

FINAL JEE-MAIN EXAMINATION – JUNE, 2022

 (Held On Monday 27th June, 2022)

TIME : 3 : 00 PM to 6 : 00 PM

MATHEMATICS
TEST PAPER WITH SOLUTION
SECTION-A

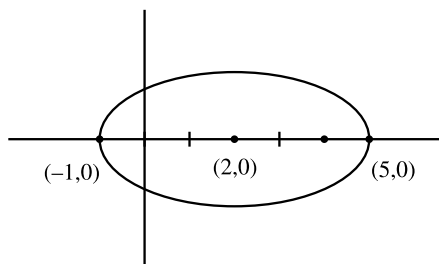
1. The number of points of intersection of $|z - (4 + 3i)| = 2$ and $|z| + |z - 4| = 6$, $z \in \mathbb{C}$ is :

- (A) 0 (B) 1
(C) 2 (D) 3

Official Ans. by NTA (C)
Allen Ans. (C)

Sol. C : $(x - 4)^2 + (y - 3)^2 = 4$

E : $\frac{(x-2)^2}{9} + \frac{y^2}{5} = 1$



Lower Extremity of vertical diameter of circle $\rightarrow (4, 1)$

Put in ellipse $\Rightarrow \frac{(4-2)^2}{9} + \frac{1}{5} - 1$

$= \frac{4}{9} + \frac{1}{5} - 1$

$= \frac{29}{45} - 1 < 0$

Two Solutions

Answer (C)

2. Let $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$, $a \in \mathbb{R}$. Then the sum of

which the squares of all the values of a for $2f'(10) - f'(5) + 100 = 0$ is :

- (A) 117 (B) 106
(C) 125 (D) 136

Official Ans. by NTA (C)
Allen Ans. (C)

Sol. $f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$

$f(x) = a \begin{vmatrix} 1 & -1 & 0 \\ x & a & -1 \\ x^2 & ax & a \end{vmatrix}$

$= a [1(a^2 + ax) + 1(ax + x^2)]$

$\Rightarrow f(x) = a(x + a)^2$

so, $f'(x) = 2a(x + a)$

as, $2f'(10) - f'(5) + 100 = 0$

$\Rightarrow 2 \times 2a(10 + a) - 2a(5 + a) + 100 = 0$

$\Rightarrow 40a + 4a^2 - 10a - 2a^2 + 100 = 0$

$2a^2 + 30a + 100 = 0$

$\Rightarrow a^2 + 15a + 50 = 0$

$(a + 10)(a + 5) = 0$

$a = -10$ or $a = -5$

Required $= (-10)^2 + (-5)^2 = 125$

3. Let for some real numbers α and β , $a = \alpha - i\beta$. If the system of equations $4ix + (1+i)y = 0$ and

$8\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)x + \bar{a}y = 0$ has more than one

solution then $\frac{\alpha}{\beta}$ is equal to :

- (A) $-2 + \sqrt{3}$ (B) $2 - \sqrt{3}$
(C) $2 + \sqrt{3}$ (D) $-2 - \sqrt{3}$

Official Ans. by NTA (B)
Allen Ans. (B)

Sol. $a = \alpha - i\beta$; $\alpha \in \mathbb{R}$; $\beta \in \mathbb{R}$

$$4ix + (1+i)y = 0 \text{ and}$$

$$8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)x + \bar{a}y = 0$$

$$\begin{vmatrix} 4i & 1+i \\ 8e^{i2\pi/3} & \bar{a} \end{vmatrix} = 0$$

$$\Rightarrow 4i\bar{a} - (1+i)8e^{i2\pi/3} = 0$$

$$\Rightarrow 4i(\alpha + i\beta) - 8(1+i)\left(\frac{-1+i\sqrt{3}}{2}\right) = 0$$

$$\Rightarrow i\alpha - \beta + 1 + \sqrt{3} + i(1 - \sqrt{3}) = 0$$

$$\Rightarrow \beta = \sqrt{3} + 1$$

$$\alpha = \sqrt{3} - 1$$

$$\text{So, } \frac{\alpha}{\beta} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$$

4. Let A and B be two 3×3 matrices such that

$AB = I$ and $|A| = \frac{1}{8}$ then $|\text{adj}(B \text{adj}(2A))|$ is equal to

- (A) 16 (B) 32
(C) 64 (D) 128

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $AB = I$

$$\begin{aligned} |\text{adj}(B \text{adj}(2A))| &= |B \text{adj}(2A)|^2 \\ &= |B|^2 |\text{adj}(2A)|^2 \\ &= |B|^2 (|2A|)^2 = |B|^2 (2^6 |A|)^2 \end{aligned}$$

$$|A| = \frac{1}{8} \text{ and } |AB| = 1 \Rightarrow |A| |B| = 1$$

$$\Rightarrow \frac{1}{8} |B| = 1$$

$$\Rightarrow |B| = 8$$

required value = 64

5. Let $S = 2 + \frac{6}{7} + \frac{12}{7^2} + \frac{20}{7^3} + \frac{30}{7^4} + \dots$ then $4S$ is equal to

- (A) $\left(\frac{7}{3}\right)^2$ (B) $\frac{7^3}{3^2}$
(C) $\left(\frac{7}{3}\right)^3$ (D) $\frac{7^2}{3^3}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $S = 2 + \frac{6}{7} + \frac{12}{7^2} + \frac{20}{7^3} + \frac{30}{7^4} + \dots$

