FINAL JEE–MAIN EXAMINATION – JUNE, 2022

(Held On Wednesday 29th June, 2022) **TIME : 9 : 00 AM to 12 : 00 PM**

MATHEMATICS **TEST PAPER WITH SOLUTION**

SECTION-A

1. Question ID: 101761

The probability that a randomly chosen 2 × 2 matrix with all the entries from the set of first 10 primes, is singular, is equal to :

(A) $\frac{133}{10^4}$ (B) $\frac{18}{10^3}$

(C) $\frac{19}{10^3}$ (D) $\frac{271}{10^4}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. Let matrix A is singular then |A| = 0

Number of singular matrix = All entries are same + only two prime number are used in matrix

$= 10 + 10 \times 9 \times 2$

$= 190$

Required probability $= \frac{190}{10^4} = \frac{19}{10^3}$

2. Question ID: 101762

Let the solution curve of the differential equation

$x\frac{dy}{dx} - y = \sqrt{y^2 + 16x^2}$, $y(1) = 3$ be $y = y(x)$.

Then y(2) is equal to :

(A) 15 (B) 11

(C) 13 (D) 17

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow x\frac{dv}{dx} = \sqrt{v^2 + 16}$

$\Rightarrow \int \frac{dv}{\sqrt{v^2+16}} = \int \frac{dx}{x}$

$\Rightarrow \ell n|v + \sqrt{v^2+16}| = \ln x + \ln C$

$\Rightarrow y + \sqrt{y^2 + 16x^2} = Cx^2$

As $y(1) = 3 \Rightarrow C = 8$

$\Rightarrow y(2) = 15$

3. Question ID: 101763

If the mirror image of the point (2, 4, 7) in the plane $3x - y + 4z = 2$ is (a, b, c), the $2a + b + 2c$ is equal to :

(A) 54 (B) 50

(C) −6 (D) −42

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $\frac{a-2}{3} = \frac{b-4}{-1} = \frac{c-7}{4} = \frac{-2(6-4+28-2)}{3^2+1^2+4^2}$

$\Rightarrow a = \frac{-84}{13} + 2, b = \frac{28}{13} + 4, C = \frac{-112}{13} + 7$

$\Rightarrow 2a + b + 2c = -6$

4. Question ID: 101764

Let $f : R \to R$ be a function defined by :

$$f(x) = \begin{cases} \max\limits_{t \le x}\{t^3 - 3t\} ; x \le 2 \\ x^2 + 2x - 6 ; 2 < x < 3 \\ [x-3] + 9 \; ; 3 \le x \le 5 \\ 2x + 1 \; ; \; x > 5 \end{cases}$$

Where [t] is the greatest integer less than or equal to t. Let m be the number of points where f is not differentiable and $I = \int_{-2}^{2} f(x)dx$. Then the ordered pair (m, I) is equal to :

(A) $\left(3, \frac{27}{4}\right)$ (B) $\left(3, \frac{23}{4}\right)$

(C) $\left(4, \frac{27}{4}\right)$ (D) $\left(4, \frac{23}{4}\right)$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $f(x)=\begin{cases} x^3-3x, & x\le -1 \\ 2, & -1<x<2 \\ x^2+2x-6, & 2<x<3 \\ 9, & 3\le x<4 \\ 10, & 4\le x<5 \\ 11, & x=5 \\ 2x+1, & x>5 \end{cases}$

Clearly f (x) is not differentiable at

x = 2, 3, 4, 5 $\Rightarrow$ m = 4

$$I=\int_{-2}^{-1}(x^3-3x)dx+\int_{-1}^{2}2\cdot dx=\frac{27}{4}$$

5. Question ID: 101765

Let $\vec{a}=\alpha\hat{i}+3\hat{j}-\hat{k}, \vec{b}=3\hat{i}-\beta\hat{j}+4\hat{k}$ and $\vec{c}=\hat{i}+2\hat{j}-2\hat{k}$ where α, β ∈ R, be three vectors. If the projection of $\vec{a}$ on $\vec{c}$ is $\frac{10}{3}$ and $\vec{b}\times\vec{c}=-6\hat{i}+10\hat{j}+7\hat{k}$, then the value of α + β equal to :

(A) 3 (B) 4

(C) 5 (D) 6

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\frac{\vec{a}\cdot\vec{c}}{|\vec{c}|}=\frac{10}{3}$

$$\Rightarrow \frac{\alpha+6+2}{\sqrt{1+4+4}}=\frac{10}{3}\Rightarrow \alpha=2$$

and $\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & -\beta & 4\\ 1 & 2 & -2\end{vmatrix}=-6\hat{i}+\hat{j}+\hat{k}$

$\Rightarrow 2\beta-8=-6\Rightarrow \beta=1$

$\Rightarrow \alpha+\beta=3$

6. Question ID : 101766

The area enclosed by y^2 = 8x and $y=\sqrt{2}x$ that lies outside the triangle formed by $y=\sqrt{2}x, x=1, y=2\sqrt{2}$, is equal to :

