FINAL JEE–MAIN EXAMINATION – JULY, 2022

(Held On Tuesday 26th July, 2022) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. The minimum value of the sum of the squares of the roots of $x^2+(3-a)x+1=2a$ is:

(A) 4 (B) 5

(C) 6 (D) 8

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$

let $f(a)=(3-a)^2-2(1-2a)$

$f(a)=a^2-2a+7$

$f(a)=(a-1)^2+6$

$f(a))_{min.} = 6$

2. If $z = x + iy$ satisfies $|z| - 2 = 0$ and $|z-i|-|z+5i|=0$, then

(A) $x+2y-4=0$ (B) $x^2+y-4=0$

(C) $x+2y+4=0$ (D) $x^2-y+3=0$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $|z-i|-|z+5i|=0$

$\Rightarrow |x+(y-1)i| = |x+(y+5)i|$

$x^2+(y-1)^2=x^2+(y+5)^2$

$(y-1)^2-(y+5)^2=0$

$(2y+4)(-6)=0$

$y=-2$

$\therefore x^2+(-2)^2=4$

$x=0$

$Z \equiv (0,-2)$, check options

3. Let $A=\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $B=\begin{bmatrix}9^2 & -10^2 & 11^2\\ 12^2 & 13^2 & -14^2\\ -15^2 & 16^2 & 17^2\end{bmatrix}$, then the value of $A'BA$ is:

(A) 1224 (B) 1042 (C) 540 (D) 539

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $A'BA=[1\ \ 1\ \ 1]\begin{bmatrix}9^2 & -10^2 & 11^2\\ 12^2 & 13^2 & -14^2\\ -15^2 & 16^2 & 17^2\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$

$=[9^2+12^2-15^2\ \ -10^2+13^2+16^2\ \ 11^2-14^2+17^2]\begin{bmatrix}1\\1\\1\end{bmatrix}$

$=[9^2+12^2-15^2-10^2+13^2+16^2+11^2-14^2+17^2]$

$=[539]$

4. $\sum\limits_{\substack{i,j=0\\ i\neq j}}^{n} {}^nC_i\ {}^nC_j$ is equal to

(A) $2^{2n}-{}^{2n}C_n$ (B) $2^{2n-1}-{}^{2n-1}C_{n-1}$

(C) $2^{2n}-\frac{1}{2}{}^{2n}C_n$ (D) $2^{n-1}+{}^{2n-1}C_n$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\sum\limits_{\substack{i,j=0\\ i\neq j}}^{n} {}^nC_i\ {}^nC_j$

$=\sum\limits_{i=0}^{n}{}^nC_i\cdot\sum\limits_{j=0}^{n}{}^nC_j-\sum\limits_{i=j=0}^{n}\left({}^nC_i\right)^2$

$=(2^n)(2^n)-{}^{2n}C_n$

$=2^{2n}-{}^{2n}C_n$

5. Let P and Q be any points on the curves $(x-1)^2+(y+1)^2=1$ and $y=x^2$, respectively. The distance between P and Q is minimum for some value of the abscissa of P in the interval

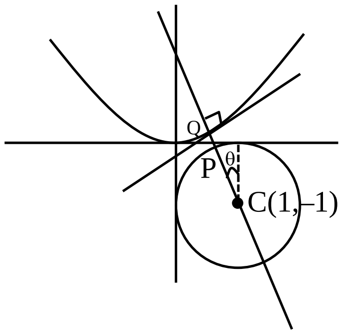
(A) $\left(0,\frac{1}{4}\right)$ (B) $\left(\frac{1}{2},\frac{3}{4}\right)$

(C) $\left(\frac{1}{4},\frac{1}{2}\right)$ (D) $\left(\frac{3}{4},1\right)$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol.



