^2 + y^2 = x^4$

(12) $\tan^{-1}\left(\dfrac{y}{x}\right) = \log(x) + c$

(13) $(3x + y)^3 (x + y)^2 = c$

(14) $c = \log(x) + \dfrac{x}{x+y}$ (15) $x^2 - y^2 = cx$

EXERCISE 6.5

1. (i) $\dfrac{x^5}{5} - \dfrac{3x^2}{2} - xy = c$

(ii) $ye^{\tan x} = e^{\tan x}(\tan x - 1) + c$

(iii) $x = y(c + y^2)$

(iv) $y(\sec x + \tan x) = \sec x + \tan x - x + c$

(v) $x^2 y = \dfrac{x^4 \log x}{4} - \dfrac{x^4}{16} + c$

(vi) $x + y + 1 = ce^y$

(vii) $2y = (x + a)^5 + 2c(x + a)^3$

(viii) $r \sin^2\theta + \dfrac{\sin^4\theta}{2} = c$

(ix) $\dfrac{y^3}{3} = xy + c$

(x) $y = \sqrt{1 - x^2} + c(1 - x^2)$

(xi) $y = \dfrac{1}{2}e^{\tan^{-1} x} + c\, e^{-\tan^{-1} x}$

2. $3(x+3y) = 2(1-e^{3x})$

3. $4x^2 + 9y^2 = 36$

4. $y = 4 - x - 2e^x$

5. $1 + y = 2e^{\frac{x^2}{2}}$

EXERCISE 6.6

1. 8 times of original.

2. 95·4 years

3. 36·36°c

4. 5656

5. $\dfrac{\log 3}{k}$

6. $\dfrac{27}{5}$ gms

7. $(3000)\left(\dfrac{4}{9}\right)^{\frac{t}{40}}$

8. 1 hour

10. $r = 3 - t$

11. 27,182

12. $\left(10 - \dfrac{p}{10}\right)^2 \%$

MISCELLANEOUS EXERCISE 6

(I)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
D	A	C	B	A	D	C	B	C	D	B	A	B	B	B

(II) (1) (i) 2, 1 (ii) 3, 10 (iii) 2 , 3 (iv) 1.4 (v) 4, not defined

(3) (i) $xy\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 - 2y\dfrac{dy}{dx} = 0$ (ii) $\dfrac{d^2y}{dx^2} + y = 0$ (iii) $(y-a)\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 0$

(iv) $2x^2y\dfrac{d^2y}{dx^2} + 2x^2\left(\dfrac{dy}{dx}\right)^2 + y = 0$ (v) $\dfrac{d^2y}{dx^2} - 9y = 0$

(4) (i) $2xy\dfrac{dy}{dx} + x^2 - y^2 = 0$ (ii) $2b\dfrac{d^2y}{dx^2} - 1 = 0$ (iii) $x + 4y\dfrac{dy}{dx} = 0$ (iv) $2\dfrac{dy}{dx} - 3 = 0$

(5) (i) $2e^{-3y} + 3e^{2x} + 6c = 0$ (ii) $\log(y) = \dfrac{x^3}{3} + x + c$ (iii) $y = \dfrac{x}{2}\log(x^2) + 2 + cx$

(iv) $y = 1 + x\log x + cx$ (v) $y = x^2 + c\cdot\operatorname{cosec} x$ (vi) $x\log y = (\log y)^2 + c$

(vii) $4xe^{2y} + 5e^{-y} = c$

(6) (i) $ex\log x - y = 0$ (ii) $x = 2y^2$ (iii) $y\operatorname{cosec}^2 x + 2 = 4\sin 2x$

(iv) $\log\sqrt{x^2+y^2} + \tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{\pi}{4}$ (v) $x + 2ye^{\frac{x}{y}} = 2$

(8) $x^2 + y^2 = 4x + 5$ (9) $r = (63t + 27)^{\frac{1}{3}}$ (10) $\dfrac{20}{9}$ years

7. PROBABILITY DISTRIBUTIONS

EXERCISE 7.1

1. $\{-6, -4, -2, 0, 2, 4, 6\}$

2. $\{0, 1, 2\}$

3. (i) p.m.f. (ii) Not p.m.f

(iii) p.m.f (iv) Not p.m.f

(v) Not p.m.f. (iv) p.m.f

4. (i)

X	0	1	2
$P(X)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(ii)

X	0	1	2	3
$P(X)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(iii)

X	0	1	2	3	4
$P(X)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

5.

