FINAL JEE–MAIN EXAMINATION – JULY, 2022

(Held On Friday 29th July, 2022) **TIME : 9 : 00 AM to 12 : 00 NOON**

MATHEMATICS — TEST PAPER WITH SOLUTION

SECTION-A

1. Let R be a relation from the set {1, 2, 3………,60} to itself such that R = {(a, b) : b = pq, where p, q ≥ 3 are prime numbers}. Then, the number of elements in R is :

(A) 600 (B) 660

(C) 540 (D) 720

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. Number of possible values of a = 60, for b = pq,

If p = 3, q = 3, 5, 7, 11, 13, 17, 19

If p = 5 q = 5, 7, 11

If p = 7 q = 7

Total cases = 60 × 11 = 660

2. If z = 2 + 3i, then $z^5 + (\overline{z})^5$ is equal to :

(A) 244 (B) 224

(C) 245 (D) 265

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $z^5 + (\overline{z})^5 = (2+3i)^5 + (2-3i)^5$

$= 2({}^5C_0 2^5 + {}^5C_2 2^3 (3i)^2 + {}^5C_4 2^1 (3i)^4)$

= 2 (32 + 10 × 8 (– 9) + 5 ×2 × 81) = 244

3. Let A and B be two 3 × 3 non-zero real matrices such that AB is a zero matrix. Then

(A) The system of linear equations AX = 0 has a unique solution

(B) The system of linear equations AX = 0 has infinitely many solutions

(C) B is an invertible matrix

(D) adj (A) is an invertible matrix

Official Ans. by NTA (B)
Allen Ans. (B)

Sol. AB = 0 ⇒ |AB| = 0

|A| |B| = 0

↙ ↘

|A| = 0 |B| = 0

If |A| ≠ 0, B = 0 (not possible)
If |B| ≠ 0, A = 0 (not possible)
Hence |A| = |B| = 0
⇒ AX = 0 has infinitely many solutions

4. If $\frac{1}{(20-a)(40-a)} + \frac{1}{(40-a)(60-a)} + \ldots\ldots + \frac{1}{(180-a)(200-a)} = \frac{1}{256}$, then the maximum value of a is :

(A) 198 (B) 202
(C) 212 (D) 218

Official Ans. by NTA (C)

Allen Ans. (C)

Sol. By splitting

$$\frac{1}{20}\left[\left(\frac{1}{20-a} - \frac{1}{40-a}\right) + \left(\frac{1}{40-a} - \frac{1}{60-a}\right) + \ldots + \left(\frac{1}{180-a} - \frac{1}{200-a}\right)\right]$$

$$\Rightarrow \frac{1}{20}\left(\frac{1}{20-a} - \frac{1}{200-a}\right) = \frac{1}{256}$$

(20 – a) (200 – a) = 256 × 9

$a^2 - 220a + 1696 = 0$

a = 8, 212

Hence maximum value of a is 212.

5. If $\lim_{x\to 0} \frac{\alpha e^x + \beta e^{-x} + \gamma \sin x}{x \sin^2 x} = \frac{2}{3}$,

where α, β,γ ∈ R, then which of the following is NOT correct ?

(A) $\alpha^2 + \beta^2 + \gamma^2 = 6$

(B) αβ + βγ + γα + 1 = 0

(C) $\alpha\beta^2 + \beta\gamma^2 + \gamma\alpha^2 + 3 = 0$

(D) $\alpha^2 - \beta^2 + \gamma^2 = 4$

Official Ans. by NTA (C)

Allen Ans. (C)

Sol.

$$\lim_{x\to 0}\frac{\alpha\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+...\right)+\beta\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+...\right)+\gamma\left(x-\frac{x^3}{3!}+...\right)}{x^3}$$

constant terms should be zero

$\Rightarrow a + \beta = 0$

coeff of x should be zero

$\Rightarrow \alpha - \beta + \gamma = 0$

coeff of x^2 should be zero

$$\lim_{x\to 0}\frac{x^3\left(\frac{\alpha}{3!}-\frac{\beta}{3!}-\frac{\gamma}{3!}\right)+x^4\left(\frac{\alpha}{3!}-\frac{\beta}{3!}-\frac{\gamma}{3!}\right)}{x^3}=\frac{2}{3}$$

