

FINAL JEE-MAIN EXAMINATION – JUNE, 2022

 (Held On Sunday 26th June, 2022)

TIME : 9 : 00 AM to 12 : 00 PM

MATHEMATICS
TEST PAPER WITH SOLUTION
SECTION-A

1. Let $f(x) = \frac{x-1}{x+1}$, $x \in \mathbb{R} - \{0, -1, 1\}$. If $f^{n+1}(x) = f(f^n(x))$

for all $n \in \mathbb{N}$, then $f^6(6) + f^7(7)$ is equal to:

- (A) $\frac{7}{6}$ (B) $-\frac{3}{2}$ (C) $\frac{7}{12}$ (D) $-\frac{11}{12}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $f(x) = \frac{x-1}{x+1}$

$$\Rightarrow f^2(x) = f(f(x)) = \frac{\frac{x-1}{x+1} - 1}{\frac{x-1}{x+1} + 1} = -\frac{1}{x}$$

$$f^3(x) = f(f^2(x)) = f\left(-\frac{1}{x}\right) = \frac{x+1}{1-x}$$

$$\Rightarrow f^4(x) = f\left(\frac{x+1}{1-x}\right) = -\frac{1}{x}$$

$$\Rightarrow f^6(x) = -\frac{1}{x} \Rightarrow f^6(6) = -\frac{1}{8}$$

$$f^7(x) = \left(-\frac{1}{x}\right) = \frac{x+1}{1-x}$$

$$\Rightarrow f^7(7) = \frac{8}{-6} = -\frac{4}{3}$$

$$\therefore -\frac{1}{6} + -\frac{4}{3} = -\frac{3}{2}$$

2. Let $A = \left\{ z \in \mathbb{C} : \left| \frac{z+1}{z-1} \right| < 1 \right\}$

and $B = \left\{ z \in \mathbb{C} : \arg\left(\frac{z-1}{z+1}\right) = \frac{2\pi}{3} \right\}$.

Then $A \cap B$ is :

(A) a portion of a circle centred at $\left(0, -\frac{1}{\sqrt{3}}\right)$ that

lies in the second and third quadrants only

(B) a portion of a circle centred at $\left(0, -\frac{1}{\sqrt{3}}\right)$ that

lies in the second quadrant only

(C) an empty set

(D) a portion of a circle of radius $\frac{2}{\sqrt{3}}$ that lies in

the third quadrant only

Official Ans. by NTA (B)

Allen Ans. (B)

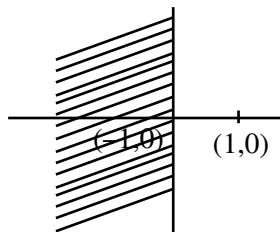
Sol. Set A

$$\Rightarrow \left| \frac{z+1}{z-1} \right| < 1$$

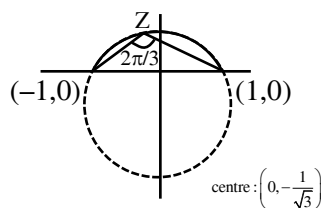
$$\Rightarrow |z+1| < |z-1|$$

$$\Rightarrow (x+1)^2 + y^2 < (x-1)^2 + y^2$$

$$\Rightarrow x < 0$$



Set B



$$\Rightarrow \arg\left(\frac{z-1}{z+1}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x-1}\right) - \tan^{-1}\left(\frac{y}{x+1}\right) = \frac{2\pi}{3}$$

$$\Rightarrow x^2 + y^2 + \frac{2y}{\sqrt{3}} - 1 = 0$$

$A \cap B$

$$\Rightarrow \text{Centre} \left(0, -\frac{1}{\sqrt{3}}\right)$$

3. Let A be a 3×3 invertible matrix. If $|\text{adj}(24A)| = \text{adj}(3\text{adj}(2A))$, then $|A|^2$ is equal to :
- (A) 6^6 (B) 2^{12} (C) 2^6 (D) 1

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $|\text{adj}(24A)| = |\text{adj}(3(\text{adj}(2A)))|$
 $\Rightarrow |24a|^2 = (3 \text{adj}(2A))^2$
 $\Rightarrow (24^3 |A|)^2 = (3^3 |\text{adj}(2A)|)^2$
 $= 3^6 (|2A|^2)^2$
 $\Rightarrow 24^6 |A|^2 = (24^3 |A|)^2 = 3^6 \times 2^{12} |A|^4$
 $\Rightarrow |A|^2 = \frac{24^6}{3^6 \times 2^{12}} = 64$

4. The ordered pair (a, b), for which the system of linear equations

$$3x - 2y + z = b$$

$$5x - 8y + 9z = 3$$

$$2x + y + az = -1$$

has no solution, is :

- (A) $\left(3, \frac{1}{3}\right)$ (B) $\left(-3, \frac{1}{3}\right)$
 (C) $\left(-3, -\frac{1}{3}\right)$ (D) $\left(3, -\frac{1}{3}\right)$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $\begin{vmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{vmatrix} = 0$

$$3(-8a - 9) + 2(5a - 18) + 1(21) = 0$$

$$\Rightarrow a = -3$$

Also $\Delta_2 = \begin{vmatrix} 3 & -2 & b \\ 5 & 8 & 3 \\ 2 & 1 & -1 \end{vmatrix}$

$$\text{If } b = \frac{1}{3}$$

$$\Delta_2 = 0$$

So b must be equal to

$$-\frac{1}{3}$$

5. The remainder when $(2021)^{2023}$ is divided by 7 is :

- (A) 1 (B) 2 (C) 5 (D) 6

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $(2021)^{2023} = (7\lambda - 2)^{2023}$
 $= {}^{2023}C_0(7A)^{2023} - \dots - {}^{2023}C_{2023}2^{2023}$
 $= 7t - 2^{2023}$
 $\therefore -2^{2023} = -2 \times 2^{2022}$
 $= -2 \times (2^3)^{674}$
 $= -2(1 + 7\mu)^{674}$
 $= -(7\alpha + 2)$
 $\Rightarrow \text{remainder} = -2 \text{ or } +5$