Considering infinite sequence,

$$S = 2 + \frac{6}{7} + \frac{12}{7^2} + \frac{20}{7^3} + \frac{30}{7^4} + \dots$$

$$\frac{S}{7} = \frac{2}{7} + \frac{6}{7^2} + \frac{12}{7^3} + \frac{20}{7^4} + \dots$$

$$\Rightarrow \frac{6S}{7} = 2 + \frac{4}{7} + \frac{6}{7^2} + \frac{8}{7^3} + \frac{10}{7^4} + \dots$$

$$\Rightarrow \frac{6S}{7^2} = \frac{2}{7} + \frac{4}{7^2} + \frac{6}{7^3} + \frac{8}{7^4} + \dots$$

$$\frac{6S}{7} \left(1 - \frac{1}{7}\right) = 2 + \frac{2}{7} + \frac{2}{7^2} + \frac{2}{7^3} + \dots$$

$$\Rightarrow \frac{6^2 S}{7^2} = \frac{2}{1 - \frac{1}{7}} = \frac{2}{6} \times 7$$

$$\Rightarrow S = \frac{2 \times 7^3}{6^3} \Rightarrow 4S = \frac{7^3}{3^3} = \left(\frac{7}{3}\right)^3$$

6. If a_1, a_2, a_3, \dots and b_1, b_2, b_3, \dots are A.P. and $a_1 = 2, a_{10} = 3, a_1 b_1 = 1 = a_{10} b_{10}$ then $a_4 b_4$ is equal to

- (A) $\frac{35}{27}$ (B) 1
(C) $\frac{27}{28}$ (D) $\frac{28}{27}$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. a_1, a_2, a_3, \dots A.P.; $a_1 = 2$; $a_{10} = 3$; $d_1 = \frac{1}{9}$

$$b_1, b_2, b_3, \dots \text{ A.P. ; } b_1 = \frac{1}{2} ; b_{10} = \frac{1}{3} ; d_2 = \frac{-1}{54}$$

[Using $a_1 b_1 = 1 = a_{10} b_{10}$; d_1 & d_2 are common differences respectively]

$$a_4 \cdot b_4 = (2 + 3d_1) \left(\frac{1}{2} + 3d_2\right)$$

$$= \left(2 + \frac{1}{3}\right) \left(\frac{1}{2} - \frac{1}{18}\right)$$

$$= \left(\frac{7}{3}\right) \left(\frac{8}{18}\right) = \frac{28}{27}$$

7. If m and n respectively are the number of local maximum and local minimum points of the

function $f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$, then the ordered

pair (m, n) is equal to

- (A) (3, 2) (B) (2, 3)
(C) (2, 2) (D) (3, 4)

Official Ans. by NTA (B)

Allen Ans. (B)

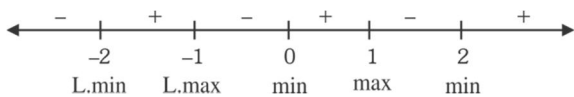
Sol. $m = L \cdot \max$

$N = L \cdot \min$

$$f(x) = \int_0^{x^2} \frac{t^2 - 5t + 4}{2 + e^t} dt$$

$$f'(x) = \frac{(x^4 - 5x^2 + 4)2x}{2 + e^{x^2}} = \frac{2x(x^2 - 1)(x^2 - 4)}{2 + e^{x^2}}$$

$$= \frac{2x(x-1)(x+1)(x-2)(x+2)}{2 + e^{x^2}}$$



so, $m = 2$ and $n = 3$

8. Let f be a differentiable function in $\left(0, \frac{\pi}{2}\right)$.

If $\int_{\cos x}^1 t^2 f(t) dt = \sin^3 x + \cos x$ then $\frac{1}{\sqrt{3}} f'\left(\frac{1}{\sqrt{3}}\right)$ is

equal to :

(A) $6 - 9\sqrt{2}$ (B) $6 - \frac{9}{\sqrt{2}}$

(C) $\frac{9}{2} - 6\sqrt{2}$ (D) $\frac{9}{\sqrt{2}} - 6$

Official Ans. by NTA (B)

Allen Ans. (Bonus)

Sol. At right hand vicinity of $x = 0$ given equation does not satisfy

$$\because \text{LHS} = \int_{\cos x}^1 t^2 f(t) dt = 0, \text{ RHS} = \lim_{x \rightarrow 0^+} (\sin^3 x + \cos x) = 1$$

LHS \neq RHS hence data given in question is wrong hence BONUS

Correct data should have been

$$\int_{\cos x}^1 t^2 f(t) dt = \sin^3 x + \cos x - 1$$

Calculation for option

differentiating both sides

$$-\cos^2 x f(\cos x) \cdot (-\sin x) = 3\sin^2 x \cdot \cos x - \sin x$$

$$\Rightarrow f(\cos x) = 3 \tan x - \sec^2 x$$

$$\Rightarrow f'(\cos x) \cdot (-\sin x) = 3\sec^2 x - 2\sec^2 x \tan x$$

$$\Rightarrow f'(\cos x) \cos x = \frac{2}{\cos^2 x} - \frac{3}{\sin x \cdot \cos x}$$