$\Rightarrow \frac{\alpha}{2}+\frac{\beta}{2}=0$

$\frac{\alpha}{6}-\frac{\beta}{6}-\frac{\gamma}{6}=2/3$

$\Rightarrow \alpha = 1, \beta = -1, \gamma = -2$

6. The integral $\int_0^{\frac{\pi}{2}}\frac{1}{3+2\sin x+\cos x}dx$ is equal to:

(A) $\tan^{-1}(2)$ (B) $\tan^{-1}(2)-\frac{\pi}{4}$

(C) $\frac{1}{2}\tan^{-1}(2)-\frac{\pi}{8}$ (D) $\frac{1}{2}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol.

$$I=\int_0^{\frac{\pi}{2}}\frac{dx}{3+2\sin x+\cos x}=\int_0^{\frac{\pi}{2}}\frac{\sec^2\frac{x}{2}.dx}{2\tan^2\frac{x}{2}+4\tan\frac{x}{2}+4}$$

Put $\tan\frac{x}{2}=t$, so

$$I=\int_0^1\frac{dt}{(t+1)^2+1}=\tan^{-1}(x+1)\Big|_0^1=\tan^{-1}2-\frac{\pi}{4}$$

7. Let the solution curve y = y (x) of the differential equation $(1 + e^{2x})\left(\frac{dy}{dx}+y\right) = 1$ pass through the point $\left(0,\frac{\pi}{2}\right)$. Then, $\lim_{x\to\infty} e^x y(x)$ is equal to :

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$

(C) $\frac{\pi}{2}$ (D) $\frac{3\pi}{2}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. $\frac{dy}{dx}+y=\frac{1}{1+e^{2x}}$

So integrating factor is $e^{\int 1.dx}=e^x$

So solution is $y\cdot e^x=\tan^{-1}\left(e^x\right)+c$

Now as curve is passing through $\left(0,\frac{\pi}{2}\right)$ so

$\Rightarrow c=\frac{\pi}{4}$

$$\Rightarrow \lim_{x\to\infty}\left(y\cdot e^x\right)=\lim_{x\to\infty}\left(\tan^{-1}\left(e^x\right)+\frac{\pi}{4}\right)=\frac{3\pi}{4}$$

8. Let a line L pass through the point of intersection of the lines bx + 10y – 8 = 0 and 2x – 3y = 0, $b \in R-\left\{\frac{4}{3}\right\}$. If the line L also passes through the point (1, 1) and touches the circle $17(x^2 + y^2) = 16$, then the eccentricity of the ellipse $\frac{x^2}{5}+\frac{y^2}{b^2}=1$ is :

(A) $\frac{2}{\sqrt{5}}$ (B) $\sqrt{\frac{3}{5}}$

(C) $\frac{1}{\sqrt{5}}$ (D) $\sqrt{\frac{2}{5}}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. Line is passing through intersection of $bx+10y-8=0$ and $2x-3y=0$ is $(bx+10y-8)+\lambda(2x-3y)=0$. As line is passing through $(1,1)$ so $\lambda=b+2$

Now line $(3b+4)x-(3b-4)y-8=0$ is tangent to circle $17\left(x^2+y^2\right)=16$

So $\dfrac{8}{\sqrt{(3b+4)^2+(3b-4)^2}}=\dfrac{4}{\sqrt{17}}$

$\Rightarrow b^2=2 \Rightarrow e=\sqrt{\dfrac{3}{5}}$

9. If the foot of the perpendicular from the point A(– 1, 4, 3) on the plane P : 2x + my + nz = 4, is $\left(-2,\frac{7}{2},\frac{3}{2}\right)$, then the distance of the point A from the plane P, measured parallel to a line with direction ratios 3, – 1, – 4, is equal to :

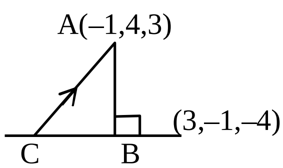
(A) 1 (B) $\sqrt{26}$

(C) $2\sqrt{2}$ (D) $\sqrt{14}$

Official Ans. by NTA (B)

Allen Ans. (B)

Sol.