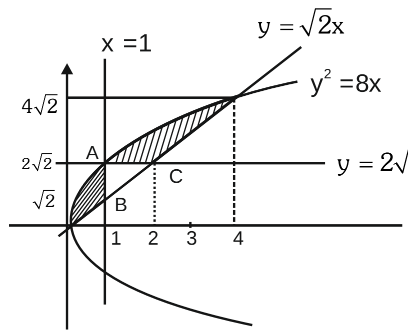
(A) $\frac{16\sqrt{2}}{6}$ (B) $\frac{11\sqrt{2}}{6}$

(C) $\frac{13\sqrt{2}}{6}$ (D) $\frac{5\sqrt{2}}{6}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol.



Area of ΔABC $=\frac{1}{2}(\sqrt{2})\cdot 1=\frac{\sqrt{2}}{2}$

So required Area $=\int_0^4\left(\sqrt{8x}-\sqrt{2}x\right)dx-\frac{\sqrt{2}}{2}$

$$=\frac{32\sqrt{2}}{3}-8\sqrt{2}-\frac{\sqrt{2}}{2}=\frac{13\sqrt{2}}{6}$$

7. Question ID: 101767

If the system of linear equations

2x + y – z = 7

x – 3y + 2z = 1

x + 4y + δz = k, where δ, k ∈ R

has infinitely many solutions, then δ + k is equal to:

(A) –3 (B) 3 (C) 6 (D) 9

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\begin{vmatrix}2 & 1 & -1\\ 1 & -3 & 2\\ 1 & 4 & \delta\end{vmatrix}=0$

$\Rightarrow \delta=-3$

And $\begin{vmatrix}7 & 1 & -1\\ 1 & -3 & 2\\ K & 4 & -3\end{vmatrix}=0\Rightarrow K=6$

$\Rightarrow$ δ + K = 3

Alternate

2x + y – z = 7 ....(1)

x – 3y + 2z = 1 ....(2)

x + 4y + δz = k ....(3)

Equation (2) + (3)

We get 2x + y + (2 + δ) z = 1 + K ......(4)

For infinitely solution

Form equation (1) and (4)

2 + δ = –1 $\Rightarrow$ $\boxed{\delta=-3}$

1 + k = 7 $\Rightarrow$ $\boxed{k=6}$

δ + k = 3

8. Question ID: 101768

Let α and β be the roots of the equation $x^2 + (2i - 1) = 0$. Then, the value of $|\alpha^8 + \beta^8|$ is equal to :

(A) 50 (B) 250

(C) 1250 (D) 1500

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $X^2 = 1 - 2i \quad \Rightarrow \alpha^2 = 1 - 2i,\ \beta^2 = 1 - 2i$

Hence $\alpha^8 = \beta^8$

$|\alpha^8 + \beta^8| = |2\alpha^8| = 2|\alpha^2|^4$

$= 2\ \sqrt{5}^4 = 50$

9. Question ID: 101769

Let $\Delta \in \{\wedge, \vee, \Rightarrow, \Leftrightarrow\}$ be such that $(p \wedge q)\Delta((p \vee q) \Rightarrow q)$ is a tautology. Then Δ is equal to :

(A) $\wedge$ (B) $\vee$

(C) $\Rightarrow$ (D) $\Leftrightarrow$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $p \vee q \Rightarrow q$

$\Rightarrow \sim(p \vee q) \vee q$

$\Rightarrow (\sim p \wedge \sim q) \vee q$

$\Rightarrow (\sim p \vee q) \wedge (\sim q \vee q)$

$\Rightarrow (\sim p \vee q) \wedge t = \sim p \vee q$

Now by taking option C

$(p \wedge q) \Rightarrow \sim p \vee q$

$\Rightarrow \sim p \vee \sim q \vee \sim p \vee q$

$\Rightarrow t$

Hence C

10. Question ID: 101770

Let $A = [a_{ij}]$ be a square matrix of order 3 such that $a_{ij} = 2^{j-i}$, for all i, j = 1, 2, 3. Then, the matrix $A^2 + A^3 + \ldots + A^{10}$ is equal to :

(A) $\left(\frac{3^{10}-3}{2}\right)A$ (B) $\left(\frac{3^{10}-1}{2}\right)A$

(C) $\left(\frac{3^{10}+1}{2}\right)A$ (D) $\left(\frac{3^{10}+3}{2}\right)A$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $A = \begin{pmatrix} 1 & 2 & 2^2 \\ 1/2 & 1 & 2 \\ 1/2^2 & 1/2 & 1 \end{pmatrix}$