$Q=\left(t,t^2\right)$

$m_{CQ}=m_{normal}$

$\frac{t^2+1}{t-1}=-\frac{1}{2t}$

Let $f\left(t\right)=2t^3+3t-1$

$f\left(\frac{1}{4}\right)f\left(\frac{1}{3}\right)<0\Rightarrow t\in\left(\frac{1}{4},\frac{1}{3}\right)$

$P\equiv\left(1+\cos\left(90+\theta\right),-1+\sin\left(90+\theta\right)\right)$

$P=\left(1-\sin\theta,-1+\cos\theta\right)$

$m_{normal}=m_{CP}\Rightarrow -\frac{1}{2t}=\frac{\cos\theta}{-\sin\theta}\Rightarrow \tan\theta=2t$

$x=1-\sin\theta=1-\frac{2t}{\sqrt{1+4t^2}}=g\left(t\right)\ \ (\text{let})$

$\Rightarrow g'\left(t\right)<0$

$g\left(t\right)\downarrow$ function

$t\in\left(\frac{1}{4},\frac{1}{3}\right)$

$\Rightarrow g\left(t\right)\in\left(0.44,0.485\right)\in\left(\frac{1}{4},\frac{1}{2}\right)$

6. If the maximum value of a, for which the function $f_a(x)=\tan^{-1}2x-3ax+7$ is non-decreasing in $\left(-\frac{\pi}{6},\frac{\pi}{6}\right)$, is $\bar{a}$, then $f_{\bar{a}}\left(\frac{\pi}{8}\right)$ is equal to

(A) $8-\frac{9\pi}{4\left(9+\pi^2\right)}$ (B) $8-\frac{4\pi}{9\left(4+\pi^2\right)}$

(C) $8\left(\frac{1+\pi^2}{9+\pi^2}\right)$ (D) $8-\frac{\pi}{4}$

Official Ans. by NTA (Drop)

Allen Ans. (Bonus)

Sol. $f_a(x)=\tan^{-1}2x-3ax+7$

$f_a'(x)=\frac{2}{1+4x^2}-3a\geq 0$

$a\leq\left(\frac{2}{3(1+4x^2)}\right)_{min.}$ at $x=\pm\frac{\pi}{6}$

$a_{max}=\bar{a}=\frac{6}{9+\pi^2}$

$f_{\bar{a}}(\frac{\pi}{8})=\tan^{-1}\frac{\pi}{4}-3\frac{6}{9+\pi^2}\frac{\pi}{8}+7=\tan^{-1}\frac{\pi}{4}-\frac{9\pi}{4\left(\pi^2+9\right)}+7$

7. Let $\beta=\lim_{x\to 0}\frac{\alpha x-\left(e^{3x}-1\right)}{\alpha x\left(e^{3x}-1\right)}$ for some $\alpha\in\mathbb{R}$. Then the value of $\alpha+\beta$ is :

(A) $\frac{14}{5}$ (B) $\frac{3}{2}$ (C) $\frac{5}{2}$ (D) $\frac{7}{2}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $\beta=\lim_{x\to 0}\frac{\alpha x-(e^{3x}-1)}{\alpha x(e^{3x}-1)}$

$\beta=\lim_{x\to 0}\frac{1+\alpha x-\left[1+3x+\frac{9x^2}{2!}+.....\right]}{(\alpha x)\frac{(e^{3x}-1)}{3x}3x}$

$\beta=\lim_{x\to 0}\frac{(\alpha x-3x)-\frac{9x^2}{2!}-.......}{3\alpha x^2}$

For existence of limit $\alpha-3=0$

$\alpha=3$

Limit $\beta=\frac{-3}{2\alpha}$

$\beta=-\frac{1}{2}$

Now,

$\alpha+\beta=\frac{5}{2}$

8. The value of $\log_e 2\frac{d}{dx}(\log_{\cos x}\operatorname{cosec}x)$ at $x=\frac{\pi}{4}$ is

(A) $-2\sqrt{2}$ (B) $2\sqrt{2}$ (C) -4 (D) 4

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $\log_e 2 \frac{d}{dx}(\log_{\cos x} \operatorname{cosec} x)$

Let,

$y = \log_{\cos x} \operatorname{cosec} x$

$y = -\frac{\ln(\sin x)}{\ln(\cos x)}$

$\frac{dy}{dx} = -\frac{[\cot x \cdot \ln(\cos x) + \tan x \cdot \ln(\sin x)]}{(\ln(\cos x))^2}$

