

FINAL JEE-MAIN EXAMINATION – JULY, 2022

 (Held On Tuesday 28th June, 2022)

TIME : 9 : 00 AM to 12 : 00 PM

MATHEMATICS
TEST PAPER WITH SOLUTION
SECTION-A

1. If

$$\sum_{k=1}^{31} \binom{31}{k} \binom{31}{k-1} - \sum_{k=1}^{30} \binom{30}{k} \binom{30}{k-1} = \frac{\alpha (60!)}{(30!)(31!)},$$

 Where $\alpha \in \mathbb{R}$, then the value of 16α is equal to

- (A) 1411 (B) 1320
 (C) 1615 (D) 1855

Official Ans. by NTA (A)
Allen Ans. (A)

$$\begin{aligned} \text{Sol. } \sum_{R=1}^{31} {}^{31}C_R \cdot {}^{31}C_{R-1} \\ &= {}^{31}C_1 \cdot {}^{31}C_0 + {}^{31}C_2 \cdot {}^{31}C_1 + \dots + {}^{31}C_{31} \cdot {}^{31}C_{30} \\ &= {}^{31}C_0 \cdot {}^{31}C_{30} + {}^{31}C_1 \cdot {}^{31}C_{29} + \dots + {}^{31}C_{30} \cdot {}^{31}C_0 \\ &= {}^{62}C_{30} \cdot \end{aligned}$$

Similarly

$$\begin{aligned} \sum_{R=1}^{30} \binom{30}{R} \binom{30}{R-1} &= {}^{60}C_{29} \\ {}^{62}C_{30} - {}^{60}C_{29} &= \frac{62!}{30!32!} - \frac{60!}{29!31!} \\ &= \frac{60!}{29!31!} \left\{ \frac{62 \cdot 61}{30 \cdot 32} - 1 \right\} \\ &= \frac{60!}{30!31!} \left(\frac{2822}{32} \right) \end{aligned}$$

$$\therefore 16\alpha = 16 \times \frac{2822}{32} = 1411$$

 2. Let a function $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined by

$$f(n) = \begin{cases} 2n, & n = 2, 4, 6, 8, \dots \\ n-1, & n = 3, 7, 11, 15, \dots \\ \frac{n+1}{2}, & n = 1, 5, 9, 13, \dots \end{cases}$$

 then, f is

- (A) one-one but not onto
 (B) onto but not one-one
 (C) neither one-one nor onto
 (D) one-one and onto

Official Ans. by NTA (D)
Allen Ans. (D)

$$\text{Sol. } f(x) = \begin{cases} 4R & ; n = 2R \\ 4R - 2 & ; n = 4R - 1 \\ 2R - 1 & ; n = 4R - 3 \end{cases}$$

 $(R \in \mathbb{N})$
Note that for any element, it will fall into exactly one of these sets.

$$\{y : y = 4R; y \in \mathbb{N}\}$$

$$\{y : y = 4R - 2; y \in \mathbb{N}\}$$

$$\{y : y = 2R - 1; y \in \mathbb{N}\}$$

 Corresponding to that y , we will get exactly one value of n .

 Thus, f is one - one & onto.

3. If the system of linear equations

$$2x + 3y - z = -2$$

$$x + y + z = 4$$

$$x - y + |\lambda|z = 4\lambda - 4$$

 where $\lambda \in \mathbb{R}$, has no solution, then

- (A) $\lambda = 7$ (B) $\lambda = -7$
 (C) $\lambda = 8$ (D) $\lambda^2 = 1$

Official Ans. by NTA (B)
Allen Ans. (B)

$$\text{Sol. } \begin{vmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & |\lambda| \end{vmatrix} = 0$$

$$\Rightarrow |\lambda| = 7 \Rightarrow \lambda = \pm 7 \quad \dots(1)$$

System :

$$2x + 3y - z = -2 \quad \dots(2)$$

$$x + y + z = 4 \quad \dots(3)$$

$$x - y + |\lambda|z = 4\lambda - 4 \quad \dots(4)$$

 Eliminating y from equal (2) & (3) we get

$$x + 4z = 14 \quad \dots(5)$$

$$(3) + (4) \Rightarrow x + \left(\frac{|\lambda|+1}{2} \right) z = 2\lambda \quad \dots(6)$$

 Clearly for $\lambda = -7$, system is inconsistent.

