FINAL JEE–MAIN EXAMINATION – JUNE, 2022

(Held On Monday 27th June, 2022) **TIME : 9 : 00 AM to 12 : 00 PM**

MATHEMATICS

TEST PAPER WITH SOLUTION

SECTION-A

1. The area of the polygon, whose vertices are the non-real roots of the equation $\bar{z} = iz^2$ is :

(A) $\frac{3\sqrt{3}}{4}$ (B) $\frac{3\sqrt{3}}{2}$

(C) $\frac{3}{2}$ (D) $\frac{3}{4}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\Rightarrow$ Let z = x + iy, x, y $\in$ R

Now $\bar{z} = iz^2$

then $x - iy = i(x^2 - y^2 + 2xyi)$

$x - iy = i(x^2 - y^2) - 2xy$

$\Rightarrow x = -2xy \;\&\; -y = x^2 - y^2$

$\Rightarrow x(1 + 2y) = 0$

$x = 0$ or $y = -\frac{1}{2}$

Put $x = 0$ in $-y = x^2 - y^2$

We get $y = y^2$

$\Rightarrow y = 0, 1$

Similarly

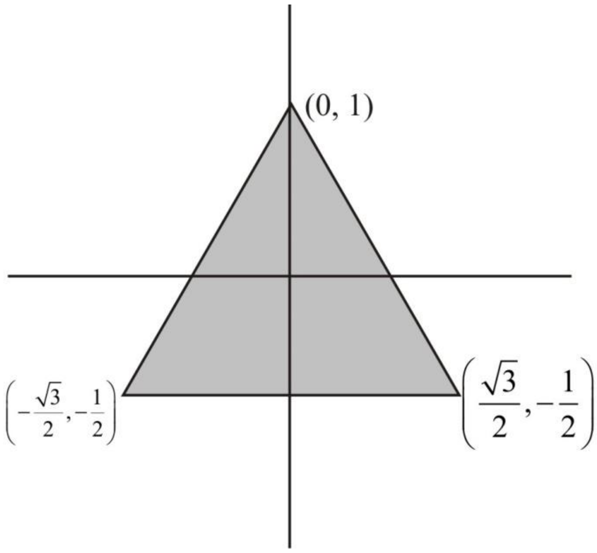
Put $y = -\frac{1}{2}$ in $-y = x^2 - y^2$

$\Rightarrow \frac{1}{2} = x^2 - \frac{1}{4}$

$\Rightarrow x^2 = \frac{3}{4}$

$x = \pm\frac{\sqrt{3}}{2}$

$z = \left(0, i, \frac{\sqrt{3}}{2} - \frac{1}{2}i, -\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$



$Area = \frac{1}{2}.(\sqrt{3})\left(\frac{3}{2}\right)$

$= \frac{3\sqrt{3}}{4}$

2. Let the system of linear equations x + 2y + z = 2, $\alpha x + 3y - z = \alpha$, $-\alpha x + y + 2z = -\alpha$ be inconsistent. Then α is equal to :

(A) $\frac{5}{2}$ (B) $-\frac{5}{2}$

(C) $\frac{7}{2}$ (D) $-\frac{7}{2}$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ -2 & 1 & 2 \end{vmatrix}$

$= (6 + y) - 2((2\alpha - \alpha) + 1(\alpha + 3\alpha)$

$= 7 - 2\alpha + 4\alpha$

$= 7 + 2\alpha$

$\Delta = 0 \Rightarrow \alpha = -\frac{7}{2}$

$$\Delta_1 = \begin{vmatrix} 2 & 2 & 1 \\ \alpha & 3 & -1 \\ -\alpha & 1 & 2 \end{vmatrix}$$

$= 14 + 2\alpha$

$\alpha = -x_2 = 7$

$\Delta_1 \neq 0$

3. If $x = \sum_{n=0}^{\infty} a^n, y = \sum_{n=0}^{\infty} b^n, z = \sum_{n=0}^{\infty} c^n$, where a, b, c are in A.P. and |a| < 1, |b| < 1, |c| < 1, abc ≠ 0, then

(A) x, y, z are in A.P.

(B) x, y, z are in G.P.

(C) $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

(D) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1 - (a+b+c)$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $x = 1 + a + a^2 = \ldots\ldots\ldots$

$$x = \frac{1}{1-a} \Rightarrow a = 1 - \frac{1}{x}$$

$$y = \frac{1}{1-b} \Rightarrow b = 1 - \frac{1}{y}$$

$$z = \frac{1}{1-c} \Rightarrow c = 1 - \frac{1}{z}$$

a, b, c are in A.P.

$\Rightarrow 1 - \frac{1}{x}, 1 - \frac{1}{y}, 1 - \frac{1}{z}$ are in A.P.

$\Rightarrow -\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$ are in A.P.

$\Rightarrow \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A.P.