$A^2 = 3A$

$A^3 = 3^2A$

$A^2 + A^3 + \ldots A^{10}$

$= 3A + 3^2A + \ldots + 3^9A = \frac{3(3^9-1)}{3-1}A$

$= \frac{3^{10}-3}{2}A$

11. Question ID: 101771

Let a set $A = A_1 \cup A_2 \cup \ldots \cup A_k$, where $A_i \cap A_j = \phi$ for $i \neq j$ $1 \le i, j \le k$. Define the relation R from A to A by $R = \{(x, y): y \in A_i$ if and only if $x \in A_i, 1 \le i \le k\}$. Then, R is :

(A) reflexive, symmetric but not transitive

(B) reflexive, transitive but not symmetric

(C) reflexive but not symmetric and transitive

(D) an equivalence relation

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. A = {1, 2, 3}

R = {(1, 1), (1, 2), (1, 3) (2, 1), (2, 2), (2, 3) (3,1), (3, 2) (3, 3)}

12. Question ID: 101772

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence such that $a_0 = a_1 = 0$ and $a_{n+2} = 2a_{n+1} - a_n + 1$ for all $n \ge 0$. Then, $\sum_{n=2}^{\infty} \frac{a_n}{7^n}$ is equal to

(A) $\frac{6}{343}$ (B) $\frac{7}{216}$

(C) $\frac{8}{343}$ (D) $\frac{49}{216}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $a_2 = 1, a_3 = 3\ a_4 = 6$

$$a_n = \frac{n(n-1)}{2}$$

$$S = \sum_{n=2}^{\infty} \frac{n(n-1)}{2(7^n)}$$

$$S = \frac{1}{7^2} + \frac{3}{7^3} + \frac{6}{7^4} + \frac{10}{7^5} + \frac{15}{7^5} + \ldots\ldots$$

$$\frac{S}{7} = \frac{1}{7^3} + \frac{3}{7^4} + \frac{6}{7^5} + \frac{10}{7^6} + \ldots$$

$$6\frac{S}{7} = \frac{1}{7^2} + \frac{2}{7^3} + \frac{3}{7^4} + \frac{4}{7^5} + \ldots$$

$$6\frac{S}{7^2} = \frac{1}{7^3} + \frac{2}{7^4} + \frac{3}{7^5} + \ldots$$

$$6\frac{S}{7} \cdot \frac{6}{7} = \frac{1}{7^2} + \frac{1}{7^3} + \ldots = \frac{1/7^2}{1-1/7}$$

$$6 \times 6\frac{S}{7^2} = \cdot\frac{1}{7 \times 6}$$

$$S = \frac{7}{6^3} = \frac{7}{216}$$

Alternate

$a_{n+2} = 2a_{n+1} - a_n + 1$

$$\Rightarrow \frac{a_{n+2}}{7^{n+2}} = \frac{2}{7}\frac{a_{n+1}}{7^{n+1}} - \frac{1}{49}\frac{a_n}{7^n} + \frac{1}{7^{n+2}}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{a_{n+2}}{7^{n+2}} = \frac{2}{7}\sum_{n=2}^{\infty} \frac{a_{n+1}}{7^{n+1}} - \frac{1}{49}\sum_{n=2}^{\infty} \frac{a_n}{7^n} + \sum_{n=2}^{\infty} \frac{1}{7^{n+2}}$$

Let $\sum_{n=2}^{\infty} \frac{a_n}{7^n} = p$

$$\Rightarrow \left(p - \frac{a_2}{7^2} - \frac{a_3}{7^3}\right) = \frac{2}{7}\left(p - \frac{a_2}{7^2}\right) - \frac{1}{49}p + \frac{1/7^4}{1-\frac{1}{7}}$$

$\because a_2 = 1, a_3 = 3$

$$\Rightarrow p - \frac{1}{49} - \frac{3}{343} = \frac{2}{7}p - \frac{2}{7^3} - \frac{p}{49} + \frac{1}{6.7^3}$$

$$\Rightarrow p = \frac{7}{216}$$

13. Question ID: 101773

The distance between the two points A and A' which lie on y = 2 such that both the line segments AB and A' B (where B is the point (2, 3)) subtend angle $\frac{\pi}{4}$ at the origin, is equal to :

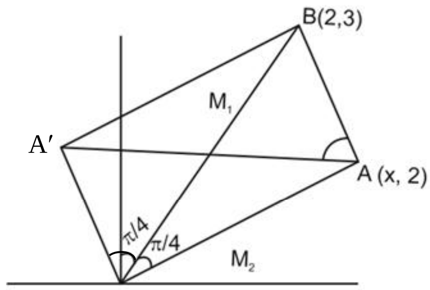
(A) 10 (B) $\frac{48}{5}$

(C) $\frac{52}{5}$ (D) 3

Official Ans. by NTA (C)

Allen Ans. (C)

Sol.