4. Let A be a matrix of order 3×3 and $\det(A) = 2$. Then $\det(\det(A) \operatorname{adj}(5 \operatorname{adj}(A^3)))$ is equal to ____.
- (A) 512×10^6 (B) 256×10^6
 (C) 1024×10^6 (D) 256×10^{11}

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $|\det(A) \operatorname{adj}(5 \operatorname{adj}(A))|$
 $= |2 \operatorname{adj}(5 \operatorname{adj}(A^3))|$
 $= 2^3 |\operatorname{adj}(5 \operatorname{adj}(A^3))|$
 $= 2^3 \cdot |5 \operatorname{adj}(A^3)|^2$
 $= 2^3 \cdot (5^3 \cdot |\operatorname{adj}(A^3)|)^2$
 $= 2^3 \cdot 5^6 \cdot |\operatorname{adj}(A^3)|^2$
 $= 2^3 \cdot 5^6 \cdot (|A|^3)^2$
 $= 2^3 \cdot 5^6 \cdot 2^{12} = 2^{15} \times 5^6$
 $= 2^9 \times 10^6$
 $= 512 \times 10^6$

5. The total number of 5-digit numbers, formed by using the digits 1, 2, 3, 5, 6, 7 without repetition, which are multiple of 6, is
- (A) 36 (B) 48
 (C) 60 (D) 72

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. To make a no. divisible by 3 we can use the digits 1,2,5,6,7 or 1,2,3,5,7.

Using 1,2,5,6,7, number of even numbers is
 $= 4 \times 3 \times 2 \times 1 \times 2 = 48$

Using 1,2,3,5,7, number of even numbers is
 $= 4 \times 3 \times 2 \times 1 \times 1 = 24$

Required answer is 72.

6. Let A_1, A_2, A_3, \dots be an increasing geometric progression of positive real numbers. If $A_1 A_3 A_5 A_7 = \frac{1}{1296}$ and $A_2 + A_4 = \frac{7}{36}$, then, the value of $A_6 + A_8 + A_{10}$ is equal to
- (A) 33 (B) 37
 (C) 43 (D) 47

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $A_1 \cdot A_3 \cdot A_5 \cdot A_7 = \frac{1}{1296}$

$$(A_4)^4 = \frac{1}{1296}$$

$$A_4 = \frac{1}{6} \quad \dots(1)$$

$$A_2 + A_4 = \frac{7}{36}$$

$$A_2 = \frac{1}{36} \quad \dots(2)$$

$$A_6 = 1$$

$$A_8 = 6$$

$$A_{10} = 36$$

$$A_6 + A_8 + A_{10} = 43$$

7. Let $[t]$ denote the greatest integer less than or equal to t . Then, the value of the integral

$$\int_0^1 [-8x^2 + 6x - 1] dx$$
 is equal to

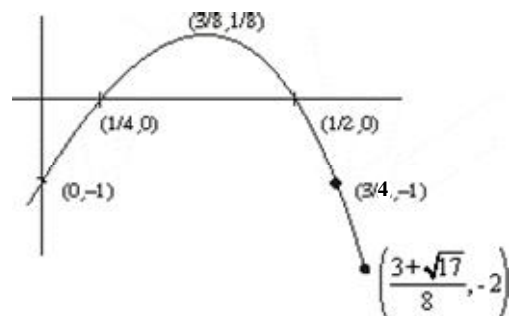
- (A) -1 (B) $-\frac{5}{4}$
 (C) $\frac{\sqrt{17}-13}{8}$ (D) $\frac{\sqrt{17}-16}{8}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $\int_0^1 [-8x^2 + 6x - 1] dx$

$$= \int_0^{1/4} -1 dx + \int_{1/4}^{1/2} 0 dx + \int_{1/2}^1 -1 dx$$



$$+ \int_{3/4}^{3+\sqrt{17}/8} -2 dx + \int_{3+\sqrt{17}/8}^1 -3 dx$$

$$= -[x]_0^{1/4} + 0 - [x]_{1/2}^{3/4} - 2[x]_{3/4}^{3+\sqrt{17}/8} - 3[x]_{3+\sqrt{17}/8}^1$$

$$\begin{aligned}
 &= -\left(\frac{1}{4}-0\right)-\left(\frac{3}{4}-\frac{1}{2}\right)-2\left(\frac{3+\sqrt{17}}{8}-\frac{3}{4}\right)-3\left(1-\frac{3+\sqrt{17}}{8}\right) \\
 &= -\frac{1}{4}-\frac{1}{4}+\frac{-6-2\sqrt{17}}{8}+\frac{3}{2}-3+\frac{9+3\sqrt{17}}{8} \\
 &= \frac{\sqrt{17}-13}{8}
 \end{aligned}$$

8. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined as

$$f(x) = \begin{cases} [e^x], & x < 0 \\ ae^x + [x-1], & 0 \leq x < 1 \\ b + [\sin(\pi x)], & 1 \leq x < 2 \\ [e^{-x}] - c, & x \geq 2 \end{cases}$$

where $a, b, c \in \mathbb{R}$ and $[t]$ denotes greatest integer less than or equal to t . Then, which of the following statements is true ?

