FINAL JEE–MAIN EXAMINATION – JUNE, 2022

(Held On Friday 24th June, 2022) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. Let $x*y = x^2 + y^3$ and $(x*1)*1 = x*(1*1)$.

Then a value of $2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right)$ is

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{6}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\because$ $(x * 1) * 1 = x * (1 * 1)$

$(x^2 + 1) * 1 = x * (2)$

$(x^2 + 1)^2 + 1 = x^2 + 8$

$x^4 + x^2 - 6 = 0 \Rightarrow (x^2 + 3)(x^2 - 2) = 0$

$x^2 = 2$

$\Rightarrow 2\sin^{-1}\left(\frac{x^4+x^2-2}{x^4+x^2+2}\right) = 2\sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{3}$

2. The sum of all the real roots of the equation $(e^{2x} - 4)(6e^{2x} - 5e^x + 1) = 0$ is

(A) $\log_e 3$ (B) $-\log_e 3$

(C) $\log_e 6$ (D) $-\log_e 6$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $(e^{2x} - 4)(6e^{2x} - 3e^x - 2e^x + 1) = 0$

$(e^{2x} - 4)(3e^x - 1)(2e^x - 1) = 0$

$e^{2x} = 4$ or $e^x = \frac{1}{3}$ or $e^x = \frac{1}{2}$

$\Rightarrow$ sum of real roots $= \frac{1}{2}\ell n4 + \ell n\frac{1}{3} + \ell n\frac{1}{2}$

$= -\ell n3$

3. Let the system of linear equations

$x + y + \alpha z = 2$

$3x + y + z = 4$

$x + 2z = 1$

have a unique solution (x^*, y^*, z^*). If (α, x^*), (y^*, α) and $(x^*, -y^*)$ are collinear points, then the sum of absolute values of all possible values of α is :

(A) 4 (B) 3

(C) 2 (D) 1

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $\Delta = \begin{vmatrix} 1 & 1 & \alpha \\ 3 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -(\alpha + 3)$

$\Delta_1 = \begin{vmatrix} 2 & 1 & \alpha \\ 4 & 1 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -(3 + \alpha)$

$\Delta_2 = \begin{vmatrix} 1 & 2 & \alpha \\ 3 & 4 & 1 \\ 1 & 1 & 2 \end{vmatrix} = -(\alpha + 3)$

$\Delta_3 = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & 4 \\ 1 & 0 & 1 \end{vmatrix} = 0$

$\alpha \neq -3$, $x = 1$, $y = 1$, $z = 0$,

Now points $(\alpha, 1)$, $(1, \alpha)$ & $(1, -1)$ are collinear

$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & -1 & 1 \end{vmatrix} = 0$

$\Rightarrow \alpha(\alpha + 1) - 1(1 - 1) + 1(-1 - \alpha) = 0$

$\alpha^2 + \alpha - 1 - \alpha = 0$

$\alpha = \pm 1$

4. Let $x, y > 0$. If $x^3y^2 = 2^{15}$, then the least value of $3x + 2y$ is

(A) 30 (B) 32

(C) 36 (D) 40

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. Using AM ≥ GM

$$\frac{x+x+x+y+y}{5} \geq \left(x^3.y^2\right)^{\frac{1}{5}}$$

$$\frac{3x+2y}{5} \geq \left(2^{15}\right)^{\frac{1}{5}}$$

$(3x + 2y)_{min} = 40$

5. Let $f(x) = \begin{cases} \frac{\sin(x-[x])}{x-[x]} & , \quad x \in (-2,-1) \\ \max\{2x, 3[|x|]\} & , \quad |x| < 1 \\ 1 & , \quad \text{otherwise} \end{cases}$

where [t] denotes greatest integer ≤ t. If m is the number of points where f is not continuous and n is the number of points where f is not differentiable, then the ordered pair (m, n) is :

(A) (3, 3) (B) (2, 4)

(C) (2, 3) (D) (3, 4)

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $f(x) = \begin{cases} \frac{\sin(x+2)}{x+2} & , \quad x \in (-2,-1) \\ \max\{2x, 0\} & , \quad x \in (-1,1) \\ 1 & , \quad \text{otherwise} \end{cases}$

$f(-2^+) = \lim\limits_{h\to 0} f(-2+h) = \lim\limits_{h\to 0} \frac{\sin h}{h} = 1$

f is continuous at x = –2

$f(-1^-) = \lim\limits_{h\to 0} \frac{\sin(-1-h+2)}{(-1-h+2)} = \sin 1$

$f(-1) = f(-1^+) = 0$

$f(1^+) = 1$ & $f(1^-) = 0 \Rightarrow f$ is not continuous at x = 1

f is continuous but not diff. at x = 0

$$\left.\begin{array}{l} \Rightarrow f \text{ is discontinuous at } x = -1 \ \& \ 1 \\ \ \& \ f \text{ is not diff. at } x = -1, 0 \ \& \ 1 \end{array}\right\} \Rightarrow \begin{array}{l} m = 2 \\ n = 3 \end{array}$$