$M_1 = 3/2$ $M_2 = 2/x$

$$\tan \pi/4 = \left|\frac{3/2 - 2/x}{1 + 6/2x}\right| = 1$$

$\Rightarrow x_1 = 10,\quad x_2 = -2/5$

$\Rightarrow AA^1 = 52/5$

14. Question ID: 101774

A wire of length 22 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into an equilateral triangle. Then, the length of the side of the equilateral triangle, so that the combined area of the square and the equilateral triangle is minimum, is :

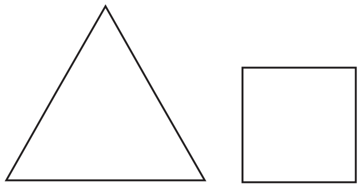
(A) $\frac{22}{9+4\sqrt{3}}$ (B) $\frac{66}{9+4\sqrt{3}}$

(C) $\frac{22}{4+9\sqrt{3}}$ (D) $\frac{66}{4+9\sqrt{3}}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol.



$3a = x$ $\quad$ $4b = 22 - x$

$a = 2/13$

$$A_T = \frac{\sqrt{3}}{4}a^2 + b^2$$

$$= \frac{\sqrt{3}}{4}x^2/9 + \frac{(22-x)^2}{16}$$

$$\frac{dA}{dx} = 0 \Rightarrow x\left(\frac{\sqrt{3}}{2\times 9} + \frac{1}{8}\right) - \frac{22}{8} = 0$$

$$\Rightarrow x\left(\frac{4\sqrt{3}+9}{36}\right) = \frac{11}{2}$$

$a = x/3$

$$a = \left(\frac{11/2}{\frac{4\sqrt{3}+9}{36}}\right)\left(\frac{1}{3}\right) = \frac{66}{4\sqrt{3}+9}$$

15. Question ID: 101775

The domain of the function $\cos^{-1}\left(\frac{2\sin^{-1}\left(\frac{1}{4x^2-1}\right)}{\pi}\right)$ is :

(A) $R - \left\{-\frac{1}{2}, \frac{1}{2}\right\}$

(B) $(-\infty, -1] \cup [1, \infty) \cup \{0\}$

(C) $\left(-\infty, \frac{-1}{2}\right) \cup \left(\frac{1}{2}, \infty\right) \cup \{0\}$

(D) $\left(-\infty, \frac{-1}{\sqrt{2}}\right] \cup \left[\frac{1}{\sqrt{2}}, \infty\right) \cup \{0\}$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $$-1 \le \frac{2\sin^{-1}\left(\frac{1}{4x^2-1}\right)}{\pi} \le 1$$

$$-\pi/2 \le \sin^{-1}\frac{1}{4x^2-1} \le \pi/2$$

Always $-1 \le \dfrac{1}{4x^2-1} \le 1$

$$x \in \left(\infty, \frac{1}{\sqrt{2}}\right) \cup \left[\frac{1}{\sqrt{2}}, \infty\right)$$

16. Question ID: 101776

If the constant term in the expansion of $\left(3x^3 - 2x^2 + \frac{5}{x^5}\right)^{10}$ is $2^k \cdot l$, where l is an odd integer, then the value of k is equal to :

(A) 6 $\qquad$ (B) 7

(C) 8 $\qquad$ (D) 9

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. General term

$$T_{r+1} = \frac{\lfloor 10}{\lfloor r_1 \lfloor r_2 \lfloor r_3}(3)^{r_1}(-2)^{r_2}(5)^{r_3}(x)^{3r_1+2r_2-5r_3}$$

$3r_1 + 2r_2 - 5r_3 = 0$ $\quad$ ...(1)

$r_1 + r_2 + r_3 = 10$ $\quad$ ...(2)

from equation (1) and (2)

$r_1 + 2(10 - r_3) - 5r_3 = 0$

$r_1 + 20 = 7r_3$

$(r_1, r_2, r_3) = (1, 6, 3)$

constant term $= \dfrac{\lfloor 10}{\lfloor 1 \lfloor 6 \lfloor 3}(3)^1(-2)^6(5)^3$

$= 2^9.3^2.5^4.7^1$

$l = 9$

17. Question ID: 101777

$\int_0^5 \cos\left(\pi\left(x - \left[\frac{x}{2}\right]\right)\right) dx,$

Where [t] denotes greatest integer less than or equal to t, is equal to :

(A) -3 $\qquad$ (B) -2

(C) 2 $\qquad$ (D) 0

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $$I = \int_0^5 \cos\left(\pi x - \pi\left[\frac{x}{2}\right]\right)dx$$

$$\Rightarrow I = \int_0^2 \cos(\pi x)dx + \int_2^4 \cos(\pi x - \pi)dx + \int_4^5 \cos(\pi x - 2\pi)dx$$

$$\Rightarrow I = \left[\frac{\sin \pi x}{\pi}\right]_0^2 + \left[\frac{\sin(\pi x - \pi)}{\pi}\right]_2^4 + \left[\frac{\sin(\pi x - 2\pi)}{\pi}\right]_4^5$$