- (A) There exists $a, b, c \in \mathbb{R}$ such that f is continuous of \mathbb{R} .
- (B) If f is discontinuous at exactly one point, then $a + b + c = 1$.
- (C) If f is discontinuous at exactly one point, then $a + b + c \neq 1$.
- (D) f is discontinuous at atleast two points, for any values of a, b and c .

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $f(x)$ is discontinuous at $x = 1$

For continuous at $x = 0$; $a = 1$

For continuous at $x = 2$; $b + c = 1$

$$a + b + c = 2$$

9. The area of the region

$$S = \{(x, y) : y^2 \leq 8x, y \geq \sqrt{2}x, x \geq 1\}$$
 is

- (A) $\frac{13\sqrt{2}}{6}$
- (B) $\frac{11\sqrt{2}}{6}$
- (C) $\frac{5\sqrt{2}}{6}$
- (D) $\frac{19\sqrt{2}}{6}$

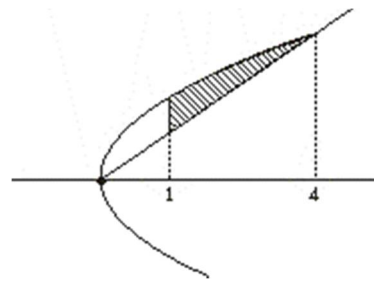
Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $y^2 = 8x$... (1)

$y = \sqrt{2}x$... (2)

$$y^2 = 2x^2$$



$$\Rightarrow 8x = 2x^2$$

$$\Rightarrow x = 0 \text{ \& \ } 4$$

$$\text{Area} = \int_1^4 2\sqrt{2}\sqrt{x} - \sqrt{2}x \, dx$$

$$= 2\sqrt{2} \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^4 - \sqrt{2} \left[\frac{x^2}{2} \right]_1^4$$

$$= \frac{4\sqrt{2}}{3}(8-1) - \frac{\sqrt{2}}{3}(16-1)$$

$$= \frac{28\sqrt{2}}{3} - \frac{15\sqrt{2}}{3} = \frac{11\sqrt{2}}{3}$$

10. Let the solution curve $y = y(x)$ of the differential equation,

$$\left[\frac{x}{\sqrt{x^2 - y^2}} + e^{\frac{y}{x}} \right] x \frac{dy}{dx} = x + \left[\frac{x}{\sqrt{x^2 - y^2}} + e^{\frac{y}{x}} \right] y$$

pass through the points $(1, 0)$ and $(2\alpha, \alpha), \alpha > 0$.

Then α is equal to

- (A) $\frac{1}{2} \exp\left(\frac{\pi}{6} + \sqrt{e} - 1\right)$
- (B) $\frac{1}{2} \exp\left(\frac{\pi}{3} + \sqrt{e} - 1\right)$
- (C) $\exp\left(\frac{\pi}{6} + \sqrt{e} + 1\right)$
- (D) $2 \exp\left(\frac{\pi}{3} + \sqrt{e} - 1\right)$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\left[\frac{x}{\sqrt{x^2 - y^2}} + e^{\frac{y}{x}} \right] x \frac{dy}{dx} = x + \left[\frac{x}{\sqrt{x^2 - y^2}} + e^{\frac{y}{x}} \right] y$

$$\Rightarrow e^{\frac{y}{x}} (x \, dy - y \, dx) + \frac{x}{\sqrt{x^2 - y^2}} (x \, dy - y \, dx) = x \, dx$$

Dividing both side by x^2

$$\frac{y}{-24} = \frac{3}{4}$$

$$y = \frac{3}{4}(-24) = -18$$

12. The number of real solutions of $x^7 + 5x^3 + 3x + 1 = 0$ is equal to _____.

- (A) 0 (B) 1
(C) 3 (D) 5

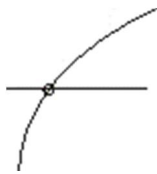
Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $f(x) = x^7 + 5x^3 + 3x + 1$

$$f'(x) = 7x^6 + 15x^2 + 3 > 0$$

$\therefore f(x)$ is strictly increasing function



$$x \rightarrow -\infty, y \rightarrow -\infty$$

$$x \rightarrow \infty, y \rightarrow \infty$$

\therefore no. of real solution = 1

13. Let the eccentricity of the hyperbola

$$H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ be } \sqrt{\frac{5}{2}} \text{ and length of its latus$$

rectum be $6\sqrt{2}$. If $y = 2x + c$ is a tangent to the hyperbola H, then the value of c^2 is equal to