When $\cos x = \frac{1}{\sqrt{3}}$; $\sin x = \frac{\sqrt{2}}{\sqrt{3}}$

$$\therefore f'\left(\frac{1}{\sqrt{3}}\right) \frac{1}{\sqrt{3}} = 6 - \frac{9}{\sqrt{2}}$$

9. The integral $\int_0^1 \frac{1}{7^{\lfloor \frac{1}{x} \rfloor}} dx$, where $\lfloor \cdot \rfloor$ denotes the greatest integer function is equal to

(A) $1 + 6 \log_e \left(\frac{6}{7}\right)$ (B) $1 - 6 \log_e \left(\frac{6}{7}\right)$

(C) $\log_e \left(\frac{7}{6}\right)$ (D) $1 - 7 \log_e \left(\frac{6}{7}\right)$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol.

$$\int_0^1 \frac{1}{7^{\lfloor \frac{1}{x} \rfloor}} dx = -\int_1^0 \frac{1}{7^{\lfloor \frac{1}{x} \rfloor}} dx$$

$$= (-1) \left[\int_1^{1/2} \frac{1}{7} dx + \int_{1/2}^{1/3} \frac{1}{7^2} dx + \int_{1/3}^{1/4} \frac{1}{7^3} dx + \dots \dots \infty \right]$$

$$= \left(\frac{1}{7} + \frac{1}{2 \cdot 7^2} + \frac{1}{3 \cdot 7^3} + \dots \dots \infty \right) - \left(\frac{1}{7 \cdot 2} + \frac{1}{7^2 \cdot 3} + \frac{1}{7^3 \cdot 4} + \dots \dots \infty \right)$$

$$= -\ln \left(1 - \frac{1}{7} \right) - 7 \left(\frac{1}{7^2 \cdot 2} + \frac{1}{7^3 \cdot 3} + \frac{1}{7^4 \cdot 4} + \dots \dots \infty \right)$$

$$\left[\text{as } \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \dots \infty \right]$$

$$\left[\text{as } \ln(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots \dots \infty \right) \right]$$

$$= -\ln \frac{6}{7} - 7 \left(-\ln \left(1 - \frac{1}{7} \right) - \frac{1}{7} \right)$$

$$= 6 \ln \frac{6}{7} + 1$$

10. If the solution curve of the differential equation $((\tan^{-1} y) - x) dy = (1 + y^2) dx$ passes through the point (1, 0) then the abscissa of the point on the curve whose ordinate is $\tan(1)$ is :

(A) $2e$ (B) $\frac{2}{e}$

(C) 2 (D) $\frac{1}{e}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol.

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}$$

$$\text{I.f} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$$x e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y} dy$$

$$x \cdot e^{\tan^{-1} y} = (\tan^{-1} y - 1) e^{\tan^{-1} y} + c$$

∴ (1, 0) lies on the curve c = 2.

$$\text{For } y = \tan^{-1} 1 \Rightarrow x = \frac{2}{e}$$

11. If the equation of the parabola, whose vertex is at (5, 4) and the directrix is $3x + y - 29 = 0$, is

$$x^2 + ay^2 + bxy + cx + dy + k = 0 \text{ then}$$

$a + b + c + d + k$ is equal to

- (A) 575 (B) -575
(C) 576 (D) -576

Official Ans. by NTA (D)

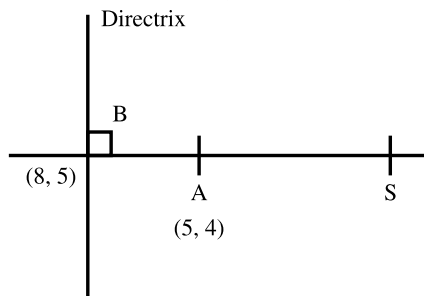
Allen Ans. (D)

Sol. Vertex (5, 4)

$$\text{Directrix : } 3x + y - 29 = 0$$

Co-ordinates of B (foot of directrix)

$$\frac{x-5}{3} = \frac{y-4}{1} = -\left(\frac{15+4-29}{10}\right) = 1$$



$$x = 8, y = 5$$

$$S = (2, 3) \text{ (focus)}$$

Equation of parabola

$$PS = PM$$

so equation is

$$x^2 + 9y^2 - 6xy + 134x - 2y - 711 = 0$$

$$a + b + c + d + k = 9 - 6 + 134 - 2 - 711 = -576$$

12. The set of values of k for which the circle

$$C : 4x^2 + 4y^2 - 12x + 8y + k = 0 \text{ lies inside the}$$

fourth quadrant and the point $\left(1, -\frac{1}{3}\right)$ lies on or inside the circle C is :

(A) An empty set (B) $\left[6, \frac{95}{9}\right]$

(C) $\left[\frac{80}{9}, 10\right]$ (D) $\left[9, \frac{92}{9}\right]$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $C : 4x^2 + 4y^2 - 12x + 8y + k = 0$

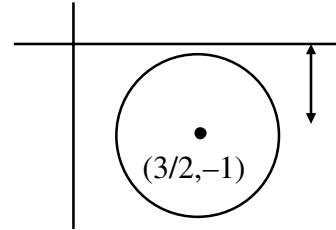
$$\Rightarrow x^2 + y^2 - 3x + 2y + \left(\frac{k}{4}\right) = 0$$

$$\text{Centre } \left(\frac{3}{2}, -1\right); r = \sqrt{\frac{13-k}{2}} \Rightarrow k \leq 13 \dots (1)$$