6. The value of the integral $\int_{-\pi/2}^{\pi/2} \frac{dx}{\left(1+e^x\right)\left(\sin^6 x + \cos^6 x\right)}$ is equal to

(A) 2π (B) 0

(C) π (D) $\frac{\pi}{2}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $I = \int_{-\pi/2}^{0} \frac{dx}{\left(1+e^x\right)\left(\sin^6 x+\cos^6 x\right)} + \int_{0}^{\pi/2} \frac{dx}{\left(1+e^x\right)\left(\sin^6 x+\cos^6 x\right)}$

Put x = –t

$$= \int_{\pi/2}^{0} \frac{-dt}{\left(1+e^{-t}\right)\left(\sin^6 t+\cos^6 t\right)} + \int_{0}^{\pi/2} \frac{dx}{\left(1+e^x\right)\left(\sin^6 x+\cos^6 x\right)}$$

$$= \int_{0}^{\pi/2} \frac{\left(e^x+1\right)dx}{\left(1+e^x\right)\left(\sin^6 x+\cos^6 x\right)}$$

$$= \int_{0}^{\pi/2} \frac{dx}{\left(\sin^2 x+\cos^2 x\right)\left(\sin^4 x - \sin^2 x\cos^2 x+\cos^4 x\right)}$$

$$= \int_{0}^{\pi/2} \frac{\left(1+\tan^2 x\right)\sec^2 x\, dx}{\left(\tan^4 x - \tan^2 x + 1\right)}$$

Put tanx = t

$$= \int_{0}^{\infty} \frac{\left(1+t^2\right)dt}{\left(t^4-t^2+1\right)}$$

$$= \int_{0}^{\infty} \frac{\left(1+\frac{1}{t^2}\right)dt}{t^2-1+\frac{1}{t^2}} = \int_{0}^{\infty} \frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+1}$$

Put $t - \frac{1}{t} = z$

$$\left(1+\frac{1}{t^2}\right)dt = dz$$

$$= \int_{-\infty}^{\infty} \frac{dz}{1+z^2} = \left(\tan^{-1} z\right)_{-\infty}^{\infty}$$

$$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$$

7. $\lim_{n\to\infty}\left(\frac{n^2}{(n^2+1)(n+1)}+\frac{n^2}{(n^2+4)(n+2)}+\frac{n^2}{(n^2+9)(n+3)}+\ldots+\frac{n^2}{(n^2+n^2)(n+n)}\right)$ is equal to

(A) $\frac{\pi}{8}+\frac{1}{4}\log_e 2$ (B) $\frac{\pi}{4}+\frac{1}{8}\log_e 2$

(C) $\frac{\pi}{4}-\frac{1}{8}\log_e 2$ (D) $\frac{\pi}{8}+\log_e \sqrt{2}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{n^2}{(n^2+r^2)(n+r)}\right)$

$$= \lim_{n\to\infty}\left(\sum_{r=1}^{n}\frac{1}{n\left(1+\left(\frac{r}{n}\right)^2\right)\left(1+\left(\frac{r}{n}\right)\right)}\right)$$

$$= \int_0^1 \frac{dx}{(1+x^2)(1+x)} = \frac{1}{2}\int_0^1 \frac{1-x}{1+x^2}dx + \frac{1}{2}\int_0^1 \frac{1}{1+x}dx$$

$$= \frac{1}{2}\int\left(\frac{1}{1+x^2}-\frac{x}{1+x^2}\right)dx + \frac{1}{2}\left(\ln(1+x)\right)_0^1$$

$$= \frac{1}{2}\left[\tan^{-1}x - \frac{1}{2}\ell n(1+x^2)\right]_0^1 + \frac{1}{2}\ell n2$$

$$= \frac{1}{2}\left[\frac{\pi}{4}-\frac{1}{2}\ell n2\right] + \frac{1}{2}\ell n2$$

$$= \frac{\pi}{8}+\frac{1}{4}\ell n\ 2$$

8. A particle is moving in the xy-plane along a curve C passing through the point (3, 3). The tangent to the curve C at the point P meets the x-axis at Q. If the y-axis bisects the segment PQ, then C is a parabola with

(A) length of latus rectum 3

(B) length of latus rectum 6

(C) focus $\left(\frac{4}{3},0\right)$

(D) focus $\left(0,\frac{3}{4}\right)$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. Let Point P(x,y)

$Y - y = y'(X - x)$

$Y = 0 \Rightarrow X = x - \frac{y}{y'}$

$Q\left(x-\frac{y}{y'},0\right)$

Mid Point of PQ lies on y axis

$x - \frac{y}{y'} + x = 0$

$y' = \frac{y}{2.x} \Rightarrow 2\frac{dy}{y} = \frac{dx}{x}$

$2\ell ny = \ell nx + \ell nk$

$y^2 = kx$

It passes through (3, 3) $\Rightarrow$ k = 3

curve c $\Rightarrow y^2 = 3x$

Length of L.R. = 3

Focus = $\left(\frac{3}{4},0\right)$ **Ans. (A)**

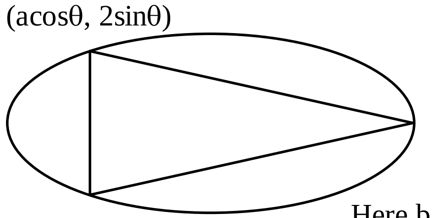
9. Let the maximum area of the triangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{4}=1$, a > 2, having one of its vertices at one end of the major axis of the ellipse and one of its sides parallel to the y-axis, be $6\sqrt{3}$. Then the eccentricity of the ellipse is :

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{\sqrt{3}}{4}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol.