- (A) 18 (B) 20
(C) 24 (D) 32

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $y = mx \pm \sqrt{a^2m^2 - b^2}$

$$m = 2, c^2 = a^2m^2 - b^2$$

$$c^2 = 4a^2 - b^2$$

$$e^2 = 1 + \frac{b^2}{a^2}$$

$$\frac{5}{2} = 1 + \frac{b^2}{a^2}$$

$$\frac{3}{2} = \frac{b^2}{a^2} \Rightarrow b^2 = \frac{3a^2}{2}$$

$$\frac{2b^2}{a} = 6\sqrt{2}$$

$$\frac{2}{a} \times \frac{3a^2}{2} = 6\sqrt{2}$$

$$3a = 6\sqrt{2}$$

$$\boxed{a = 2\sqrt{2}}$$

$$b^2 = \frac{3}{2} \times 8 = 12$$

$$b = 2\sqrt{3}$$

$$\therefore c^2 = 4 \times 8 - 12$$

$$c^2 = 20$$

14. If the tangents drawn at the point $O(0, 0)$ and $P(1 + \sqrt{5}, 2)$ on the circle $x^2 + y^2 - 2x - 4y = 0$ intersect at the point Q, then the area of the triangle OPQ is equal to

- (A) $\frac{3 + \sqrt{5}}{2}$ (B) $\frac{4 + 2\sqrt{5}}{2}$
(C) $\frac{5 + 3\sqrt{5}}{2}$ (D) $\frac{7 + 3\sqrt{5}}{2}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. Tangent at O

$$-(x + 0) - 2(y + 0) = 0$$

$$\Rightarrow \boxed{x + 2y = 0}$$

Tangent at P

$$x(1 + \sqrt{5}) + y \cdot 2 - (x + 1 + \sqrt{5}) - 2(y + 2) = 0$$

$$\text{Put } x = -2y$$

$$-2y(1 + \sqrt{5}) + 2y + 2y - 1 - \sqrt{5} - 2y - 4 = 0$$

$$-2\sqrt{5}y = 5 + \sqrt{5} \Rightarrow y = \left(\frac{\sqrt{5} + 1}{2}\right)$$

$$Q\left(\sqrt{5}+1, -\frac{\sqrt{5}+1}{2}\right)$$

$$\text{Length of tangent OQ} = \frac{5+\sqrt{5}}{2}$$

$$\text{Area} = \frac{RL^3}{R^2+L^2}$$

$$R = \sqrt{5}$$

$$= \frac{\sqrt{5} \times \left(\frac{5+\sqrt{5}}{2}\right)^3}{5 + \left(\frac{5+\sqrt{5}}{2}\right)^2}$$

$$= \frac{\sqrt{5}}{2} \times \frac{4 \times (125 + 75 + 75\sqrt{5} + 5\sqrt{5})}{(20 + 25 + 10\sqrt{5} + 5)}$$

$$= \frac{5+3\sqrt{5}}{2}$$

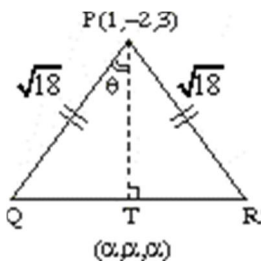
15. If two distinct point Q, R lie on the line of intersection of the planes $-x + 2y - z = 0$ and $3x - 5y + 2z = 0$ and $PQ = PR = \sqrt{18}$ where the point P is $(1, -2, 3)$, then the area of the triangle PQR is equal to

- (A) $\frac{2}{3}\sqrt{38}$ (B) $\frac{4}{3}\sqrt{38}$
(C) $\frac{8}{3}\sqrt{38}$ (D) $\sqrt{\frac{152}{3}}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol.



$$-x + 2y - z = 0$$

$$3x - 5y + 2z = 0$$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & -1 \\ 3 & -5 & 2 \end{vmatrix}$$

$$= \hat{i}(-1) - \hat{j}(1) + \hat{k}(-1)$$

$$\vec{n} = -\hat{i} - \hat{j} - \hat{k}$$

$$\text{Equation of LOI is } \frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$$\text{DR: of PT} \rightarrow \alpha - 1, \alpha + 2, \alpha - 3$$

$$\text{DR: of QR} \rightarrow 1, 1, 1$$

$$\Rightarrow (\alpha - 1) \times 1 + (\alpha + 2) \times 1 + (\alpha - 3) \times 1 = 0$$

$$3\alpha = 2$$

$$\alpha = \frac{2}{3}$$

$$PT^2 = \frac{1}{9} + \frac{64}{9} + \frac{49}{9}$$

$$PT^2 = \frac{114}{9}$$

$$PT = \frac{\sqrt{114}}{3}$$

$$\cos \theta = \frac{\sqrt{114}}{3} \times \frac{1}{3\sqrt{2}} = \frac{\sqrt{57}}{9} = \frac{\sqrt{19 \times 3}}{3 \times 3} = \frac{\sqrt{19}}{3\sqrt{3}}$$

$$\cos 2\theta = \frac{2 \times 19}{27} - 1 = \frac{11}{27}$$

$$\sin 2\theta = \sqrt{1 - \left(\frac{11}{27}\right)^2} = \frac{\sqrt{38}\sqrt{16}}{27} = \frac{4}{27}\sqrt{38}$$

$$\text{Area} = \frac{1}{2} \times \sqrt{18}\sqrt{18} \times \frac{4}{27}\sqrt{38}$$

$$= \frac{18}{2} \times \frac{4}{27}\sqrt{38} = \frac{36}{27}\sqrt{38} = \frac{4}{3}\sqrt{38}$$