$\left.\frac{dy}{dx}\right)_{x=\frac{\pi}{4}} = \frac{4}{\ln 2}$

Now,

$\Rightarrow \log_e 2 \cdot \frac{4}{\ln 2} = 4$

9. $\int_0^{20\pi} (|\sin x| + |\cos x|)^2 dx$ is equal to :-

(A) $10(\pi+4)$ (B) $10(\pi+2)$

(C) $20(\pi-2)$ (D) $20(\pi+2)$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $I = \int_0^{20\pi} (|\sin x| + |\cos x|)^2 dx$; (Jack property)

$I = 40\int_0^{\pi/2} (\sin x + \cos x)^2 dx$

$I = 40\int_0^{\pi/2} (1 + \sin 2x) dx$

$I = 20[\pi + 2]$

10. Let the solution curve $y = f(x)$ of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}, x \in (-1,1)$ pass through the origin. Then $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx$ is equal to

(A) $\frac{\pi}{3} - \frac{1}{4}$ (B) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$

(C) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (D) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$

I.F $= e^{\int \frac{x}{x^2-1} dx}$

I.F $= \sqrt{1-x^2}$

Solution of D.E.

$y \cdot \sqrt{1-x^2} = \int \frac{x^4+2x}{\sqrt{1-x^2}} \cdot \sqrt{1-x^2} dx$

$y \cdot \sqrt{1-x^2} = \int (x^4+2x) dx$

$y \cdot \sqrt{1-x^2} = \frac{x^5}{5} + x^2 + C$

At $x = 0$, $y = 0$, get $C = 0$

$y = \frac{x^5}{5\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}}$

Now,

$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx = \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{x^5}{5\sqrt{1-x^2}} dx + \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$

$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx = 0 + 2\int_0^{\frac{\sqrt{3}}{2}} \frac{x^2}{\sqrt{1-x^2}} dx$

$\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$

11. The acute angle between the pair of tangents drawn to the ellipse $2x^2 + 3y^2 = 5$ from the point (1,3) is

(A) $\tan^{-1}\left(\frac{16}{7\sqrt{5}}\right)$ (B) $\tan^{-1}\left(\frac{24}{7\sqrt{5}}\right)$

(C) $\tan^{-1}\left(\frac{32}{7\sqrt{5}}\right)$ (D) $\tan^{-1}\left(\frac{3+8\sqrt{5}}{35}\right)$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. Equation of tangent to the ellipse $2x^2 + 3y^2 = 5$ is

$y = mx \pm \sqrt{\frac{5}{2}m^2 + \frac{5}{3}}$

It pass through (1, 3)

$$3 = m \pm \sqrt{\frac{5}{2}m^2 + \frac{5}{3}}$$

$$3m^2 + 12m - \frac{44}{3} = 0$$

Let θ be the angle between the tangents

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\theta = \left|\frac{3\sqrt{320}}{-35}\right|$$

$$\theta = \tan^{-1}\left(\frac{24}{7\sqrt{5}}\right)$$

12. The equation of a common tangent to the parabolas $y = x^2$ and $y = -(x-2)^2$ is

(A) y = 4(x-2) (B) y = 4 (x-1)

(C) y = 4 (x+1) (D) y = 4 (x+2)

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. Equation of tangent of $y = x^2$ be

$tx = y + at^2$ ....(1)

$$y = tx - \frac{t^2}{4}$$

Solve with $y = -(x-2)^2$

$$tx - \frac{t^2}{4} = -(x-2)^2$$

$$x^2 + x(t-4) - \frac{t^2}{4} + 4 = 0$$

D = 0

$$(t-4)^2 - 4 \cdot \left(4 - \frac{t^2}{4}\right) = 0$$

$t^2 - 4t = 0$

t = 0 or t = 4

From eq. (1), required common tangent is

y = 4 (x-1)

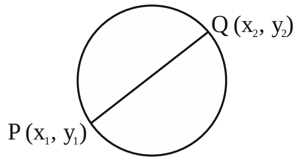
13. Let the abscissae of the two points P and Q on a circle be the roots of $x^2 - 4x - 6 = 0$ and the ordinates of P and Q be the roots of $y^2 + 2y - 7 = 0$. If PQ is a diameter of the circle $x^2 + y^2 + 2ax + 2by + c = 0$, then the value of (a+b-c) is

(A) 12 (B) 13 (C) 14 (D) 16

Official Ans. by NTA (A)

Allen Ans. (A)

Sol.