Let B be foot of ⊥ coordinates of $B=\left(-2,\frac{7}{2},\frac{3}{2}\right)$

Direction ratio of line AB is $<2,1,3>$ so $m=1,\ n=3$

So equation of AC is $\dfrac{x+1}{3}=\dfrac{y-4}{-1}=\dfrac{z-3}{-4}=\lambda$

So point C is $(3\lambda-1,-\lambda+4,-4\lambda+3)$. But C lies on the plane, so

$6\lambda-2-\lambda+4-12\lambda+9=4$

$\Rightarrow \lambda=1 \Rightarrow C(2,3,-1)$

$\Rightarrow AC=\sqrt{26}$

10. Let $\vec{a}=3\hat{i}+\hat{j}$ and $\vec{b}=\hat{i}+2\hat{j}+\hat{k}$. Let $\vec{c}$ be a vector satisfying $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\vec{b}+\lambda\vec{c}$. If $\vec{b}$ *and* $\vec{c}$ are non-parallel, then the value of λ is :

(A) – 5 (B) 5

(C) 1 (D) – 1

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\vec{a}=3\hat{i}+\hat{j},\ \vec{b}=\hat{i}+2\hat{j}+\hat{k}$

As $\vec{a}\times\left(\vec{b}\times\vec{c}\right)=\vec{b}+\lambda\vec{c}$

$\Rightarrow \vec{a}.\vec{c}\left(\vec{b}\right)-\left(\vec{a}.\vec{b}\right)\vec{c}=\vec{b}+\lambda\vec{c}$

$\Rightarrow \vec{a}\cdot\vec{c}=1, \vec{a}\cdot\vec{b}=-\lambda$

$\Rightarrow \left(3\hat{i}+\hat{j}\right).\left(\hat{i}+2\hat{j}+\hat{k}\right)=-\lambda$

$\Rightarrow \lambda=-5$

11. The angle of elevation of the top of a tower from a point A due north of it is α and from a point B at a distance of 9 units due west of A is $\cos^{-1}\left(\frac{3}{\sqrt{13}}\right)$. If the distance of the point B from the tower is 15 units, then cot α is equal to :

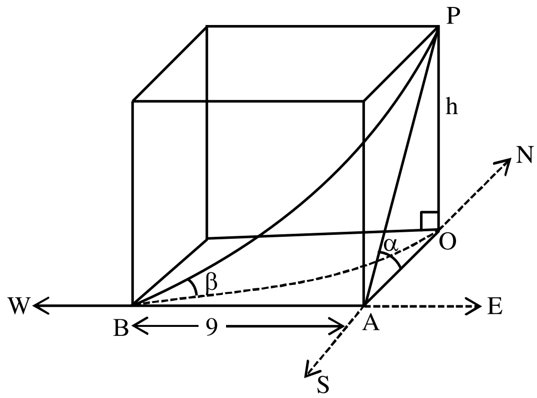
(A) $\frac{6}{5}$ (B) $\frac{9}{5}$

(C) $\frac{4}{3}$ (D) $\frac{7}{3}$

Official Ans. by NTA (A)

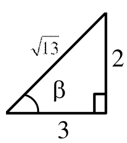
Allen Ans. (A)

Sol.

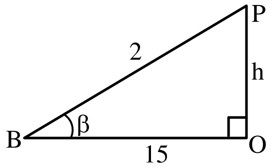


given OB = 15

$\cos\beta=\dfrac{3}{\sqrt{13}}$



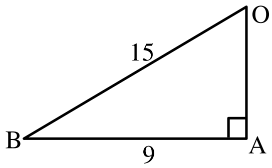
$\tan\beta=\dfrac{2}{3}$



$$\tan\beta = \frac{h}{15}$$

$$\frac{2}{3} = \frac{h}{15}$$

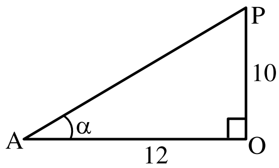
$$\boxed{10 = h}$$



$OA^2 + AB^2 = 225$

$OA^2 + 81 = 225$

$$\boxed{OA = 12}$$



$$\tan\alpha = \frac{10}{12}$$

$$\cot\alpha = \frac{12}{10} = \frac{6}{5}$$

12. The statement $(p \wedge q) \Rightarrow (p \wedge r)$ is equivalent to :

(A) $q \Rightarrow (p \wedge r)$ (B) $p \Rightarrow (p \wedge r)$

(C) $(p \wedge r) \Rightarrow (p \wedge q)$ (D) $(p \wedge q) \Rightarrow r$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $(p \wedge q) \Rightarrow (p \wedge r)$