(i) Point $\left(1, -\frac{1}{3}\right)$ lies on or inside circle C

$$\Rightarrow S_1 \leq 0 \Rightarrow k \leq \frac{92}{9} \dots (2)$$

(ii) C lies in 4th quadrant



$$r < 1$$

$$\Rightarrow \frac{\sqrt{13-k}}{2} < 1$$

$$\Rightarrow k < 9 \dots (3)$$

$$\text{Hence } (1) \cap (2) \cap (3) \Rightarrow k \in \left[9, \frac{92}{9}\right]$$

13. Let the foot of the perpendicular from the point

$$(1, 2, 4) \text{ on the line } \frac{x+2}{4} = \frac{y-1}{2} = \frac{z+1}{3} \text{ be P. Then}$$

the distance of P from the plane $3x + 4y + 12z + 23 = 0$

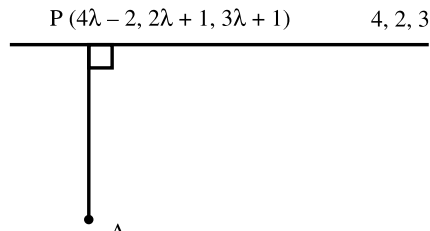
(A) 5 (B) $\frac{50}{13}$

(C) 4 (D) $\frac{63}{13}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol.



$$\frac{x+2}{4} = \frac{y-1}{2} = \frac{z+1}{3} = \lambda$$

$$(x, y, z) = (4\lambda - 2, 2\lambda + 1, 3\lambda - 1)$$

$$\overline{AP} = (4\lambda - 3)\hat{i} + (2\lambda - 1)\hat{j} + (3\lambda - 5)\hat{k}$$

$$\vec{b} = 4\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overline{AP} \cdot \vec{b} = 0$$

$$4(4\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 5) = 0$$

$$29\lambda = 12 + 2 + 15 = 29$$

$$\lambda = 1$$

$$P = (2, 3, 2)$$

$$3x + 4y + 12z + 23 = 0$$

$$d = \frac{|6 + 12 + 24 + 23|}{\sqrt{9 + 16 + 144}}$$

$$d = \frac{65}{13} = 5$$

14. The shortest distance between the lines $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{-1}$ and $\frac{x+3}{2} = \frac{y-6}{1} = \frac{z-5}{3}$ is :

(A) $\frac{18}{\sqrt{5}}$ (B) $\frac{22}{3\sqrt{5}}$

(C) $\frac{46}{3\sqrt{5}}$ (D) $6\sqrt{3}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{-1}$

$$\frac{x+3}{2} = \frac{y-6}{1} = \frac{z-5}{3}$$

$$A = (3, 2, 1) \quad B = (-3, 6, 5)$$

$$\vec{n}_1 = 2\hat{i} + 3\hat{j} - \hat{k}$$

$$\vec{n}_2 = 2\hat{i} + \hat{j} - 3\hat{k}$$

$$\overline{BA} = 6\hat{i} - 4\hat{j} - 4\hat{k}$$

$$\text{SHORTEST DISTANCE} = \frac{[\overline{BA} \vec{n}_1 \vec{n}_2]}{|\vec{n}_1 \times \vec{n}_2|}$$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -1 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= 10\hat{i} - 8\hat{j} - 4\hat{k}$$

$$[\overline{BA} \vec{n}_1 \vec{n}_2] = 60 + 32 + 16 = 108$$

$$|\vec{n}_1 \times \vec{n}_2| = \sqrt{100 + 64 + 16} = \sqrt{180}$$

$$S.D = \frac{108}{\sqrt{180}} = \frac{108}{6\sqrt{5}} = \frac{18}{\sqrt{5}}$$

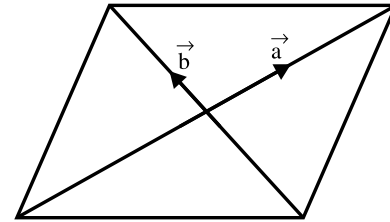
15. Let \vec{a} and \vec{b} be the vectors along the diagonal of a parallelogram having area $2\sqrt{2}$. Let the angle between \vec{a} and \vec{b} be acute. $|\vec{a}| = 1$ and $|\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$. If $\vec{c} = 2\sqrt{2}(\vec{a} \times \vec{b}) - 2\vec{b}$, then an angle between \vec{b} and \vec{c} is :

(A) $\frac{\pi}{4}$ (B) $-\frac{\pi}{4}$

(C) $\frac{5\pi}{6}$ (D) $\frac{3\pi}{4}$

Official Ans. by NTA (D)

Allen Ans. (D)



Sol.