Equation of circle diameter form

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$

(where x_1, x_2 are the roots of $x^2 - 4x - 6 = 0$ and y_1, y_2 are the roots of $y^2 + 2y - 7 = 0$)

$x^2 + y^2 - 4x + 2y - 13 = 0$

Now,

Compare it with the given equation, we get

a = –2, b = 1, c = –13

Now

a + b – c = 12

14. If the line x-1 = 0, is a directrix of the hyperbola $kx^2 - y^2 = 6$, then the hyperbola passes through the point

(A) $\left(-2\sqrt{5}, 6\right)$ (B) $\left(-\sqrt{5}, 3\right)$

(C) $\left(\sqrt{5}, -2\right)$ (D) $\left(2\sqrt{5}, 3\sqrt{6}\right)$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $\frac{x^2}{6/k} - \frac{y^2}{6} = 1$ ......(1)

$$e^2 = 1 + \frac{6}{6/k}$$

$$e = \sqrt{1+k}$$

$$a = \sqrt{\frac{6}{k}}$$

Eq. of directrix $x = \frac{a}{e} \Rightarrow x = \sqrt{\frac{6}{k(k+1)}}$

$$\frac{6}{k(k+1)} = 1$$

k = 2

From eq. (1), we get $2x^2 - y^2 = 6$

Check options

15. A vector $\vec{a}$ is parallel to the line of intersection of the plane determined by the vectors $\hat{i}, \hat{i}+\hat{j}$ and the plane determined by the vectors $\hat{i}-\hat{j}, \hat{i}+\hat{k}$. The obtuse angle between $\vec{a}$ and the vector $\vec{b}=\hat{i}-2\hat{j}+2\hat{k}$ is

(A) $\frac{3\pi}{4}$ (B) $\frac{2\pi}{3}$

(C) $\frac{4\pi}{5}$ (D) $\frac{5\pi}{6}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\vec{n}_1=\hat{i}\times(\hat{i}+\hat{j})=\hat{k}$

$\vec{n}_2=(\hat{i}+\hat{k})\times(\hat{i}-\hat{j})$

$=\hat{i}+\hat{j}-\hat{k}$

Line of intersection along $\vec{n}_1\times\vec{n}_2$

$=\hat{k}\times(\hat{i}+\hat{j}-\hat{k})=-\hat{i}+\hat{j}$

D.R of $\vec{a}=-\hat{i}+\hat{j}$

D.R of $\vec{b}=\hat{i}-2\hat{j}+2\hat{k}$

$\vec{a}\cdot\vec{b}=-3$ and $\left(\vec{a}\wedge\vec{b}\right)=\theta$

$\cos\ \theta=\frac{-3}{\sqrt{2}\times 3}$

$\theta=\frac{3\pi}{4}$

16. If $0<x<\frac{1}{\sqrt{2}}$ and $\frac{\sin^{-1}x}{\alpha}=\frac{\cos^{-1}x}{\beta}$, then a value of $\sin\left(\frac{2\pi\alpha}{\alpha+\beta}\right)$ is

(A) $4\sqrt{(1-x^2)}(1-2x^2)$

(B) $4x\sqrt{(1-x^2)}(1-2x^2)$

(C) $2x\sqrt{(1-x^2)}(1-4x^2)$

(D) $4\sqrt{(1-x^2)}(1-4x^2)$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\frac{\sin^{-1}x}{\alpha}=\frac{\cos^{-1}x}{\beta}=k$

$\sin^{-1}x=k\alpha$

$\cos^{-1}x=k\beta$

$k=\frac{\pi}{2(\alpha+\beta)}$ ....(i)

$\sin\left(\frac{2\pi\alpha}{\alpha+\beta}\right)=\sin(4\sin^{-1}x)$

$=2\sin(2\sin^{-1}x)\cos(2\sin^{-1}x)$

$=4x\sqrt{1-x^2}(1-2x^2)$

17. Negation of the Boolean expression $p \Leftrightarrow (q\Rightarrow p)$ is

(A) $(\sim p)\wedge q$ (B) $p\wedge(\sim q)$

(C) $(\sim p)\vee(\sim q)$ (D) $(\sim p)\wedge(\sim q)$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $\sim\left(p\leftrightarrow(q\to p)\right)$

