FINAL JEE–MAIN EXAMINATION – JUNE, 2022

(Held On Friday 24th June, 2022) **TIME : 9 : 00 AM to 12 : 00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. Let $A=\{z\in C:1\le|z-(1+i)|\le 2\}$ and $B=\{z\in A:|z-(1-i)|=1\}$. Then, B :

(A) is an empty set

(B) contains exactly two elements

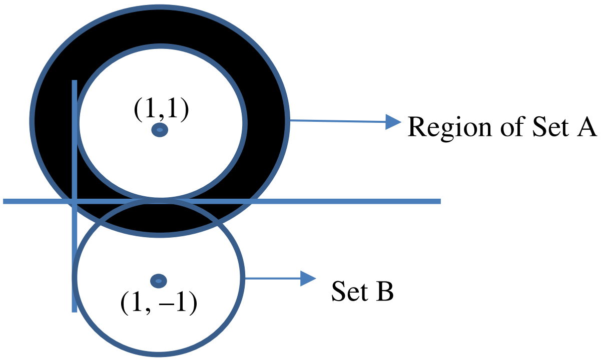
(C) contains exactly three elements

(D) is an infinite set

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $A=\{z\in C:1\le|z-(1+i)|\le 2\}$



$B=\{z\in A:|z-(1-i)|=1\}$.

$A\cap B$ has infinite set.

2. The remainder when 3^{2022} is divided by 5 is

(A) 1 (B) 2

(C) 3 (D) 4

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $3^{2022}=9^{1011}=(10-1)^{1011}=10\,m-1=10\,m-5+4$

$=5\,(2m-1)+4$ (m is integer)

Remainder = 4

3. The surface area of a balloon of spherical shape being inflated, increases at a constant rate. If initially, the radius of balloon is 3 units and after 5 seconds,, it becomes 7 units, then its radius after 9 seconds is :

(A) 9 (B) 10

(C) 11 (D) 12

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. Let r be the radius of spherical balloon

S = Surface area

$S=4\pi r^2$

$\frac{dS}{dt}=8\pi r\times\frac{dr}{dt}=k$ (constant)

$4\pi r^2=kt+C$ (C is constant of integration)

For $t=0$, $r=3$ $\Rightarrow 36\pi=C$

For $t=5$, $r=7$ $\Rightarrow K=32\pi$

$4\pi r^2=32\pi t+36\pi$

$r^2=8t+9$

for $t=9$

$r^2=81$

$r=9$

4. Bag A contains 2 white, 1 black and 3 red balls and bag B contains 3 black, 2 red and n white balls. One bag is chosen at random and 2 balls drawn from it at random, are found to be 1 red and 1 black. If the probability that both balls come from Bag A is $\frac{6}{11}$, then n is equal to _____ .

(A) 13 (B) 6

(C) 4 (D) 3

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. E_1 = denotes selection for 1st bag

E_2 = denotes selection for 2nd bag

$P(E_1)=\frac{1}{2}$, $P(E_2)=\frac{1}{2}$

A = selected balls are 1 red & 1 black

$P\left(\frac{A}{E_1}\right)=\frac{{}^3C_1\times{}^1C_1}{{}^6C_2}=\frac{1}{5}$

$$P\left(\frac{A}{E_1}\right) = \frac{{}^3C_1 \times {}^2C_1}{(n+5)_{C_2}} = \frac{12}{(n+5)(n+4)}$$

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1)\times P\left(\frac{A}{E_1}\right)}{P(E_1)\times P\left(\frac{A}{E_1}\right) + P(E_2)\times P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{10}}{\frac{1}{10} + \frac{6}{(n+5)(n+4)}} = \frac{6}{11}$$

$\Rightarrow n = 4$

5. Let $x^2 + y^2 + Ax + By + C = 0$ be a circle passing through (0, 6) and touching the parabola $y = x^2$ at (2, 4). Then A + C is equal to______ .