$\Rightarrow I = 0$

18. Question ID: 101778

Let PQ be a focal chord of the parabola $y^2 = 4x$ such that it subtends an angle of $\frac{\pi}{2}$ at the point (3, 0). Let the line segment PQ be also a focal chord of the ellipse E: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, a^2 > b^2$. If e is the eccentricity of the ellipse E, then the value of $\frac{1}{e^2}$ is equal to :

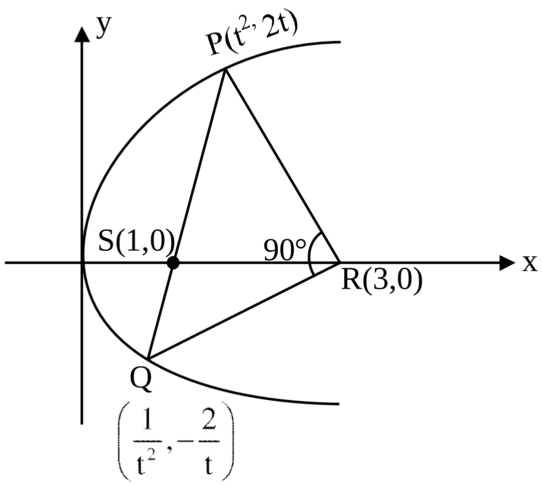
(A) $1+\sqrt{2}$ (B) $3+2\sqrt{2}$

(C) $1+2\sqrt{3}$ (D) $4+5\sqrt{3}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. PQ is focal chord



$m_{PR} \cdot m_{PQ} = -1$

$$\frac{2t}{t^2-3}\times\frac{-2/t}{\frac{1}{t^2}-3}=-1$$

$(t^2-1)^2 = 0$

$\Rightarrow t = 1$

$\Rightarrow$ P & Q must be end point of latus rectum: P(1, 2) & Q(1, –2)

$\therefore \frac{2b^2}{a} = 4$ & ae = 1

$\because$ We know that $b^2 = a^2(1-e^2)$

$\therefore a = 1+\sqrt{2}$

$\because e^2 = 1-\frac{b^2}{a^2}$

$\therefore e^2 = 3-2\sqrt{2}$

$$\frac{1}{e^2} = 3+2\sqrt{2}$$

19. Question ID: 101779

Let the tangent to the circle $C_1 : x^2+y^2 = 2$ at the point $M(-1,1)$ intersect the circle $C_2 : (x-3)^2+(y-2)^2 = 5$, at two distinct points A and B. If the tangents to C_2 at the points A and B intersect at N, then the area of the triangle ANB is equal to :

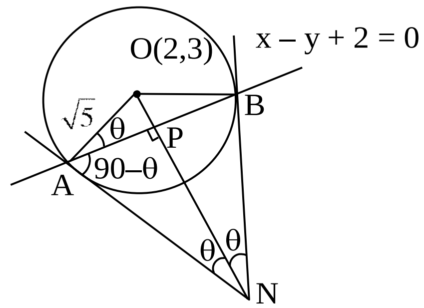
(A) $\frac{1}{2}$ (B) $\frac{2}{3}$

(C) $\frac{1}{6}$ (D) $\frac{5}{3}$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. $OP = \left|\frac{2-3+2}{\sqrt{2}}\right|$



$OP = \frac{3}{\sqrt{2}}$

$AP = \sqrt{OA^2 - OP^2}$

$= \frac{1}{\sqrt{2}}$

$\tan\theta = 3$

$\therefore \sin\theta = \frac{3}{\sqrt{10}} = \frac{AP}{AN}$

$\Rightarrow AN = \frac{\sqrt{5}}{3} = BN$

Area of $\Delta ANB = \frac{1}{2}.(AN^2)\sin 2\theta = \frac{1}{6}$

20. Question ID: 101780

Let the mean and the variance of 5 observations x_1, x_2, x_3, x_4, x_5 be $\frac{24}{5}$ and $\frac{194}{25}$ respectively. If the mean and variance of the first 4 observation are $\frac{7}{2}$ and a respectively, then $(4a + x_5)$ is equal to:

(A) 13 (B) 15

(C) 17 (D) 18

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\bar{x} = \frac{\sum x_i}{5} = \frac{24}{5} \Rightarrow \sum x_i = 24$

$\sigma^2 = \frac{\sum x_i^2}{5} - \left(\frac{24}{5}\right)^2 = \frac{194}{25}$

$\Rightarrow \sum x_i^2 = 154$

$x_1 + x_2 + x_3 + x_4 = 14$

$\Rightarrow x_5 = 10$

$\sigma^2 = \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2}{4} - \frac{49}{4} = a$