4. Let $\frac{dy}{dx} = \frac{ax - by + a}{bx + cy + a}$, where a, b, c are constants, represent a circle passing through the point (2, 5). Then the shortest distance of the point (11, 6) from this circle is :

(A) 10 (B) 8

(C) 7 (D) 5

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. Let equation of circle is

$x^2 + y^2 + 2gx + 2fy + c = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-(2x+2g)}{(2y+2f)}$$

Comparing with $\frac{dy}{dx} = \frac{ax - by + a}{bx + cy + a}$

$\Rightarrow$ b = 0, a = - 2, c = 2

$\Rightarrow$ -2g = -2 $\Rightarrow$ g = 1 2f = -2

f = -1

Now circle will be

$x^2 + y^2 + 2x - 2y + c = 0$

its passes through (2, 5)

which will give c = -23

so circle will be $x^2 + y^2 + 2x - 2y - 23 = 0$

centre C = (-1, 1)

and radius 5

Now P is (11, 6)

So minimum distance of P from circle will be

$$= \sqrt{(11+1)^2 + (6-1)^2} - 5$$

= 13 – 5

= 8

5. Let a be an integer such that $\lim_{x \to 7} \frac{18 - [1-x]}{[x - 3a]}$ exists, where [t] is greatest integer ≤ t. Then a is equal to :

(A) -6 (B) -2

(C) 2 (D) 6

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\lim_{x\to 7}\frac{18-[1-x]}{[x]-3a}$

L.H.L. $\lim_{x\to 7-}\frac{18-[1-x]}{[x]-3a}$

$=\frac{18-(-6)}{6-3a}$

$=\frac{24}{6-3a}$

R.H.L. $\lim_{x\to 7+}\frac{18-[1-x]}{[x]-3a}$

$=\frac{18-(-7)}{7-3a}$

$=\frac{25}{7-3a}$

Now L.H.L. = R.H.L.

$\frac{24}{6-3a}=\frac{25}{7-3a}$

$\Rightarrow 168-72a=150-75a$

$\Rightarrow 18=-3a$

$\Rightarrow$ a = -6

6. The number of distinct real roots of $x^4 - 4x + 1 = 0$ is :

(A) 4 (B) 2

(C) 1 (D) 0

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. Let $f(x)=x^4-4x+1$

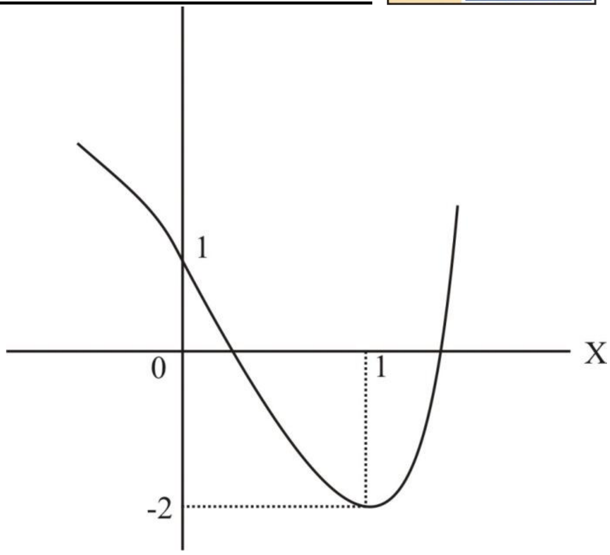
$f'(x) = 4x^3 - 4$

$f'(x) = 0 \Rightarrow x = 1$

x = 1 is point of minima.

f(1) = - 2

f(0) = 1



Hence 2 solutions.

7. The lengths of the sides of a triangle are $10 + x^2$, $10 + x^2$ and $20 - 2x^2$. If for x = k, the area of the triangle is maximum, then $3k^2$ is equal to :

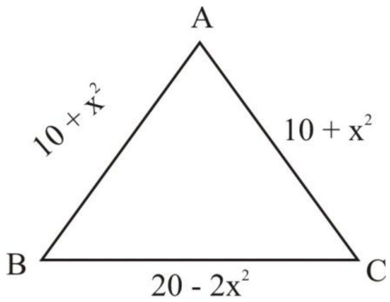
(A) 5 (B) 8

(C) 10 (D) 12

Official Ans. by NTA (C)

Allen Ans. (C)

Sol.