X	0	1	2
$P(X)$	$\frac{2}{3}$	$\frac{2}{9}$	$\frac{1}{9}$

6.

X	0	1	2	3	4
$P(X)$	$\left(\frac{4}{5}\right)^4$	$\left(\frac{4}{5}\right)^3\frac{1}{5}$	$\left(\frac{4}{5}\right)^2\left(\frac{1}{5}\right)^2$	$\frac{4}{5}\left(\frac{1}{5}\right)^3$	$\left(\frac{1}{5}\right)^4$

7.

X	0	1	2
$P(X)$	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$

8. (i) $\frac{1}{10}$ (ii) $\frac{3}{10}$ (iii) $\frac{1}{5}$

9. $-0{\cdot}05,\ 2{\cdot}2475$ 10. $\frac{7}{3},\ \frac{524}{54}$ 11. $1{\cdot}5$

12. $\frac{1}{3}$ 13. $4{\cdot}67$ 14. $2{\cdot}41$

15. $17{\cdot}53,\ 4{\cdot}9,\ 2{\cdot}21$ 16. $0{\cdot}7,\ 0{\cdot}21$

EXERCISE 7.2

1. (i) p.d.f. (ii) Not a p.d.f

(iii) Not a p.d.f

2. (a) $\frac{2{\cdot}25}{16}$, (b) $\frac{3}{16}$, (c) $\frac{3}{4}$

3. (i) p.d.f. (ii) $\frac{1}{9}$ (iii) $\frac{1}{9}$

4. (i) $\frac{1}{2}, \frac{35}{64}$ (ii) $6, \frac{11}{32}, \frac{1}{2}$

5. (i) $\frac{1}{4}$ (ii) $\frac{1}{2}$ (iii) $\frac{7}{16}$

6. (i) $\frac{2}{5}$ (ii) $\frac{1}{5}$

7. (i) $\frac{1}{2}$ (ii) $\frac{11}{16}$ (iii) $0{\cdot}6328$

8. (i) $\frac{x^2}{16}$ (ii) $\frac{1}{64}, 0{\cdot}18, 1$

9. $\frac{2}{9}, 0, \frac{8}{9}, \frac{7}{9}$

10. $\frac{1}{\log 3}, \frac{4}{\log 3}, \frac{4(\log 3 - 1)}{(\log 3)^2}$

MISCELLANEOUS EXERCISE 7

(I)

1	2	3	4	5	6	7	8	9	10
B	C	A	B	C	B	B	A	D	B

(II) Solve the following :

(1) (i) Discrete $\{1, 2, 3, ..., 100000\}$ (ii) Continuous. (iii) Continuous.

(iv) Discrete $\{0, 1, 2, 3, 4, 5\}$ (v) Continuous

(2) (i) $\frac{1}{21}$ (ii) $\frac{10}{21}, \frac{1}{7}, \frac{6}{7}$ (3) (i) $0{\cdot}5$ (ii) $0{\cdot}7$ (iii) $0{\cdot}55$ (iv) $0{\cdot}45$

(5) $\frac{1}{4}$

x	1	2	3	4	5
$P(X)$	$\frac{1}{20}$	$\frac{3}{20}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{20}$
$F(x)$	$\frac{1}{20}$	$\frac{1}{5}$	$\frac{9}{20}$	$\frac{19}{20}$	1

(6)

X	0	1	2	3	4
$P(X)$	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

$\frac{{}^4C_x}{2^4}$

(7) (i)

X	0	1	2
$P(X)$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

(ii)

X	0	1	2
$P(X)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

(8) (i) $\frac{1}{10}$ (ii) $\frac{17}{100}$ (iii) $\frac{3}{10}$

(9)

X	-3	-2	-1	0	1	2	3	4
$F(X)$	0·1	0·3	0·5	0·65	0·75	0·85	0·9	1
$P(x)$	0·1	0·2	0·2	0·15	0·10	0·10	0·05	0·10

(i) 0·55 (ii) 0·25

(10) (i) $\frac{11}{5}, \frac{14}{25}, \frac{\sqrt{14}}{5}$ (ii) $\frac{1}{5}, \frac{14}{25}, \frac{\sqrt{14}}{5}$ (iii) $\frac{n+1}{2}, \frac{n^2-1}{12}, \sqrt{\frac{n^2-1}{12}}$ (iv) $\frac{5}{2}, \frac{5}{4}, \frac{\sqrt{5}}{2}$

