FINAL JEE–MAIN EXAMINATION – JULY, 2022

(Held On Wednesday 27th July, 2022) **TIME : 3 : 00 PM to 6 : 00 PM**

MATHEMATICS | TEST PAPER WITH SOLUTION

SECTION-A

1. The domain of the function

$$f(x)=\sin^{-1}[2x^2-3]+\log_2\left(\log_{\frac{1}{2}}\left(x^2-5x+5\right)\right),$$

where [t] is the greatest integer function, is :

(A) $\left(-\sqrt{\frac{5}{2}},\frac{5-\sqrt{5}}{2}\right)$ (B) $\left(\frac{5-\sqrt{5}}{2},\frac{5+\sqrt{5}}{2}\right)$

(C) $\left(1,\frac{5-\sqrt{5}}{2}\right)$ (D) $\left[1,\frac{5+\sqrt{5}}{2}\right)$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $f(x)=\sin^{-1}[2x^2-3]+\log_2\left(\log_{\frac{1}{2}}\left(x^2-5x+5\right)\right)$

$P_1: -1\le\left[2x^2-3\right]<1$

$\Rightarrow -1\le 2x^2-3<2$

$\Rightarrow 2<2x^2<5$

$\Rightarrow 1<x^2<\frac{5}{2}$

$\Rightarrow P_1:\ x\in\left(-\sqrt{\frac{5}{2}},-1\right)\cup\left(1,\sqrt{\frac{5}{2}}\right)$

$P_2: x^2-5x+5>0$

$\Rightarrow \left(x-\left(\frac{5-\sqrt{5}}{2}\right)\right)\left(x-\left(\frac{5+\sqrt{5}}{2}\right)\right)>0$

$P_3:\ \log_{\frac{1}{2}}\left(x^2-5x+5\right)>0$

$\Rightarrow x^2-5x-5<1$

$\Rightarrow x^2-5x+4<0$

$\Rightarrow P_3:\ x\in(1,4)$

So, $P_1\cap P_2\cap P_3=\left(1,\frac{5-\sqrt{5}}{2}\right)$

2. Let S be the set of all (α,β), $\pi<\alpha,\beta<2\pi$, for which the complex number $\frac{1-i\sin\alpha}{1+2i\sin\alpha}$ is purely imaginary and $\frac{1+i\cos\beta}{1-2i\cos\beta}$ is purely real. Let $Z_{\alpha\beta}=\sin 2\alpha+i\cos 2\beta,(\alpha,\beta)\in S$.

Then $\sum_{(\alpha,\beta)\in S}\left(iZ_{\alpha\beta}+\frac{1}{i\bar{Z}_{\alpha\beta}}\right)$ is equal to :

(A) 3 (B) 3 i

(C) 1 (D) 2 – i

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $\pi<\alpha,\beta<2\pi$

$\frac{1-i\sin\alpha}{1+i(2\sin\alpha)}$ = Purely imaginary

$\Rightarrow \frac{(1-i\sin\alpha)\ (1-i(2\sin\alpha))}{1+4\sin^2\alpha}$ = Purely imaginary

$\Rightarrow \frac{1-2\sin^2\alpha}{1+4\sin^2\alpha}=0$

$\Rightarrow \sin^2\alpha=\frac{1}{2}$

$\Rightarrow \alpha=\left\{\frac{5\pi}{4},\frac{7\pi}{4}\right\}$

& $\frac{1+i\cos\beta}{1+i(-2\cos\beta)}$ = Purely real

$\Rightarrow \frac{(1+i\cos\beta)\ (1+2i\cos\beta)}{1+4\cos^2\beta}$ = Purely real

$\Rightarrow 3\cos\beta=0$

$\Rightarrow \boxed{\beta=\frac{3\pi}{2}}$

$\Rightarrow Z_{\alpha\beta}=\sin\frac{5\pi}{2}+i\cos 3\pi=1-i$

or

$Z_{\alpha\beta}=\sin\frac{7\pi}{2}+i\cos 3\pi=-1-i$

Required value = $\left[i(1-i)+\frac{1}{i(1+i)}\right]+\left[i(-1-i)+\frac{1}{i(-1+i)}\right]$

$=i(-2i)+\frac{1}{i}\frac{2i}{(-2)}\Rightarrow 2-1=1$

3. If α, β are the roots of the equation

$$x^2-\left(5+3^{\sqrt{\log_3 5}}-5^{\sqrt{\log_5 3}}\right)+3\left(3^{(\log_3 5)^{\frac{1}{3}}}-5^{(\log_5 3)^{\frac{2}{3}}}-1\right)=0$$

then the equation, whose roots are

$\alpha+\dfrac{1}{\beta}$ and $\beta+\dfrac{1}{\alpha}$,

(A) $3x^2-20x-12=0$

(B) $3x^2-10x-4=0$

(C) $3x^2-10x+2=0$

(D) $3x^2-20x+16=0$

Official Ans. by NTA (Drop)

Allen Ans. (Bonus)

Sol. Bonus because 'x' is missing the correct will be,

$$x^2-\left(5+3^{\sqrt{\log_3 5}}-5^{\sqrt{\log_5 3}}\right)x+3\left(3^{(\log_3 5)^{\frac{1}{3}}}-5^{(\log_5 3)^{\frac{2}{3}}}-1\right)=0$$

$$3^{\sqrt{\log_3 5}}=3^{\sqrt{\log_3 5}.\sqrt{\log_3 5}.\sqrt{\log_5 3}}=3^{\log_3 5.\sqrt{\log_5 3}}$$

$$=\left(3^{\log_3 5}\right)^{\sqrt{\log_5 3}}=5^{\sqrt{\log_5 3}}$$

$$3^{\sqrt[3]{\log_3 5}}=3^{\log_3 5.\sqrt[3]{(\log_5 3)^2}}=\left(3^{\log_3 5}\right)^{(\log_5 3)^{2/3}}$$

$$=5^{(\log_5 3)^{2/3}}$$

So, equation is $x^2-5x-3=0$ and roots are α & β

$\{\alpha+\beta=5;\ \alpha\beta=-3\}$

New roots are $\alpha+\dfrac{1}{\beta}$ & $\beta+\dfrac{1}{\alpha}$

i.e., $\dfrac{\alpha\beta+1}{\beta}$ & $\dfrac{\alpha\beta+1}{\alpha}$ i.e., $\dfrac{-2}{\beta}$ & $\dfrac{-2}{\alpha}$