(A) 16 (B) 88/5

(C) 72 (D) –8

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $x^2 + y^2 + Ax + By + C = 0$ is passing through (0,6)

$\Rightarrow 6B + C = -36$

The tangent of the parabola $y = x^2$ at (2, 4) is

$4x - y - 4 = 0$ ----(1)

The tangent of circle $x^2 + y^2 + Ax + By + C = 0$ at (2, 4) is

$(4 + A)x + (8 + B)y + 2A + 4B + 2C = 0$ -----(2)

From Equation (1) and (2)

$$\frac{4+A}{4} = \frac{8+B}{-1} = \frac{2A+4B+2C}{-4}$$

$A + 4B = -36$ ---(3)

$3A + 4B + 2C = -4$ ---(4)

From equation (3) and (4)

A+ C = 16

6. The number of values of α for which the system of equations :

$x + y + z = \alpha$

$\alpha x + 2\alpha y + 3z = -1$

$x + 3\alpha y + 5z = 4$

is inconsistent, is

(A) 0 (B) 1

(C) 2 (D) 3

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $x + y + z = \alpha$

$\alpha x + 2\alpha y + 3z = -1$

$x + 3\alpha y + 5z = 4$

Has inconsistent solution

$$D = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & 2\alpha & 3 \\ 1 & 3\alpha & 5 \end{vmatrix} = 0$$

$\Rightarrow (\alpha - 1)^2 = 0$

$\alpha = 1$

For $\alpha = 1$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 2 & 3 \\ 4 & 3 & 5 \end{vmatrix}$$

$= (10 - 9) - (-5-12) + (-3-8)$

$= 1 + 17 - 11 \neq 0$

For $\alpha = 1$ the system of equation has Inconsistent solution

7. If the sum of the squares of the reciprocals of the roots α and β of the equation $3x^2 + \lambda x - 1 = 0$ is 15, then $6(\alpha^3 + \beta^3)^2$ is equal to :

(A) 18 (B) 24

(C) 36 (D) 96

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. **Here** α, β roots of equation $3x^2 + \lambda x - 1 = 0$

$$\alpha + \beta = \frac{-\lambda}{3}, \quad \alpha\beta = \frac{-1}{3}$$

$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha^2\beta^2} = 15$$

$\lambda^2 = 9$

Now $6(\alpha^3 + \beta^3)^2 = 6\left((\alpha+\beta)\left((\alpha+\beta)^2 - 3\alpha\beta\right)\right)^2$

$$= 6\left(\frac{\lambda^2}{9}\right)\left\{\frac{\lambda^2}{9} + 1\right\}^2 = 24$$

8. The set of all values of k for which $(\tan^{-1}x)^3+(\cot^{-1}x)^3=k\pi^3, x\in R$, is the interval :

(A) $\left[\frac{1}{32},\frac{7}{8}\right)$ (B) $\left(\frac{1}{24},\frac{13}{16}\right)$

(C) $\left[\frac{1}{48},\frac{13}{16}\right]$ (D) $\left[\frac{1}{32},\frac{9}{8}\right)$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. **Let S =** $(\tan^{-1}x)^3+(\cot^{-1}x)^3$

$$=\left(\tan^{-1}x+\cot^{-1}x\right)-3\tan^{-1}x.\cot^{-1}x\left(\tan^{-1}x+\cot^{-1}x\right)$$

$$=\frac{\pi^3}{8}-\frac{3\pi}{2}\tan^{-1}x\left(\frac{\pi}{2}-\tan^{-1}x\right)$$

$$=\frac{3\pi}{2}\left(\tan^{-1}x-\frac{\pi}{4}\right)^2+\frac{\pi^3}{32}$$

$$\Rightarrow\frac{\pi^3}{32}\le S<\frac{7}{8}\pi^3$$

$$=\frac{\pi^3}{32}\le K\pi^3<\frac{7}{8}\pi^3$$

$$\frac{1}{32}\le K<\frac{7}{8}$$

9. Let $S=\{\sqrt{n}:1\le n\le 50 \text{ and n is odd}\}$

Let $a\in S$ and $A=\begin{bmatrix}1&0&a\\-1&1&0\\-a&0&1\end{bmatrix}$

If $\sum\limits_{a\in S}\det(\text{adj}A)=100\lambda$, then λ is equal to

(A) 218 (B) 221

(C) 663 (D) 1717

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $S=\{\sqrt{n}:1\le n\le 50 \text{ and n is odd}\}$

$=\left\{\sqrt{1},\sqrt{3},\sqrt{5}..........\sqrt{49}\right\}$, 25 terms

$|A| = 1 + a^2$

$$\sum_{a\in S}\det(\text{adj}A)=\sum_{a\in S}|A|^2=\sum(1+a^2)^2$$

$= 22100 = 100\lambda$

$\lambda = 221$

10. $f(x) = 4\log_e(x-1)-2x^2+4x+5, x>1$, which one of the following is NOT correct ?