$a = 20 - 2x^2$, $b = 10 + x^2$, $c = 10 + x^2$

$=\frac{a+b+c}{2}$

= 20

$\Delta=\sqrt{s(s-a)(s-b)(s-c)}$

$=\sqrt{20(2x^2)(10-x^2)(10-x^2)}$

$=2\sqrt{10}\sqrt{x^2(10-x^2)^2}$

$=2\sqrt{10}\left|x(10-x^2)\right|$

$= 2\sqrt{10}\,\left|10x - x^3\right|$

S = 10x – x³

$\frac{ds}{dx} = 10 - 3x^2$

$\frac{ds}{dx} = 0 \Rightarrow x^2 = \frac{10}{3}$

3x² = 10

8. If $\cos^{-1}\left(\frac{y}{2}\right) = \log_e\left(\frac{x}{5}\right)^5, |y| < 2$, then :

(A) $x^2y'' + xy' - 25y = 0$

(B) $x^2y'' - xy' - 25y = 0$

(C) $x^2y'' - xy' + 25y = 0$

(D) $x^2y'' + xy' + 25y = 0$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $\cos^{-1}\left(\frac{y}{2}\right) = \log_e\left(\frac{x}{5}\right)^5$

$\cos^{-1}\left(\frac{y}{2}\right) = 5\log_e\left(\frac{x}{5}\right)$

$\frac{-1}{\sqrt{1-\frac{y^2}{4}}}.\frac{y'}{2} = 5.\frac{1}{\frac{x}{5}} \times \frac{1}{5}$

$\Rightarrow \frac{-y'}{\sqrt{4-y^2}} = \frac{5}{x}$

$-xy' = 5\sqrt{4-y^2}$

$-xy'' - y' = 5.\frac{1}{2\sqrt{4-y^2}}(-2y\ y')$

$\Rightarrow xy'' + y' = \frac{5y'.y}{\sqrt{4-y^2}}$

$xy'' + y' = 5.\left(\frac{-5}{x}\right)y$

$x^2y'' + xy' = -25y$

9. $\int \frac{(x^2+1)e^x}{(x+1)^2}dx = f(x)e^x + C$, Where C is a constant, then $\frac{d^3f}{dx^3}$ at x = 1 is equal to :

(A) $-\frac{3}{4}$

(B) $\frac{3}{4}$

(C) $-\frac{3}{2}$

(D) $\frac{3}{2}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\int \left(\frac{x^2+1}{(x+1)^2}\right)e^x.dx$

$= \int \left(\frac{x^2-1+2}{(x+1)^2}\right)e^x\ dx$

$= \int \left(\frac{x-1}{x+1} + \frac{2}{(x+1)^2}\right)e^x\ dx$

$= \int (f(x) + f'(x))e^x\ dx$

$= f(x)\ e^x + c$

Where $f(x) = \frac{x-1}{x+1}$

$f'(x) = \frac{2}{(x+1)^2}$

$f''(x) = \frac{-4}{(x+1)^3}$

$= \frac{12}{(x+1)^4}$

$f''(1) = \frac{12}{16}$

$= \frac{3}{4}$

10. The value of the integral $\int_{-2}^{2}\frac{|x^3+x|}{\left(e^{x|x|}+1\right)}dx$ is equal to :

(A) $5e^2$ (B) $3e^{-2}$

(C) 4 (D) 6

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $f(x)=\frac{|x^3+x|}{\left(e^{x|x|}+1\right)}dx$

$$\int_{-2}^{2} f(x)dx=\int_{0}^{2}\left(f(x)+f(-x)\right)dx$$

$$=\int_0^2\left(\frac{|x^3+x|}{\left(e^{x|x|}+1\right)}+\frac{|-x^3-x|}{\left(e^{-x|-x|}+1\right)}\right)dx$$

$$=\int_0^2\left(\frac{|x^3+x|}{\left(e^{x|x|}+1\right)}+\frac{|x^3+x|}{\left(e^{-x|x|}+1\right)}\right)dx$$

$$=\int_0^2\left(\frac{x^3+x}{\left(e^{x^2}+1\right)}+\frac{x^3+x}{\left(e^{-x^2}+1\right)}\right)dx$$

$$I=\int_0^2\left(\frac{x^3+x}{1+e^{x^2}}+\frac{e^{x^2}\left(x^3+x\right)}{1+e^{x^2}}\right)dx$$

$$=\int_0^2\left(x^3+x\right)dx$$

$$=\left[\frac{x^4}{4}+\frac{x^2}{2}\right]_0^2$$

$= 4 + 2 = 6$

11. If $\frac{dy}{dx}+\frac{2^{x-y}\left(2^y-1\right)}{2^x-1}=0, x, y>0, y(1)=1$, then y(2) is equal to :

(A) $2 + \log_2 3$ (B) $2 + \log_2 2$

(C) $2 - \log_2 3$ (D) $2 - \log_2 3$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $\frac{dy}{dx}+\frac{2^{x-y}\left(2^y-1\right)}{2^x-1}=0$,

x, y > 0, y(1) = 1, y (2) = ?