Let $\dfrac{-2}{\alpha}=t\ \Rightarrow\ \alpha=\dfrac{-2}{t}$

As $\alpha^2-5\alpha-3=0$

$$\Rightarrow\left(\frac{-2}{t}\right)^2-5\left(\frac{-2}{t}\right)-3=0$$

$$\Rightarrow\frac{4}{t^2}+\frac{10}{t}-3=0$$

$\Rightarrow 4+10t-3t^2=0$

$\Rightarrow 3t^2-10t-4=0$

i.e., $3x^2-10x-4=0$

4. Let $A=\begin{pmatrix}4 & -2\\ \alpha & \beta\end{pmatrix}$

If $A^2+\gamma A+18I=O$, then det (A) is equal to ________.

(A) –18 (B) 18 (C) –50 (D) 50

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. The characteristic equation for A is $|A-\lambda I|=0$

$$\Rightarrow\begin{vmatrix}4-\lambda & -2\\ \alpha & \beta-\lambda\end{vmatrix}=0$$

$$\Rightarrow(4-\lambda)(\beta-\lambda)+2\alpha=0$$

$$\Rightarrow\lambda^2-(\beta+4)\lambda+4\beta+2\alpha=0$$

Put $\lambda=A$

$$A^2-(\beta+4)A+(4\beta+2\alpha)\,I=0$$

On comparison

$-9(\beta+4)=\gamma$ & $4\beta+2\alpha=18$

and $|A|=4\beta+2\alpha=18$

5. If for $p\neq q\neq 0$, then function

$f(x)=\dfrac{\sqrt[7]{p(729+x)}-3}{\sqrt[3]{729+qx}-9}$ is continuous at x = 0, then:

(A) $7pq\,f(0)-1=0$ (B) $63q\,f(0)-p^2=0$

(C) $21q\,f(0)-p^2=0$ (D) $7pq\,f(0)-9=0$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $f(0)=\lim\limits_{x\to0}f(x)$

Limit should be $\dfrac{0}{0}$ form

So, $\sqrt[7]{p.729}-3=0\Rightarrow p.3^6=3^7\Rightarrow p=3$

$$\text{Now, } f(0)=\lim_{x\to0}\frac{\sqrt[7]{3(3^6+x)}-3}{\sqrt[3]{3^6+qx}-9}$$

$$=\lim_{x\to0}\frac{3\left[\left(1+\frac{x}{3^6}\right)^{1/7}-1\right]}{9\left[\left(1+\frac{qx}{3^6}\right)^{1/3}-1\right]}=\frac{3}{9}\times\frac{\frac{1}{7.3^6}}{\frac{q}{3.3^6}}$$

$$\Rightarrow f(0)=\frac{1}{3}\times\frac{3}{7q}=\frac{1}{7q}$$

$\Rightarrow 7qf(0)-1=0$

$\Rightarrow 7.p^2.qf(0)-p^2=0$ (for option)

$\Rightarrow 63qf(0)-p^2=0$

6. Let $f(x) = 2 + |x| - |x - 1| + |x + 1|$, $x \in \mathbf{R}$. Consider

$$(S1): f'\left(-\frac{3}{2}\right)+f'\left(-\frac{1}{2}\right)+f'\left(\frac{1}{2}\right)+f'\left(\frac{3}{2}\right)=2$$

$$(S2): \int_{-2}^{2} f(x)dx = 12$$

Then,

(A) both (S1) and (S2) are correct

(B) both (S1) and (S2) are wrong

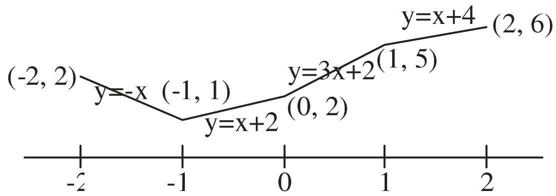
(C) only (S1) is correct

(D) only (S2) is correct

Official Ans. by NTA (D)

Allen Ans. (D)

Sol.



$$(S1): f'\left(-\frac{3}{2}\right)+f'\left(-\frac{1}{2}\right)+f'\left(\frac{1}{2}\right)+f'\left(\frac{3}{2}\right)=4$$

$$(S2): \int_{-2}^{2} f(x)dx = 12$$

$\therefore$ (D)

7. Let the sum of an infinite G.P., whose first term is a and the common ratio is r, be 5. Let the sum of its first five terms be $\frac{98}{25}$. Then the sum of the first 21 terms of an AP, whose first term is 10ar, n^{th} term is a_n and the common difference is $10ar^2$, is equal to :

(A) $21\, a_{11}$ (B) $22\, a_{11}$

(C) $15\, a_{16}$ (D) $14\, a_{16}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $S_{21} = \frac{21}{2}[20ar + 20.10ar^2]$

$= 21\,[10\,ar + 100\,ar^2]$

$= 21.\, a_{11}$

8. The area of the region enclosed by

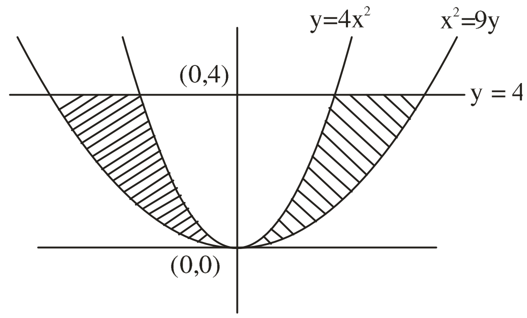
$y \le 4x^2, x^2 \le 9y$ and $y \le 4$, is equal to :