16. The acute angle between the planes P_1 and P_2 , when P_1 and P_2 are the planes passing through the intersection of the planes $5x + 8y + 13z - 29 = 0$ and $8x - 7y + z - 20 = 0$ and the points $(2, 1, 3)$ and $(0, 1, 2)$, respectively, is

- (A) $\frac{\pi}{3}$ (B) $\frac{\pi}{4}$
 (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{12}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. Equation of plane passing through the intersection of planes $5x + 8y + 13z - 29 = 0$ and $8x - 7y + z - 20 = 0$ is

$$5x + 8y + 13z - 29 + \lambda(8x - 7y + z - 20) = 0 \quad \text{and}$$

if it is passing through $(2, 1, 3)$ then $\lambda = \frac{7}{2}$

P_1 : Equation of plane through intersection of $5x + 8y + 13z - 29 = 0$ and $8x - 7y + z - 20 = 0$ and the point $(2, 1, 3)$ is

$$5x + 8y + 13z - 29 + \frac{7}{2}(8x - 7y + z - 20) = 0$$

$$\Rightarrow 2x - y + z = 6$$

Similarly P_2 : Equation of plane through intersection of

$$5x + 8y + 13z - 29 = 0 \quad \text{and} \quad 8x - 7y + z - 20 = 0$$

and the point $(0, 1, 2)$ is

$$\Rightarrow x + y + 2z = 5$$

$$\text{Angle between planes} = \theta = \cos^{-1} \left(\frac{3}{\sqrt{6}\sqrt{6}} \right) = \frac{\pi}{3}$$

17. Let the plane $P: \vec{r} \cdot \vec{a} = d$ contain the line of intersection of two planes $\vec{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) = 6$ and $\vec{r} \cdot (-6\hat{i} + 5\hat{j} - \hat{k}) = 7$. If the plane P passes through the point $\left(2, 3, \frac{1}{2}\right)$, then the value of $\frac{13|\vec{a}|^2}{d^2}$ is equal to

- (A) 90 (B) 93
 (C) 95 (D) 97

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. Equation of plane passing through line of intersection of planes $P_1: \vec{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) = 6$ and

$$P_2: \vec{r} \cdot (-6\hat{i} + 5\hat{j} - \hat{k}) = 7 \quad \text{is}$$

$$P_1 + \lambda P_2 = 0$$

$$\left(\vec{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) - 6 \right) + \lambda \left(\vec{r} \cdot (-6\hat{i} + 5\hat{j} - \hat{k}) - 7 \right) = 0$$

and it passes through point $\left(2, 3, \frac{1}{2}\right)$

$$\Rightarrow \left(2 + 9 - \frac{1}{2} - 6 \right) + \lambda \left(-12 + 15 - \frac{1}{2} - 7 \right) = 0$$

$$\Rightarrow \lambda = 1$$

$$\text{Equation of plane is } \vec{r} \cdot (-5\hat{i} + 8\hat{j} - 2\hat{k}) = 13$$

$$|\vec{a}|^2 = 25 + 64 + 4 = 93; \quad d = 13$$

$$\text{Value of } \frac{13|\vec{a}|^2}{d^2} = 93$$

18. The probability, that in a randomly selected 3-digit number at least two digits are odd, is

- (A) $\frac{19}{36}$ (B) $\frac{15}{36}$
 (C) $\frac{13}{36}$ (D) $\frac{23}{36}$

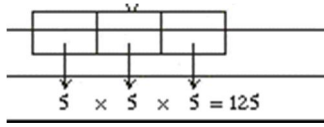
Official Ans. by NTA (A)

Allen Ans. (A)

Sol. Atleast two digits are odd

= exactly two digits are odd + exactly there 3 digits are odd

For exactly three digits are odd



For exactly two digits odd :

If 0 is used then : $2 \times 5 \times 5 = 50$

If 0 is not used then : ${}^3C_1 \times 4 \times 5 \times 5 = 300$

$$\text{Required Probability} = \frac{475}{900} = \frac{19}{36}$$

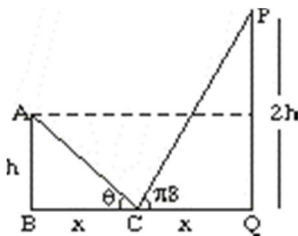
19. Let AB and PQ be two vertical poles, 160 m apart from each other. Let C be the middle point of B and Q, which are feet of these two poles. Let $\frac{\pi}{8}$ and θ be the angles of elevation from C to P and A, respectively. If the height of pole PQ is twice the height of pole AB, then $\tan^2 \theta$ is equal to

- (A) $\frac{3-2\sqrt{2}}{2}$ (B) $\frac{3+\sqrt{2}}{2}$
(C) $\frac{3-2\sqrt{2}}{4}$ (D) $\frac{3-\sqrt{2}}{4}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol.