$\sim(p\leftrightarrow q)=(p\wedge\sim q)\vee(q\wedge\sim p)$

$\sim\left(p\leftrightarrow(q\to p)\right)=\left(p\wedge\sim(q\to p)\right)\vee\left((q\to p)\wedge\sim p\right)$

$\left(p\wedge\sim(q\to p)\right)=p\wedge(q\wedge\sim p)=(p\wedge\sim p)\wedge q=c$

$(q\to p)\wedge\sim p=(\sim q\vee p)\wedge\sim p=\sim p\wedge(\sim q\vee p)$

$=(\sim p\wedge\sim q)\vee(\sim p\wedge p)=\sim p\wedge\sim q$

$\sim\left(p\leftrightarrow(q\to p)\right)=c\vee(\sim p\wedge\sim q)=\sim p\wedge\sim q$

18. Let X be a binomially distributed random variable with mean 4 and variance $\frac{4}{3}$. Then 54 P(X$\le$ 2) is equal to

(A) $\frac{73}{27}$ (B) $\frac{146}{27}$

(C) $\frac{146}{81}$ (D) $\frac{126}{81}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. np = 4

npq = 4/3

n = 6, p = 2/3, q = 1/3

54(P(X = 2) + P (X = 1) + P(X = 0))

$54\left({}^6C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^4+{}^6C_1\left(\frac{2}{3}\right)^1\left(\frac{1}{3}\right)^5+{}^6C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^6\right)$

$=\frac{146}{27}$

19. The integral $\int \frac{\left(1-\frac{1}{\sqrt{3}}\right)(\cos x - \sin x)}{\left(1+\frac{2}{\sqrt{3}}\sin 2x\right)}dx$ is equal to

(A) $\frac{1}{2}\log_e\left|\frac{\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)}{\left(\frac{x}{2}+\frac{\pi}{6}\right)}\right|+C$

(B) $\frac{1}{2}\log_e\left|\frac{\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)}{\left(\frac{x}{2}+\frac{\pi}{3}\right)}\right|+C$

(C) $\log_e\left|\frac{\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)}{\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)}\right|+C$

(D) $\frac{1}{2}\log_e\left|\frac{\tan\left(\frac{x}{2}-\frac{\pi}{12}\right)}{\tan\left(\frac{x}{2}-\frac{\pi}{6}\right)}\right|+C$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $I=\int \frac{\left(1-\frac{1}{\sqrt{3}}\right)(\cos x - \sin x)}{\left(1+\frac{2}{\sqrt{3}}\sin 2x\right)}dx$

$$\frac{\sqrt{3}}{2}\int \frac{\left(1-\frac{1}{\sqrt{3}}\right)(\cos x - \sin x)}{\left(\frac{\sqrt{3}}{2}+\sin 2x\right)}dx$$

$$\int \frac{\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)(\cos x - \sin x)}{\sin 60^\circ + \sin 2x}dx$$

$$\int \frac{\left(\frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\cos x-\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\sin x\right)}{2\sin\left(x+\frac{\pi}{6}\right)\cos\left(x-\frac{\pi}{6}\right)}dx$$

$$\int \frac{\left(\cos\left(x-\frac{\pi}{6}\right)-\sin\left(x+\frac{\pi}{6}\right)\right)}{2\sin\left(x+\frac{\pi}{6}\right)\cos\left(x-\frac{\pi}{6}\right)}dx$$

$$\frac{1}{2}\left(\int\frac{dx}{\sin\left(x+\frac{\pi}{6}\right)}-\int\frac{dx}{\cos\left(x-\frac{\pi}{6}\right)}\right)$$

$$\frac{1}{2}\ln\left|\frac{\tan\left(\frac{x}{2}+\frac{\pi}{12}\right)}{\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)}\right|$$