$$\frac{dy}{dx}=-\frac{2^x\left(2^y-1\right)}{2^y\left(2^x-1\right)}$$

$$\int\frac{2^y}{2^y-1}dy=-\int\frac{2^x}{2^x-1}dx$$

$$\frac{1}{\ln 2}\int\frac{2^y\ln 2}{2^y-1}dy=-\frac{1}{\ln^2}\int\frac{2^x\ln 2}{2^x-1}dx$$

$$\frac{1}{\ln 2}\ln\left|2^y-1\right|=\frac{-1}{\ln 2}\ln\left|2^x-1\right|+C$$

At x= 1, y = 1

Putting this values in above relation we get C = 0

$$\ln\left|2^y-1\right|+\ln\left|2^x-1\right|=0$$

$$\left(2^x-1\right)\left(2^y-1\right)=1$$

$$2^y-1=\frac{1}{2^x+1}$$

At x = 2

$$2^y=\frac{1}{3}+1=\frac{4}{3}$$

$$y=\log_2\frac{4}{3}=\log_2 4-\log_2 3=2-\log_2 3$$

12. In an isosceles triangle ABC, the vertex A is (6, 1) and the equation of the base BC is 2x + y = 4. Let the point B lie on the line x + 3y = 7. If (α, β) is the centroid ΔABC, then $15(\alpha+\beta)$ is equal to :

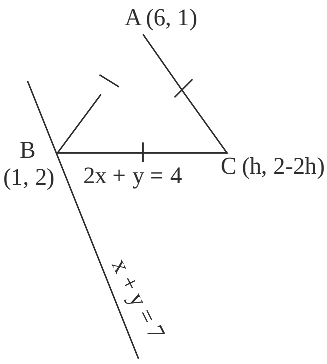
(A) 39 (B) 41

(C) 51 (D) 63

Official Ans. by NTA (C)

Allen Ans. (C)

Sol.



Point B (1, 2)

Now let C be (h, 4 - 2h)

(As C lies on 2x + y = 4)

$\because \Delta$ is isosceles with base BC

$\therefore AB = AC$

$$\sqrt{25+1} = \sqrt{(6-h)^2 + (2h-3)^2}$$

$$\sqrt{26} = \sqrt{36 + h^2 - 12h + 4h^2 + 9 - 12h}$$

$$26 = 5h^2 - 24h + 45 \Rightarrow 5h^2 - 24h + 19 = 0$$

$$\Rightarrow 5h^2 - 5h - 19h + 19 = 0$$

$h = \frac{19}{5}$ or h = 1

Thus $C\left(\frac{19}{5}, \frac{-18}{5}\right)$

Centroid $\left(\frac{6+1+\frac{19}{5}}{3}, \frac{1+2-\frac{18}{5}}{3}\right)$

$$\left(\frac{35+19}{15}, \frac{15-18}{15}\right)$$

$$\left(\frac{54}{15}, \frac{-3}{15}\right)$$

$$\alpha = \frac{54}{15}; \beta = \frac{-3}{15}$$

$$15(\alpha + \beta) = 51$$

13. Let the eccentricity of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a > b,$ be $\frac{1}{4}$. If this ellipse passes through the point $\left(-4\sqrt{\frac{2}{5}}, 3\right)$, then $a^2 + b^2$ is equal to :

(A) 29 (B) 31

(C) 32 (D) 34

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad a > b$

$$e^2 = 1 - \frac{b^2}{a^2}$$

$$\frac{1}{16} = 1 - \frac{b^2}{a^2}$$

$$\frac{b^2}{a^2} = 1 - \frac{1}{16} = \frac{15}{16} \Rightarrow b^2 = \frac{15}{16}a^2$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{16 \times \frac{2}{5}}{a^2} + \frac{9}{b^2} = 1$$

$$\frac{32}{5a^2} + \frac{9}{b^2} = 1$$

$$\frac{32}{5a^2} + \frac{9}{\frac{15}{16}a^2} = 1$$

$$\frac{80}{5a^2} = 1$$

$16 = a^2$

$b^2 = 15$

14. If two straight lines whose direction cosines are given by the relations l + m – n = 0, $3l^2 + m^2 + cnl = 0$ are parallel, then the positive value of c is :

(A) 6 (B) 4

(C) 3 (D) 2

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. l + m – n = 0

$3l^2 + m^2 + cl\,(l + m) = 0$

n = l + m

$3l^2 + m^2 + cl^2 + clm = 0$

$(3 + c)\, l^2 + clm + m^2 = 0$

$(3+c)\left(\frac{l}{m}\right)^2 + c\left(\frac{l}{m}\right) + 1 = 0$ …..(1)

$\because$ lies are parallel.

Roots of (1) must be equal

$\Rightarrow D = 0$

$c^2 - 4(3 + c) = 0$

$c^2 - 4c - 12 = 0$

$(c - 6)(c + 2) = 0$

c = 6 or c = - 2

+ve value of c = 6

15. Let $\vec{a} = \hat{i} + \hat{j} - \hat{k}$ and $\vec{c} = 2\hat{i} - 3\hat{j} + 2\hat{k}$. Then the number of vectors $\vec{b}$ such that $\vec{b} \times \vec{c} = \vec{a}$ and $|\vec{b}| \in \{1, 2, ..., 10\}$ is :

(A) 0 (B) 1

(C) 2 (D) 3

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\vec{a} = i + j - k$

$\vec{c} = 2i - 3j + 2k$

$\vec{b} \times \vec{c} = \vec{a}$

$|\vec{b}| \in \{1, 2.......10\}$

$\because \vec{b} \times \vec{c} = \vec{a}$

$\Rightarrow \vec{a}$ is perpendicular to $\vec{b}$ as well as $\vec{a}$ is perpendicular to $\vec{c}$

Now $\vec{a}.\vec{c} = 2 - 3 - 2 = -3 \neq 0$

This $\vec{b} \times \vec{c} = \vec{a}$ is not possible.