6. $\lim_{x \rightarrow \frac{1}{\sqrt{2}}} \frac{\sin(\cos^{-1} x) - x}{1 - \tan(\cos^{-1} x)}$ is equal to :

- (A) $\sqrt{2}$ (B) $-\sqrt{2}$
 (C) $\frac{1}{\sqrt{2}}$ (D) $-\frac{1}{\sqrt{2}}$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $\lim_{x \rightarrow \frac{1}{\sqrt{2}}} \frac{\sin(\cos^{-1} x) - x}{1 - \tan(\cos^{-1} x)}$

$$\lim_{x \rightarrow \frac{1}{\sqrt{2}}} \frac{\sin(\sin^{-1} \sqrt{1-x^2}) - x}{1 - \tan\left(\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right)}$$

$$\lim_{x \rightarrow \frac{1}{\sqrt{2}}} \frac{\sqrt{1-x^2} - x}{1 - \left(\frac{\sqrt{1-x^2}}{x}\right)}$$

$$\lim_{x \rightarrow \frac{1}{\sqrt{2}}} (-x) = -\frac{1}{\sqrt{2}}$$

7. Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be two real valued functions

$$\text{defined as } f(x) = \begin{cases} -|x+3| & , x < 0 \\ e^x & , x \geq 0 \end{cases} \text{ and}$$

$$g(x) = \begin{cases} x^2 + k_1x & , x < 0 \\ 4x + k_2 & , x \geq 0 \end{cases}, \text{ where } k_1 \text{ and } k_2 \text{ are}$$

real constants. If $(g \circ f)$ is differentiable at $x = 0$, then $(g \circ f)(-4) + (g \circ f)(4)$ is equal to :

- (A) $4(e^4 + 1)$ (B) $2(2e^4 + 1)$
 (C) $4e^4$ (D) $2(2e^4 - 1)$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $f(x) = \begin{cases} x+3 & ; x < -3 \\ -(x+3) & ; -3 \leq x < 0 \\ e^x & ; x \geq 0 \end{cases}$

$$g(x) = \begin{cases} x^2 + k_1x & ; x < 0 \\ 4x + k_2 & ; x \geq 0 \end{cases}$$

$$g(f(x)) = \begin{cases} f(x)^2 + k_1f(x) & ; f(x) < 0 \\ 4f(x) + k_2 & ; f(x) \geq 0 \end{cases}$$

$$g(f(x)) = \begin{cases} (x+3)^2 + k_1(x+3) & ; x < -3 \\ (x+3)^2 - k_1(x+3) & ; -3 \leq x < 0 \\ 4e^x + k_2 & ; x > 0 \end{cases}$$

check continuity at $x = 0$

$$g \circ f(0) = g(f(0^-)) = g(f(0^+))$$

$$4 + k_2 = 9 - 3k_1 = 4 + k_2$$

$$3k_1 + k_2 = 5 \quad \dots(a)$$

differentiate

$$(g(f(x)))' = \begin{cases} 2(x+3) + k_1 & ; x < -3 \\ 2(x+3) - k_1 & ; -3 \leq x < 0 \\ 4e^x & ; x \geq 0 \end{cases}$$

$$6 - k_1 = 4$$

$$k_1 = 2 \quad \dots(b)$$

$$\therefore k_1 = 2, k_2 = -1$$

$$g \circ f(x) = \begin{cases} (x+3)^2 + 2(x+3) & ; x < -3 \\ (x+3)^2 - 2(x+3) & ; -3 \leq x < 0 \\ 4e^x - 1 & ; x \geq 0 \end{cases}$$

$$g \circ f(-4) + g \circ f(4) = 4e^4 - 2$$

$$\Rightarrow 2(2e^4 - 1)$$

8. The sum of the absolute minimum and the absolute maximum values of the function $f(x) = |3x - x^2 + 2| - x$ in the interval $[-1, 2]$ is :

- (A) $\frac{\sqrt{17} + 3}{2}$ (B) $\frac{\sqrt{17} + 5}{2}$
 (C) 5 (D) $\frac{9 - \sqrt{17}}{2}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $f(x) = \begin{cases} x^2 - 4x - 2, & \forall x \in \left(-1, \frac{3 - \sqrt{17}}{2}\right) \\ -x^2 + 2x + 2, & \forall x \in \left(\frac{3 - \sqrt{17}}{2}, 2\right) \end{cases}$

$$f'(x) \text{ when } x \in \left(-1, \frac{3 - \sqrt{17}}{2}\right)$$

$$f'(x) = 2x - 4 = 0 \Rightarrow x = 2$$

$$f'(x) = 2(x - 2) \Rightarrow f'(x) \text{ is always } \downarrow$$

$$f(2) = 2$$

$$f(-1) = 3$$

$$f\left(\frac{3 - \sqrt{17}}{2}\right) = \frac{\sqrt{17} - 3}{2}$$

$$f'(x) \text{ when } x \in \left(\frac{3 - \sqrt{17}}{2}, 2\right)$$

$$f'(x) = -2x + 2$$

$$f'(x) = -2(x - 1)$$

$$f'(x) = 0 \text{ when } x = 1$$

$$f(1) = 3$$

$$\text{absolute minimum value} = \frac{\sqrt{17} - 3}{2}$$

$$\text{absolute maximum value} = 3$$

$$\text{Sum} = \frac{\sqrt{17}-3}{2} + 3 = \frac{\sqrt{17}+3}{2}$$

9. Let S be the set of all the natural numbers, for which the line $\frac{x}{a} + \frac{y}{b} = 2$ is a tangent to the curve