$A = \frac{1}{2}a\ (1 - \cos\theta)\ (4\sin\theta)$

A = 2a(1–cosθ) sinθ

$$\frac{dA}{d\theta} = 2a\left(\sin^2\theta + \cos\theta - \cos^2\theta\right)$$

$$\frac{dA}{d\theta} = 0 \Rightarrow 1 + \cos\theta - 2\cos^2\theta = 0$$

cosθ = 1 (Reject)

OR

$$\cos\theta = \frac{-1}{2} \Rightarrow \theta = \frac{2\pi}{3}$$

$$\frac{d^2A}{d\theta^2} = 2a\left(2\sin^2\theta - \sin\theta\right)$$

$$\frac{d^2A}{d\theta^2} < 0 \text{ for } \theta = \frac{2\pi}{3}$$

$$\text{Now, } A_{max} = \frac{3\sqrt{3}}{2}a = 6\sqrt{3}$$

$$\boxed{a = 4}$$

$$\text{Now, } e = \sqrt{\frac{a^2 - b^2}{a^2}} = \frac{\sqrt{3}}{2}$$ **Ans. (A)**

10. Let the area of the triangle with vertices A(1, α), B(α, 0) and C(0, α) be 4 sq. units. If the point (α, –α), (–α, α) and (α^2, β) are collinear, then β is equal to

(A) 64 (B) –8

(C) –64 (D) 512

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $$\frac{1}{2}\begin{vmatrix} \alpha & 0 & 1 \\ 1 & \alpha & 1 \\ 0 & \alpha & 1 \end{vmatrix} = \pm 4$$

α = ± 8

Now given points (8, –8), (–8, 8), (64, β)

OR (–8, 8), (8, –8), (64, β)

are collinear ⇒ Slope = –1.

$\boxed{\beta = -64}$ **Ans. (C)**

11. The number of distinct real roots of the equation $x^7 - 7x - 2 = 0$ is

(A) 5 (B) 7 (C) 1 (D) 3

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $x^7 - 7x - 2 = 0$

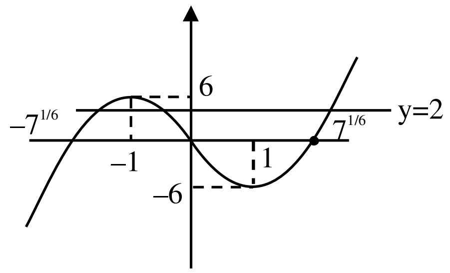
$x^7 - 7x = 2$

f(x) = x^7 – 7x (odd) & y = 2

f(x) = x ($x^2 - 7^{1/3}$) ($x^4 + x^2 \cdot 7^{1/3} + 7^{2/3}$)

f'(x) = 7(x^6 – 1) = 7 (x^2 – 1) ($x^4 + x^2$ + 1)

f'(x) = 0 ⇒ x = ± 1



f(x) = 2 has 3 real distinct solution.

12. A random variable X has the following probability distribution :

X	0	1	2	3	4
P(X)	k	2k	4k	6k	86

The value of P(1 < X < 4 | X ≤ 2) is equal to :

(A) $\frac{4}{7}$ (B) $\frac{2}{3}$

(C) $\frac{3}{7}$ (D) $\frac{4}{5}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $$P\left(\frac{1<x<4}{x\le 2}\right) = \frac{P(1<x<4 \cap x\le 2)}{P(x\le 2)}$$

$$= \frac{P(1<x\le 2)}{P(x\le 2)} = \frac{P(x=2)}{P(x\le 2)}$$

$$= \frac{4k}{k+2k+4k} = \frac{4}{7}$$

13. The number of solutions of the equation $\cos\left(x+\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}-x\right)=\frac{1}{4}\cos^2 2x$, $x \in [-3\pi, 3\pi]$ is :

(A) 8 (B) 5

(C) 6 (D) 7

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $\cos\left(\frac{\pi}{3}+x\right)\cos\left(\frac{\pi}{3}-x\right)=\frac{1}{4}\cos^2 2x$

$x\in[-3\pi,3\pi]$

$4\left(\cos^2\left(\frac{\pi}{3}\right)-\sin^2 x\right)=\cos^2 2x$

$4\left(\frac{1}{4}-\sin^2 x\right)=\cos^2 2x$

$1 - 4\sin^2 x = \cos^2 2x$

$1 - 2(1 - \cos 2x) = \cos^2 2x$

let $\cos 2x = t$

$-1 + 2\cos 2x = \cos^2 2x$

$t^2 - 2t + 1 = 0$

$(t - 1)^2 = 0$

$\boxed{t=1}$ $\boxed{\cos 2x = 1}$

$2x = 2n\pi$

$\boxed{x = n\pi}$

$n = -3, -2, -1, 0, 1, 2, 3$

(D) option is correct.