20. The area bounded by the curves $y = |x^2-1|$ and $y = 1$ is

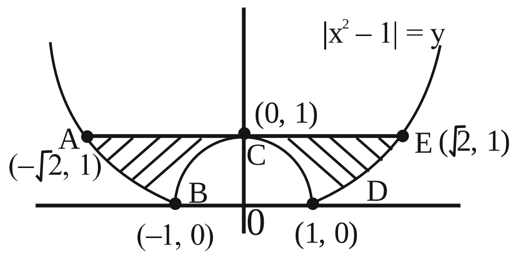
(A) $\frac{2}{3}\left(\sqrt{2}+1\right)$ (B) $\frac{4}{3}\left(\sqrt{2}-1\right)$

(C) $2\left(\sqrt{2}-1\right)$ (D) $\frac{8}{3}\left(\sqrt{2}-1\right)$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $y = |x^2 - 1|$



Area = ABCDEA

$$=2\left(\int_0^1\left(1-(1-x^2)\right)dx+\int_1^{\sqrt{2}}\left(1-(x^2-1)\right)dx\right)$$

$$=\frac{8}{3}(\sqrt{2}-1)$$

SECTION-B

1. Let A = {1,2,3,4,5,6,7} and B = {3,6,7,9}. Then the number of elements in the set $\{C\subseteq A : C\cap B\neq\phi\}$ is________

Official Ans. by NTA (112)

Allen Ans. (112)

Sol. A = {1,2,3,4,5,6,7} and

B = {3,6,7,9}

Total subset of A = 2^7 = 128

$C\cap B=\phi$ when set C contains the element 1, 2, 4, 5

$\therefore S = \{C \subseteq A; C \cap B \neq \phi\}$

$= \text{Total} - (C \cap B = \phi)$

$= 128 - 2^4 = 112$

2. The largest value of a, for which the perpendicular distance of the plane containing the lines $\vec{r} = (\hat{i}+\hat{j}) + \lambda(\hat{i} + a\hat{j} - \hat{k})$ and $\vec{r} = (\hat{i}+\hat{j}) + \mu(-\hat{i} + \hat{j} - a\hat{k})$ from the point (2,1,4) is $\sqrt{3}$, is__________.

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\vec{r} = (\hat{i}+\hat{j}) + \lambda(\hat{i} + a\hat{j} - \hat{k})$

$\vec{r} = (\hat{i}+\hat{j}) + \mu(-\hat{i} + \hat{j} - a\hat{k})$

D.R's of plane containing these lines is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & a & -1 \\ -1 & 1 & -a \end{vmatrix} = \hat{i}(1-a^2) - \hat{j}(-a-1) + \hat{k}(1+a)$$

$\vec{n} = (1-a)\hat{i} + \hat{j} + \hat{k}$

One point in plane : (1, 1, 0)

$\therefore$ equation of plane is

$(1 - a)(x - 1) + (y - 1) + (z - 0) = 0$

$(1 - a)x + y + z + a - 2 = 0$

$$\therefore D = \frac{|(1-a)2+1+4+a-2|}{\sqrt{(1-a)^2+1+1}}$$

$\Rightarrow |5 - a| = \sqrt{3}.\sqrt{a^2 - 2a + 3}$

$\Rightarrow a^2 + 2a - 8 = 0$

$\Rightarrow a = 2, -4$

$\therefore$ largest value of a = 2

3. Numbers are to be formed between 1000 and 3000, which are divisible by 4, using the digits 1,2,3,4,5 and 6 without repetition of digits. Then the total number of such numbers is ______________.

Official Ans. by NTA (30)

Allen Ans. (30)

Sol. Here 1st digit is 1 or 2 only

Case-I

If first digit is 1

Then last two digits can be 24, 32, 36, 52, 56, 64

1			

$1 \times 3 \times 6 = 18$ ways

Case – II

If first digit is 2 then last two digit can be 16, 36, 56, 64

2			

$1 \times 3 \times 4 = 12$ ways

Total ways = 12 + 18 = 30 ways

4. If $\sum_{k=1}^{10} \frac{k}{k^4+k^2+1} = \frac{m}{n}$, where m and n are co-prime, then m + n is equal to

Official Ans. by NTA (166)

Allen Ans. (166)