No. of vectors $\vec{b} = 0$

16. Five numbers x_1, x_2, x_3, x_4, x_5 are randomly selected from the numbers 1, 2, 3,……, 18 and are arranged in the increasing order ($x_1 < x_2 < x_3 < x_4 < x_5$). The probability that $x_2 = 7$ and $x_4 = 11$ is :

(A) $\frac{1}{136}$ (B) $\frac{1}{72}$

(C) $\frac{1}{68}$ (D) $\frac{1}{34}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. No. of ways to select and arrange x_1, x_2, x_3, x_4, x_5 from 1, 2, 3……………….18

$n(s) = {}^{18}C_5$

$x_1 \quad (x_2) \quad x_3 \quad (x_4) \quad x_5$

$\quad\quad 7 \quad\quad\quad 11$

$n(E) = {}^6C_1 \times {}^3C_1 \times {}^7C_1$

$P(E) = \frac{6 \times 3 \times 7}{{}^{18}C_5}$

$\frac{1}{17 \times 4} = \frac{1}{68}$

17. Let X be a random variable having binomial distribution B(7, p). If P(X = 3) = 5P(X = 4), then the sum of the mean and the variance of X is :

(A) $\frac{105}{16}$ (B) $\frac{7}{16}$

(C) $\frac{77}{36}$ (D) $\frac{49}{16}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. B (7, p)

n = 7 p = p

given

$P(x = 3) = 5P(x = 4)$

$${}^{7}C_{3}\times p^{3}(1-p)^{4}=5.{}^{7}C_{4}p^{4}(1-p)^{3}$$

$$\frac{{}^{7}C_{3}}{5\times {}^{7}C_{4}}=\frac{p}{1-p}$$

1 – p = 5p

6p = 1

$$p=\frac{1}{6}\Rightarrow q=\frac{5}{6}$$

n = 7

Mean = np $=7\times\frac{1}{6}=\frac{7}{6}$

Var = npq $=7\times\frac{1}{6}\times\frac{5}{6}=\frac{35}{36}$

Sum

$$=\frac{7}{6}+\frac{35}{36}$$

$$=\frac{42+35}{36}$$

$$=\frac{77}{36}$$

18. The value of $\cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)$ is equal to :

(A) -1 (B) $-\frac{1}{2}$

(C) $-\frac{1}{3}$ (D) $-\frac{1}{4}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}$

$$=\frac{\sin\left(3\times\frac{\pi}{7}\right)}{\sin\frac{\pi}{7}}\times\cos\left(\frac{\frac{2\pi}{7}+\frac{6\pi}{7}}{2}\right)$$

$$=\frac{2\sin\left(\frac{3\pi}{7}\right)}{2\sin\frac{\pi}{7}}\times\cos\left(\frac{4\pi}{7}\right)$$

$$=\frac{\sin\left(\frac{7\pi}{7}\right)+\sin\left(\frac{-\pi}{7}\right)}{2\sin\frac{\pi}{7}}$$

$$=\frac{-\sin\frac{\pi}{7}}{2\sin\frac{\pi}{7}}$$

$$=-\frac{1}{2}$$

19. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)+\cos^{-1}\left(\cos\frac{7\pi}{6}\right)+\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ is equal to :

(A) $\frac{11\pi}{12}$ (B) $\frac{17\pi}{12}$

(C) $\frac{31\pi}{12}$ (D) $-\frac{3\pi}{4}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)+\cos^{-1}\left(\cos\frac{7\pi}{6}\right)+\tan^{-1}\tan\left(\frac{3\pi}{4}\right)$

$$\sin^{-1}\sin\left(\frac{2\pi}{3}\right)=\pi-\frac{2\pi}{3}=\frac{\pi}{3}$$

$$\cos^{-1}\left(\cos\frac{2\pi}{6}\right)=2\pi-\frac{7\pi}{6}=\frac{5\pi}{6}$$

$$\tan^{-1}\tan\left(\frac{3\pi}{4}\right)=\frac{3\pi}{4}-\pi=\frac{-\pi}{4}$$

$$\sin^{-1}\left(\sin\frac{2\pi}{3}\right)+\cos^{-1}\cos\frac{7\pi}{6}+\tan^{-1}\tan\frac{3\pi}{4}$$

$$=\frac{11\pi}{12}$$

20. The Boolean expression $(\sim(p \wedge q)) \vee q$ is equivalent to :

(A) $q \rightarrow (p \wedge q)$ (B) $p \rightarrow q$

(C) $p \rightarrow (p \rightarrow q)$ (D) $p \rightarrow (p \vee q)$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. (~(p^q)) ∨ q

= (~p ∨ ~q) ∨ q

= ~p ∨ ~ q ∨ q

= ~ p ∨ t

= this statement is a tautology option D

p ⇒ (p ∨ q) is also a tautology.