$$\text{Area} = \frac{1}{2} |\vec{a} \times \vec{b}| = 2\sqrt{2} \Rightarrow |\vec{a} \times \vec{b}| = 4\sqrt{2}$$

$$|\vec{a}| = 1 \text{ and } |\vec{a} \cdot \vec{b}| = |\vec{a} \times \vec{b}|$$

$$\Rightarrow \cos \theta = \sin \theta$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

$$\therefore |\vec{a} \times \vec{b}| = 4\sqrt{2} \Rightarrow |\vec{a}| |\vec{b}| \sin \frac{\pi}{4} = 4\sqrt{2}$$

$$\Rightarrow |\vec{b}| = 8$$

$$\text{Now, } \vec{c} = 2\sqrt{2}(\vec{a} \times \vec{b}) - 2\vec{b}$$

$$|\vec{c}| = \sqrt{(2\sqrt{2})^2 |\vec{a} \times \vec{b}|^2 + (2|\vec{b}|)^2} = 16\sqrt{2}$$

$$\text{Now, } \vec{b} \cdot \vec{c} = -2|\vec{b}|^2$$

$$\Rightarrow 8 \times 16\sqrt{2} \times \cos \alpha = -2.64$$

$$\Rightarrow \cos \alpha = -\frac{1}{\sqrt{2}} \Rightarrow \alpha = \frac{3\pi}{4}$$

16. The mean and variance of the data 4, 5, 6, 6, 7, 8, x, y where $x < y$ are 6, and $\frac{9}{4}$ respectively. Then

$x^4 + y^2$ is equal to

- (A) 162 (B) 320
 (C) 674 (D) 420

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. mean $\bar{x} = \frac{4+5+6+6+7+8+x+y}{8} = 6$

$\Rightarrow x + y = 48 - 36 = 12$

Variance

$= \frac{1}{8} (16+25+36+36+49+64+x^2+y^2) - 36 = \frac{9}{4}$

$\Rightarrow x^2 + y^2 = 80$

$\therefore x = 4; y = 8$

$x^4 + y^2 = 256 + 64 = 320$

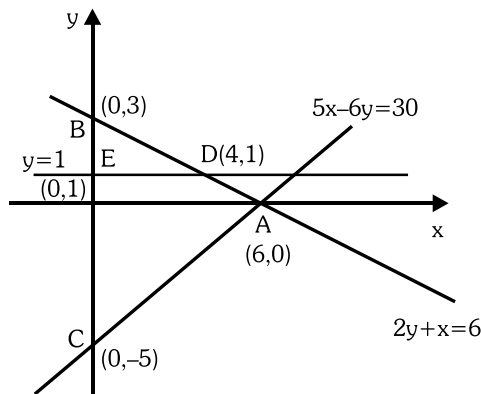
17. If a point A(x, y) lies in the region bounded by the y-axis, straight lines $2y + x = 6$ and $5x - 6y = 30$, then the probability that $y < 1$ is :

- (A) $\frac{1}{6}$ (B) $\frac{5}{6}$
 (C) $\frac{2}{3}$ (D) $\frac{6}{7}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. Required probability = $\frac{\text{ar}(ADEC)}{\text{ar}(ABC)}$



$= 1 - \frac{\text{ar}(BDE)}{\text{ar}(ABC)}$

$= 1 - \frac{\frac{1}{2} \times 2 \times 4}{\frac{1}{2} \times 8 \times 6} = 1 - \frac{1}{6} = \frac{5}{6}$

18. The value of $\cot \left(\sum_{n=1}^{50} \tan^{-1} \left(\frac{1}{1+n+n^2} \right) \right)$ is

- (A) $\frac{26}{25}$ (B) $\frac{25}{26}$
 (C) $\frac{50}{51}$ (D) $\frac{52}{51}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\tan^{-1} \frac{1}{1+n+n^2} = \tan^{-1} \left(\frac{(n+1)-n}{1+n(n+1)} \right)$

$= \tan^{-1} (n+1) - \tan^{-1} n$

so, $\sum_{n=1}^{50} (\tan^{-1} (n+1) - \tan^{-1} n)$

$= \tan^{-1} 51 - \tan^{-1} 1$

$\cot \left(\sum_{n=1}^{50} \tan^{-1} \left(\frac{1}{1+n+n^2} \right) \right) = \cot (\tan^{-1} 51 + \tan^{-1} 1)$

$= \frac{1}{\tan(\tan^{-1} 51 + \tan^{-1} 1)} = \frac{1+51 \times 1}{51-1} = \frac{52}{50} = \frac{26}{25}$

19. $\alpha = \sin 36^\circ$ is a root of which of the following equation

- (A) $10x^4 - 10x^2 - 5 = 0$ (B) $16x^4 + 20x^2 - 5 = 0$
 (C) $16x^4 - 20x^2 + 5 = 0$ (D) $16x^4 - 10x^2 + 5 = 0$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $\cos 72^\circ = \frac{\sqrt{5}-1}{4}$

$\Rightarrow 1 - 2 \sin^2 36^\circ = \frac{\sqrt{5}-1}{4}$

$\Rightarrow 4 - 8\alpha^2 = \sqrt{5} - 1$

$\Rightarrow 5 - 8\alpha^2 = \sqrt{5}$

$\Rightarrow (5 - 8\alpha^2)^2 = 5$

$\Rightarrow 25 + 64\alpha^4 - 80\alpha^2 = 5$

$\Rightarrow 64\alpha^4 - 80\alpha^2 + 20 = 0$

$\Rightarrow 16\alpha^4 - 20\alpha^2 + 5 = 0$

20. Which of the following statement is a tautology?

- (A) $(\sim q) \wedge p \wedge q$
 (B) $(\sim q) \wedge p \wedge (p \wedge \sim p)$
 (C) $(\sim q) \wedge p \vee (p \vee \sim p)$
 (D) $(p \wedge q) \wedge (\sim (p \wedge q))$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. (A) $(\sim q \wedge p) \wedge q = (\sim q \wedge q) \wedge p = f$