Sol. $\sum_{k=1}^{10} \frac{k}{k^4+k^2+1} = \frac{m}{n}$

$$\Rightarrow \frac{1}{2}\sum_{k=1}^{10} \frac{(k^2+k+1)-(k^2-k+1)}{(k^2+k+1)(k^2-k+1)}$$

$$\Rightarrow \frac{1}{2}\left(\sum_{k=1}^{10}\left(\frac{1}{(k^2-k+1)} - \frac{1}{k^2+k+1}\right)\right)$$

$$\Rightarrow \frac{55}{111} = \frac{m}{n}$$

m + n = 166

5. If the sum of solutions of the system of equations $2\sin^2\theta - \cos 2\theta = 0$ and $2\cos^2\theta + 3\sin\theta = 0$ in the interval $[0, 2\pi]$ is $k\pi$, then k is equal to ________.

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $2\sin^2\theta - \cos 2\theta = 0$

$2\sin^2\theta - (1 - 2\sin^2\theta) = 0$

$\Rightarrow \sin^2\theta = \left(\frac{1}{2}\right)^2$

$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

$2\cos^2\theta + 3\sin\theta = 0$

$\Rightarrow 2\sin^2\theta - 3\sin\theta - 2 = 0$

$\therefore \sin\theta = -\frac{1}{2}$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

So, the common solution is

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

Sum = $\frac{7\pi + 11\pi}{6} = 3\pi = k\pi$

K = 3

6. The mean and standard deviation of 40 observations are 30 and 5 respectively. It was noticed that two of these observations 12 and 10 were wrongly recorded. If σ is the standard deviation of the data after omitting the two wrong observations from the data, then $38\sigma^2$ is equal to___________.

Official Ans. by NTA (238)

Allen Ans. (238)

Sol. Wrong mean = $\mu_1 = 30$

Wrong S.D = $\sigma_1 = 5$

$\frac{\sum x_i}{40} = 30$

$\Rightarrow \sum x_i = 1200$

$\sigma_1^2 = 25$

$\Rightarrow \frac{\sum x_i^2}{40} - 30^2 = 25$

$\Rightarrow \sum x_i^2 = 925 \times 40 = 37000$

New sum = $\sum x_i' = 1200 - 10 - 12 = 1178$

New mean = $\mu_1' = \frac{1178}{38} = 31$

New $\sum x_i^2 = 37000 - (10)^2 - (12)^2 = 36756$

New S.D, $\sigma_1' = \sqrt{\frac{36756}{38} - (31)^2} = \sigma$

$36756 - (31)^2 \times 38 = 38\sigma^2$

$\Rightarrow 38\sigma^2 = 238$

7. The plane passing through the line L: ℓ x-y+3(1-ℓ) z = 1, x+2y - z = 2 and perpendicular to the plane 3x+2y+z = 6 is 3x-8y+7z=4. If θ is the acute angle between the line L and the y-axis, then 415 $\cos^2\theta$ is equal to_____.

Official Ans. by NTA (125)

Allen Ans. (125)

Sol. $\vec{n}_1 = \ell\hat{i} - \hat{j} + 3(1-\ell)\hat{k}$

$\vec{n}_2 = \hat{i} + 2\hat{j} - \hat{k}$

Direction ratio of line = $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \ell & -1 & 3(1-\ell) \\ 1 & 2 & -1 \end{vmatrix}$

$= (6\ell - 5)\hat{i} + (3 - 2\ell)\hat{j} + (2\ell + 1)\hat{k}$

3x – 8y + 7z = 4 will contain the line $(6\ell - 5)\hat{i} + (3 - 2\ell)\hat{j} + (2\ell + 1)\hat{k}$

Normal of 3x – 8y + 7z = 4 will be perpendicular to the line

$= 3(6\ell - 5) + (3 - 2\ell)(-8) + 7(2\ell + 1) = 0$

$\Rightarrow \ell = \frac{2}{3}$

$\therefore$ direction ratio of line $\left(-1, \frac{5}{3}, \frac{7}{3}\right)$

Angle with y axis

$\cos\theta = \frac{5/3}{\sqrt{1 + \frac{25}{9} + \frac{49}{9}}}$

$\cos\theta = \frac{5}{\sqrt{83}}$

$\therefore 415\cos^2\theta = \frac{25}{83} \times 415 = 125$

8. Suppose $y = y(x)$ be the solution curve to the differential equation $\frac{dy}{dx} - y = 2 - e^{-x}$ such that $\lim_{x\to\infty} y(x)$ is finite. If a and b are respectively the x- and y- intercepts of the tangent to the curve at x=0, then the value of a-4b is equal to __________.