14. If the shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{\lambda}$ and $\frac{x-2}{1}=\frac{y-4}{4}=\frac{z-5}{5}$ is $\frac{1}{\sqrt{3}}$, then the sum of all possible values of λ is :

(A) 16 (B) 6

(C) 12 (D) 15

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. SHORTEST distance $\frac{|(a_2-a_1)\cdot(b_1\times b_2)|}{|b_1\times b_2|}$

$a_1 = (1, 2, 3)$

$a_2 = (2, 4, 5)$

$\vec{b}_2 = 2\hat{i}+3\hat{j}+\lambda\hat{k}$

$\vec{b}_2 = \hat{i}+4\hat{j}+5\hat{k}$

S.D. $= \frac{\left|\left((2-1)\hat{i}+(4-2)\hat{j}+(5-3)\hat{k}\right)\cdot(\vec{b}_1\times\vec{b}_2)\right|}{|b_1\times b_2|}$

$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & \lambda\\ 1 & 4 & 5\end{vmatrix}$

$= \hat{i}(15 - 4\lambda) + \hat{j}(\lambda - 10) + \hat{k}(5)$

$= (15 - 4\lambda)\hat{i} + (\lambda - 10)\hat{j} + 5\hat{k}$

$|\vec{b}_1\times\vec{b}_2|=\sqrt{(15-4\lambda)^2+(\lambda-10)^2+25}$

Now

S.D. $= \frac{\left|\left(\hat{i}+2\hat{j}+2\hat{k}\right)\cdot\left[(15-4\lambda)\hat{i}+(\lambda-10)\hat{j}+5\hat{k}\right]\right|}{\sqrt{(15-4\lambda)^2+(\lambda-10)^2+25}}$

$\frac{|15-4\lambda+2\lambda-20+10|}{\sqrt{(15-4\lambda)^2+(\lambda-10)^2+25}}=\frac{1}{\sqrt{3}}$

square both side

$3(5-2\lambda)^2 = 225 + 16\lambda^2 - 120\lambda + \lambda^2 + 100 - 20\lambda + 25$

$12\lambda^2 + 75 - 60\lambda = 17\lambda^2 - 140\lambda + 350$

$5\lambda^2 - 80\lambda + 275 = 0$

$\lambda^2 - 16\lambda + 55 = 0$

$(\lambda - 5)(\lambda - 11) = 0$

$\Rightarrow \lambda = 5, 11$

(A) is correct option.

15. Let the points on the plane P be equidistant from the points (–4, 2, 1) and (2, –2, 3). Then the acute angle between the plane P and the plane 2x + y + 3z = 1 is

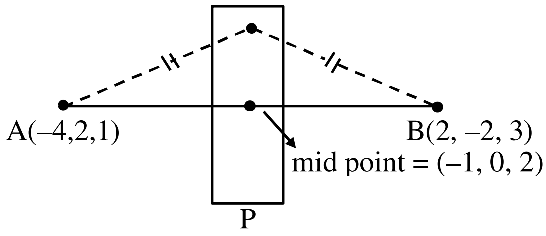
(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$ (D) $\frac{5\pi}{12}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol.



Normal vector = $\overrightarrow{AB} = \left(\overrightarrow{OB} - \overrightarrow{OA}\right)$

$= \left(6\hat{i} - 4\hat{j} + 2\hat{k}\right)$

or $2\left(3\hat{i} - 2\hat{j} + \hat{k}\right)$

$P \equiv 3(x + 1) - 2(y) + 1\,(z - 2) = 0$

$P \equiv 3x - 2y + z + 1 = 0$

$P' \equiv 2x + y + 3z - 1 = 0$

angle between P & P' $= \left|\dfrac{\hat{n}_1 \cdot \hat{n}_2}{|n_1||n_2|}\right| = \cos\theta$

$\theta = \cos^{-1}\left(\dfrac{6 - 2 + 3}{\sqrt{14} \times \sqrt{14}}\right)$

$\theta = \cos^{-1}\left(\dfrac{7}{14}\right) = = \cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$

Option C is correct.