(A) $\frac{40}{3}$ (B) $\frac{56}{3}$ (C) $\frac{112}{3}$ (D) $\frac{80}{3}$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol.



$$\Delta = 2\cdot\int_0^4\left(3\sqrt{y}-\frac{\sqrt{y}}{2}\right)dy$$

$$= 2\cdot\int_0^4 \frac{5}{2}\sqrt{y}\,dy = \frac{80}{3}$$

9. $\displaystyle\int_0^2\left(\left|2x^2-3x\right|+\left[x-\frac{1}{2}\right]\right)dx$,

where [t] is the greatest integer function, is equal to:

(A) $\frac{7}{6}$ (B) $\frac{19}{12}$ (C) $\frac{31}{12}$ (D) $\frac{3}{2}$

Official Ans. by NTA (B)

Allen Ans. (B)

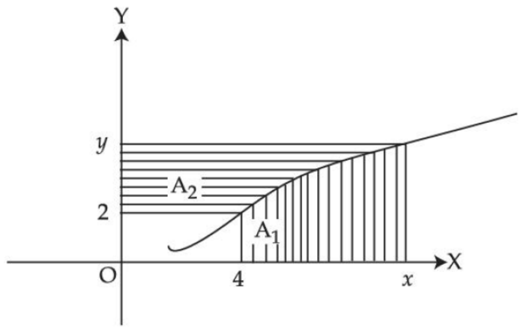
Sol. $\displaystyle\int_0^2\left|2x^2-3x\right|dx$

$$= \int_0^{\frac{3}{2}}(3x-2x^2)dx + \int_{\frac{3}{2}}^{2}(2x^2-3x)dx = \frac{19}{12}.$$

$$\int_0^2\left[x-\frac{1}{2}\right]dx = \int_{-\frac{1}{2}}^{\frac{3}{2}}[t]dt$$

$$= \int_{-\frac{1}{2}}^{0}(-1)dt + \int_0^1 0\cdot dt + \int_1^{\frac{3}{2}} 1\cdot dt = 0.$$

10. Consider a curve $y = y(x)$ in the first quadrant as shown in the figure. Let the area A_1 is twice the area A_2. Then the normal to the curve perpendicular to the line $2x - 12y = 15$ does **NOT** pass through the point.



(1) (6, 21) (2) (8, 9)

(3) (10, –4) (4) (12, –15)

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. Given that $A_1 = 2A_2$

from the graph $A_1 + A_2 = xy - 8$

$$\Rightarrow \frac{3}{2}A_1 = xy - 8$$

$$\Rightarrow A_1 = \frac{2}{3}xy - \frac{16}{3}$$

$$\Rightarrow \int_4^x f(x)dx = \frac{2}{3}xy - \frac{16}{3}$$

$$\Rightarrow f(x) = \frac{2}{3}\left(x\frac{dy}{dx} + y\right)$$

$$\Rightarrow \frac{2}{3}x\frac{dy}{dx} = \frac{y}{3}$$

$$\Rightarrow 2\int\frac{dy}{y} = \int\frac{dx}{x}$$

$$\Rightarrow 2\ell ny = \ell nx + \ell nc$$

$$\Rightarrow y^2 = cx$$

As $f(4) = 2 \Rightarrow c = 1$

so $y^2 = x$

slope of normal = –6

$$y = -6(x) - \frac{1}{2}(-6) - \frac{1}{4}(-6)^3$$

$\Rightarrow y = -6x + 3 + 54$

$\Rightarrow y + 6x = 57$

Now check options and (C) will not satisfy.

11. The equations of the sides AB, BC and CA of a triangle ABC are $2x + y = 0$, $x + py = 39$ and $x - y = 3$ respectively and P(2, 3) is its circumcentre. Then which of the following is NOT true :

(A) $(AC)^2 = 9p$

(B) $(AC)^2 + p^2 = 136$

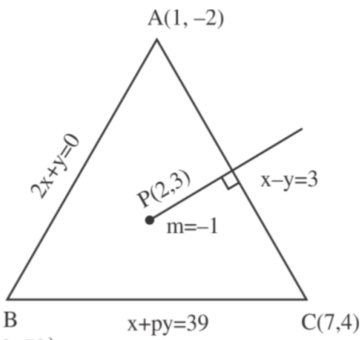
(C) $32 < \text{area}(\Delta ABC) < 36$

(D) $34 < \text{area}(\Delta ABC) < 38$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol.



$\left(\frac{-39}{15}, \frac{78}{15}\right)$

Perpendicular bisector of AB

$x + y = 5$

Take image of A

$$\frac{x-1}{1} = \frac{y+2}{1} = \frac{-2(-6)}{2} = 6$$

(7, 4)

$7 + 4p = 39$

$p = 8$

solving $x + 8y = 39$ and $y = -2x$

$x = \frac{-39}{15}$ $y = \frac{78}{15}$

$AC^2 = 72 = 9p$

$AC^2 + p^2 = 72 + 64 = 136$

$$\Delta ABC = \frac{1}{2}\begin{vmatrix} 1 & -2 & 1 \\ 7 & 4 & 1 \\ \frac{-39}{15} & \frac{78}{15} & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[4 - \frac{78}{15} + 2\left(7 + \frac{39}{15}\right) + 7\left(\frac{78}{15}\right) + \frac{4\times 39}{15}\right]$$

$$= \frac{1}{2}\left[18 + 18\times\frac{13}{5}\right]$$

$$= 9\left[\frac{18}{5}\right] = \frac{162}{5} = 32.4$$

Ans. (D)