OR

p	q	P^q	~(p^q)	~(p^q) ∨ q	p ∨ q	$p \rightarrow (p \vee q)$
T	T	T	F	T	T	T
T	F	F	T	T	T	T
F	T	F	T	T	T	T
F	F	F	T	T	F	T

SECTION-B

1. Let $f : R \rightarrow R$ be a function defined $f(x)=\frac{2e^{2x}}{e^{2x}+e}$.

Then $f\left(\frac{1}{100}\right)+f\left(\frac{2}{100}\right)+f\left(\frac{3}{100}\right)+.....+f\left(\frac{99}{100}\right)$ is equal to____________.

Official Ans. by NTA (99)

Allen Ans. (99)

Sol.

$$f(x)+f(1-x)=\frac{2e^{2x}}{e^{2x}+e}+\frac{2e^{2-2x}}{e^{2-ex}+e}=\left[\frac{e^{2x}}{e^{2x}+e}+\frac{e^{2}}{e^{2}+e^{2x+1}}\right]$$

$$=2\left[\frac{e^{2x-1}}{e^{2x-1}+1}+\frac{1}{1+e^{2x-1}}\right]=2$$

$$f\left(\frac{1}{100}\right)+f\left(\frac{2}{100}\right)+f\left(\frac{3}{100}\right)+.....+f\left(\frac{99}{100}\right)$$

$$=\left\{f\left(\frac{1}{100}\right)+f\left(\frac{99}{100}\right)\right\}+\left\{f\left(\frac{2}{100}\right)+f\left(\frac{98}{100}\right)\right\}+.....+f\left\{\left(\frac{49}{100}\right)+f\left(\frac{51}{100}\right)\right\}+f\left(\frac{1}{2}\right)$$

$$=(2+2+2+----49\,times)+\frac{2e}{e+e}$$

$$=98+1=99$$

2. If the sum of all the roots of the equation $e^{2x}-11e^{x}-45e^{-x}+\frac{81}{2}=0$ is $\log_e$ P, then p is equal to _______.

Official Ans. by NTA (45)

Allen Ans. (45)

Sol. $e^{2x}-11e^{x}-45e^{-x}+\frac{81}{2}=0\,]$

$(e^{x})^{3}-11(e^{x})^{2}-45+\frac{81e^{x}}{2}=0\,]$

$e^{x}=t$

$2t^{3}-22t^{2}+81t-90=0$

$t_1t_2t_3=45$

$e^{x_1}.e^{x_2}.e^{x_3}=45$

$e^{x_1+x_2+x_3}=45$

$\log_e e^{x_1+x_2+x_3}=\log_e 45$

$x_1+x_2+x_3=\log_e 45$

$\log_e P=\log_e 45$

P = 45

3. The positive value of the determinant of the matrix A, whose $Adj\left(Adj\left(A\right)\right)=\begin{pmatrix}14 & 28 & -14\\ -14 & 14 & 28\\ 28 & -14 & 14\end{pmatrix}$, is ___________.

Official Ans. by NTA (14)

Allen Ans. (14)

Sol. $Adj\left(AdjA\right)=\begin{bmatrix}14 & 18 & -14\\ -14 & 14 & 28\\ 28 & -14 & 14\end{bmatrix}$

$$\left|Adj\left(AdjA\right)\right|=\begin{vmatrix}14 & 28 & -14\\ -14 & 14 & 28\\ 28 & -14 & 14\end{vmatrix}=14\times14\times14\begin{vmatrix}1 & 2 & -1\\ -1 & 1 & 2\\ 2 & -1 & 1\end{vmatrix}$$

$$=(14)^3\left[3-2(-5)-1(-1)\right]=(14)^3[14]=(14)^4$$

$$|A|^4=(14)^4\Rightarrow|A|=14$$

4. The number of ways, 16 identical cubes, of which 11 are blue and rest are red, can be placed in a row so that between any two red cubes there should be at least 2 blue cubes, is _______.