Let $BC = CQ = x$ & $AB = h$ and $PQ = 2h$

$$\tan \theta = \frac{h}{x}, \tan \frac{\pi}{8} = \frac{2h}{x}$$

$$\frac{\tan \theta}{\tan \left(\frac{\pi}{8} \right)} = \frac{1}{2}$$

$$\tan \theta = \frac{1}{2} \tan \left(\frac{\pi}{8} \right) = \frac{1}{2} (\sqrt{2} - 1)$$

$$\tan^2 \theta = \frac{1}{4} (3 - 2\sqrt{2})$$

20. Let p, q, r be three logical statements. Consider the compound statements

$$S_1 : ((\sim p) \vee q) \vee ((\sim p) \vee r) \text{ and}$$

$$S_2 : p \rightarrow (q \vee r)$$

Then, which of the following is **NOT** true ?

- (A) If S_2 is True, then S_1 is True
(B) If S_2 is False, then S_1 is False
(C) If S_2 is False, then S_1 is True
(D) If S_1 is False, then S_2 is False

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $s_1 : (\sim p \vee q) \vee (\sim p \vee r)$

$$\equiv \sim p \vee (q \vee r)$$

$$s_2 : p \rightarrow (q \vee r)$$

$$\equiv \sim p \vee (q \vee r) \rightarrow \text{By conditional law}$$

$$s_1 \equiv s_2$$

SECTION-B

1. Let R_1 and R_2 be relations on the set $\{1, 2, \dots, 50\}$ such that

$$R_1 = \{(p, p^n) : p \text{ is a prime and } n \geq 0 \text{ is an integer}\}$$

$$\text{and } R_2 = \{(p, p^n) : p \text{ is a prime and } n = 0 \text{ or } 1\}.$$

Then, the number of elements in $R_1 - R_2$ is _____.

Official Ans. by NTA (8)

Allen Ans. (8)

Sol. Here, $p, p^n \in \{1, 2, \dots, 50\}$

Now p can take values

2,3,5,7,11,13,17,23,29,31,37,41,43 and 47.

\therefore we can calculate no. of elements in R, as

$$(2, 2^0), (2, 2^1) \dots (2, 2^5)$$

$$(3, 3^0), \dots (3, 3^3)$$

$$(5, 5^0), \dots (5, 5^2)$$

$$(7, 7^0), \dots (7, 7^2)$$

$$(11, 11^0), \dots (11, 11^1)$$

And rest for all other two elements each

$$\therefore n(R_1) = 6 + 4 + 3 + 3 + (2 \times 10) = 36$$

Similarly for R_2

$$(2, 2^0), (2, 2^1)$$

$$(47, 47^0), (47, 47^1)$$

$$\therefore n(R_2) = 2 \times 14 = 28$$

$$\therefore n(R_1) - n(R_2) = 36 - 28 = 8$$

2. The number of real solutions of the equation $e^{4x} + 4e^{3x} - 58e^{2x} + 4e^x + 1 = 0$ is _____.

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $e^{4x} + 4e^{3x} - 58e^{2x} + 4e^x + 1 = 0$

$$\text{Let } f(x) = e^{2x} \left(e^{2x} + \frac{1}{e^{2x}} + 4 \left(e^x + \frac{1}{e^x} \right) - 58 \right)$$

$$e^x + \frac{1}{e^x}$$

$$\text{Let } h(t) = t^2 + 4t - 58 = 0$$

$$t = \frac{-4 \pm \sqrt{16 + 4 \cdot 58}}{2}$$

$$\frac{-4 \pm 2\sqrt{62}}{2}$$

$$t_1 = -2 + 2\sqrt{62}$$

$$t_2 = -2 - 2\sqrt{62} \text{ (not possible)}$$

$$t \geq 2$$

$$e^x + \frac{1}{e^x} = -2 + 2\sqrt{62}$$

$$e^{2x} - (-2 + 2\sqrt{62})e^x + 1 = 0$$

$$(-2 + 2\sqrt{62}) - 4$$

$$4 + 4 \cdot 62 - 8\sqrt{62} - 4$$

$$248 - 8\sqrt{62} > 0$$

$$\frac{-b}{2a} > 0$$

both roots are positive

2 real roots

3. The mean and standard deviation of 15 observations are found to be 8 and 3 respectively. On rechecking it was found that, in the observations, 20 was misread as 5. Then, the correct variance is equal to _____.