12. A circle C_1 passes through the origin O and has diameter 4 on the positive x-axis. The line $y = 2x$ gives a chord OA of a circle C_1. Let C_2 be the circle with OA as a diameter. If the tangent to C_2 at the point A meets the x-axis at P and y-axis at Q, then QA : AP is equal to :

(A) 1 : 4 (B) 1 : 5

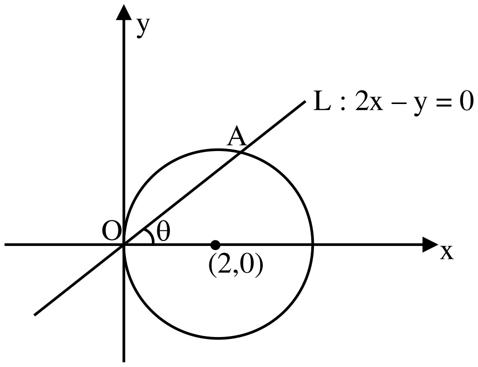
(C) 2 : 5 (D) 1 : 3

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $C_1 : x^2 + y^2 - 4x = 0$

$\tan\theta = 2$



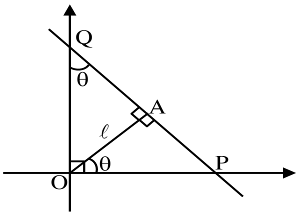
C_2 is a circle with OA as diameter.

So, tangent at A on C_2 is perpendicular to OR

Let $OA = \ell$

$$\therefore \frac{QA}{AP} = \frac{\ell\cot\theta}{\ell\tan\theta}$$

$$= \frac{1}{\tan^2\theta} = \frac{1}{4}$$



13. If the length of the latus rectum of a parabola, whose focus is (a, a) and the tangent at its vertex is $x + y = a$, is 16, then $|a|$ is equal to :

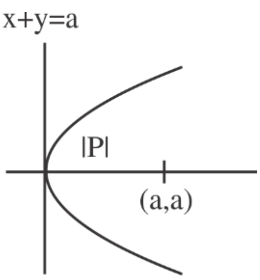
(A) $2\sqrt{2}$ (B) $2\sqrt{3}$

(C) $4\sqrt{2}$ (D) 4

Official Ans. by NTA (C)

Allen Ans. (C)

Sol.



$$|P| = \left|\frac{a}{\sqrt{2}}\right| = \frac{16}{4} = 4$$

$$|a| = 4\sqrt{2}$$

Ans. (C)

14. If the length of the perpendicular drawn from the point P(a, 4, 2), a > 0 on the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z-1}{-1}$ is $2\sqrt{6}$ units and $Q(\alpha_1, \alpha_2, \alpha_3)$ is the image of the point P in this line, then $a + \sum_{i=1}^{3}\alpha_i$ is equal to :

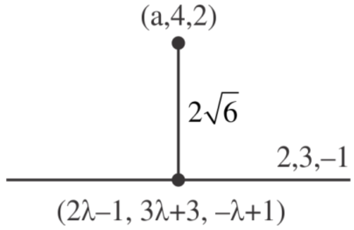
(A) 7 (B) 8

(C) 12 (D) 14

Official Ans. by NTA (B)

Allen Ans. (B)

Sol.



$(2\lambda - 1 - a)2 + (3\lambda - 1)3 + (-\lambda - 1)(-1) = 0$

$\Rightarrow 4\lambda - 2 - 2a + 9\lambda - 3 + \lambda + 1 = 0$

$\Rightarrow 14\lambda - 4 - 2a = 0$

$\Rightarrow 7\lambda - 2 - a = 0$

and,

$(2\lambda - 1 - a)^2 + (3\lambda - 1)^2 + (\lambda + 1)^2 = 24$

$\Rightarrow (5\lambda - 1)^2 + (3\lambda - 1)^2 + (\lambda + 1)^2 = 24$

$\Rightarrow 35\lambda^2 - 14\lambda - 21 = 0$

$\Rightarrow (\lambda - 1)(35\lambda + 21) = 0$

For, $\lambda = 1 \quad \Rightarrow a = 5$

Let $(\alpha_1, \alpha_2, \alpha_3)$ be reflection of point P

$\alpha_1 + 5 = 2 \qquad \alpha_2 + 4 = 12 \qquad \alpha_3 + 2 = 0$

$\alpha_1 = -3 \qquad \alpha_2 = 8 \qquad \alpha_3 = -2$

$a + \alpha_1 + \alpha_2 + \alpha_3 = 8$

15. If the line of intersection of the planes $ax + by = 3$ and $ax + by + cz = 0$, a > 0 makes an angle 30° with the plane $y - z + 2 = 0$, then the direction cosines of the line are :

(A) $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$ (B) $\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0$

(C) $\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}, 0$ (D) $\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0$

Official Ans. by NTA (A or B)

Allen Ans. (A or B or both)

Sol. $\vec{n}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & 0 \\ a & b & c \end{vmatrix}$

$= bc\hat{i}-ac\hat{j}$

Direction ratios of line are **b, –a, 0**

Direction ratios of normal of the plane are **0, 1, – 1**

$$\cos 60^\circ=\left|\frac{-a}{\left|\sqrt{2}\right|\sqrt{b^2+a^2}}\right|=\frac{1}{2}$$

$$\Rightarrow \left|\frac{a}{\sqrt{a^2+b^2}}\right|=\frac{1}{\sqrt{2}}$$

$\Rightarrow b = \pm a$

So, D.R.'s can be (±a, –a, 0)

$\therefore$ D.C.'s can be $\pm\left(\frac{\pm 1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0\right)$

16. Let X have a binomial distribution B(n, p) such that the sum and the product of the mean and variance of X are 24 and 128 respectively. If $P(X>n-3)=\frac{k}{2^n}$, then k is equal to