(B) $(\sim q \wedge p) \wedge (p \wedge \sim p) = \sim q \wedge (p \wedge \sim p) = f$

(C) $(\sim q \wedge p) \vee (p \vee \sim p) = (\sim q \wedge p) \vee (t) = t$

(D) $(p \wedge q) \wedge (\sim (p \wedge q)) = f$

SECTION-B

1. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define $f : S \rightarrow S$ as $f(n) = \begin{cases} 2n, & \text{if } n=1,2,3,4,5 \\ 2n-11 & \text{if } n=6,7,8,9,10 \end{cases}$.

Let $g : S \rightarrow S$ be a function such that $f \circ g(n) = \begin{cases} n+1 & \text{,if } n \text{ is odd} \\ n-1 & \text{,if } n \text{ is even} \end{cases}$, then

$g(10) ((g(1) + g(2) + g(3) + g(4) + g(5)))$ is equal to:

Official Ans. by NTA (190)

Allen Ans. (190)

Sol. $f^{-1}(n) = \begin{cases} \frac{n}{2} & ; n=2,4,6,8,10 \\ \frac{n+11}{2} & ; n=1,3,5,7,9 \end{cases}$

$f(g(n)) = \begin{cases} n+1 & ; n \in \text{odd} \\ n-1 & ; n \in \text{even} \end{cases}$

$\Rightarrow g(n) = \begin{cases} f^{-1}(n+1) & ; n \in \text{odd} \\ f^{-1}(n-1) & ; n \in \text{even} \end{cases}$

$\therefore g(n) = \begin{cases} \frac{n+1}{2} & ; n \in \text{odd} \\ \frac{n+10}{2} & ; n \in \text{even} \end{cases}$

$g(10) \cdot [g(1) + g(2) + g(3) + g(4) + g(5)] = 10 \cdot [1 + 6 + 2 + 7 + 3] = 190$

2. Let α, β be the roots of the equation $x^2 - 4\lambda x + 5 = 0$ and α, γ be the roots of the equation $x^2 - (3\sqrt{2} + 2\sqrt{3})x + 7 + 3\lambda\sqrt{3} = 0$.

If $\beta + \gamma = 3\sqrt{2}$, then $(\alpha + 2\beta + \gamma)^2$ is equal to :

Official Ans. by NTA (98)

Allen Ans. (98)

Sol. $x^2 - 4\lambda x + 5 = 0 \left\langle \begin{matrix} \alpha \\ \beta \end{matrix} \right.$
 $x^2 - (3\sqrt{2} + 2\sqrt{3})x + (7 + 3\lambda\sqrt{3}) = 0 \left\langle \begin{matrix} \alpha \\ \gamma \end{matrix} \right.$
 $\alpha + \beta = 4\lambda$
 $\alpha + \gamma = 3\sqrt{2} + 2\sqrt{3}$

$\beta + \lambda = 3\sqrt{2}$ $\alpha\gamma = 7 + 3\lambda\sqrt{3}$
 $\therefore \alpha = 2\lambda + \sqrt{3}$ $\alpha\beta = 5$
 $\beta = 2\lambda - \sqrt{3}$ $4\lambda^2 = 8 \Rightarrow \lambda = \sqrt{2}$

$\therefore (\alpha + 2\beta + \lambda)^2 = (4\alpha + 3\sqrt{2})^2 = (7\sqrt{2})^2 = 98$

3. Let A be a matrix of order 2×2 , whose entries are from the set $\{0, 1, 2, 3, 4, 5\}$. If the sum of all the entries of A is a prime number p, $2 < p < 8$, then the number of such matrices A is :

Official Ans. by NTA (180)

Allen Ans. (180)

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; $a, b, c, d \in \{0, 1, 2, 3, 4, 5\}$

$a + b + c + d = p, p \in \{3, 5, 7\}$

Case-(i)

$a + b + c + d = 3; a, b, c, d \in \{0, 1, 2, 3\}$

No. of ways = ${}^{3+4-1}C_{4-1} = {}^6C_3 = 56$ (1)

Case-(ii)

$a + b + c + d = 5; a, b, c, d \in \{0, 1, 2, 3, 4, 5\}$

No. of ways = ${}^{5+4-1}C_{4-1} = {}^8C_3 = 56$ (2)

Case-(iii)

$a + b + c + d = 7$

No. of ways = total ways when $a, b, c, d \in \{0, 1, 2, 3, 4, 5, 6, 7\}$ – total ways when $a, b, c, d \notin \{6, 7\}$

No of ways = ${}^{7+4-1}C_{4-1} = \left(\frac{14}{3} + \frac{14}{2} \right) = {}^{10}C_3 - 16 = 104$ (3)

Hence total no. of ways = 180

4. If the sum of the coefficients of all the positive powers of x, in the binomial expansion of $\left(x^n + \frac{2}{x^5}\right)^7$ is 939, then the sum of all the possible integral values of n is :

Official Ans. by NTA (57)

Allen Ans. (57)

Sol. coefficients and there cumulative sum are :