$\sim(p \wedge q) \vee (p \wedge r)$

$(\sim p \vee \sim q) \vee (p \wedge r)$

$(\sim p \vee (p \wedge r)) \vee \sim q$

$(\sim p \vee p) \wedge (\sim p \vee r) \vee \sim q$

$(\sim p \vee r) \vee \sim q$

$(\sim p \vee \sim q) \vee r$

$\sim(p \wedge q) \vee r$

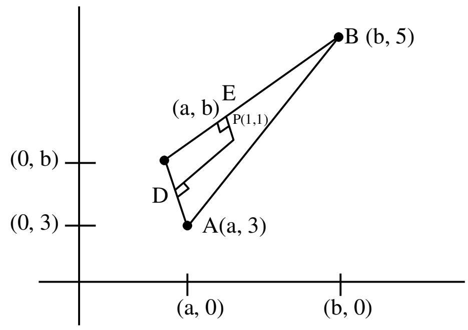
$(p \wedge q) \Rightarrow r$

13. Let the circumcentre of a triangle with vertices A(a, 3), B(b, 5) and C(a, b), ab > 0 be P(1, 1). If the line AP intersects the line BC at the point $Q(k_1, k_2)$, then $k_1 + k_2$ is equal to :

(A) 2 (B) $\frac{4}{7}$ (C) $\frac{2}{7}$ (D) 4

Official Ans. by NTA (B)

Allen Ans. (B)



$m_{AC} \longrightarrow \infty$

$m_{PD} = 0$

$$D\left(\frac{a+a}{2}, \frac{b+3}{2}\right)$$

$$D\left(a, \frac{b+3}{2}\right)$$

$m_{PD} = 0$

$$\frac{b+3}{2} - 1 = 0$$

$b + 3 - 2 = 0$

$$\boxed{b = -1}$$

$$E\left(\frac{b+a}{2}, \frac{5+b}{2}\right) = \left(\frac{af}{2}, 2\right)$$

$m_{CB} \cdot m_{EP} = -1$

$$\left(\frac{5-b}{b-a}\right) = \left(\frac{2-1}{\frac{a-1}{2}-1}\right) = -1$$

$$\left(\frac{6}{-1-a}\right) = \left(\frac{2}{a-3}\right) = -1$$

$12 = (1 + a)(a - 3)$

$12 = a^2 - 3a + a - 3$

$\Rightarrow a^2 - 2a - 15 = 0$

$(a - 5)(a + 3) = 0$

a = 5 or a = –3

Given ab > 0

a(–1) > 0

–a > 0

a < 0

$\boxed{a=-3}$ Accept

AP line A (–3, 3) P(1, 1)

$$y-1=\left(\frac{3-1}{-3-1}\right)(x-1)$$

–2y + 2 = x – 1

$\Rightarrow$ $\boxed{x+2y=3}$ Appling .....(1)

Line BC B(–1, 5)

C(–3, –1)

$$(y-5)=\frac{6}{2}(x+1)$$

y –5 = 3x + 3

$\boxed{y=3x+8}$ .....(2)

Solving (1) & (2)

x + 2 (3x + 8) = 3

$\Rightarrow$ 7x + 16 = 3

7x = –13

$$x=-\frac{13}{7}$$

$$y=3\left(-\frac{13}{7}\right)+8$$

$$=\frac{-39+56}{7}$$

$$y=\frac{17}{7}$$

$$x+y=\frac{-13+17}{7}=\frac{4}{7}$$

14. Let $\hat{a}$ and $\hat{b}$ be two unit vectors such that the angle between them is $\frac{\pi}{4}$. If θ is the angle between the vectors $\left(\hat{a}+\hat{b}\right)$ and $\left(\hat{a}+2\hat{b}+2\left(\hat{a}\times\hat{b}\right)\right)$, then the value of $164\cos^2\theta$ is equal to :

(A) $90+27\sqrt{2}$ (B) $45+18\sqrt{2}$

(C) $90+3\sqrt{2}$ (D) $54+90\sqrt{2}$

Official Ans. by NTA (A)