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2 \text{ at the point } (a, b), ab \neq 0. \text{ Then:}$$

- (A) $S = \phi$ (B) $n(S) = 1$
(C) $S = \{2k : k \in \mathbb{N}\}$ (D) $S = \mathbb{N}$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol.
$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$$

Slope of tangent at (a, b)

$$n \cdot \left(\frac{x}{a}\right)^{n-1} \cdot \frac{1}{a} + n \left(\frac{y}{b}\right)^{n-1} \cdot \frac{1}{b} \frac{dy}{dx} = 0$$

$$\left. \frac{dy}{dx} \right|_{(a,b)} = -\frac{b}{a}$$

\therefore Equation of tangent

$$y - b = -\frac{b}{a}(x - a)$$

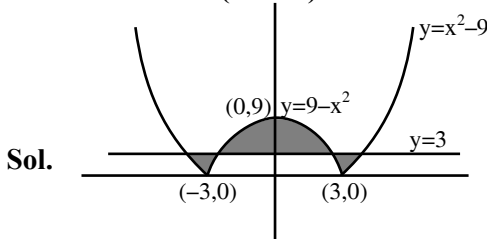
$$\frac{x}{a} + \frac{y}{b} = 2 \quad \forall n \in \mathbb{N}$$

10. The area bounded by the curve $y = |x^2 - 9|$ and the line $y = 3$ is :

- (A) $4(2\sqrt{3} + \sqrt{6} - 4)$ (B) $4(4\sqrt{3} + \sqrt{6} - 4)$
(C) $8(4\sqrt{3} + 3\sqrt{6} - 9)$ (D) $8(4\sqrt{3} + \sqrt{6} - 9)$

Official Ans. by NTA (DROP)

Allen Ans. (Bonus)



Area of shaded region

$$\begin{aligned} &= 2 \int_0^3 (\sqrt{9+y} - \sqrt{9-y}) dy + 2 \int_3^9 (\sqrt{9-y}) dy \\ &= 2 \left[\int_0^3 (9+y)^{1/2} dy - \int_0^3 (9-y)^{1/2} dy + \int_3^9 (9-y)^{1/2} dy \right] \\ &= 2 \left[\frac{2}{3} [(9+y)^{3/2}]_0^3 + \frac{2}{3} [(9-y)^{3/2}]_0^3 - \frac{2}{3} [(9-y)^{3/2}]_3^9 \right] \\ &= \frac{4}{3} [12\sqrt{12} - 27 + 6\sqrt{6} - 27 - (0 - 6\sqrt{6})] \end{aligned}$$

$$\begin{aligned} &= \frac{4}{3} [24\sqrt{3} + 12\sqrt{6} - 54] \\ &= 8(4\sqrt{3} + 2\sqrt{6} - 9) \end{aligned}$$

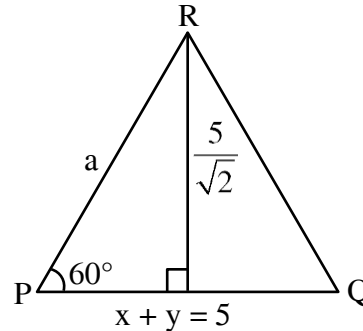
11. Let R be the point (3, 7) and let P and Q be two points on the line $x + y = 5$ such that PQR is an equilateral triangle. Then the area of ΔPQR is :

- (A) $\frac{25}{4\sqrt{3}}$ (B) $\frac{25\sqrt{3}}{2}$ (C) $\frac{25}{\sqrt{3}}$ (D) $\frac{25}{2\sqrt{3}}$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol.



$$\sin 60^\circ = \frac{5/\sqrt{2}}{a}$$

$$a = \frac{5\sqrt{2}}{3}$$

$$\text{Area of } \Delta PQR = \frac{\sqrt{3}}{4} a^2 = \frac{25}{2\sqrt{3}}$$

12. Let C be a circle passing through the points A(2, -1) and B(3, 4). The line segment AB is not a diameter of C. If r is the radius of C and its centre

lies on the circle $(x - 5)^2 + (y - 1)^2 = \frac{13}{2}$, then r^2 is

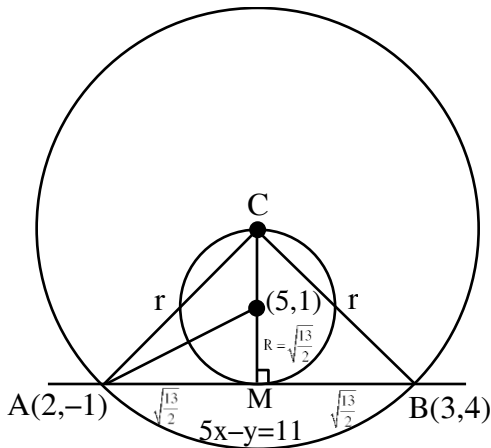
equal to :

- (A) 32 (B) $\frac{65}{2}$ (C) $\frac{61}{2}$ (D) 30

Official Ans. by NTA (B)

Allen Ans. (B)

Sol.



$$AB = \sqrt{26}$$

$$r^2 = CM^2 + AM^2$$

$$= \left(2 \times \frac{\sqrt{13}}{2}\right)^2 + \left(\frac{\sqrt{13}}{2}\right)^2$$