Official Ans. by NTA (56)

Allen Ans. (56)

Sol. 16 cubes < 11 Blue, 5 Red

$$x_1+x_2+x_3+x_4+x_5+x_6=11$$

$$x_1,x_6\ge0,\qquad x_2,x_3,x_4,x_5\ge2$$

$$x_2=t_1+2$$

$$x_3=t_3+2$$

$$x_4=t_4+2$$

$$x_5=t_5+2$$

$$x_1,t_2,t_3,t_4,t_5,x_6\ge0$$

No. of solutions $={}^{6+3-1}C_3={}^8C_3=56$

5. If the coefficient of x^{10} in the binomial expansion of $\left(\frac{\sqrt{x}}{5^{\frac{1}{4}}}+\frac{\sqrt{5}}{x^{\frac{1}{3}}}\right)^{60}$ is $5^k l$, where l, k ∈ N and l is co-prime to 5, then k is equal to _____.

Official Ans. by NTA (5)

Allen Ans. (5)

Sol. $\left(\frac{\sqrt{x}}{5^{1/4}}+\frac{\sqrt{5}}{x^{1/3}}\right)^{60}$

$$T_{r+1}={}^{60}C_r\left(\frac{x^{1/2}}{5^{1/4}}\right)^{60-r}\left(\frac{5^{1/2}}{x^{1/3}}\right)^r$$

$$={}^{60}C_r\,5\frac{3r-60}{4}.x\frac{180-5r}{6}$$

$$\frac{180-5r}{6}=10\Rightarrow r=24$$

Coeff. of $x^{10}={}^{60}C_{24}5^3=\frac{\underline{|60}}{\underline{|24}\,\underline{|36}}5^3$

Powers of 5 in $={}^{60}C_{24}.5^3=\frac{5^{14}}{5^4\times5^8}\times5^3=5^5$

6. Let

$A_1=\left\{(x,y):|x|\le y^2,|x|+2y\le8\right\}$ and

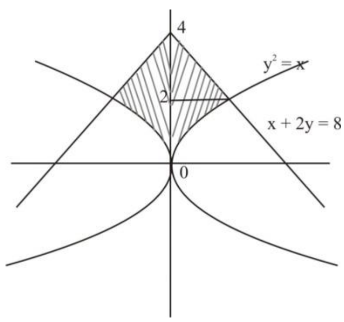
$A_2=\left\{(x,y):|x|+|y|\le k\right\}$. If 27 (Area A_1) = 5 (Area A_2), then k is equal to :

Official Ans. by NTA (6)

Allen Ans. (6)

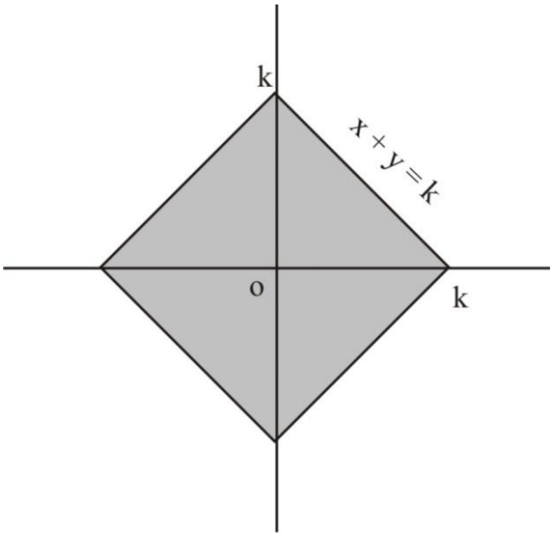
Sol. $A_1=\left\{(x,y):|x|\le y^2,|x|+2y\le8\right\}$ and

$A_2=\left\{(x,y):|x|+|y|\le k\right\}$.



$$area(A_1) = 2\left[\int_0^2 y^2 dy + \int_2^4 (8-2y)dy\right]$$

$$= 2\left[\left(\frac{y^3}{3}\right)_0^2 + \left(8y - y^2\right)_2^4\right]$$



$$area(A_1) = 2 \times \frac{20}{3} = \frac{40}{3}$$

Area $(A_2) = 4 \times \frac{1}{2}k^2$

Area $(A_2) = 2k^2$

Now

27 (Area A_1) = 5 (Area A_2)

$9 \times 4 = k^2$

k = 6

7. If the sum of the first ten terms of the series $\frac{1}{5} + \frac{2}{65} + \frac{3}{325} + \frac{4}{1025} + \frac{5}{2501} + ...$ is $\frac{m}{n}$, where m and n are co-prime numbers, then m + n is equal to ___________.