(A) 528 (B) 529 (C) 629 (D) 630

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. Let α = Mean & β = Variance $(\alpha>\beta)$

So, $\alpha+\beta=24$, $\alpha\beta=128$

$\Rightarrow \alpha=16$ & $\beta=8$

$\Rightarrow np=16$ $\quad npq=8 \Rightarrow q=\frac{1}{2}$

$\therefore \quad p=\frac{1}{2}, n=32$

$p(x > n-3) = \frac{1}{2^n}({}^nC_{n-2} + {}^nC_{n-1} + {}^nC_n)$

$\therefore k = {}^{32}C_{30} + {}^{32}C_{31} + {}^{32}C_{32} = \frac{32\times 31}{2}+32+1$

$= 496 + 33 = 529$

17. A six faced die is biased such that 3 × P(a prime number) = 6 × P(a composite number) = 2 × P(1). Let X be a random variable that counts the number of times one gets a perfect square on some throws of this die. If the die is thrown twice, then the mean of X is :

(A) $\frac{3}{11}$ (B) $\frac{5}{11}$

(C) $\frac{7}{11}$ (D) $\frac{8}{11}$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. Let $\frac{P(\text{a prime number})}{2}=\frac{P(\text{a composite})}{1}=\frac{P(1)}{3}=k$

So, P(a prime number) = 2k,

P(a composite number) = k,

& P(1) = 3k

& $3\times 2k+2\times k+3k=1$

$\Rightarrow \; k=\frac{1}{11}$

P(success) = P(1 or 4) = 3k + k = $\frac{4}{11}$

Number of trials, n = 2

$\therefore$ mean = np = $2\times\frac{4}{11}=\frac{8}{11}$

18. The angle of elevation of the top P of a vertical tower PQ of height 10 from a point A on the horizontal ground is 45°. Let R be a point on AQ and from a point B, vertically above R, the angle of elevation of P is 60°. If ∠BAQ = 30°, AB = d and the area of the trapezium PQRB is α, then the ordered pair (d, α) is :

(A) $\left(10\left(\sqrt{3}-1\right),25\right)$ (B) $\left(10\left(\sqrt{3}-1\right),\frac{25}{2}\right)$

(C) $\left(10\left(\sqrt{3}+1\right),25\right)$ (D) $\left(10\left(\sqrt{3}+1\right),\frac{25}{2}\right)$

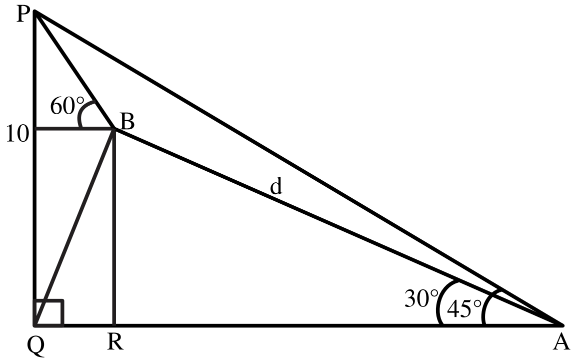
Official Ans. by NTA (A)

Allen Ans. (A)

Sol. QA = 10 $\quad RA = d\cos 30^\circ = \dfrac{\sqrt{3}d}{2}$

$$QR = 10 - \frac{\sqrt{3}d}{2}$$

$$BR = d\sin 30^\circ = \frac{d}{2}$$



$$\tan 60^\circ = \frac{PQ - BR}{QR} = \frac{10 - \frac{d}{2}}{10 - \frac{\sqrt{3}d}{2}}$$

$$\Rightarrow \sqrt{3} = \frac{20 - d}{20 - \sqrt{3}d}$$

$$\Rightarrow 20\sqrt{3} - 3d = 20 - d$$

$$\Rightarrow 2d = 20(\sqrt{3} - 1)$$

$$\Rightarrow d = 10(\sqrt{3} - 1)$$

$$ar(PQRB) = \alpha = \frac{1}{2}(PQ + BR)\cdot QR$$

$$= \frac{1}{2}\left(10 + \frac{d}{2}\right)\cdot\left(10 - \frac{\sqrt{3}d}{2}\right)$$

$$= \frac{1}{2}\left(10 + 5\sqrt{3} - 5\right)\left(10 - 15 + 5\sqrt{3}\right)$$

$$= \frac{1}{2}\left(5\sqrt{3} + 5\right)\left(5\sqrt{3} - 5\right) = \frac{1}{2}(75 - 25) = 25$$

19. Let $S = \left\{\theta \in \left(0, \frac{\pi}{2}\right) : \sum_{m=1}^{9} \sec\left(\theta + (m-1)\frac{\pi}{6}\right)\sec\left(\theta + \frac{m\pi}{6}\right) = -\frac{8}{\sqrt{3}}\right\}$

Then

(A) $S = \left\{\frac{\pi}{12}\right\}$ (B) $S = \left\{\frac{2\pi}{3}\right\}$

(C) $\sum_{\theta \in S} \theta = \frac{\pi}{2}$ (D) $\sum_{\theta \in S} \theta = \frac{3\pi}{4}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. Let $\alpha = \theta + (m-1)\dfrac{\pi}{6}$

& $\beta = \theta + m\dfrac{\pi}{6}$

So, $\beta - \alpha = \dfrac{\pi}{6}$

Here, $\displaystyle\sum_{m=1}^{9} \sec\alpha \cdot \sec\beta = \sum_{m=1}^{9} \frac{1}{\cos\alpha \cdot \cos\beta}$

$$= 2\sum_{m=1}^{9} \frac{\sin(\beta - \alpha)}{\cos\alpha \cdot \cos\beta} = 2\sum_{m=1}^{9} (\tan\beta - \tan\alpha)$$

$$= 2\sum_{m=1}^{9}\left(\tan\left(\theta + m\frac{\pi}{6}\right) - \tan\left(\theta + (m-1)\frac{\pi}{6}\right)\right)$$

$$= 2\left(\tan\left(\theta + \frac{9\pi}{6}\right) - \tan\theta\right) = 2(-\cot\theta - \tan\theta) = -\frac{8}{\sqrt{3}}$$