(11) ₹ 5·5, 8·25 (12) 0, 1 (13) (i) $\frac{1}{2}$ (ii) $\frac{11}{16}$ (iii) $\frac{81}{128}$

(15) $k = \frac{1}{\theta}, \frac{1}{e}$ (16) $k = \frac{1}{4}, F(x) = \frac{\sqrt{x}}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$

8. BINOMIAL DISTRIBUTION

EXERCISE 8.1

1. (i) $\frac{3}{2^5}$ (ii) $\frac{7}{2^6}$ (iii) $\frac{63}{64}$

2. $\frac{25}{216}$ 3. $29\left(\frac{19^9}{20^{10}}\right)$

4. (i) $\frac{1}{1024}$ (ii) $\frac{45}{1024}$

5. (i) $(0·95)^5$ (ii) $(1·2)(0·95)^4$
   (iii) $1-(1·2)(0·95)^4$ (iv) $1-(0·95)^5$

6. $\left(\frac{9}{10}\right)^4$ 7. $\frac{11}{243}$

8. (i) $1-\left(\frac{99}{100}\right)^{50}$ (ii) $50\left(\frac{99^{49}}{100^{50}}\right)$
   (iii) $1-149\left(\frac{99^{49}}{100^{50}}\right)$

9. (i) $\frac{1}{20^3}$ (ii) $3\left(\frac{19}{20^3}\right)$ (iii) $3\left(\frac{19^2}{20^3}\right)$ (iv) $\left(\frac{19}{20}\right)^3$

10. $\frac{7}{3}\left(\frac{5}{6}\right)^5$ 11. $22\left(\frac{9^3}{10^{11}}\right)$

12. (i) 4, 2·4 (ii) 10, 2·4 (iii) $\frac{2}{5}; \sqrt{6}$ (iv) $\frac{8}{5}$

MISCELLANEOUS EXERCISE 8

(I)

1	2	3	4	5	6	7
B	D	D	C	B	C	B

(II) Solve the following :

(1) (i) $2 \times (0 \cdot 8)^9$ (ii) $1 - (0 \cdot 8)^{10}$

(iii) $1 - (8 \cdot 2)(0 \cdot 2)^9$

(2) (i) $p = \dfrac{1}{2}$, $Var(X) = 2 \cdot 5$

(ii) $n = 10, p = \dfrac{1}{2}$

(3) (i) $\dfrac{63}{256}$ (ii) $\dfrac{105}{512}$

(4) $45\left(\dfrac{2^{26}}{2^{10}}\right)$

(5) (i) $0 \cdot 65 \times (0 \cdot 95)^{16}$

(ii) $(2 \cdot 0325) \times (0 \cdot 95)^{14}$

(iii) $1 - (1 \cdot 6) \times (0 \cdot 95)^{16}$

(6) $0 \cdot 2114$ (7) $1 \cdot 4 \times (0 \cdot 9)^4$

(8) $6 \cdot 97 \times (0 \cdot 97)^{19}$

(9) $0 \cdot 3456$ (10) $\dfrac{30 \cdot 44}{5^8}$

(11) (i) $(0 \cdot 998)^8$ (ii) $1 \cdot 014 \times (0 \cdot 998)^7$

(iii) $1 - 1 \cdot 014 \times (0 \cdot 998)^7$

(12) $775 \cdot 44 \times (0 \cdot 003)^{38}$

(13) (i) $0 \cdot 9^{10}$ (ii) $0 \cdot 9^9$

(iii) $0 \cdot 45 \times (0 \cdot 9)^8$

(iv) $1 - 2 \cdot 16 \times (0 \cdot 9)^8$

(14) (i) $\dfrac{1}{5^4}, \dfrac{16}{5^4}, \dfrac{96}{5^4}, \dfrac{256}{5^4}, \dfrac{256}{5^4}$

(ii) (a) $\dfrac{608}{5^4}$ (b) $1 - \dfrac{33}{5^8}$

(15) (i) $35 \times 8 \times \dfrac{81}{5^7}$ (ii) $1 - \dfrac{12393}{5^7}$

(16) (i) $\dfrac{\log 0 \cdot 5}{\log 0 \cdot 99}$

(17) $\dfrac{1}{5}$

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