$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 4a + 49$

$x_5^2 = 154 - 4a - 49$

$\Rightarrow 100 = 105 - 4a \Rightarrow 4a = 5$

$4a + x_5 = 15$

SECTION-B

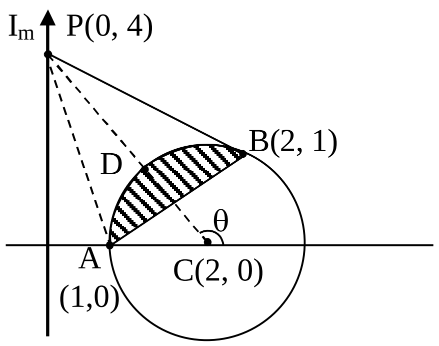
1. Question ID: 101781

Let $S = \{z \in \mathbb{C} : |z-2| \le 1, z(1+i) + \bar{z}(1-i) \le 2\}$. Let $|z - 4i|$ attains minimum and maximum values, respectively, at $z_1 \in S$ and $z_2 \in S$. If $5(|z_1|^2 + |z_2|^2) = \alpha + \beta\sqrt{5}$, where α and β are integers, then the value of α + β is equal to ______.

Official Ans. by NTA (26)

Allen Ans. (26)

Sol. $|z-2| \le 1$



$(x-2)^2 + y^2 \le 1$ ….. (1)

&

$z(1+i) + \bar{z}(1-i) \le 2$

Put $z = x + iy$

$\therefore x - y \le 1$ ….. (2)

$PA = \sqrt{17}$, $PB = \sqrt{13}$

Maximum is PA & Minimum is PD

Let D $(2 + \cos\theta, 0 + \sin\theta)$

$\therefore m_{cp} = \tan\theta = -2$

$\cos\theta = -\frac{1}{\sqrt{5}}$, $\sin\theta = \frac{2}{\sqrt{5}}$

$\therefore D\left(2 - \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

$\Rightarrow z_1 = \left(2 - \frac{1}{\sqrt{5}}\right) + \frac{2i}{\sqrt{5}}$

$|z_1| = \frac{25 - 4\sqrt{5}}{5}$ & $z_2 = 1$

$\therefore |z_2|^2 = 1$

$\therefore 5\left(|z_1|^2 + |z_2|^2\right) = 30 - 4\sqrt{5}$

$\therefore \alpha = 30$

$\beta = -4$

$\therefore \alpha + \beta = 26$

2. Question ID: 101782

Let y = y(x) be the solution of the differential equation

$\frac{dy}{dx} + \frac{\sqrt{2}y}{2\cos^4 x - \cos 2x} = xe^{\tan^{-1}(\sqrt{2}\cot 2x)}, 0 < x < \pi/2$ with $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{32}$.

If $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{18} e^{-\tan^{-1}(\alpha)}$, then the value of $3\alpha^2$ is equal to ______.

Official Ans. by NTA (2)

Allen Ans. (2)

Sol. $\frac{dy}{dx} + \frac{\sqrt{2}}{2\cos^4 x - \cos 2x} y = xe^{\tan^{-1}(\sqrt{2}\cot 2x)}$

$\int \frac{dx}{2\cos^4 x - \cos 2x}$

$= \int \frac{dx}{\cos^4 x + \sin^4 x} = \int \frac{\text{cosec}^4 x\, dx}{1 + \cot^4 x}$

$= -\int \frac{t^2 + 1}{t^4 + 1} dt = -\int \frac{\left(1 + \frac{1}{t^2}\right)}{\left(t - \frac{1}{t}\right)^2 + 2} dt = \frac{-1}{\sqrt{2}} \tan^{-1}\left(\frac{t - \frac{1}{t}}{\sqrt{2}}\right)$

Cotx = t

$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\sqrt{2}\cot 2x\right)$

$\therefore\ IF = e^{-\tan^{-1}(\sqrt{2}\cot 2x)}$

$ye^{-\tan^{-1}(\sqrt{2}\cot 2x)} = \int x\,dx$

$ye^{-\tan^{-1}(\sqrt{2}\cot 2x)} = \frac{x^2}{2} + c$

$y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{32} + c \Rightarrow c = 0$

$y = \frac{x^2}{2} e^{\tan^{-1}(\sqrt{2}\cot 2x)}$

$y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{18} e^{\tan^{-1}\left(\sqrt{2}\cot\frac{2\pi}{3}\right)}$

$= \frac{\pi^2}{18} e^{-\tan^{-1}\left(\sqrt{\frac{2}{3}}\right)}$

$\alpha = \sqrt{\frac{2}{3}} \Rightarrow 3\alpha^2 = 2$

3. Question ID: 101783

Let d be the distance between the foot of perpendiculars of the points P(1, 2 – 1) and Q(2, – 1, 3) on the plane – x + y + z = 1. Then d^2 is equal to ________.

Official Ans. by NTA (26)

Allen Ans. (26)

Sol. Points P(1, 2, –1) and Q (2, –1, 3) lie on same side of the plane.