16. Let $\hat{a}$ and $\hat{b}$ be two unit vectors such that $\left|\left(\hat{a} + \hat{b}\right) + 2\left(\hat{a} \times \hat{b}\right)\right| = 2$. If $\theta \in (0, \pi)$ is the angle between $\hat{a}$ and $\hat{b}$, then among the statements :

(S1) : $2\left|\hat{a} \times \hat{b}\right| = \left|\hat{a} - \hat{b}\right|$

(S2) : The projection of $\hat{a}$ on $\left(\hat{a} + \hat{b}\right)$ is $\dfrac{1}{2}$

(A) Only (S1) is true

(B) Only (S2) is true

(C) Both (S1) and (S2) are true

(D) Both (S1) and (S2) are false

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $\left|\left(\hat{a} + \hat{b}\right) + 2\left(\hat{a} \times \hat{b}\right)\right| = 2$, $\theta \in (0, \pi)$

$\left(\left(\hat{a} + \hat{b}\right) + 2\left(\hat{a} \times \hat{b}\right)\right).\left(\left(\hat{a} + \hat{b}\right) + 2\left(\hat{a} \times \hat{b}\right)\right) = 4$

$\left|\hat{a} + \hat{b}\right|^2 + 4\left|\left(\hat{a} \times \hat{b}\right)\right|^2 + 0 = 4$

Let the angle be θ between $\hat{a}$ and $\hat{b}$

$2 + 2\cos\theta + 4\sin^2\theta = 4$

$2 + 2\cos\theta - 4\cos^2\theta = 0$

Let $\cos\theta = t$ then

$2t^2 - t - 1 = 0$

$2t^2 - 2t + t - 1 = 0$

$2t(t - 1) + (t - 1) = 0$

$(2t + 1)(t - 1) = 0$

$t = -\dfrac{1}{2}$ or $t = 1$

$\cos\theta = -\dfrac{1}{2}$ | not possible as $\theta \in (0, \pi)$

$\boxed{\theta = \dfrac{2\pi}{3}}$

Now,

S_1 $2\left|\vec{a} \times \vec{b}\right| = 2\sin\left(\dfrac{2\pi}{3}\right)$

$\left|\hat{a} - \hat{b}\right| = \sqrt{1 + 1 - 2\cos\left(\dfrac{2\pi}{3}\right)}$

$= \sqrt{2 - 2 \times \left(-\dfrac{1}{2}\right)}$

$= \sqrt{3}$

S_1 is correct.

S_2 projection of $\hat{a}$ on $\left(\hat{a} + \hat{b}\right)$.

$\dfrac{\hat{a}.\left(\hat{a} + \hat{b}\right)}{\left|\hat{a} + \hat{b}\right|} = \dfrac{1 + \cos\left(\dfrac{2\pi}{3}\right)}{\sqrt{2 + 2\cos\dfrac{2\pi}{3}}}$

$= \dfrac{1 - \dfrac{1}{2}}{\sqrt{1}}$

$= \dfrac{1}{2}$

C Option is true.

17. If $y = \tan^{-1}(\sec x^3 - \tan x^3)$, $\frac{\pi}{2} < x^3 < \frac{3\pi}{2}$, then

(A) $xy'' + 2y' = 0$

(B) $x^2y'' - 6y + \frac{3\pi}{2} = 0$

(C) $x^2y'' - 6y + 3\pi = 0$

(D) $xy'' - 4y' = 0$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $y = \tan^{-1}(\sec x^3 - \tan x^3)$

$= \tan^{-1}\left(\frac{1-\sin x^3}{\cos x^3}\right)$

$= \tan^{-1}\left(\frac{1-\cos\left(\frac{\pi}{2}-x^3\right)}{\sin\left(\frac{\pi}{2}-x^3\right)}\right)$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\frac{x^3}{2}\right)\right)$

Since $\frac{\pi}{4}-\frac{x^3}{2} \in \left(-\frac{\pi}{2},0\right)$

$y = \left(\frac{\pi}{4}-\frac{x^3}{2}\right)$

$y' = \frac{-3x^2}{2}$, $y'' = -3x$

$4y = \pi - 2x^3$

$4y = \pi - 2x^2\left(\frac{-y''}{3}\right)$

$12y = 3\pi + 2x^2y''$

$x^2y'' - 6y + \frac{3\pi}{2} = 0$

18. Consider the following statements :

A : Rishi is a judge.

B : Rishi is honest.

C : Rishi is not arrogant.

The negation of the statement "if Rishi is a judge and he is not arrogant, then he is honest" is

(A) $B \to (A \vee C)$

(B) $(\sim B) \wedge (A \wedge C)$

(C) $B \to ((\sim A) \vee (\sim C))$

(D) $B \to (A \wedge C)$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\sim((A \wedge C) \to B)$

$\sim(\sim(A \wedge C) \vee B)$

Using De-Morgan's law

$(A \wedge C) \wedge (\sim B)$

Option B is correct.

19. The slope of normal at any point (x, y), x > 0, y > 0 on the curve y = y(x) is given by $\frac{x^2}{xy - x^2y^2 - 1}$.