(A) f is increasing in (1, 2) and decreasing in $(2,\infty)$

(B) $f(x)=-1$ has exactly two solutions

(C) $f'(e)-f''(2)<0$

(D) $f(x) = 0$ has a root in the interval $(e, e+1)$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $f(x) = 4\log_e(x-1)-2x^2+4x+5, x>1$

$$f'(x)=\frac{4}{x-1}-4(x-1)$$

For $1<x<2\Rightarrow f'(x)>0$

For $x>2\Rightarrow f'(x)<0$ (option 1 is correct)

$f(x)=-1$ has two solution (option 2 is correct)

$f(e)>0$

$f(e+1)<0$

$f(e).f(e+1)<0$ (option 4 is correct)

$$f'(e)-f''(2)=\frac{4}{e-1}-4(e-1)+8>0$$

(option C is incorrect)

11. the tangent at the point (x_1, y_1) on the curve $y = x^3+3x^2+5$ passes through the origin, then (x_1, y_1) does NOT lie on the curve :

(A) $x^2+\frac{y^2}{81}=2$ (B) $\frac{y^2}{9}-x^2=8$

(C) $y=4x^2+5$ (D) $\frac{x}{3}-y^2=2$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. **The tangent at (x_1, y_1) to the curve**

$y=x^3+3x^2+5$

$y-y_1=\left(3x_1^2+6x_1\right)\left(x-x_1\right)$ passing through origin

$-y_1=\left(3x_1^3+6x_1\right)\left(-x_1\right)$

$y_1=\left(3x_1^3+6x_1^2\right)$ --------(1)

And (x_1, y_1) lies on the curve

$$y = x^3 + 3x^2 + 5$$

$$y_1 = x_1^3 + 3x_1^2 + 5 \text{ ----(2)}$$

From equation (1) and (2)

$$2y_1 = 3x_1^2 + \frac{15}{2}$$

Hence the equation of curve $y = \frac{3}{2}x^2 + \frac{15}{2}$

This curve does not intersect $\frac{x}{3} - y^2 = 2$

12. The sum of absolute maximum and absolute minimum values of the function $f(x) = |2x^2 + 3x - 2| + \sin x \cos x$ in the interval $[0, 1]$ is :

(A) $3 + \frac{\sin(1)\cos^2(\frac{1}{2})}{2}$ (B) $3 + \frac{1}{2}(1 + 2\cos(1))\sin(1)$

(C) $5 + \frac{1}{2}(\sin(1) + \sin(2))$ (D) $2 + \sin\left(\frac{1}{2}\right)\cos\left(\frac{1}{2}\right)$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $f(x) = |2x^2 + 3x - 2| + \sin x \cos x$

$f(x) = |(2x - 1)(x + 2)| + \sin x \cos x$

$$f'(x) = \begin{cases} 4x + 3 + \frac{\cos 2x}{4}, & \frac{1}{2} < x < 1 \\ -(4x+3) + \frac{\cos 2x}{4}, & 0 \le x < \frac{1}{2} \end{cases}$$

For $0 \le x < \frac{1}{2} \Rightarrow f'(x) < 0$

For $\frac{1}{2} < x \le 1 \Rightarrow f'(x) > 0$

f (x) local minima at $x = \frac{1}{2}$ and

local maxima at $x = 1$

$$f\left(\frac{1}{2}\right) + f(1) = 3 + \frac{1}{2}(1 + 2\cos 1)\sin 1$$

13. If $\{a_i\}_{i=1}^{n}$ where n is an even integer , is an arithmetic progression with common difference 1, and $\sum_{i=1}^{n} a_i = 192$, $\sum_{i=1}^{n/2} a_{2i} = 120$, then n is equal to:

(A) 48 (B) 96

(C) 92 (D) 104

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\sum_{i=1}^{n} a_i = \frac{n}{2}\{2a_1 + (n+1)\} = 192$

$$\Rightarrow 2a_1 + (n-1) = \frac{384}{n} \text{ ----(1)}$$

$$\sum_{i=1}^{n/2} a_{2i} = \frac{n}{4}\left[2a_1 + 2 + \left(\frac{n}{2} - 1\right)2\right] = 120$$

$$2a_1 + n = \frac{480}{n} \text{ ----(2)}$$

From equation (2) and (1)

$$1 = \frac{480}{n} - \frac{384}{n}$$

$n = 480 - 384 = 96$

14. If $x = x(y)$ is the solution of the differential equation $y\frac{dx}{dy} = 2x + y^3(y+1)e^y, x(1) = 0$; then $x(e)$ is equal to :

(A) $e^3(e^e - 1)$ (B) $e^e(e^3 - 1)$

(C) $e^2(e^e + 1)$ (D) $e^e(e^2 - 1)$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $y\frac{dx}{dy} = 2x + y^3(y+1)e^y, x(1) = 0$

$$\frac{dx}{dy} - \frac{2}{y}x = y^2(y+1)e^y$$

$$\text{I.f} = e^{\int \frac{-2}{y}dy} = \frac{1}{y^2}$$

$$x.\frac{1}{y^2} = \int (y+1)e^y dy$$

$$\frac{x}{y^2} = (y+1)e^y - e^y + c = y.e^y + c$$

$$x = y^3e^y + cy^2$$

For $x = 0$, $y = 1 \Rightarrow c = -e$

$$x = y^3e^y - e.y^2$$

$x(e) = e^3(e^e - 1)$

15. Let $\lambda x - 2y = \mu$ be a tangent to the hyperbola $a^2x^2 - y^2 = b^2$. Then $\left(\frac{\lambda}{a}\right)^2 - \left(\frac{\mu}{b}\right)^2$ is equal to:

(A) –2 (B) –4

(C) 2 (D) 4

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $\lambda x - 2y = \mu$ is a tangent to the curve

$a^2x^2 - y^2 = b^2$ then

$$a^2x^2 - \left(\frac{\lambda x - \mu}{2}\right)^2 = b^2$$

$$\left(4a^2 - \lambda^2\right)x^2 + 2\lambda\mu x - \mu^2 - 4b^2 = 0$$

Disc. = 0

$$4\lambda^2\mu^2 + 4\left(4a^2 - \lambda^2\right)\left(\mu^2 + 4b^2\right) = 0$$

$$4\lambda^2b^2 - 4a^2\mu^2 = 16a^2b^2$$

$$\frac{\lambda^2}{a^2} - \frac{\mu^2}{b^2} = 4$$

16. Let $\hat{a}, \hat{b}$ be unit vectors. If $\vec{c}$ be a vector such that the angle between $\hat{a}$ and $\vec{c}$ is $\frac{\pi}{12}$, and $\hat{b} = \vec{c} + 2(\vec{c} \times \hat{a})$, then $|6\vec{c}|^2$ is equal to

(A) $6(3 - \sqrt{3})$ (B) $3 + \sqrt{3}$

(C) $6(3 + \sqrt{3})$ (D) $6(\sqrt{3} + 1)$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $|\hat{b}|^2 = |\vec{c} + 2(\vec{c} \times \hat{a})|^2$

$$|\hat{b}|^2 = |c|^2 + 4|\vec{c} \times \hat{a}|^2 + 4\vec{c}.(\vec{c} \times \hat{a})$$

$$1 = |c|^2 + 4|c|^2 \sin^2\frac{\pi}{12} + 0$$

$$1 = |c|^2 + 4|c|^2\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)^2$$

$$|c|^2 = \frac{1}{3-\sqrt{3}} = \frac{3+\sqrt{3}}{6}$$

So $6^2|c|^2 = 6(3 + \sqrt{3})$

17. If a random variable X follows the Binomial distribution B (33, p) such that 3P(X = 0) = P(X = 1), then the value of $\frac{P(X=15)}{P(X=18)} - \frac{P(X=16)}{P(X=17)}$ is equal to