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $\frac{dy}{dx} - y = 2 - e^{-x}$

I.F. = $e^{-\int dx} = e^{-x}$

$\therefore$ solution of D.E

$y \cdot e^{-x} = \int \left(2e^{-x} - e^{-2x}\right) dx$

$\Rightarrow y = -2 + \frac{e^{-x}}{2} + C.e^{x}$

$\because \lim_{x\to\infty} y$ is finite

$\therefore \lim_{x\to\infty}\left(-2 + \frac{e^{-x}}{2} + C.e^{x}\right) \to \text{finite}$

This is possible only when C = 0

$\therefore y = y(x) = -2 + \frac{e^{-x}}{2}$

$\frac{dy}{dx} = -\frac{1}{2}e^{-x}$

$\left.\frac{dy}{dx}\right|_{x=0} = -\frac{1}{2} = m, \; y(0) = -2 + \frac{1}{2} = \frac{-3}{2}$

$\therefore$ equation of tangent

$y + \frac{3}{2} = -\frac{1}{2}(x - 0)$

$\Rightarrow x + 2y = -3$

$a = -3, \; b = \frac{-3}{2}$

$a - 4b = -3 + 6 = 3$

9. Different A.P.'s are constructed with the first term 100, the last term 199, And integral common differences. The sum of the common differences of all such, A.P's having at least 3 terms and at most 33 terms is.

Official Ans. by NTA (53)

Allen Ans. (53)

Sol. 1^{st} term = 100 = a

Last term = 199 = ℓ

If 3 term

a, a + d, a + 2d

$a_n = \ell = a + (n-1)d$

$d_i = \frac{\ell - a}{n-1}$

n → number of terms

$n=3,\ d_1 = \frac{199-100}{2}$

$= \frac{99}{2} \notin I$

$n = 4,\ d_2 = \frac{99}{3} = 33 \in I$

$n = 10,\ d_3 = \frac{99}{9} = 11 \in I$

$n = 12,\ d_4 = \frac{99}{11} = 9 \in I$

$\therefore \sum d_i = 33 + 11 + 9 = 53$

10. The number of matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $a,b,c,d \in \{-1,0,1,2,3,\ldots\ldots,10\}$, such that $A = A^{-1}$, is______.

Official Ans. by NTA (50)

Allen Ans. (50)

Sol. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Given $A = A^{-1}$

$\therefore A^2 = A \cdot A^{-1} = I$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore a^2 + bc = 1$ ....(1)

$ab + bd = 0$ .....(2)

$ac + cd = 0$ ...(3)

$bc + d^2 = 1$ ....(4)

(1) – (4) gives

$a^2 - d^2 = 0$

$\Rightarrow (a + d) = 0$ or $a - d = 0$

Case – I

$a + d = 0 \Rightarrow (a, d) = (-1, 1), (0, 0), (1, -1)$

(a) $(a, d) = (-1, 1)$

$\therefore$ from equation (1)

$1 + bc = 1 \Rightarrow bc = 0$

b = 0 C = 12 possibilities

c = 0 b = 12 possibilities

but (0, 0) is repeated

$\therefore 2 \times 12 = 24$

24 – 1 (repeated) = 23 pairs

(b) $(a, d) = (1, -1) \Rightarrow bc = 0 \rightarrow 23$ pairs

(c) $(a, d) = (0, 0) \Rightarrow bc = 1$

$\Rightarrow (b, c) = (1, 1)$ & $(-1, -1)$, 2 pairs

Case – II

$a = d$

from (2) and (3)

$a \neq 0$ then $b = c = 0$

$a^2 = 1$

$a = \pm 1 = d$

$(a, d) = (1, 1), (-1, -1) \rightarrow 2$ pairs

$\therefore$ Total = 23 + 23 + 2 + 2

= 50 pairs