Allen Ans. (A)

Sol. $\hat{a}\wedge\hat{b}=\frac{\pi}{4}=\phi$

$$\hat{a}.\hat{b}=|\hat{a}||\hat{b}|\cos\phi$$

$$\hat{a}.\hat{b}=\cos\phi=\frac{1}{\sqrt{2}}$$

$$\cos\theta=\frac{\left(\hat{a}+\hat{b}\right).\left(\hat{a}+2\hat{b}+2\left(\hat{a}\times\hat{b}\right)\right)}{\left|\hat{a}+\hat{b}\right|\left|\hat{a}+2\hat{b}+2\left(\hat{a}\times\hat{b}\right)\right|}$$

$$\left|\hat{a}+\hat{b}\right|^2=\left(\hat{a}+\hat{b}\right).\left(\hat{a}+\hat{b}\right)$$

$$\left|\hat{a}+\hat{b}\right|^2=2+2\hat{a}.\hat{b}$$

$$=2+\sqrt{2}$$

$$\hat{a}\times\hat{b}=|\hat{a}||\hat{b}|\sin\phi\,\hat{n}$$

$\hat{a}\times\hat{b}=\frac{\hat{n}}{\sqrt{2}}$ when $\hat{n}$ is vector $\perp$ $\hat{a}$ and $\hat{b}$

let $\vec{c}=\hat{a}\times\hat{b}$

We know.

$\vec{c}.\vec{a}=0$

$\vec{c}.\vec{b}=0$

$$\left|\hat{a}+2\hat{b}+2\vec{c}\right|^2$$

$$=1+4+\frac{(4)}{2}+4\,\hat{a}.\hat{b}+8\hat{b}.\vec{c}+4\vec{c}.\hat{a}$$

$$=7+\frac{4}{\sqrt{2}}=7+2\sqrt{2}$$

Now

$$\left(\hat{a}+\hat{b}\right).\left(\hat{a}+2\hat{b}+2\vec{c}\right)$$

$$=|\hat{a}|^2+2\hat{a}.\hat{b}+0+\hat{b}.\hat{a}+2|\hat{b}|^2+0$$

$$=1+\frac{2}{\sqrt{2}}+\frac{1}{\sqrt{2}}+2$$

$$=3+\frac{3}{\sqrt{2}}$$

$$\cos\theta=\frac{3+\frac{3}{\sqrt{2}}}{\sqrt{2+\sqrt{2}}\sqrt{7+2\sqrt{2}}}$$

$$\cos^2\theta=\frac{9\left(\sqrt{2}+1\right)^2}{2\left(2+\sqrt{2}\right)\left(7+2\sqrt{2}\right)}$$

$\cos^2\theta = \left(\frac{9}{2\sqrt{2}}\right)\frac{\left(\sqrt{2}+1\right)}{\left(7+2\sqrt{2}\right)}$

$164\cos^2\theta = \frac{(82)(9)}{\sqrt{2}}\frac{\left(\sqrt{2}+1\right)\left(7-2\sqrt{2}\right)}{\left(7+2\sqrt{2}\right)\left(7-2\sqrt{2}\right)}$

$= \frac{(82)}{\sqrt{2}}\ \frac{(9)\left[7\sqrt{2}-4+7-2\sqrt{2}\right]}{(41)}$

$= \left(9\sqrt{2}\right)\left[5\sqrt{2}+3\right]$

$= 90 + 27\sqrt{2}$

15. If $f(\alpha) = \int_1^{\alpha}\frac{\log_{10}t}{1+t}dt, \alpha > 0$, then $f(e^3) + f(e^{-3})$ is equal to :

(A) 9 (B) $\frac{9}{2}$

(C) $\frac{9}{\log_e(10)}$ (D) $\frac{9}{2\log_e(10)}$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $f(e^3) = \int_1^{e^3}\frac{\ell n\, t}{\ell n 10(1+t)}dt$......(1)

$f(\alpha) = \int_1^{\alpha}\frac{\ell n\, t}{(\ell n 10)(1+t)}dt$

$t = \frac{1}{x} \Rightarrow x = \frac{1}{t}$

$dt = \frac{-1}{x^2}dx$

$= \int_1^{\frac{1}{\alpha}}\frac{-\ell n\, x}{(\ell n 10)\left(1+\frac{1}{x}\right)}\left(-\frac{1}{x^2}\right)dx$