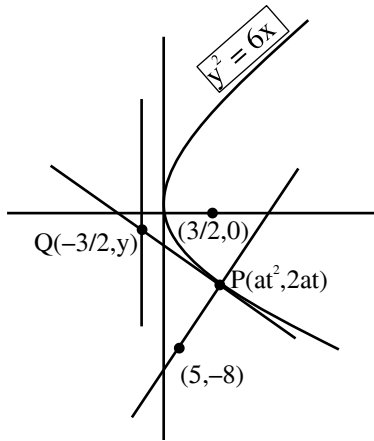
$$r^2 = \frac{65}{2}$$

13. Let the normal at the point P on the parabola $y^2 = 6x$ pass through the point $(5, -8)$. If the tangent at P to the parabola intersects its directrix at the point Q, then the ordinate of the point Q is :

- (A) -3 (B) $-\frac{9}{4}$ (C) $-\frac{5}{2}$ (D) -2

Official Ans. by NTA (B)
Allen Ans. (B)

Sol.



Equation of normal : $y = -tx + 2at + at^3$ $\left(a = \frac{3}{2}\right)$

since passing through $(5, -8)$, we get $t = -2$

Co-ordinate of Q : $(-3/2, -6)$

Equation of tangent at Q : $x + 2y + 6 = 0$

Put $x = -\frac{3}{2}$ to get R $\left(\frac{-3}{2}, \frac{-9}{4}\right)$

14. If the two lines $l_1 : \frac{x-2}{3} = \frac{y+1}{-2}, z = 2$ and $l_2 : \frac{x-1}{1} = \frac{2y+3}{\alpha} = \frac{z+5}{2}$ perpendicular, then an angle between the lines l_2 and $l_3 : \frac{1-x}{3} = \frac{2y-1}{-4} = \frac{z}{4}$ is :

- (A) $\cos^{-1}\left(\frac{29}{4}\right)$ (B) $\sec^{-1}\left(\frac{29}{4}\right)$
(C) $\cos^{-1}\left(\frac{2}{29}\right)$ (D) $\cos^{-1}\left(\frac{2}{\sqrt{29}}\right)$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $l_1 : \frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-2}{0}$

$$l_2 : \frac{x-1}{1} = \frac{y+3/2}{\alpha/2} = \frac{z+5}{2}$$

$$l_3 : \frac{x-1}{-3} = \frac{y-1/2}{-2} = \frac{z-0}{4}$$

$$l_1 \perp l_2 \Rightarrow \frac{|3-\alpha+0|}{\sqrt{13}\sqrt{1+\frac{\alpha^2}{4}+4}} = 0 \Rightarrow \alpha = 3$$

angle between l_2 & l_3

$$\cos \theta = \frac{|1 \times (-3) + (-2)(\alpha/2) + 2 \times 4|}{\sqrt{1+4+\frac{\alpha^2}{4}} \sqrt{9+16+4}}$$

$$\cos \theta = \frac{|-3-\alpha+8|}{\sqrt{5+\frac{\alpha^2}{4}} \sqrt{29}}$$

put $\alpha = 3$

$$\cos \theta = \frac{2}{\sqrt{\frac{29}{4}} \sqrt{29}} = \frac{4}{29}$$

$$\theta = \cos^{-1}\left(\frac{4}{29}\right) \Rightarrow \theta = \sec^{-1}\left(\frac{29}{4}\right)$$

15. Let the plane $2x + 3y + z + 20 = 0$ be rotated through a right angle about its line of intersection

with the plane $x - 3y + 5z = 8$. If the mirror image

of the point $\left(2, -\frac{1}{2}, 2\right)$ in the rotated plane is

B(a, b, c), then :

(A) $\frac{a}{8} = \frac{b}{5} = \frac{c}{-4}$ (B) $\frac{a}{4} = \frac{b}{5} = \frac{c}{-2}$

(C) $\frac{a}{8} = \frac{b}{-5} = \frac{c}{4}$ (D) $\frac{a}{4} = \frac{b}{5} = \frac{c}{2}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. Let equation of rotated plane be :

$$(2x + 3y + z + 20) + \lambda(x - 3y + 5z - 8) = 0$$

$$(2 + \lambda)x + (3 - 3\lambda)y + (1 + 5\lambda)z + 20 - 8\lambda = 0$$

Above plane is perpendicular to $2x + 3y + z + 20 = 0$

$$\text{So, } (2 + \lambda) \cdot 2 + (3 - 3\lambda) \cdot 3 + (1 + 5\lambda) \cdot 1 = 0 \Rightarrow \lambda = 7$$

$$\Rightarrow \text{Equation of rotated plane : } x - 2y + 4z - 4 = 0$$

Mirror image of $A\left(2, -\frac{1}{2}, 2\right)$ in rotated plane is

B(a, b, c)

$$\text{Equation of AB : } \frac{x-2}{1} = \frac{y+1/2}{-2} = \frac{z-2}{4} = k$$

Let coordinate of B be $(2+k, \frac{-1}{2}-2k, 2+4k)$

midpoint of AB is $\left(2 + \frac{k}{2}, \frac{-1}{2} - k, 2 + 2k\right)$ which

will lie on the plane $x - 2y + 4z - 4 = 0$

$$\text{Hence } k = \frac{-2}{3}$$

$$\text{Therefore B is } \left(\frac{4}{3}, \frac{5}{6}, \frac{-2}{3}\right) \equiv \left(\frac{8}{6}, \frac{5}{6}, \frac{-4}{6}\right)$$

$$\text{So, } \frac{a}{8} = \frac{b}{5} = \frac{c}{-4}$$

16. If $\vec{a} \cdot \vec{b} = 1$, $\vec{b} \cdot \vec{c} = 2$ and $\vec{c} \cdot \vec{a} = 3$, then the value

of $\left[\vec{a} \times (\vec{b} \times \vec{c}), \vec{b} \times (\vec{c} \times \vec{a}), \vec{c} \times (\vec{b} \times \vec{a})\right]$ is :