Coefficient	Commulative sum
$x^{7n} \rightarrow {}^7C_0$	1
$x^{6n-5} \rightarrow 2 \cdot {}^7C_1$	1+14
$x^{5n-10} \rightarrow 2^2 \cdot {}^7C_2$	1+14+84
$x^{4n-15} \rightarrow 2^3 \cdot {}^7C_3$	1+14+84+280
$x^{3n-20} \rightarrow 2^4 \cdot {}^7C_4$	1+4+84+280+560 = 939
$x^{2n-25} \rightarrow 2^5 \cdot {}^7C_5$	

$$3n-20 \geq 0 \cap 2n-25 < 0 \cap n \in \mathbb{I}$$

$$\therefore 7 \leq n \leq 12$$

$$\text{Sum} = 7 + 8 + 9 + 10 + 11 + 12 = 57$$

5. Let $[t]$ denote the greatest integer $\leq t$ and $\{t\}$ denote the fractional part of t . Then integral value of α for which the left hand limit of the function

$$f(x) = [1+x] + \frac{\alpha^{2[x]+\{x\}} + [x] - 1}{2[x] + \{x\}} \text{ at } x = 0 \text{ is equal to}$$

$$\alpha - \frac{4}{3} \text{ is } \underline{\hspace{2cm}}$$

Official Ans. by NTA (3)

Allen Ans. (3)

Sol. $f(x) = [1+x] + \frac{\alpha^{2[x]+\{x\}} + [x] - 1}{2[x] + \{x\}}$

$$\lim_{x \rightarrow 0^-} f(x) = \alpha - \frac{4}{3} \Rightarrow 0 + \frac{\alpha^{-1} - 2}{-1} = \alpha - \frac{4}{3}$$

$$\Rightarrow 2 - \frac{1}{\alpha} = \alpha - \frac{4}{3}$$

$$\Rightarrow \alpha + \frac{1}{\alpha} = \frac{10}{3}$$

$$\Rightarrow \alpha = 3; \alpha \in \mathbb{I}$$

6. If $y(x) = (x^{x^x})$, $x > 0$ then $\frac{d^2x}{dy^2} + 20$ at $x = 1$ is equal to:

Official Ans. by NTA (16)

Allen Ans. (16)

Sol. $y = (x)^x = (x^x)^x$

$$\ln y(x) = x^2 \cdot \ln x$$

$$\frac{1}{y(x)} \cdot y'(x) = \frac{x^2}{x} + 2x \cdot \ln x$$

$$y'(x) = y(x) [x + 2x \ln x]$$

$$y(1) = 1; y'(1) = 1$$

$$y''(x) = y'(x) [x + 2x \cdot \ln(x)]$$

$$+ y(x) [1 + 2(1 + \ln x)]$$

$$y''(1) = 1 [1 + 0] + 1 (1 + 2) = 4$$

$$\frac{d^2y}{dx^2} = - \left(\frac{dy}{dx} \right)^3 \cdot \frac{d^2x}{dy^2}$$

$$\Rightarrow 4 = - \frac{d^2x}{dy^2}$$

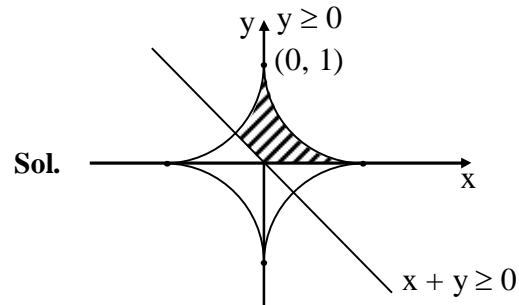
$$\frac{d^2x}{dy^2} = -4$$

$$\text{Ans. } -4 + 20 = 16$$

7. If the area of the region $\{(x, y) : x^{\frac{2}{3}} + y^{\frac{2}{3}} \leq 1, x + y \geq 0, y \geq 0\}$ is A , then $\frac{256A}{\pi}$ is

Official Ans. by NTA (36)

Allen Ans. (36)



$$A = \frac{3}{2} \int_0^1 (1 - x^{2/3})^{3/2} dx$$

$$\text{Let } x = \sin^3 \theta$$

$$A = \frac{3}{2} \int_0^{\pi/2} (1 - \sin^2 \theta)^{3/2} \cdot 3 \sin^2 \theta \cos \theta d\theta$$

$$= \frac{3}{2} \int_0^{\pi/2} 3 \sin^2 \theta \cos^4 \theta d\theta$$

$$= \frac{9}{2} \int_0^{\pi/2} \sin^2 \theta \cos^4 \theta d\theta$$

$$A = \frac{9}{2} \times \frac{1.3.1}{(2+4)(4)(2)} \cdot \frac{\pi}{2}$$

$$\Rightarrow A = \frac{9\pi}{64} \Rightarrow \frac{64A}{\pi} = 9$$

$$\Rightarrow \frac{256A}{\pi} = 36 \text{ Ans.}$$

8. Let v be the solution of the differential equation $(1-x^2)dy = (xy + (x^3 + 2)\sqrt{1-x^2})dx$, $-1 < x < 1$

$$\text{and } y(0) = 0 \text{ if } \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1-x^2} y(x) dx = k \text{ then } k^{-1} \text{ is}$$

equal to :

Official Ans. by NTA (320)

Allen Ans. (320)