$f(\alpha) = \frac{1}{\ell n 10}\int_1^{\frac{1}{\alpha}}\frac{\ell n x}{x(x+1)}dx$

$f(e^{-3}) = \frac{1}{\ell n 10}\int_1^{e^3}\frac{\ell n t}{t(t+1)}dt$........(2)

Add (1) & (2)

$f(e^3) + f(e^{-3})$

$= \left(\frac{1}{\ell n 10}\right)\int_1^{e^3}\frac{\ell n t}{(1+t)}\left[1+\frac{1}{t}\right]dt$

$= \left(\frac{1}{\ell n 10}\right)\int_1^{3}\frac{\ell n t}{t}dt$

$\ell n t = r$

$\frac{dt}{t} = dr$

$= \frac{1}{\ell n 10}\int_0^3 r\,dr$

$= \left(\frac{1}{\ell n 10}\right)\left(\frac{r^2}{2}\right)\Bigg|_0^3$

$= \left(\frac{1}{\log 10}\right)\left(\frac{9}{2}\right)$

$= \frac{9}{2\log_e 10}$

16. The area of the region $\left\{(x,y): |x-1| \le y \le \sqrt{5-x^2}\right\}$ is equal to :

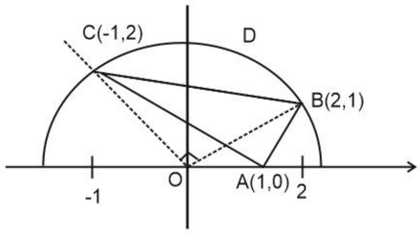
(A) $\frac{5}{2}\sin^{-1}\left(\frac{3}{5}\right)-\frac{1}{2}$ (B) $\frac{5\pi}{4}-\frac{3}{2}$

(C) $\frac{3\pi}{4}+\frac{3}{2}$ (D) $\frac{5\pi}{4}-\frac{1}{2}$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol.



$|x-1| < y < \sqrt{5-x^2}$

When $|x-1| = \sqrt{5-x^2}$

$\Rightarrow (x-1)^2 = 5-x^2$

$\Rightarrow x^2 - x - 2 = 0$

$\Rightarrow x = 2, -1$

Required Area = Area of ΔABC + Area of region BCD

$= \frac{1}{2}\begin{vmatrix}1 & 0 & 1\\ 2 & 1 & 1\\ -1 & 2 & 1\end{vmatrix} + \frac{\pi}{4}\left(\sqrt{5}\right)^2 - \frac{1}{2}\left(\sqrt{5}\right)^2$

$= \frac{5\pi}{4} - \frac{1}{2}$

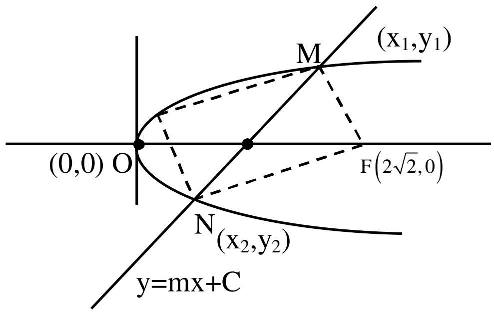
17. Let the focal chord of the parabola P : $y^2 = 4x$ along the line L : y = mx + c, m > 0 meet the parabola at the points M and N. Let the line L be a tangent to the hyperbola H : $x^2 - y^2 = 4$. If O is the vertex of P and F is the focus of H on the positive x-axis, then the area of the quadrilateral OMFN is :

(A) $2\sqrt{6}$ (B) $2\sqrt{14}$

(C) $4\sqrt{6}$ (D) $4\sqrt{14}$

Official Ans. by NTA (B)

Allen Ans. (B)



Sol.

$$H : \frac{x^2}{4} - \frac{y^2}{4} = 1$$

Focus (ae, 0)

$$F(2\sqrt{2}, 0)$$

Line L : y = mx + c pass (1,0)

$\boxed{o = m + C}$ ....(1)