Official Ans. by NTA (276)

Allen Ans. (276)

Sol. $\frac{1}{5} + \frac{2}{65} + \frac{3}{325} + \frac{4}{1025} + \frac{5}{2501} + ...$

$$T_n = \frac{n}{4n^4 + 1}$$

$$= \frac{n}{(2n^2+1)^2 - (2n)^2} = \frac{n}{(2n^2+2n+1)(2n^2-2n+1)}$$

$$= \frac{1}{4}\left[\frac{1}{2n^2-2n+1} - \frac{1}{2n^2+2n+1}\right]$$

$$S_{10} = \sum_{n=1}^{10} T_n = \frac{1}{4}\left[\frac{1}{1} - \frac{1}{5} + \frac{1}{5} - \frac{1}{13} + ...\frac{1}{200+20+1}\right]$$

$$= \frac{1}{4}\left[1 - \frac{1}{221}\right] = \frac{1}{4} \times \frac{220}{221} - \frac{55}{221} = \frac{m}{n}$$

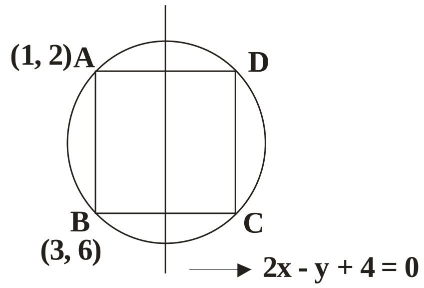
m + n = 55 + 221 = 276

8. A rectangle R with end points of the one of its dies as (1, 2) and (3, 6) is inscribed in a circle. If the equation of a diameter of the circle is 2x – y + 4 = 0, then the area of R is _________.

Official Ans. by NTA (16)

Allen Ans. (16)

Sol.



Eq. of line AB

y = 2x

Slope of AB = 2

Slope of given diameter = 2

So the diameter is parallel to AB

Distance between diameter and line AB

$$= \left(\frac{4}{\sqrt{2^2+12}}\right) = \frac{4}{\sqrt{5}}$$

Thus BC = $2 \times \frac{4}{\sqrt{5}} = \frac{8}{\sqrt{5}}$

$$AB = \sqrt{(1-3)^2 + (2-6)^2} = \sqrt{20} = 2\sqrt{5}$$

Area = AB × BC = $\frac{8}{\sqrt{5}} \times 2\sqrt{5} = 16$ Ans.

9. A circle of radius 2 unit passes through the vertex and the focus of the parabola $y^2 = 2x$ and touches the parabola $y=\left(x-\frac{1}{4}\right)^2+\alpha$, where $\alpha > 0$. Then $(4\alpha - 8)^2$ is equal to ___________.

Official Ans. by NTA (63)

Allen Ans. (63)

Sol. Vertex and focus of parabola $y^2 = 2x$

are V (0, 0) and $S\left(\frac{1}{2},0\right)$ resp.

Let equation of circle be

$(x - h)^2 + (y - k)^2 = 4$

$\because$ Circle passes through (0, 0)

$\Rightarrow h^2 + k^2 = 4$ ……(1)

$\because$ Circle passes through $\left(\frac{1}{2},0\right)$

$$\left(\frac{1}{2}-h\right)^2+k^2=4$$

$$\Rightarrow h^2+k^2-h=\frac{15}{4} \quad \text{……(2)}$$

On solving (1) and (2)

$$4-h=\frac{15}{4}$$

$$h=4-\frac{15}{4}=\frac{1}{4}$$

$$k=+\frac{\sqrt{63}}{4}$$

$k=-\frac{\sqrt{63}}{4}$ is rejected as circle with centre $\left(\frac{1}{4},-\frac{\sqrt{63}}{4}\right)$ can't touch given parabola.

Equation of circle is

$$\left(x-\frac{1}{4}\right)^2+\left(k-\frac{\sqrt{63}}{4}\right)^2=4$$

From figure

$$\alpha=2+\frac{\sqrt{63}}{4}=\frac{8+\sqrt{63}}{4}$$

$$4\alpha-8=\sqrt{63}$$

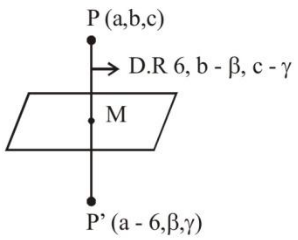
$$(4\alpha-8)^2=63$$

10. Let the mirror image of the point (a, b, c) with respect to the plane $3x - 4y + 12z + 19 = 0$ be (a- 6, β, γ). If a + b + c = 5, then 7 β - 9 γ is equal to ____________.

Official Ans. by NTA (137)

Allen Ans. (137)

Sol.



$$M=\left(a-3,\frac{\beta+b}{2},\frac{\gamma+c}{2}\right)$$

Since M lies on $3x+4y+12z+19=0$

$$\Rightarrow 6a-4b+12c-4\beta+12\gamma+20=0 \quad \text{….(1)}$$

Since PP' is parallel to normal of the plane then

$$\frac{6}{3}=\frac{b-\beta}{-4}=\frac{c-\gamma}{12}$$

$$\Rightarrow \beta=b+8, \quad \gamma=c-24$$

$$a+b+c=5 \Rightarrow a+\beta-8+\gamma+24=5$$

$$\Rightarrow a=-\beta-\gamma-11$$

Now putting these values in (1) we get

$$6(-\beta-\gamma-11)-4(\beta-8)+12(\gamma+24)-4\beta+12\gamma+20=0$$

$$\Rightarrow 7\beta-9\gamma=170-33=137$$