If the curve passes through the point (1, 1), then e.y(e) is equal to

(A) $\frac{1-\tan(1)}{1+\tan(1)}$ (B) $\tan(1)$

(C) 1 (D) $\frac{1+\tan(1)}{1-\tan(1)}$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. Slope of normal $= \frac{-dx}{dy} = \frac{x^2}{xy - x^2y^2 - 1}$

$x^2y^2dx + dx - xydx = x^2dy$

$x^2y^2dx + dx = x^2dy + xydx$

$x^2y^2dx + dx = x(xdy + ydx)$

$x^2y^2dx + dx = xd(xy)$

$$\frac{dx}{x} = \frac{d(xy)}{1+x^2y^2}$$

$\ln kx = \tan^{-1}(xy)$ … (i)

passes though (1, 1)

$$\ln k = \frac{\pi}{4} \Rightarrow k = e^{\frac{\pi}{4}}$$

equation (i) be becomes

$$\frac{\pi}{4} + \ln x = \tan^{-1}(xy)$$

$$xy = \tan\left(\frac{\pi}{4} + \ell n\, x\right)$$

$$xy = \left(\frac{1+\tan(\ell n\, x)}{1-\tan(\ell n\, x)}\right) \ \ldots \text{(ii)}$$

put x = e in (ii)

$$\therefore\ ey(e) = \frac{1+\tan 1}{1-\tan 1}$$

20. Let λ^* be the largest value of λ for which the function $f_\lambda(x) = 4\lambda x^3 - 36\lambda x^2 + 36x + 48$ is increasing for all $x \in R$. Then $f_{\lambda^*}(1) + f_{\lambda^*}(-1)$ is equal to :

(A) 36 (B) 48

(C) 64 (D) 72

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $f_\lambda(x) = 4\lambda x^3 - 36\lambda x^2 + 36x + 48$

$f_\lambda'(x) = 12\lambda x^2 - 72\lambda x + 36$

$f_\lambda'(x) = 12(\lambda x^2 - 6\lambda x + 3) \geq 0$

$\therefore\ \lambda > 0\ \&\ D \leq 0$

$36\lambda^2 - 4 \times \lambda \times 3 \leq 0$

$9\lambda^2 - 3\lambda \leq 0$

$3\lambda(3\lambda - 1) \leq 0$

$$\lambda \in \left[0, \frac{1}{3}\right]$$

$$\therefore\ \lambda_{largest} = \frac{1}{3}$$

$$f(x) = \frac{4}{3}x^3 - 12x^2 + 36x + 48$$

$\therefore\ f(1) + f(1) = 72$

SECTION-B

1. Let $S = \{z \in \mathbb{C} : |z-3| \leq 1 \text{ and } z(4+3i) + \bar{z}(4-3i) \leq 24\}$. If $\alpha + i\beta$ is the point in S which is closest to 4i, then $25(\alpha + \beta)$ is equal to _____.

Official Ans. by NTA (80)

Allen Ans. (80)

Sol. $|z - 3| \leq 1$

represent pt. i/s circle of radius 1 & centred at (3, 0)

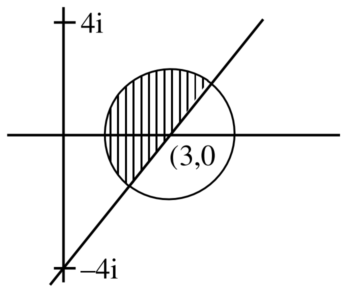
$z(4 + 3i) + \bar{z}(4 - 3i) \leq 24$

$(x + iy)(4 + 3i) + (x - iy)(4 - 3i) \leq 24$

$4x + 3xi + 4iy - 3y + 4x - 3ix - 4iy - 3y \leq 24$

$8x - 6y \leq 24$

$4x - 3y \leq 12$



minimum of (0, 4) from circle $= \sqrt{3^2 + 4^2} - 1 = 4$

will lie along line joining (0, 4) & (3, 0)

$\therefore$ equation line

$$\frac{x}{3} + \frac{y}{4} = 1 \Rightarrow 4x + 3y = 12 \ \ldots \text{(i)}$$

equation circle $(x - 3)^2 + y^2 = 1$ … (ii)

$$\left(\frac{12-3y}{4} - 3\right)^2 + y^2 = 1$$

$\left(\frac{-3y}{4}\right)^2 + y^2 = 1$

$\frac{25y^2}{16} = 1 \Rightarrow y = \pm\frac{4}{5}$

for minimum distance $y = \frac{4}{5}$

$\therefore x = \frac{12}{5}$

$\therefore 25(\alpha + \beta) = 25\left(\frac{4}{5} + \frac{12}{5}\right)$

$= 16 \times 5 = 80$

2. Let $S = \left\{\begin{pmatrix} -1 & a \\ 0 & b \end{pmatrix}; a, b \in \{1, 2, 3, \ldots 100\}\right\}$ and let $T_n = \{A \in S : A^{n(n+1)} = I\}$. Then the number of elements in $\bigcap_{n=1}^{100} T_n$ is _____.