Line L is tangent to Hyperbola. $\frac{x^2}{4} - \frac{y^2}{4} = 1$

$$C = \pm\sqrt{a^2m^2 - \ell^2}$$

$$C = \pm\sqrt{4m^2 - 4}$$

From (1)

$$-m = \pm\sqrt{4m^2 - 4}$$

Squaring

$m^2 = 4m^2 - 4$

$4 = 3m^2$

$\boxed{\frac{2}{\sqrt{3}} = m}$ (as m > 0)

$C = -m$

$$C = \frac{-2}{\sqrt{3}}$$

$$y = \frac{2x}{\sqrt{3}} - \frac{2}{\sqrt{3}}$$

$y^2 = 4x$

$$\Rightarrow \left(\frac{2x-2}{\sqrt{3}}\right)^2 = 4x$$

$\Rightarrow x^2 + 1 - 2x = 3x$

$\Rightarrow \boxed{x^2 - 5x + 1 = 0}$

$$y^2 = 4\left(\frac{\sqrt{3}y + 2}{2}\right)$$

$$y^2 = 2\sqrt{3}\,y + 4$$

$\Rightarrow \boxed{y^2 - 2\sqrt{3}y - 4 = 0}$

Area

$$\left|\frac{1}{2}\begin{vmatrix} 0 & x_1 & 2\sqrt{2} & x_2 & 0 \\ 0 & y_1 & 0 & y_2 & 0 \end{vmatrix}\right|$$

$$= \left|\frac{1}{2}\left[-2\sqrt{2}y_1 + 2\sqrt{2}y_2\right]\right|$$

$$= \sqrt{2}\left|y_2 - y_1\right| = \frac{\left(\sqrt{2}\right)\sqrt{12+16}}{111}$$

$$= \sqrt{56}$$

$$= 2\sqrt{14}$$

18. The number of points, where the function f : **R** → **R**, f (x) = |x – 1| cos |x – 2| sin |x – 1| + (x – 3) $|x^2 - 5x + 4|$, is **NOT** differentiable, is :

(A) 1 (B) 2 (C) 3 (D) 4

Official Ans. by NTA (B)

Allen Ans. (B)

Sol. f(x) = |x – 1| cos |x – 2| sin |x – 1| + (x – 3)$|x^2 - 5x + 4|$

= |x – 1| cos |x – 2| sin |x – 1| + (x – 3)| x – 1||x – 4|

= |x – 1| [cos |x – 2| sin |x – 1| + (x – 3) |x – 4|]

Non differentiable at x = 1 and x = 4.

19. Let S = {1, 2, 3, …, 2022}. Then the probability, that a randomly chosen number n from the set S such that HCF (n, 2022) = 1, is :

(A) $\frac{128}{1011}$ (B) $\frac{166}{1011}$

(C) $\frac{127}{337}$ (D) $\frac{112}{337}$

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. Total number of elements = 2022

$2022 = 2\times3\times337$

HCF (n, 2022) = 1

is feasible when the value of 'n' and 2022 has no common factor.

A = Number which are divisible by 2 from {1,2,3.....2022}

n(A) = 1011

B = Number which are divisible by 3 by 3 from {1,2,3......2022}

n(B) = 674

$A \cap B =$ Number which are divisible by 6

from {1,2,3........2022}

6,12,18........., 2022

$\boxed{337 = n(A \cap B)}$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

= 1011+ 674 –337

= 1348

C= Number which divisible by 337 from {1,........1022}

C= {337,674,1011,1348,1685,20222}

Already counted in Set (A∪B)   Already counted in Set (A∪B)   Already counted in Set (A∪B)

Total elements which are divisible by 2 or 3 or 337

= 1348 +2 = 1350

Favourable cases = Element which are neither divisible by 2, 3 or 337

= 2022 – 1350

= 672

Required probability $= \dfrac{672}{2022} = \dfrac{112}{337}$

20. Let $f(x) = 3^{(x^2-2)^3+4}$, x ∈ **R**. Then which of the following statements are true ?

P : x = 0 is a point of local minima of f

Q : $x = \sqrt{2}$ is a point of inflection of f

R : f' is increasing for $x > \sqrt{2}$

(A) Only P and Q   (B) Only P and R

(C) Only Q and R   (D) All, P, Q and R

Official Ans. by NTA (D)