(Given)

$$\therefore \quad \tan\theta + \cot\theta = \frac{4}{\sqrt{3}}$$

$$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}} \text{ or } \sqrt{3}$$

So, $S = \left\{\dfrac{\pi}{6}, \dfrac{\pi}{3}\right\}$

$$\sum_{\theta \in S} \theta = \frac{\pi}{6} + \frac{\pi}{3} = \frac{\pi}{2}$$

20. If the truth value of the statement $\left(P \wedge (\sim R)\right) \rightarrow \left((\sim R) \wedge Q\right)$ is F, then the truth value of which of the following is F ?

(A) $P \vee Q \rightarrow \sim R$ (B) $R \vee Q \rightarrow \sim P$

(C) $\sim(P \vee Q) \rightarrow \sim R$ (D) $\sim(R \vee Q) \rightarrow \sim P$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $X \Rightarrow Y$ is a false

when X is true and Y is false

So, $P \rightarrow T$, $Q \rightarrow F$, $R \rightarrow F$

(A) $P \vee Q \rightarrow \sim R$ is T

(B) $R \vee Q \rightarrow \sim P$ is T

(C) $\sim(P \vee Q) \rightarrow \sim R$ is T

(D) $\sim(R \vee Q) \rightarrow \sim P$ is F

SECTION-B

1. Consider a matrix $A=\begin{bmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ \beta+\gamma & \gamma+\alpha & \alpha+\beta\end{bmatrix}$,

where α, β, γ are three distinct natural numbers.

If $\dfrac{\det\left(\text{adj}\left(\text{adj}\left(\text{adj}\left(\text{adj}A\right)\right)\right)\right)}{(\alpha-\beta)^{16}(\beta-\gamma)^{16}(\gamma-\alpha)^{16}}=2^{32}\times3^{16}$, then the number of such 3 – tuples (α, β, γ) is ________ .

Official Ans. by NTA (42)

Allen Ans. (42)

Sol. $A=\begin{bmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ \beta+\gamma & \gamma+\alpha & \alpha+\beta\end{bmatrix}$

$R_3 \to R_3 + R_1$

$\Rightarrow |A| = |\alpha+\beta+\gamma|\begin{vmatrix}\alpha & \beta & \gamma\\ \alpha^2 & \beta^2 & \gamma^2\\ 1 & 1 & 1\end{vmatrix}$

$\Rightarrow |A| = (\alpha+\beta+\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

$\because$ $|\text{adj } A| = |A|^{n-1}$

$|\text{adj (adj } A)| = |A|^{(n-1)^2}$

$\left|\text{adj}\left(\text{adj}\left(\text{adj}\left(\text{adj}A\right)\right)\right)\right| = |A|^{(n-1)^4} = |A|^{2^4} = |A|^{16}$

$\therefore (\alpha+\beta+\gamma)^{16} = 2^{32}\,.\,3^{16}$

$\Rightarrow (\alpha+\beta+\gamma)^{16} = (2^2\,.\,3)^{16} = (12)^{16}$

$\Rightarrow \alpha+\beta+\gamma = 12$

$\because \alpha, \beta, \gamma \in N$

$(\alpha-1)+(\beta-1)+(\gamma-1) = 9$

number all tuples (α, β, γ) = ${}^{11}C_2 = 55$

1 case for α = β = γ

& 12 case when any two of these are equal

So, No. of distinct tuples (α, β, γ)

$= 55 - 13 = 42$

2. The number of functions f, from the set

$A=\left\{x\in N : x^2-10x+9\le 0\right\}$ to the set

$B=\left\{n^2 : n\in N\right\}$ such that $f(x)\le(x-3)^2+1$, for every $x\in A$, is __________ .

Official Ans. by NTA (1440)

Allen Ans. (1440)

Sol. $(x^2 - 10x + 9) \le 0 \Rightarrow (x-1)(x-9) \le 0$

$\Rightarrow x \in [1, 9] \Rightarrow A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$f(x) \le (x-3)^2 + 1$

$x = 1 : f(1) \le 5 \Rightarrow 1^2, 2^2$

$x = 2 : f(2) \le 2 \Rightarrow 1^2$

$x = 3 : f(3) \le 1 \Rightarrow 1^2$

$x = 4 : f(4) \le 2 \Rightarrow 1^2$

$x = 5 : f(5) \le 5 \Rightarrow 1^2, 2^2$

$x = 6 : f(6) \le 10 \Rightarrow 1^2, 2^2, 3^2$

$x = 7 : f(7) \le 17 \Rightarrow 1^2, 2^2, 3^2, 4^2$

$x = 8 : f(8) \le 26 \Rightarrow 1^2, 2^2, 3^2, 4^2, 5^2$

$x = 9 : f(9) \le 37 \Rightarrow 1^2, 2^2, 3^2, 4^2, 5^2, 6^2$

Total number of such function

$= 2(6!) = 2(720) = 1440$

3. Let for the 9^{th} term in the binomial expansion of $(3 + 6x)^n$, in the increasing powers of 6x, to be the greatest for $x=\dfrac{3}{2}$, the least value of n is n_0. If k is the ratio of the coefficient of x^6 to the coefficient of x^3, then $k + n_0$ is equal to:

Official Ans. by NTA (24)

Allen Ans. (24)

Sol. $(3+6x)^n = {}^nC_0 3^n + {}^nC_1 3^{n-1}(6x)^1 + \ldots$

$T_{r+1}\ {}^nC_r 3^{n-r}.(6x)r = {}^nC_r 3^{n-r}.6^r.\ x^r$

$= {}^nC_r 3^{n-r}.3^r.2^r.\left(\frac{3}{2}\right)^r = {}^nC_r 3^n.3^r \qquad$ [for $x = \frac{3}{2}$]