Official Ans. by NTA (17)

Allen Ans. (17)

Sol. We have

$$\text{Variance} = \frac{\sum_{r=1}^{15} x_r^2}{15} - \left(\frac{\sum_{r=1}^{15} x_r}{15} \right)^2$$

Now, as per information given in equation

$$\frac{\sum x_r^2}{15} - 8^2 = 3^2 \Rightarrow \sum x_r^2 = \log 5$$

$$\text{Now, the new } \sum x_r^2 = \log 5 - 5^2 + 20^2 = 1470$$

$$\text{And, new } \sum x_r = (15 \times 8) - 5 + (20) = 135$$

$$\therefore \text{Variance} = \frac{1470}{15} - \left(\frac{135}{15} \right)^2 = 98 - 81 = 17$$

4. If $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = 3\hat{i} + 3\hat{j} + \hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are coplanar vectors and $\vec{a} \cdot \vec{c} = 5$, $\vec{b} \perp \vec{c}$, then $122(c_1 + c_2 + c_3)$ is equal to _____.

Official Ans. by NTA (150)

Allen Ans. (150)

Sol. $\vec{a} \cdot \vec{c} = 5 \Rightarrow 2c_1 + c_2 + 3c_3 = 5 \quad \dots(1)$

$$\vec{b} \cdot \vec{c} = 0 \Rightarrow 3c_1 + 3c_2 + c_3 = 0 \quad \dots(2)$$

$$\text{And } [\vec{a} \ \vec{b} \ \vec{c}] = 0 \Rightarrow \begin{vmatrix} c_1 & c_2 & c_3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 8c_1 - 7c_2 - 3c_3 = 0 \quad \dots(3)$$

By solving (1), (2), (3) we get

$$c_1 = \frac{10}{122}, c_2 = \frac{-85}{122}, c_3 = \frac{225}{122}$$

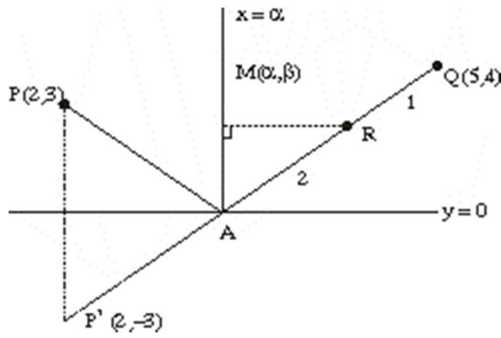
$$\therefore 122(c_1 + c_2 + c_3) = 150$$

5. A ray of light passing through the point P(2, 3) reflects on the x-axis at point A and the reflected ray passes through the point Q(5, 4). Let R be the point that divides the line segment AQ internally into the ratio 2 : 1. Let the co-ordinates of the foot of the perpendicular M from R on the bisector of the angle PAQ be (α, β) . Then, the value of $7\alpha + 3\beta$ is equal to _____.

Official Ans. by NTA (31)

Allen Ans. (31)

Sol.



By observation we see that $A(\alpha, 0)$.

And $\beta = y$ -coordinate of R

$$= \frac{2 \times 4 + 1 \times 0}{2 + 1} = \frac{8}{3} \dots(1)$$

Now P' is image of P in $y = 0$ which will be $P'(2, -3)$

$$\therefore \text{Equation of } P'Q \text{ is } (y + 3) = \frac{4 + 3}{5 - 2}(x - 2)$$

$$\text{i.e. } 3y + 9 = 7x - 14$$

$$A \equiv \left(\frac{23}{7}, 0 \right) \text{ by solving with } y = 0$$

$$\therefore \alpha = \frac{23}{7} \dots(2)$$

By (1), (2)

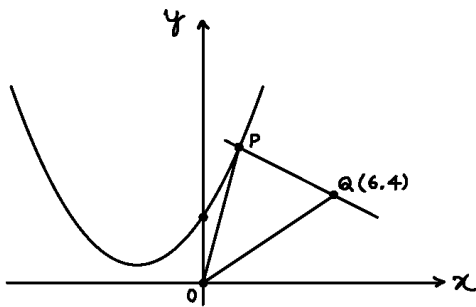
$$7\alpha + 3\beta = 23 + 8 = 31$$

6. Let ℓ be a line which is normal to the curve $y = 2x^2 + x + 2$ at a point P on the curve. If the point Q(6, 4) lies on the line ℓ and O is origin, then the area of the triangle OPQ is equal to _____.