Sol. $(1 - x^2) \frac{dy}{dx} = xy + (x^3 + 2) \sqrt{1 - x^2}$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{-x}{1-x^2} \right) y = \frac{x^3 + 2}{\sqrt{1-x^2}}$$

$$\text{IF} = e^{\int \frac{-x}{1-x^2} dx} = \sqrt{1-x^2}$$

$$y(x) \cdot \sqrt{1-x^2} = \frac{x^4}{4} + 2x + c$$

$$y(0) = 0 \Rightarrow c = 0$$

$$\sqrt{1-x^2} y(x) = \frac{x^4}{4} + 2x$$

$$\text{required value} = \int_{-1/2}^{1/2} \left(\frac{x^4}{4} + 2x \right) dx - \frac{1}{4} \cdot 2 \int_0^{1/2} x^4 dx$$

$$= \frac{1}{10} (x^5)_0^{1/2} = \frac{1}{320}$$

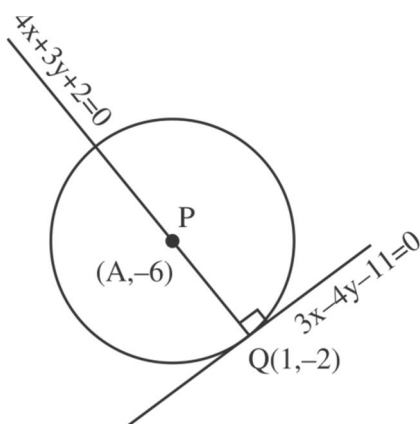
$$k^{-1} = 320$$

- 9.** Let a circle C of radius 5 lie below the x-axis. The line $L_1 = 4x + 3y - 2$ passes through the centre P of the circle C and intersects the line $L_2 : 3x - 4y - 11 = 0$ at Q. The line L_2 touches C at the point Q. Then the distance of P from the line $5x - 12y + 51 = 0$ is

Official Ans. by NTA (11)

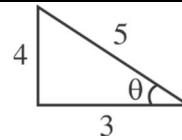
Allen Ans. (11)

Sol.



$$4x + 3y + 2 = 0$$

$$3x - 4y - 11 = 0$$



$$\frac{x}{-25} = \frac{y}{50} = \frac{1}{-25}$$

$$\frac{x-1}{\cos \theta} = \frac{y+2}{\sin \theta} = \pm 5$$

$$y = -2 + 5 \left(-\frac{4}{5} \right) = -6$$

$$x = 1 + 5 \left(\frac{3}{5} \right) = 4$$

Req. distance

$$\left| \frac{5(4) - 12(-6) + 51}{13} \right|$$

$$= \left| \frac{20 + 72 + 51}{13} \right|$$

$$= \frac{143}{13} = 11$$

- 10.** Let $S = \{E_1, E_2, \dots, E_8\}$ be a sample space of random experiment such that $P(E_n) = \frac{n}{36}$ for every $n = 1, 2, \dots, 8$. Then the number of elements in the set $\left\{ A \subset S : P(A) \geq \frac{4}{5} \right\}$ is _____

Official Ans. by NTA (19)

Allen Ans. (19)

Sol. $P(A') < \frac{1}{5} = \frac{36}{180}$

5 times the sum of missing number should be less than 36.

If 1 digit is missing = 7

If 2 digit is missing = 9

If 3 digit is missing = 2

If 0 digit is missing = 1

Alternate

A is subset of S hence

A can have elements:

type 1 : { }

type 2: $\{E_1\}, \{E_2\}, \dots, \{E_8\}$

type 3: $\{E_1, E_2\}, \{E_1, E_3\}, \dots, \{E_1, E_8\}$

⋮
⋮

type 6: $\{E_1, E_2, \dots, E_5\}, \dots, \{E_4, E_5, E_6, E_7, E_8\}$

type 7: $\{E_1, E_2, \dots, E_6\}, \dots, \{E_3, E_4, \dots, E_8\}$

type 8: $\{E_1, E_2, \dots, E_7\}, \{E_2, E_3, \dots, E_8\}$

type 9: $\{E_1, E_2, \dots, E_8\}$

As $P(A) \geq \frac{4}{5}$;

Note : Type 1 to Type 4 elements can not be in set A as maximum probability of type 4 elements.

$\{E_5, E_6, E_7, E_8\}$ is $\frac{5}{36} + \frac{6}{36} + \frac{7}{36} + \frac{8}{36} = \frac{13}{18} < \frac{4}{5}$

Now for Type 5 acceptable elements let's call probability as P_5

$$P_5 = \frac{n_1 + n_2 + n_3 + n_4 + n_5}{36} \leq \frac{4}{5}$$

$$\Rightarrow n_1 + n_2 + n_3 + n_4 + n_5 \geq 28.8$$

Hence, 2 possible ways $\{E_5, E_6, E_7, E_8, \mathbf{E_3}$ or $\mathbf{E_4}\}$

$$P_6 = n_1 + n_2 + n_3 + n_4 + n_5 + n_6 \geq 28.8$$

\Rightarrow 9 possible ways

$$P_7 \Rightarrow n_1 + n_2 + \dots + n_7 \geq 288$$

\Rightarrow 7 possible ways

$$P_8 \Rightarrow n_1 + n_2 + \dots + n_8 \geq 28.8$$

\Rightarrow 1 possible way

Total = 19