Official Ans. by NTA (100)

Allen Ans. (100)

Sol. $A = \begin{bmatrix} -1 & a \\ 0 & b \end{bmatrix}$

$A^2 = \begin{bmatrix} -1 & a \\ 0 & b \end{bmatrix}\begin{bmatrix} -1 & a \\ 0 & b \end{bmatrix}$

$= \begin{bmatrix} 1 & -a + ab \\ 0 & b^2 \end{bmatrix}$

$\therefore T_n = \{A \in S; A^{n(n+1)} = I\}$

$\therefore$ b must be equal to 1

$\therefore$ In this case A^2 will become identity matrix and a can take any value from 1 to 100

$\therefore$ Total number of common element will be 100.

3. The number of 7-digit numbers which are multiples of 11 and are formed using all the digits 1, 2, 3, 4, 5, 7 and 9 is _____.

Official Ans. by NTA (576)

Allen Ans. (576)

Sol. Digits are 1, 2, 3, 4, 5, 7, 9

Multiple of 11 $\rightarrow$ Difference of sum at even & odd place is divisible by 11.

Let number of the form abcdefg

$\therefore (a + c + e + g) - (b + d + f) = 11x$

$a + b + c + d + e + f = 31$

$\therefore$ either $a + c + e + g = 21$ or 10

$\therefore b + d + f = 10$ or 21

<u>Case- 1</u>

$a + c + e + g = 21$

$b + d + f = 10$

$(b, d, f) \in \{(1, 2, 7)\ (2, 3, 5)\ (1, 4, 5)\}$

$(a, c, e, g) \in \{(1, 4, 7, 9), (3, 4, 5, 9), (2, 3, 7, 9)\}$

$\therefore$ Total number in case-1 $= (3! \times 3)(4!) = 432$

<u>Case- 2</u>

$a + c + e + g = 10$

$b + d + f = 21$

$(a, b, e, g) \in \{1, 2, 3, 4)\}$

(b, d, f) & $\{(5, 7, 9)\}$

$\therefore$ Total number in case $2 = 3! \times 4! = 144$

$\therefore$ Total numbers $= 144 + 432 = 576$

4. The sum of all the elements of the set $\{\alpha \in \{1, 2, \ldots, 100\} : \text{HCF}(\alpha, 24) = 1\}$ is _____.

Official Ans. by NTA (1633)

Allen Ans. (1633)

Sol. HCF $(\alpha, 24) = 1$

Now, $24 = 2^2.3$

$\rightarrow \alpha$ is not the multiple of 2 or 3

Sum of values of α

= S(U) $-${S(multiple of 2) + S (multiple of 3) $-$ S(multiple of 6)}

$= (1 + 2 + 3 + \ldots\ldots 100) - (2 + 4 + 6 \ldots\ldots + 100) - (3 + 6 + \ldots\ldots 99) + (6 + 12 + \ldots\ldots + 96)$

$= \frac{100 \times 101}{2} - 50 \times 51 - \frac{33}{2} \times (3 + 99) + \frac{16}{2}(6 + 96)$

$= 5050 - 2550 - 1683 + 816 = 1633$ **Ans.**

5. The remainder on dividing $1 + 3 + 3^2 + 3^3 + \ldots + 3^{2021}$ by 50 is _____.

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $\frac{1.(3^{2022}-1)}{2} = \frac{9^{1011}-1}{2}$

$= \frac{(10-1)^{1011}-1}{2}$

$= \frac{100\lambda + 10110 - 1 - 1}{2}$

$= 50\lambda + \frac{10108}{2}$

$= 50\lambda + 5054$

$= 50\lambda + 50 \times 101 + 4$

Rem (50) = 4.

6. The area (in sq. units) of the region enclosed between the parabola $y^2 = 2x$ and the line $x + y = 4$ is _____.

Official Ans. by NTA (18)

Allen Ans. (18)

Sol. $x = 4 - y$

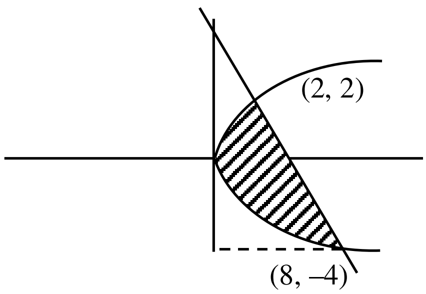
$y^2 = 2(4 - y)$

$y^2 = 8 - 2y$

$y^2 + 2y - 8 = 0$

$y = -4,\ y = 2$

$x = 8,\ x = 2$



$\int_{-4}^{2}\left[(4-y) - \frac{y^2}{2}\right]dy$

$= \left[4y - \frac{y^2}{2} - \frac{y^3}{6}\right]_{-4}^{2}$

$= 8 - 2 - \frac{8}{6} + 16 + \frac{16}{2} - \frac{64}{6}$

$= 22 + 8 - \frac{72}{6}$

$= 30 - 12 = 18$

7. Let a circle C : $(x-h)^2 + (y-k)^2 = r^2$, $k > 0$, touch the x-axis at (1, 0). If the line $x + y = 0$ intersects the circle C at P and Q such that the length of the chord PQ is 2, then the value of $h + k + r$ is equal to _____.