(A) 1320 (B) 1088

(C) $\frac{120}{1331}$ (D) $\frac{1088}{1089}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. n = 33, let probability of success is p and q = 1 – p

$3p(x = 0) = p(x = 1)$

$$3.\,{}^{33}C_0(q)^{33} = {}^{33}C_1pq^{32}$$

$$p = \frac{1}{12},\ q = \frac{11}{12},\ \frac{q}{p} = 11$$

$$\frac{p(x=15)}{p(x=18)} - \frac{p(x=16)}{p(x=17)}$$

$$\frac{{}^{33}C_{15}p^{15}q^{18}}{{}^{33}C_{18}p^{18}q^{15}} - \frac{{}^{33}C_{16}p^{16}q^{17}}{{}^{33}C_{17}p^{17}q^{16}} = \left(\frac{q}{p}\right)^3 - \left(\frac{q}{p}\right)$$

$$= (11)^3 - 11$$

= 1320

18. The domain of the function

$$f(x) = \frac{\cos^{-1}\left(\frac{x^2 - 5x + 6}{x^2 - 9}\right)}{\log_e(x^2 - 3x + 2)}$$ is

(A) $(-\infty, 1) \cup (2, \infty)$

(B) $(2, \infty)$

(C) $\left[-\frac{1}{2}, 1\right) \cup (2, \infty)$

(D) $\left[-\frac{1}{2}, 1\right) \cup (2, \infty) - \left\{\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right\}$

Official Ans. by NTA (DROP)

Allen Ans. (D)

Sol. $-1 \le \frac{x^2-5x+6}{x^2-9} \le 1$

$\frac{x^2-5x+6}{x^2-9} - 1 \le 0$

$\frac{1}{x+3} \ge 0$

$x \in (-3, \infty)$ **......(1)**

$\frac{x^2-5x+6}{x^2-9} + 1 \ge 0$

$\frac{2x+1}{x+3} \ge 0$

$x \in (-\infty, -3) \cup \left[-\frac{1}{2}, \infty\right)$ **......(2)**

after taking intersection

$x \in \left[-\frac{1}{2}, \infty\right)$

$x^2 - 3x + 2 > 0$

$x \in (-\infty, 1) \cup (2, \infty)$

$x^2 - 3x + 2 \ne 1$

$x \ne \frac{3 \pm \sqrt{5}}{2}$

after taking intersection of each solution

$\left[-\frac{1}{2}, 1\right) \cup (2, \infty) - \left\{\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\right\}$

19. Let

$S = \left\{\theta \in [-\pi, \pi] - \left\{\pm\frac{\pi}{2}\right\} : \sin\theta\tan\theta + \tan\theta = \sin 2\theta\right\}.$

If $T = \sum_{\theta \in S} \cos 2\theta$, then T + n(S) is equal

(A) $7+\sqrt{3}$ (B) 9

(C) $8+\sqrt{3}$ (D) 10

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\sin\theta\tan\theta + \tan\theta = \sin 2\theta$

$\tan\theta(\sin\theta + 1) = \frac{2\tan\theta}{1+\tan^2\theta}$

$\tan\theta = 0 \Rightarrow \theta = -\pi, 0, \pi$

$(\sin\theta + 1) = 2.\cos^2\theta = 2(1+\sin\theta)(1-\sin\theta)$

$\sin\theta = -1$ which is not possible

$\sin\theta = \frac{1}{2} \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

n(s) = 5

$T = \cos 0 + \cos 2\pi + \cos 2\pi + \cos\frac{\pi}{3} + \cos\frac{5\pi}{3}$

T = 4

T + n(s) = 9

20. The number of choices of $\Delta \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$, such that $(p\Delta q) \Rightarrow ((p\Delta \sim q) \vee ((\sim p)\Delta q))$ is a tautology, is

(A) 1 (B) 2

(C) 3 (D) 4

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. For tautology $((p\Delta \sim q) \vee ((\sim p)\Delta q))$ must be true.