(A) 0 (B) $-6\vec{a} \cdot (\vec{b} \times \vec{c})$

(C) $12\vec{c} \cdot (\vec{a} \times \vec{b})$ (D) $-12\vec{b} \cdot (\vec{c} \times \vec{a})$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = 3\vec{b} - \vec{c}$

$$\vec{b} \times (\vec{c} \times \vec{a}) = (\vec{b} \cdot \vec{a})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a} = \vec{c} - 2\vec{a}$$

$$\vec{c} \times (\vec{b} \times \vec{a}) = (\vec{c} \cdot \vec{a})\vec{b} - (\vec{c} \cdot \vec{b})\vec{a} = 3\vec{b} - 2\vec{a}$$

$$\left[3\vec{b} - \vec{c}, \vec{c} - 2\vec{a}, 3\vec{b} - 2\vec{a}\right]$$

$$(3\vec{b} - \vec{c}) \cdot [(\vec{c} - 2\vec{a}) \times (3\vec{b} - 2\vec{a})]$$

$$(3\vec{b} - \vec{c}) \cdot [3(\vec{c} \times \vec{b}) - 2(\vec{c} \times \vec{a}) - 6(\vec{a} \times \vec{b})]$$

$$-6[\vec{b} \cdot \vec{c} \vec{a}] + 6[\vec{c} \cdot \vec{a} \vec{b}]$$

17. Let a biased coin be tossed 5 times. If the probability of getting 4 heads is equal to the probability of getting 5 heads, then the probability of getting atmost two heads is:

(A) $\frac{275}{6^5}$ (B) $\frac{36}{5^4}$ (C) $\frac{181}{5^5}$ (D) $\frac{46}{6^4}$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $P(H) = x, P(T) = 1 - x$

$$P(4H, 1T) = P(5H)$$

$${}^5C_1(x)^4(1-x)^1 = {}^5C_5 x^5$$

$$5(1-x) = x$$

$$6x = 5 = 0 \quad x = \frac{5}{6}$$

P(atmost 2H)

$$= P(0H, 5T) + P(1H, 4T) + P(2H, 3T)$$

$$= {}^5C_0 \left(\frac{1}{6}\right)^5 + {}^5C_1 \frac{5}{6} \cdot \left(\frac{1}{6}\right)^4 + {}^5C_2 \left(\frac{5}{6}\right)^3 \left(\frac{1}{6}\right)^3$$

$$= \frac{1}{6^5} (1 + 25 + 250) = \frac{276}{6^5}$$

$$= \frac{46}{6^4}$$

18. The mean of the numbers a, b, 8, 5, 10 is 6 and their variance is 6.8. If M is the mean deviation of the numbers about the mean, then 25 M is equal to:

- (A) 60 (B) 55 (C) 50 (D) 45

Official Ans. by NTA (A)

Allen Ans. (A)

Sol.
$$\sigma^2 = \frac{\sum_{i=1}^5 (x_i - \bar{x})^2}{n}$$

Mean = 6

$$\frac{a + b + 8 + 5 + 10}{5} = 6$$

$a + b = 7$
 $b = 7 - a$

$$6.8 = \frac{(a-6)^2 + (b-6)^2 + (8-6)^2 + (5-6)^2 + (10-6)^2}{5}$$

$34 = (a-6)^2 + (7-a-6)^2 + 4 + 1 + 18$

$a^2 - 7a + 12 = 0 \Rightarrow a = 4 \text{ or } a = 3$

$a = 4 \quad a = 3$
 $b = 3 \quad b = 4$

$$M = \frac{\sum_{i=1}^5 |x_i - \bar{x}|}{n}$$

$$M = \frac{|a-6| + |b-6| + |8-6| + |5-6| + |10-6|}{5}$$

when $a = 3, b = 4$

$$M = \frac{3 + 2 + 2 + 1 + 4}{5}$$

$$M = \frac{12}{5}$$

$$25M = 25 \times \frac{12}{5} = 60$$

19. Let $f(x) = 2\cos^{-1}x + 4\cot^{-1}x - 3x^2 - 2x + 10, x \in [-1, 1]$. If $[a, b]$ is the range of the function then $4a - b$ is equal to:

- (A) 11 (B) $11 - \pi$ (C) $11 + \pi$ (D) $15 - \pi$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol.
$$f'(x) = \frac{-2}{\sqrt{1-x^2}} - \frac{4}{1+x^2} - 6x - 2$$

$$= -2 \left[\frac{1}{\sqrt{1-x^2}} + \frac{2}{1+x^2} + 3x + 1 \right]$$

$f'(x) < 0 \Rightarrow f(x)$ is a dec. function

$f(1) = \pi + 5$

$f(-1) = 5\pi + 5$

Range : $[a, b] \equiv [\pi + 5, 5\pi + 5]$

$a = \pi + 5, b = 5\pi + 5 \Rightarrow 4a - b = 11 - \pi.$

20. Let $\Delta, \nabla \in \{\wedge, \vee\}$ be such that

$p \nabla q \Rightarrow ((p \Delta q) \nabla r)$ is a tautology.