Perpendicular distance of point P from plane is

$$\left|\frac{-1+2-1-1}{\sqrt{1^2+1^2+1^2}}\right| = \frac{1}{\sqrt{3}}$$

Perpendicular distance of point Q from plane is

$$= \left|\frac{-2-1+3-1}{\sqrt{1^2+1^2+1^2}}\right| = \frac{1}{\sqrt{3}}$$

$\Rightarrow \overrightarrow{PQ}$ is parallel to given plane. So, distance between P and Q = distance between their foot of perpendiculars.

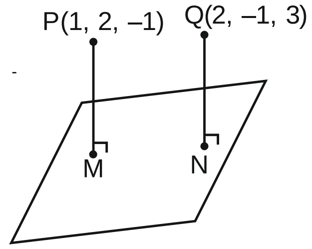
$$\Rightarrow |\overrightarrow{PQ}| = \sqrt{(1-2)^2 + (2+1)^2 + (-1-3)^2}$$

$= \sqrt{26}$

$|\overrightarrow{PQ}|^2 = 26 = d^2$

Alternate

$-x + y + z - 1 = 0$



$M(x_1, y_1, z_1)$

$\frac{x_1-1}{-1} = \frac{y_1-2}{1} = \frac{z_1+1}{1} = \frac{1}{3}$

$x_1 = \frac{2}{3}, y_1 = \frac{7}{3}, z_1 = \frac{-2}{3}$

$M\left(\frac{2}{3}, \frac{7}{3}, \frac{-2}{3}\right)$

$N(x_2, y_2, z_2)$

$\frac{x_2-2}{-1} = \frac{y_2+1}{1} = \frac{z_2-3}{1} = \frac{1}{3}$

$x_2 = \frac{5}{3}, y_2 = \frac{-2}{3},\ z_2 = \frac{10}{3}$

$N = \left(\frac{5}{3}, \frac{-2}{3}, \frac{10}{3}\right)$

$d^2 = 1^2 + 3^2 + 4^2 = 26$

4. Question ID: 101784

The number of elements in the set $S = \{\theta \epsilon [-4\pi, 4\pi] : 3\cos^2 2\theta + 6\cos 2\theta - 10\cos^2\theta + 5 = 0\}$ is ________.

Official Ans. by NTA (32)

Allen Ans. (32)

Sol. $3\cos^2 2\theta + 6\cos 2\theta - 10\cos^2\theta + 5 = 0$

$3\cos^2 2\theta + 6\cos 2\theta - 5(1 + \cos 2\theta) + 5 = 0$

$3\cos^2 2\theta + \cos 2\theta = 0$

$\cos 2\theta = 0$ OR $\cos 2\theta = -1/3$

$\theta \in [-4\pi, 4\pi]$

$2\theta = (2n+1).\frac{\pi}{2}$

$\therefore\ \theta = \pm\pi/4. \pm 3\pi/4 .... \pm 15\pi/4$

Similarly $\cos 2\theta = -1/3$ gives 16 solution

5. Question ID: 101785

The number of solutions of the equation $2\theta - \cos^2\theta + \sqrt{2} = 0$ is R is equal to ______.

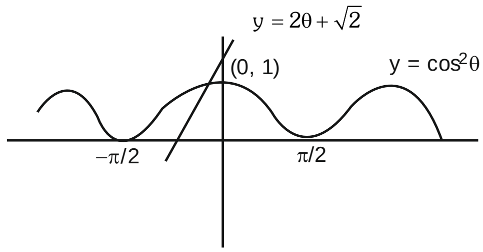
Official Ans. by NTA (1)

Allen Ans. (1)

Sol. $2\theta - \cos^2\theta + \sqrt{2} = 0$

$\Rightarrow \cos^2\theta = 2\theta + \sqrt{2}$

$y = 2\theta + \sqrt{2}$



Both graphs intersect at one point.

6. Question ID: 101786

$50\, tan\left(3tan^{-1}\left(\frac{1}{2}\right) + 2cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right) +$ $4\sqrt{2}\tan\left(\frac{1}{2}tan^{-1}(2\sqrt{2})\right)$ is equal to ______.

Official Ans. by NTA (29)

Allen Ans. (29)

Sol. $50\tan\left(3\tan^{-1}\frac{1}{2} + 2\cos^{-1}\frac{1}{\sqrt{5}}\right)$

$+4\sqrt{2}\tan\left(\frac{1}{2}\tan^{-1}2\sqrt{2}\right)$

$= 50\tan\left(\tan^{-1}\frac{1}{2} + 2(\tan^{-1}\frac{1}{2} + \tan^{-1}2)\right)$

$+4\sqrt{2}\tan\left(\frac{1}{2}\tan^{-1}2\sqrt{2}\right)$

$= 50\tan\left(\tan^{-1}\frac{1}{2} + 2.\frac{\pi}{2})\right) + 4\sqrt{2}\times\frac{1}{\sqrt{2}}$

$= 50\left(\tan\tan^{-1}\frac{1}{2}\right) + 4$

$= 25 + 4 = 29$

7. Question ID: 101787

Let c, k ∈ R. If $f(x) = (c+1)x^2 + (1-c^2)x + 2k$ and $f(x+y) = f(x) + f(y) - xy$, for all x, y ∈ R, then the value of $|2(f(1) + f(2) + f(3) + \ldots\ldots + f(20))|$ is equal to ______.