This is possible only when $\Delta = \vee \,\&\, \Rightarrow$

SECTION-B

1. The number of one-one function f : {a, b, c, d} → {0, 1, 2, … .,10} such that 2f(a) – f(b) + 3f(c) + f(d) = 0 is _____ .

Official Ans. by NTA (31)

Allen Ans. (31)

Sol. 2f(a) + 3f(c) = f(d) – f(b)

Using fundamental principle of counting

Number of one-one function is 31

2. In an examination, there are 5 multiple choice questions with 3 choices, out of which exactly one is correct There are 3 marks for each correct answer, -2 marks for each wrong answer and 0 mark if the question is not attempted. Then, the number of ways a student appearing in the examination gets 5 marks is_.

Official Ans. by NTA (40)

Allen Ans. (40)

Sol. $x_1 + x_2 + x_3 + x_4 + x_5 = 5$

Only one possibilities 3, 3, 3, -2, -2

Number of ways is = $\frac{5!}{3!2!} \times 2 \times 2 = 40$

3. Let $A\left(\frac{3}{\sqrt{a}}, \sqrt{a}\right) a > 0$, be a fixed point in the xy-plane. The image of A in y-axis be B and the image of B in x-axis be C. If D(3 cos θ, a sin θ) is a point in the fourth quadrant such that the maximum area of ΔACD is 12 square units, then a is equal to ________ .

Official Ans. by NTA (8)

Allen Ans. (8)

Sol. $A=\left(\frac{3}{\sqrt{a}},\sqrt{a}\right)$

$B=\left(\frac{-3}{\sqrt{a}},\sqrt{a}\right)$

$C=\left(-\frac{3}{\sqrt{a}},-\sqrt{a}\right)$

Area of ACD

$$\frac{1}{2}\begin{vmatrix}\frac{3}{\sqrt{a}} & \sqrt{a}\\ -\frac{3}{\sqrt{a}} & -\sqrt{a}\\ 3\cos\theta & a\sin\theta\\ \frac{3}{\sqrt{a}} & \sqrt{a}\end{vmatrix}$$

$\frac{1}{2}6\sqrt{a}(\cos\theta-\sin\theta)$

$3\sqrt{a}(\cos\theta-\sin\theta)$

max values of function is $3\sqrt{a}\sqrt{2}$

$3\sqrt{a}\sqrt{2}=12$

2a = 16

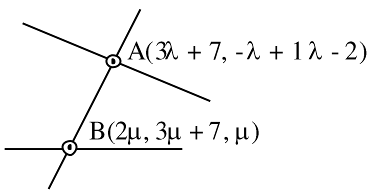
a = 8

4. Let a line having direction ratios 1, -4, 2 intersect the lines $\frac{x-7}{3}=\frac{y-1}{-1}=\frac{z+2}{1}$ and $\frac{x}{2}=\frac{y-7}{3}=\frac{z}{1}$ at the point A and B. Then $(AB)^2$ is equal to ____ .

Official Ans. by NTA (84)

Allen Ans. (84)

Sol.



DR's of AB

$(3\lambda-2\mu+7,-\lambda-3\mu-6,\lambda-\mu-2)$

$\frac{3\lambda-2\mu+7}{1}=\frac{-\lambda-3\mu-6}{-4}=\frac{\lambda-\mu-2}{2}$

Taking first (2) $-12\lambda+8\mu-28=-\lambda-3\mu-6$

$\lambda-\mu+2=0$

Taking second & third

$-2\lambda-6\mu-12=-4\lambda+4\mu+8$

$\lambda-5\mu-10=0$

After solving above two equation $\lambda=-5,\mu=-3$

A = (-8, 6, 7)

B = (-6, -2, -3)

$(AB)^2$ = 4 + 64 + 16 = 84

5. The number of points where the function

$$f(x)=\begin{cases}|2x^2-3x-7| & \text{if} \quad x\le-1\\ [4x^2-1] & \text{if} \quad -1<x<1\\ |x+1|+|x-2| & \text{if} \quad x\ge1\end{cases}$$