Then $(p \nabla q) \Delta r$ is logically equivalent to :

- (A) $(p \Delta r) \vee q$ (B) $(p \Delta r) \wedge q$

- (C) $(p \wedge r) \Delta q$ (D) $(p \nabla r) \wedge q$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. **Case-I** If $\Delta \equiv \nabla \equiv \wedge$

$$(p \wedge q) \rightarrow ((p \wedge q) \wedge r)$$

it can be false if r is false,

so not a tautology

Case-II If $\Delta \equiv \nabla \equiv \vee$

$$(p \vee q) \rightarrow ((p \vee q) \vee r) \equiv \text{tautology}$$

then $(p \vee q) \vee r \equiv (p \Delta r) \vee q$

Case-III if $\Delta = \vee, \nabla = \wedge$

then $(p \wedge q) \rightarrow \{(p \vee q) \wedge r\}$

Not a tautology

(Check $p \rightarrow T, q \rightarrow T, r \rightarrow F$)

Case-IV if $\Delta = \wedge, \nabla = \vee$

$$(p \wedge q) \rightarrow \{(p \wedge q) \vee r\}$$

Not a tautology

SECTION-B

1. The sum of the cubes of all the roots of the equation $x^4 - 3x^3 - 2x^2 + 3x + 1 = 10$ is _____.

Official Ans. by NTA (36)

Allen Ans. (36)

Sol. $x^4 - 3x^3 - 2x^2 + 3x + 1 = 10$

$x = 0$ is not the root of this equation so divide it by x^2

$$x^2 - 3x - 2 + \frac{3}{x} + \frac{1}{x^2} = 0$$

$$x^2 + \frac{1}{x^2} - 2 + 2 - 3 \left(x - \frac{1}{x} \right) - 2 = 0$$

$$\left(x - \frac{1}{x} \right)^2 - 3 \left(x - \frac{1}{x} \right) = 0$$

$$x - \frac{1}{x} = 0, \quad x - \frac{1}{x} = 3$$

$$x^2 - 1 = 0 \quad x^2 - 3x - 1 = 0$$

$$x = \pm 1 \quad \gamma + \delta = 3$$

$$\alpha = 1, \beta = -1 \quad \gamma\delta = -1$$

$$\alpha^3 + \beta^3 + \gamma^3 + \delta^3$$

$$1 - 1 + (\gamma + \delta)((\gamma + \delta)^2 - 3\gamma\delta)$$

$$0 + 3(9 - 3(-1))$$

$$+ 3(12) = 36$$

2. There are ten boys B_1, B_2, \dots, B_{10} and five girls G_1, G_2, \dots, G_5 in a class. Then the number of ways of forming a group consisting of three boys and three girls, if both B_1 and B_2 together should not be the members of a group, is _____.

Official Ans. by NTA (1120)

Allen Ans. (1120)

Sol. $n(B) = 10$

$$n(a) = 5$$

The number of ways of forming a group of 3 girls of 3 boys.

$$= {}^{10}C_3 \times {}^5C_3$$

$$= \frac{10 \times 9 \times 8}{3 \times 2} \times \frac{5 \times 4}{2} = 1200$$

The number of ways when two particular boys B_1 of B_2 be the member of group together

$$= {}^8C_1 \times {}^5C_3 = 8 \times 10 = 80$$

Number of ways when boys B_1 of B_2 hot in the same group together

$$= 1200 \times 80 = 1120$$

3. Let the common tangents to the curves $4(x^2 + y^2) = 9$ and $y^2 = 4x$ intersect at the point Q. Let an ellipse, centered at the origin O, has lengths of semi-minor and semi-major axes equal to OQ and 6, respectively. If e and l respectively denote the eccentricity and the length of the latus rectum of this ellipse, then $\frac{l}{e^2}$ is equal to _____.

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $x^2 + y^2 = \frac{9}{4} \quad y = 4x$

Equation tangent in slope form

$$y = mx \pm \frac{3}{2} \sqrt{(1+m^2)} \quad \dots(1)$$

$$y = mx + \frac{1}{m} \quad \dots(2)$$

compare (1) & (2)

$$\pm \frac{3}{2} \sqrt{(1+m^2)} = \frac{1}{m^2}$$

$$9m^2(1+m^2) = 4$$

$$9m^4 + 9m^2 - 4 = 0$$

$$9m^4 + 12m^2 - 3m^2 - 4 = 0$$

$$3m^2(3m^2 + 4) - (3m^2 + 4) = 0$$

$$m^2 = -\frac{4}{3} \text{ (Rejected)}$$

$$m^2 = \frac{1}{3} \Rightarrow m = \pm \frac{1}{\sqrt{3}}$$

Equation of common tangent

$$y = \frac{1}{\sqrt{3}}x + \sqrt{3}$$

on X axis $y = 0$

$$OQ = -3$$

$$b = |OQ| = 3$$

$$a = 6$$

$$b^2 = a^2(1 - e^2) \Rightarrow e^2 = 1 - \frac{9}{36} = \frac{3}{4}$$

$$e = \frac{2b^2}{a} = \frac{2 \times 9}{6} = 3$$

$$\frac{e}{e^2} = \frac{3}{3/4} = 4$$

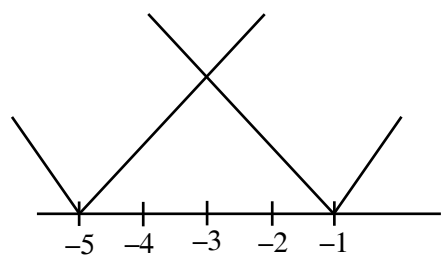
4. Let $f(x) = \max\{|x + 1|, |x + 2|, \dots, |x + 5|\}$. Then

$$\int_{-6}^0 f(x) dx \text{ is equal to } \underline{\hspace{2cm}}.$$

4. **Official Ans. by NTA (21)**

Allen Ans. (21)