T_9 is greatest of $x = \frac{3}{2}$

So, $T_9 > T_{10}$ and $T_9 > T_8$

(concept of numerically greatest term)

Here, $\frac{T_9}{T_{10}} > 1$ and $\frac{T_9}{T_8} > 1$

$\Rightarrow \frac{{}^nC_8 3^n.3^8}{{}^nC_9 3^n.3^9} > 1$ and $\frac{{}^nC_8 3^n.3^8}{{}^nC_7 3^n.3^7} > 1$

and $\frac{{}^nC_8}{{}^nC_7} > \frac{1}{3}$

and $\frac{n-7}{8} > \frac{1}{3}$

$\Rightarrow \frac{29}{3} < n < 11 \Rightarrow n = 10 = n_0$

So, in $(3+6x)^n$ for $n = n_0 = 10$

i.e., in $(3+6x)^{10}$, here $T_{r+1} = {}^{10}C_r 3^{10-r} 6^r x^r$

$T_7 = {}^{10}C_6 3^4.6^6.x^6 = 210.3^{10}.2^6 x^6$

$T_4 = {}^{10}C_3 3^7 6^3 x^3 = 120.3^{10}.2^3 x^3$

Ratio of coefficient of x^6 and coefficient of $x^3 = k$

$\therefore k = \frac{210.3^{10} 2^6}{120.3^{10}.2^3} = \frac{7}{4} \times 2^3 = 14$

So, $k + n_0 = 14 + 10 = 24$

4. $\frac{2^3-1^3}{1\times 7} + \frac{4^3-3^3+2^3-1^3}{2\times 11} + \frac{6^3-5^3+4^3-3^3+2^3-1^3}{3\times 15} + \ldots.. + \frac{30^3-29^3+28^3-27^3+\ldots+2^3-1^3}{15\times 63}$

is equal to _____.

Official Ans. by NTA (120)

Allen Ans. (120)

Sol. $T_n = \frac{2\sum_{r=1}^{n}(2r)^3 - \left(\sum_{r=1}^{2n} r^3\right)}{n(4n+3)}$

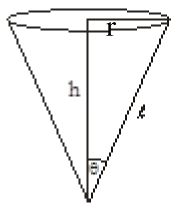
$\Rightarrow T_n = n$

So, $\sum_{n=1}^{15} T_n = 120$

5. A water tank has the shape of a right circular cone with axis vertical and vertex downwards. Its semi-vertical angle is $\tan^{-1}\frac{3}{4}$. Water is poured in it at a constant rate of 6 cubic meter per hour. The rate (in square meter per hour), at which the wet curved surface area of the tank is increasing, when the depth of water in the tank is 4 meters, is ________.

Official Ans. by NTA (5)

Allen Ans. (5)



$\tan\theta = \frac{3}{4} = \frac{r}{h}$

$\frac{dV}{dt} = 6$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3 \tan^2\theta = \frac{9\pi}{48}h^3 = \frac{3\pi}{16}h^3$

$\Rightarrow \frac{dV}{dt} = \frac{3\pi}{16}.3h^2.\frac{dh}{dt} = 6 \Rightarrow \left(\frac{dh}{dt}\right)_{h=4} = \frac{2}{3\pi} m/hr$

Now, $S = \pi r\ell = \frac{15}{16}\pi h^2$

$\Rightarrow \frac{dS}{dt} = \frac{15\pi}{16}.2h\frac{dh}{dt}$

$\Rightarrow \left(\frac{dS}{dt}\right)_{h=4} = 5m^2/hr$

6. For the curve $C : (x^2 + y^2 - 3) + (x^2 - y^2 - 1)^5 = 0$, the value of $3y' - y^3y''$, at the point (α, α), $\alpha > 0$, on C, is equal to ________.

Official Ans. by NTA (16)

Allen Ans. (16)

Sol. (α, α) lies on

$C:\ x^2+y^2-3 + x^2-y^2-1^5 = 0$

Put (α, α), $2\alpha^2-3 + -1^5 = 0$

$\Rightarrow \alpha = \sqrt{2}$

Now, differentiate C

$2x + 2y\cdot y' + 5\left(x^2-y^2-1\right)^4\left(2x-2yy'\right) = 0 \ \ldots (1)$

At $\left(\sqrt{2}, \sqrt{2}\right)$

$$\sqrt{2}+\sqrt{2}y'+5(-1)^4\left(\sqrt{2}-\sqrt{2}y'\right)=0$$

$$\Rightarrow \quad y'=\frac{3}{2} \qquad \ldots.(2)$$

Diff. (1) w.r.t. x

Again, Diff. (1) w.r.t. x

$$1+(y')^2+yy''+20\left(x^2-y^2-1\right)^3(x-yy')^2.2$$
$$+5\left(x^2-y^2-1\right)^4\left(1-(y')^2-yy''\right)=0$$

At $\left(\sqrt{2},\sqrt{2}\right)$ and $y'=\frac{3}{2}$

We have,

$$\left(1+\frac{9}{4}\right)+\sqrt{2}y''-40\left(\sqrt{2}-\sqrt{2}\cdot\frac{3}{2}\right)^2$$
$$+5(1)\left(1-\frac{9}{4}-\sqrt{2}y''\right)=0$$

$$\Rightarrow \quad 4\sqrt{2}y''=-23$$

$$\therefore \quad 3y'-y^3y''=\frac{9}{2}+\frac{23}{2}=16$$

7. Let f(x) = min{[x – 1], [x – 2], …, [x – 10]} where [t] denotes the greatest integer ≤ t. Then $\int_0^{10} f(x)dx+\int_0^{10}(f(x))^2dx+\int_0^{10}|f(x)|dx$ is equal to ____.