Official Ans. by NTA (3395)

Allen Ans. (3395)

Sol. $f(x) = (c+1)x^2 + (1-c^2)x + 2k$ ....(1)

$\&\ f(x+y) = f(x) + f(y) - xy \quad \forall\, xy \in R$

$$\lim_{y\to 0}\frac{f(x+y)-f(x)}{y} = \lim_{y\to 0}\frac{f(y)-xy}{y} \Rightarrow f'(x) = f'(0) - x$$

$f(x) = -\frac{1}{2}x^2 + f'(0).x + \lambda$ but $f(0) = 0 \Rightarrow \lambda = 0$

$f(x) = -\frac{1}{2}x^2 + (1-c^2).x$ ....(2)

∴ as $f'(0) = 1 - c^2$

Comparing equation (1) and (2)

We obtain, $c = -\frac{3}{2}$

∴ $f(x) = -\frac{1}{2}x^2 - \frac{5}{4}x$

Now $\left|2\sum_{x=1}^{20} f(x)\right| = \sum_{x=1}^{20} x^2 + \frac{5}{2}.\sum_{x=1}^{20} x$

$= 2870 + 525$

$= 3395$

8. Question ID: 101788

Let $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, a > 0, b > 0, be a hyperbola such that the sum of lengths of the transverse and the conjugate axes is $4\left(2\sqrt{2} + \sqrt{14}\right)$. If the eccentricity H is $\frac{\sqrt{11}}{2}$, then value of $a^2 + b^2$ is equal to __________.

Official Ans. by NTA (88)

Allen Ans. (88)

Sol. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Given $e^2=1+\frac{b^2}{a^2}\Rightarrow\frac{11}{4}=1+\frac{b^2}{a^2}\ \Rightarrow b^2=\frac{7}{4}a^2$

$\therefore\ \frac{x^2}{(a)^2}-\frac{y^2}{\left(\frac{\sqrt{7}}{2}a\right)^2}=1$ Now given

$2a+2.\frac{\sqrt{7}a}{2}=4\left(2\sqrt{2}+\sqrt{14}\right)$

$a\left(2+\sqrt{7}\right)=4\sqrt{2}(2+\sqrt{7})$

$a=4\sqrt{2}\Rightarrow a^2=32$

$b^2=\frac{7}{4}\times16\times2=56$

9. Question ID: 101789

Let $P_1:\vec{r}.(2\hat{i}+\hat{j}-3\hat{k})=4$ be a plane. Let P_2 be another plane which passes through the points (2, -3, 2) (2, − 2, − 3) and (1, −4, 2). If the direction ratios of the line of intersection of P_1 and P_2 be 16, α, β, then the value of α + β is equal to _____.

Official Ans. by NTA (28)

Allen Ans. (28)

Sol. $P_1:\vec{r}.(2\hat{i}+\hat{j}-3\hat{k})=4$

P_1: $2x+y-3z=4$

$$P_2\begin{vmatrix}x-2 & y+3 & z-2\\ 0 & 1 & -5\\ -1 & -1 & 0\end{vmatrix}=0$$

$\Rightarrow -5x+5y+z+23=0$

Let a, b, c be the d's of line of intersection

Then $a=\frac{16\lambda}{15};\ b=\frac{13\lambda}{15};\ c=\frac{15\lambda}{15}$

$\therefore\ \alpha=13:\ \beta=15$

10. Question ID: 101790

Let $b_1b_2b_3b_4$ be a 4-element permutation with $b_i\in\{1, 2, 3, \ldots\ldots, 100\}$ for $1\le i\le4$ and $b_i\ne b_j$ for $i\ne j$, such that either b_1, b_2, b_3 are consecutive integers or b_2, b_3, b_4 are consecutive integers.

Then the number of such permutations $b_1b_2b_3b_4$ is equal to ________.

Official Ans. by NTA (18915)

Allen Ans. (18915)

Sol. $b_i\in\{1,2,3.........100\}$

Let A = set when b_1 b_2 b_3 are consecutive

$n(A)=\frac{97+97+.......+97}{98\text{times}}=97\times98$

Similarly when b_2b_3 b_4 are consecutive

$N(A)=97\times98$

$n(A\cap B)=\frac{97+97+-------97}{98\text{times}}=97\times98$

Similarly when b_2 b_3 b_4 are consecutive

$n(B)=97\times98$

$n(A\cap B)=97$

$n(A\cup B)=n(A)+n(B)-n(A\cap B)$

Number of permutation = 18915