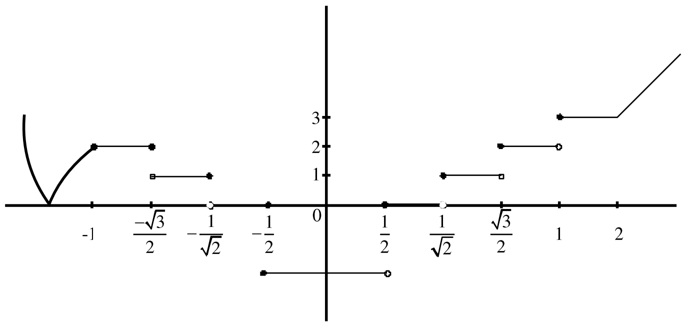
[t] denotes the greatest integer $\le t$, is

discontinuous is _______ .

Official Ans. by NTA (7)

Allen Ans. (7)

Sol.



6. Let $f(\theta)=\sin\theta+\int_{-\pi/2}^{\pi/2}(\sin\theta+t\cos\theta)f(t)dt$. Then the value of $\left|\int_0^{\pi/2}f(\theta)d\theta\right|$ is _______ .

Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $f(\theta)=\sin\theta+\int_{-\pi/2}^{\pi/2}(\sin\theta+t\cos\theta)\,f(t)\,dt$

$f(\theta)=\sin\theta+\sin\theta\int_{-\pi/2}^{\pi/2}f(t)dt+\cos\theta\int_{-\pi/2}^{\pi/2}tf(t)dt$

Let $A=\int_{-\pi/2}^{\pi/2}f(t)dt,\ B=\int_{-\pi/2}^{\pi/2}tf(t)dt$

$f(\theta)=\sin\theta+A\sin\theta+B\cos\theta$

$f(\theta)=(A+1)\sin\theta+B\cos\theta$

$A=\int_{-\pi/2}^{\pi/2}(A+1)\sin t+B\cos t\ dt$

$A = 2B \quad ......(1)$

$B=\int_{-\pi/2}^{\pi/2}t((A+1)\sin t+B\cos t)$

$B=\int_{-\pi/2}^{\pi/2}t(A+1)\sin t$

$B=(A+1)2\int_{0}^{\pi/2}t\sin t\ \ dt$

$B = (A + 1)\ 2.1$

$2A + 2 - B = 0\ .....(2)$

After solving

$B=-\frac{2}{3},A=-\frac{4}{3}$

$\left|\int_{0}^{\pi/2}f(\theta)d\theta\right|=\left|\int_{0}^{\pi/2}-\frac{1}{3}\sin\theta-\frac{2}{3}\cos\theta\right|$

$= 1$

7. Let $\underset{0\le x\le 2}{\text{Max}}\left\{\frac{9-x^2}{5-x}\right\}=\alpha$ and $\underset{0\le x\le 2}{\text{Min}}\left\{\frac{9-x^2}{5-x}\right\}=\beta$

If $\int_{\beta-\frac{8}{3}}^{2\alpha-1}\text{Max}\left\{\frac{9-x^2}{5-x},x\right\}dx=\alpha_1+\alpha_2\log_e\left(\frac{8}{15}\right)$ then

$\alpha_1+\alpha_2$ is equal to _______

Official Ans. by NTA (34)

Allen Ans. (34)

Sol. $y=\frac{9-x^2}{5-x}=5+x+\frac{16}{x-5}$

$\frac{dy}{dx}=1-\frac{16}{(x-5)^2}$

So critical point is $x = 1$ in $[0, 2]$

$y(0)=\frac{9}{5},\ y(1)=2,\ y(2)=\frac{5}{3}$

So $\alpha=2$ and $\beta=\frac{5}{3}$

$I=\int_{-1}^{3}\max\left(\frac{9-x^2}{5-x},x\right)$

$I=\int_{-1}^{9/5}\frac{9-x^2}{5-x}dx+\int_{9/5}^{3}x\,dx$

$I=\int_{-1}^{9/5}5+x+\frac{16}{x-5}dx+\int_{9/5}^{3}x\,dx$

After solving

$I=14+\frac{28}{25}+16\ln\left(\frac{8}{15}\right)+\frac{72}{25}$

$\alpha_1=18$ and $\alpha_2=16$

8. If two tangents drawn from a point (α, β) lying on the ellipse $25x^2+4y^2=1$ to the parabola $y^2=4x$ are such that the slope of one tangent is four times the other, then the value of $(10\alpha+5)^2+(16\beta^2+50)^2$ equals ___________