Official Ans. by NTA (385)

Allen Ans. (385)

Sol. f(x) = [x] – 10

$$\int_0^{10} f(x)\cdot dx=-10-9-8-\ldots..-1$$

$$=-\frac{10\cdot 11}{2}=-55$$

$$\int_0^{10}(f(x))^2dx=10^2+9^2+8^2+\ldots+1^2$$

$$=\frac{10\cdot 11\cdot 21}{6}=385$$

$$\int_0^{10}|f(x)|=10+9+8+\ldots.+1$$

$$=\frac{10\cdot 11}{2}=55$$

= – 55 + 385 + 55 = 385

8. Let f be a differentiable function satisfying $f(x)=\frac{2}{\sqrt{3}}\int_0^{\sqrt{3}} f\left(\frac{\lambda^2x}{3}\right)d\lambda$, x > 0 and f(1) = $\sqrt{3}$. If y=f(x) passes through the point (α, 6), then α is equal to ______.

Official Ans. by NTA (12)

Allen Ans. (12)

Sol. Let, $\frac{\lambda^2x}{3}=t$

$$\Rightarrow \frac{2\lambda x}{3}d\lambda=dt$$

$$\Rightarrow d\lambda=\frac{3}{2}\cdot\frac{1\sqrt{x}}{x\cdot\sqrt{3}\sqrt{t}}dt$$

$$\Rightarrow d\lambda=\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{x}}\cdot\frac{dt}{\sqrt{t}}$$

So, $f(x)=\frac{1}{\sqrt{x}}\int_0^x\frac{f(t)}{\sqrt{t}}dt$

$$\Rightarrow \sqrt{x}\cdot f'(x)+\frac{f(x)}{2\sqrt{x}}=\frac{f(x)}{\sqrt{x}}$$

$$\Rightarrow \sqrt{x}\cdot f'(x)=\frac{f(x)}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{y}=\frac{dx}{2x}$$

$$\Rightarrow \ln y=\frac{1}{2}\ln x+c\Rightarrow f(x)=\sqrt{x}$$

$$\Rightarrow y=\sqrt{3x} \quad \{\text{as } f(1)=\sqrt{3}\}$$

So, $f(x)=\sqrt{3x}$

Now, f(α) = 6 ⇒ 36 = 3α

⇒ α = 12

9. A common tangent T to the curves $C_1:\frac{x^2}{4}+\frac{y^2}{9}=1$ and $C_2:\frac{x^2}{42}-\frac{y^2}{143}=1$ does not pass through the fourth quadrant. If T touches C_1 at (x_1, y_1) and C_2 at (x_2, y_2), then $|2x_1 + x_2|$ is equal to _______.

Official Ans. by NTA (20)

Allen Ans. (20)

Sol. Let common tangents are

$T_1 : y = mx \pm \sqrt{4m^2 + 9}$

& $T_2 : y = mx \pm \sqrt{42m^2 - 13}$

So, $4m^2 + 9 = 42m^2 - 143$

$\Rightarrow 38m^2 = 152$

$\Rightarrow m = \pm 2$

& $c = \pm 5$

For given tangent not pass through 4th quadrant

$T : y = 2x + 5$

Now, comparing with $\frac{xx_1}{4} + \frac{yy_1}{9} = 1$

We get, $\frac{x_1}{8} = -\frac{1}{5} \Rightarrow x_1 = -\frac{8}{5}$

$\frac{xx_2}{42} - \frac{yy_2}{143} = 1$

$2x - y = -5$ we have

$x_2 = -\frac{84}{5}$

So, $|2x_1 + x_2| = \left|\frac{-100}{5}\right| = 20$

10. Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplanar vectors such that

$\vec{a} \times \vec{b} = 4\vec{c}$, $\vec{b} \times \vec{c} = 9\vec{a}$ and $\vec{c} \times \vec{a} = \alpha\vec{b}, \alpha > 0$.

If $|\vec{a}| + |\vec{b}| + |\vec{c}| = \frac{1}{36}$, then α is equal to ______.

Official Ans. by NTA (Drop)

Allen Ans. (Bonus)

Sol. $\vec{a} \times \vec{b} = 4\vec{c} \Rightarrow \vec{a} \cdot \vec{c} = 0 = \vec{b} \cdot \vec{c}$

$\vec{b} \times \vec{c} = 9\vec{a} \Rightarrow \vec{a} \cdot \vec{b} = 0 = \vec{a} \cdot \vec{c}$

$\therefore \vec{a}, \vec{b}, \vec{c}$ are mutually $\perp$ set of vectors.

$\Rightarrow |\vec{a}||\vec{b}| = 4|\vec{c}|, |\vec{b}||\vec{c}| = 9|\vec{a}|$ & $|\vec{c}||\vec{a}| = \alpha|\vec{b}|$

$\Rightarrow \frac{|\vec{a}|}{|\vec{c}|} = \frac{4|\vec{c}|}{9|\vec{a}|}$

$\Rightarrow \frac{|\vec{c}|}{|\vec{a}|} = \frac{3}{2}$

$\therefore$ If $|a| = \lambda$, $|c| = \frac{3\lambda}{2}$ & $|b| = 6$

Now $|a| + |b| + |c| = \frac{1}{36}$

$\Rightarrow \frac{5}{2}\lambda + 6 = \frac{1}{36}, \lambda = \frac{-43}{18} = |a|$

which gives negative value of λ or $|a|$ which is NOT possible & hence data seems to be wrong.

But if $|\vec{a}| + |\vec{b}| + |\vec{c}| = 36$

$\frac{5}{2}\lambda + 6 = 36$

$\lambda = 12$

$\alpha = \frac{|\vec{c}||\vec{a}|}{|\vec{b}|} = \frac{3 \times 12}{2} \times \frac{12}{6}$

$\alpha = 36$