Official Ans. by NTA (7)

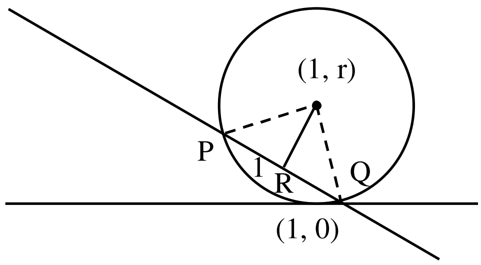
Allen Ans. (7)

Sol. $k = r$

$h = 1$

$OP = r,\ PR = 1$

$OR = \left|\frac{r+1}{\sqrt{2}}\right|$



$r^2 = 1 + \frac{(r+1)^2}{2}$

$2r^2 = 2 + r^2 + 1 + 2r$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$\boxed{r = 3}, -1$

$h + k + r = 1 + 3 + 3$

$= 7$

8. In an examination, there are 10 true-false type questions. Out of 10, a student can guess the answer of 4 questions correctly with probability $\frac{3}{4}$ and the remaining 6 questions correctly with probability $\frac{1}{4}$. If the probability that the student guesses the answers of exactly 8 questions correctly out of 10 is $\frac{27k}{4^{10}}$, then k is equal to _____.

Official Ans. by NTA (479)

Allen Ans. (479)

Sol. $A = \{1, 2, 3, 4\}$: $P(A) = \frac{3}{4} \rightarrow$ Correct

$B = \{5, 6, 7, 8, 9, 10\}$; $P(B) = \frac{1}{4}$ Correct

8 Correct Ans.:

(4, 4): $^4C_4\left(\frac{3}{4}\right)^4 \cdot {}^6C_4 \cdot \left(\frac{1}{4}\right)^4 \cdot \left(\frac{3}{4}\right)^2$

(3, 5): $^4C_3\left(\frac{3}{4}\right)^3 \cdot \left(\frac{1}{4}\right)^1 \cdot {}^6C_5\left(\frac{1}{4}\right)^5 \cdot \left(\frac{3}{4}\right)$

(2, 6): $^4C_2\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2 \cdot {}^6C_6\left(\frac{1}{4}\right)^6$

Total $= \frac{1}{4^{10}}[3^4 \times 15 \times 3^2 + 4 \times 3^3 \times 6 \times 3 + 6 \times 3^2]$

$= \frac{27}{4^{10}}[2.7 \times 15 + 72 + 2]$

$\Rightarrow K = 479$

9. Let the hyperbola H : $\frac{x^2}{a^2} - y^2 = 1$ and the ellipse E : $3x^2 + 4y^2 = 12$ be such that the length of latus rectum of H is equal to the length of latus rectum of E. If e_H and e_E are the eccentricities of H and E respectively, then the value of $12\left(e_H^2 + e_E^2\right)$ is equal to _____.

Official Ans. by NTA (42)

Allen Ans. (42)

Sol. $\frac{x^2}{a^2} - \frac{y^2}{1} = 1$ $\qquad$ $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$e_H = \sqrt{1 + \frac{1}{a^2}}$ $\qquad$ $e_E = \sqrt{1 - \frac{3}{4}} = \frac{1}{2}$

$\ell.R. = \frac{2}{a}$ $\qquad$ $\ell R = \frac{2 \times 3}{2} = 3$

$\frac{2}{a} = 3$

$\boxed{a = \frac{2}{3}}$

$e_H = \sqrt{1 + \frac{9}{4}} = \frac{\sqrt{13}}{2}$

$12\left(e_H^2 + e_E^2\right) = 12\left(\frac{13}{4} + \frac{1}{4}\right)$

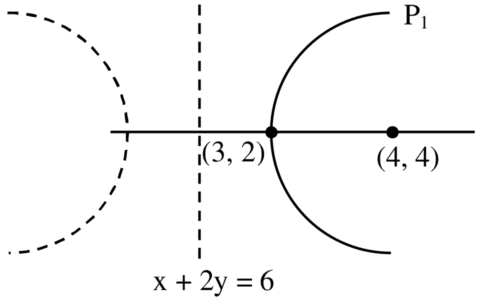
$= \frac{12 \times 14}{4} = 42$

10. Let P_1 be a parabola with vertex (3, 2) and focus (4, 4) and P_2 be its mirror image with respect to the line $x + 2y = 6$. Then the directrix of P_2 is $x + 2y =$ _____.

Official Ans. by NTA (10)

Allen Ans. (10)

Sol.



P_1: Directorix :

$x + 2y = k$

$x + 2y - k = 0$

$\left|\frac{3 + 4 - K}{\sqrt{5}}\right| = \sqrt{5}$

$|7 - k| = 5$

$7 - K = 5 \qquad 7 - K = -5$

$\boxed{k = 2} \qquad \boxed{k = 12}$

Accepted $\qquad$ Rejected

Passes through focus

$D_1 = x + 2y = 2$
$\ell = x + 2y = 6$ $\quad \Rightarrow d$
$D_2 = x + 2y = C$ $\quad \Rightarrow d$

$\Rightarrow \boxed{c = 10}$