Official Ans. by NTA (2929)

Allen Ans. (2929)

Sol. $\alpha=\frac{1}{5}\cos\theta,\ \beta=\frac{1}{2}\sin\theta$

Equation of tangent to $y^2=4x$

$y=mx+\frac{1}{m}$

It passes through (α,β)

$\frac{1}{2}\sin\theta=m\frac{1}{5}\cos\theta+\frac{1}{m}$

$m^2\left(\frac{\cos\theta}{5}\right)-m\left(\frac{1}{2}\sin\theta\right)+1=0$

It has two roots m_1 and m_2 where $m_1=4m_2$

$m_1+m_2=\frac{\frac{1}{2}\sin\theta}{\frac{\cos\theta}{5}}$

$m_1m_2=\frac{5}{\cos\theta}$

After eliminating m_1 and m_2

$$\cos\theta = \frac{-5 \pm \sqrt{29}}{2}$$

$$\alpha = \frac{-5 \pm \sqrt{29}}{10} \Rightarrow 10\alpha + 5 = \pm\sqrt{29}$$

$$\beta^2 = \frac{1}{4}\sin^2\theta \Rightarrow 16\beta^2 = -50 \pm 10\sqrt{29}$$

$$\left(10\alpha + 5\right)^2 + \left(16\beta^2 + 50\right)^2 = 2929$$

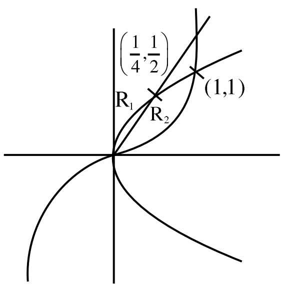
9. Let S be the region bounded by the curves $y = x^3$ and $y^2 = x$. The curve $y = 2|x|$ divides S into two regions of areas R_1 and R_2.

If $\max\{R_1, R_2\} = R_2$, then $\frac{R_2}{R_1}$ is equal to ____ .

Official Ans. by NTA (19)

Allen Ans. (19)

Sol.



$$S = \int_0^1 \sqrt{x} - x^3$$

$$= \left[\frac{2x^{3/2}}{3} - \frac{x^4}{4}\right]_1^0$$

$$= \frac{5}{12}$$

$$R_1 = \int_0^{1/4} (\sqrt{x} - 2x)dx$$

$$= \left[\frac{2x^{3/2}}{3} - x^2\right]_0^{1/4} = \frac{1}{48}$$

$$\therefore\ R_2 = \frac{19}{48}$$

So, $\frac{R_2}{R_1} = 19$

10. If the shortest distance between the line $\vec{r} = \left(-\hat{i} + 3k\right) + \lambda\left(\hat{i} - a\hat{j}\right)$ and $\vec{r} = \left(-\hat{j} + 2k\right) + \mu\left(\hat{i} - \hat{j} + k\right)$ is $\sqrt{\frac{2}{3}}$, then the integral value of a is equal to

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $a_1 = (-1, 0, 3)$

$a_2 = (0, -1, 2)$

$b_1 = (1, -a, 0)$ dr's of line (1)

$b_2 = (1, -1, 1)$ dr's of line (2)

$\bar{a}_2 - \bar{a}_1 = (1, -1, -1)$

$$\bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & k \\ 1 & -a & 0 \\ 1 & -1 & 1 \end{vmatrix}$$

$$\bar{b}_1 \times \bar{b}_2 = \hat{i}(-a) - \hat{j} + k(a-1)$$

$$\left|\bar{b}_1 \times \bar{b}_2\right| = \sqrt{a^2 + 1 + (a-1)^2}$$

$$a_2 - a_1 . \bar{b}_1 \times \bar{b}_2 = 2 - 2a$$

$$\frac{2(1-a)}{\sqrt{a^2 + 1 + (a-1)^2}} = \sqrt{\frac{2}{3}}$$

Squaring an both the side

After solving $a = 2, \frac{1}{2}$