Official Ans. by NTA (13)

Allen Ans. (13)

Sol. $y = 2x^2 + x + 2$



$$\frac{dy}{dx} = 4x + 1$$

Let P be (h, k) , then normal at P is

$$y - k = -\frac{1}{4h + 1}(x - h)$$

This passes through Q (6,4)

$$\therefore 4 - k = -\frac{1}{4h + 1}(6 - h)$$

$$\Rightarrow (4h + 1)(4 - k) + 6 - h = 0$$

$$\text{Also } k = 2h^2 + h + 2$$

$$\therefore (4h + 1)(4 - 2h^2 - h - 2) + 6 + h = 0$$

$$\Rightarrow 4h^3 - 3h^2 + 3h - 8 = 0$$

$$\Rightarrow h = 1, k = 5$$

$$\text{Now area of } \Delta OPQ \text{ will be } = \frac{1}{2} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 5 \\ 1 & 6 & 4 \end{vmatrix} = 13$$

7. Let $A = \{1, a_1, a_2, \dots, a_{18}, 77\}$ be a set of integers with $1 < a_1 < a_2 < \dots < a_{18} < 77$. Let the set $A + A = \{x + y : x, y \in A\}$ contain exactly 39 elements. Then, the value of $a_1 + a_2 + \dots + a_{18}$ is equal to _____.

Official Ans. by NTA (702)

Allen Ans. (702)

Sol. $a_1, a_2, a_3, \dots, a_{18}, 77$

are in AP i.e. 1, 5, 9, 13, ..., 77.

$$\text{Hence } a_1 + a_2 + a_3 + \dots + a_{18} = 5 + 9 + 13 + \dots + 77 = 702$$

8. The number of positive integers k such that the constant term in the binomial expansion of $\left(2x^3 + \frac{3}{x^k}\right)^{12}$, $x \neq 0$ is $2^8 \cdot \ell$, where ℓ is an odd integer, is _____.

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\left(2x^3 + \frac{3}{x^k}\right)^{12}$

$$t_{r+1} = {}^{12}C_r (2x^3)^r \left(\frac{3}{x^k}\right)^{12-r}$$

$$x^{3r-(12-r)k} \rightarrow \text{constant}$$

$$\therefore 3r - 12k + rk = 0$$

$$\Rightarrow k = \frac{3r}{12-r}$$

\therefore possible values of r are 3,6,8,9,10 and corresponding values of k are 1,3,6,9,15

$$\text{Now } {}^{12}C_r = 220, 924, 495, 220, 66$$

\therefore possible values of k for which we will get 2^8 are 3, 6

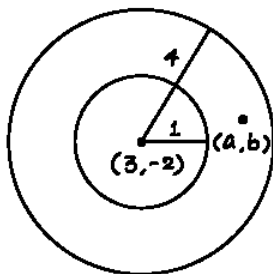
9. The number of elements in the set

$$\{z = a + ib \in \mathbb{C} : a, b \in \mathbb{Z} \text{ and } 1 < |z - 3 + 2i| < 4\} \text{ is } \underline{\hspace{2cm}}$$

Official Ans. by NTA (40)

Allen Ans. (40)

Sol. $1 < |z - 3 + 2i| < 4$



$$1 < (a - 3)^2 + (b + 2)^2 < 16$$

$$(0, \pm 2), (\pm 2, 0), (\pm 1, \pm 2), (\pm 2, \pm 1)$$

$$(\pm 2, \pm 3), (3 \pm, \pm 2), (\pm 1, \pm 1), (2 \pm, \pm 2)$$

$$(\pm 3, 0), (0, \pm 3), (\pm 3 \pm 1), (\pm 1, \pm 3)$$

Total 40 points

10. Let the lines $y + 2x = \sqrt{11} + 7\sqrt{7}$ and $2y + x = 2\sqrt{11} + 6\sqrt{7}$ be normal to a circle $C: (x - h)^2 + (y - k)^2 = r^2$. If the line

$$\sqrt{11}y - 3x = \frac{5\sqrt{77}}{3} + 11 \text{ is tangent to the circle } C,$$

then the value of $(5h - 8k)^2 + 5r^2$ is equal to ____.

Official Ans. by NTA (816)

Allen Ans. (816)

Sol. Normal are

$$y + 2x = \sqrt{11} + 7\sqrt{7},$$

$$2y + x = 2\sqrt{11} + 6\sqrt{7}$$

Center of the circle is point of intersection of normals i.e.

$$\left(\frac{8\sqrt{7}}{3}, \sqrt{11} + \frac{5\sqrt{7}}{3}\right)$$

$$\text{Tangent is } \sqrt{11}y - 3x = \frac{5\sqrt{77}}{3} + 11$$

Radius will be \perp distance of tangent from center

$$\text{i.e. } 4\sqrt{\frac{7}{5}}$$

$$\text{Now } (5h - 8k)^2 + 5r^2 = 816$$