Sol. $f(x) = \max\{|x+1|, |x+2|, |x+3|, |x+4|, |x+5|\}$



$$\int_{-6}^0 f(x) dx = \int_{-6}^{-3} |x+1| dx + \int_{-3}^0 |x+5| dx$$

$$= -\int_{-6}^{-3} (x+1) dx + \int_{-3}^0 (x+5) dx$$

$$= -\left[\frac{x^2}{2} + x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 5x\right]_{-3}^0$$

$$= -\left[\left(\frac{9}{2} - 3\right) - (18 - 6)\right] + \left[0 - \left(\frac{9}{2} - 15\right)\right]$$

$$= -\left[\frac{3}{2} - 12\right] + \frac{21}{2} = \frac{21}{2} + \frac{21}{2} = 21$$

5. Let the solution curve $y = y(x)$ of the differential equation $(4 + x^2)dy - 2x(x^2 + 3y + 4)dx = 0$ pass through the origin. Then $y(2)$ is equal to _____.

Official Ans. by NTA (12)

Allen Ans. (12)

Sol. $(4 + x^2)dy - 2x(x^2 + 3y + 4)dx = 0$

$$(x^2 + 4) \frac{dy}{dx} = 2x^3 + 6xy + 8x$$

$$(x^2 + 4) \frac{dy}{dx} - 6xy = 2x^3 + 8x$$

$$\frac{dy}{dx} - \frac{6x}{x^2 + 4} y = \frac{2x^3 + 8x}{x^2 + 4}$$

L.I. $\frac{dy}{dx} + py = \phi$

$$p = \frac{-6x}{x^2 + 4} \quad \phi = \frac{2x^3 + 8x}{x^2 + 4}$$

$$\text{I.F.} = e^{-\int \frac{6x}{x^2 + 4} dx} = e^{-3 \log_e(x^2 + 4)}$$

$$= e^{\log_e(x^2 + 4)^{-3}} = \frac{1}{(x^2 + 4)^3}$$

Sol.

$$y \cdot \frac{1}{(x^2 + 4)^3} = \int \frac{2x^3 + 8x}{(x^2 + 4)^3(x^2 + 4)} dx$$

$$\frac{y}{(x^2 + 4)^3} = \int \frac{2x(x^2 + 4)}{(x^2 + 4)^3(x^2 + 4)} dx$$

$$x^2 + 4 = t$$

$$2x dx = dt$$

$$\frac{y}{(x^2 + 4)^3} = \int \frac{dt}{t^3}$$

$$\frac{y}{(x^2 + 4)^3} = \frac{-1}{2(x^2 + 4)^2} + C$$

passes through origin $(0, 0)$

$$0 = \frac{-1}{2 \times 16} + C$$

$$\frac{y}{(x^2 + 4)^3} = \frac{-1}{2(x^2 + 4)^2} + \frac{1}{32}$$

$$y = \frac{-(x^2 + 4)}{2} + \frac{(x^2 + 4)^3}{32}$$

$$y(2) = -\frac{8}{2} + \frac{8 \times 8 \times 8}{32} = 12$$

6. If $\sin^2(10^\circ)\sin(20^\circ)\sin(40^\circ)\sin(50^\circ)\sin(70^\circ) = \alpha - \frac{1}{16} \sin(10^\circ)$, then $16 + \alpha^{-1}$ is equal to _____.

Official Ans. by NTA (80)

Allen Ans. (80)

Sol. $\sin 10^\circ \left(\frac{1}{2} \cdot 2 \sin 20^\circ \sin 40^\circ \right) \cdot \sin 10^\circ \sin(60^\circ - 10^\circ) \sin(60^\circ + 10^\circ)$

$$\sin 10^\circ \frac{1}{2} (\cos 20^\circ - \cos 60^\circ) \cdot \frac{1}{4} \sin 30^\circ$$

$$\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \sin 10^\circ \left(\cos 20^\circ - \frac{1}{2} \right)$$

$$= \frac{1}{32} (2 \sin 10^\circ \cos 20^\circ - \sin 10^\circ)$$

$$= \frac{1}{32} (\sin 30^\circ - \sin 10^\circ - \sin 10^\circ)$$

$$= \frac{1}{32} \left(\frac{1}{2} - 2 \sin 10^\circ \right)$$

$$= \frac{1}{64} (1 - 4 \sin 10^\circ)$$

$$= \frac{1}{64} - \frac{1}{16} \sin 10^\circ$$

Hence $\alpha = \frac{1}{64}$

$$16 + \alpha^{-1} = 80$$

7. Let $A = \{n \in \mathbb{N} : \text{H.C.F.}(n, 45) = 1\}$ and

Let $B = \{2k : k \in \{1, 2, \dots, 100\}\}$. Then the sum of all the elements of $A \cap B$ is _____.

Official Ans. by NTA (5264)

Allen Ans. (5264)

Sol. Sum of elements in $A \cap B$

$$= \underbrace{(2 + 4 + 6 + \dots + 200)}_{\text{Multiple of 2}} - \underbrace{(6 + 12 + \dots + 198)}_{\text{Multiple of 2 \& 3 i.e. 6}}$$

$$- \underbrace{(10 + 20 + \dots + 200)}_{\text{Multiple of 5 \& 2 i.e. 10}} + \underbrace{(30 + 60 + \dots + 180)}_{\text{Multiple of 2, 5 \& 3 i.e. 30}}$$

$$= 5264$$

8. The value of the integral

$$\frac{48}{\pi^4} \int_0^\pi \left(\frac{3\pi x^2}{2} - x^3 \right) \frac{\sin x}{1 + \cos^2 x} dx \text{ is equal to}$$

_____.

Official Ans. by NTA (6)

Allen Ans. (6)

Sol. $I = \frac{48}{\pi^4} \int_0^\pi x^2 \left(\frac{3\pi}{2} - x \right) \frac{\sin x}{1 + \cos^2 x} dx \dots(1)$

Apply king property

$$I = \frac{48}{\pi^4} \int_0^\pi (\pi - x)^2 \left(\frac{\pi}{2} + x \right) \frac{\sin x}{1 + \cos^2 x} dx \dots(2)$$

$$(1) + (2)$$

$$I = \frac{12}{\pi^3} \int_0^\pi \frac{\sin x}{1 + \cos^2 x} \left[\pi^2 + (\pi - 2)x(\pi - 2x) \right] dx \dots(3)$$

Apply king again

$$I = \frac{12}{\pi^3} \int_0^\pi \frac{\sin x}{1 + \cos^2 x} \left[\pi^2 + (\pi - 2)(\pi - x)(2x - \pi) \right] dx \dots(4)$$

$$(3) + (4)$$

$$I = \frac{6}{\pi^2} \int_0^\pi \frac{\sin x}{1 + \cos^2 x} \left[2\pi + (\pi - 2)(\pi - 2x) \right] dx \dots(5)$$

Apply king

$$I = \frac{6}{\pi^2} \int_0^\pi \frac{\sin x}{1 + \cos^2 x} \left[2\pi + (\pi - 2)(2x - \pi) \right] dx \dots(6)$$

$$(5) + (6)$$

$$I = \frac{12}{\pi} \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx$$

Let $\cos x = t \Rightarrow \sin x dx = -dt$

$$I = \frac{12}{\pi} \int_1^{-1} \frac{-dt}{1 + t^2} = 6$$

9. Let $A = \sum_{i=1}^{10} \sum_{j=1}^{10} \min\{i, j\}$ and

$B = \sum_{i=1}^{10} \sum_{j=1}^{10} \max\{i, j\}$. Then $A + B$ is equal to

_____.

Official Ans. by NTA (1100)

Allen Ans. (1100)

Sol. $A = \sum_{i=1}^{10} \sum_{j=1}^{10} \min\{i, j\}$

$$B = \sum_{i=1}^{10} \sum_{j=1}^{10} \max\{i, j\}$$

$$A = \sum_{j=1}^{10} \min(i, 1) + \min(j, 2) + \dots + \min(i, 10)$$

$$= \underbrace{(1+1+1+\dots+1)}_{19 \text{ times}} + \underbrace{(2+2+2+\dots+2)}_{17 \text{ times}} + \underbrace{(3+3+3+\dots+3)}_{15 \text{ times}}$$

$$+ \dots (1) \text{ 1 times}$$

$$B = \sum_{j=1}^{10} \max(i, 1) + \max(j, 2) + \dots + \max(i, 10)$$

$$= \underbrace{(10+10+\dots+10)}_{19 \text{ times}} + \underbrace{(9+9+\dots+9)}_{17 \text{ times}} + \dots + 1 \text{ 1 times}$$

$$A + B = 20(1 + 2 + 3 + \dots + 10)$$

$$= 20 \times \frac{10 \times 11}{2} = 10 \times 110 = 1100$$

10. Let $S = (0, 2\pi) - \left\{ \frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4} \right\}$. Let $y =$

$y(x)$, $x \in S$, be the solution curve of the differential equation $\frac{dy}{dx} = \frac{1}{1 + \sin 2x}$, $y\left(\frac{\pi}{4}\right) = \frac{1}{2}$. If the sum

of abscissas of all the points of intersection of the curve $y = y(x)$ with the curve $y = \sqrt{2} \sin x$ is $\frac{k\pi}{12}$,

then k is equal to _____.

Official Ans. by NTA (42)

Allen Ans. (42)

Sol. $\frac{dy}{dx} = \frac{1}{1 + \sin 2x}$

$$\int dy = \int \frac{dx}{(\sin x + \cos x)^2}$$

$$\int dy = \int \frac{\sec^2 x}{(1 + \tan x)^2}$$

$$y(x) = -\frac{1}{1 + \tan x} + C$$

$$y\left(\frac{\pi}{4}\right) = \frac{1}{2} = -\frac{1}{2} + C$$

$$C = 1$$

$$y(x) = \frac{-1}{1 + \tan x} + 1$$

$$y(x) = \frac{-1 + 1 + \tan x}{1 + \tan x}$$

$$y(x) = \frac{\tan x}{1 + \tan x}$$

Solving with $y = \sqrt{2} \sin x$

$$\frac{\tan x}{1 + \tan x} = \sqrt{2} \sin x$$

$$\sin x = 0, \quad \frac{1}{\sqrt{2}} = \sin x + \cos x$$

$$x = \pi, \quad \frac{1}{2} = \sin\left(x + \frac{\pi}{4}\right)$$

$$\sin \frac{\pi}{6} = \sin\left(x + \frac{\pi}{4}\right)$$

$$x + \frac{\pi}{4} = \pi - \frac{\pi}{6}, \quad 2\pi + \frac{\pi}{6}$$

$$x = \frac{5\pi}{6} - \frac{\pi}{4}, \quad x = \frac{13\pi}{6} - \frac{\pi}{4}$$

$$x = \frac{7\pi}{12}, \quad x = \frac{23\pi}{12}$$

sum of sol.

$$= \pi + \frac{7\pi}{12} + \frac{23\pi}{12}$$

$$= \frac{12\pi + 7\pi + 23\pi}{12} = \frac{42\pi}{12} = \frac{k\pi}{12}$$

$$\Rightarrow k = 42$$