Allen Ans. (D)

Sol. $f(x) = 81.3^{(x^2-2)^3}$

$f'(x) = 81.3^{(x^2-2)^3}.\ell n3.3(x^2-2)^2.2x$

$= (81\times6)3^{(x^2-2)^3}x(x^2-2)^2\ell n3$

$$\begin{array}{ccccccc} + & & - & & + & & + \\ \hline & -\sqrt{2} & & 0 & & \sqrt{2} & \end{array}$$

x = 6 is point of local min

$f'(x) = \underbrace{(486.\ell n3)}_{k}\underbrace{3^{(x^2-2)^3}x(x^2-2)^2}_{g(x)}$

$g'(x) = 3^{(x^2-2)^3}(x^2-2)^2 + x.3^{(x^2-2)^3}.4x.(x^2-2)$

$+x.(x^2-2)^2.3^{(x^2-2)^3}\ell n3.3(x^2-2)^2.2x$

$= 3^{(x^2-2)^3}(x^2-2)\left[x^2-2+4x^2+6x^2\ell n3(x^2-2)^3\right]$

$g'(x) = 3^{(x^2-2)^3}(x^2-2)\left[5x^2-2+6x^2\ell n3(x^2-2)^3\right]$

$f''(x) = k.g'(x)$

$f''(\sqrt{2}) = 0, f''(\sqrt{2}^+) > 0, f''(\sqrt{2}^-) < 0$

$x = \sqrt{2}$ is point of inflection

$f''(x) > 0$ for $x > \sqrt{2}$ so f'(x) is increasing

SECTION-B

1. Let S = {θ ∈ (0, 2π) : 7 $\cos^2\theta$ – 3 $\sin^2\theta$ – 2 $\cos^2 2\theta$ = 2}. Then, the sum of roots of all the equations $x^2 - 2(\tan^2\theta + \cot^2\theta)x + 6\sin^2\theta = 0$ θ ∈ S, is________.

Official Ans. by NTA (16)

Allen Ans. (16)

Sol. $7\cos^2\theta - 3\sin^2\theta - 2\cos^2 2\theta = 2$

$4\cos^2\theta + 3\cos2\theta - 2\cos^2 2\theta = 2$

$2(1 + \cos2\theta) + 3\cos2\theta - 2\cos^2 2\theta = 2$

$2\cos^2 2\theta - 5\cos2\theta = 0$

$\cos2\theta(2\cos2\theta - 5) = 0$

$\cos2\theta = 0$

$2\theta = (2n+1)\frac{\pi}{2}$

$\theta = (2n+1)\frac{\pi}{4}$

$S = \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$

For all four values of θ

$x^2 - 2(\tan^2\theta + \cot^2\theta)x + 6\sin^2\theta = 0$

$\Rightarrow x^2 - 4x + 3 = 0$

Sum of roots of all four equations = $4 \times 4 = 16$.

2. Let the mean and the variance of 20 observations $x_1, x_2, \ldots x_{20}$ be 15 and 9, respectively. For $\alpha \in R$, if the mean of $(x_1 + \alpha)^2, (x_2 + \alpha)^2, \ldots, (x_{20} + \alpha)^2$ is 178, then the square of the maximum value of α is equal to ______.

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $\sum x_1 = 15 \times 20 = 300 \qquad \ldots(i)$

$\frac{\sum x_1^2}{20} - (15)^2 = 9 \qquad \ldots(ii)$

$\sum x_1^2 = 234 \times 20 = 4680$

$\frac{\sum (x_1+\alpha)^2}{20} = 178 \Rightarrow \sum (x_1+\alpha)^2 = 3560$

$\Rightarrow \sum x_1^2 + 2\alpha \sum x_1 + \sum \alpha^2 = 3560$

$4680 + 600\alpha + 20\alpha^2 = 3560$

$\Rightarrow \alpha^2 + 30\alpha + 56 = 0$

$\Rightarrow (\alpha+28)(\alpha+2) = 0$

$\alpha = -2, -28$

Square of maximum value of α is 4

3. Let a line with direction ratios a, –4a, –7 be perpendicular to the lines with direction ratios 3, –1, 2b and b, a, –2. If the point of intersection of the line $\frac{x+1}{a^2+b^2} = \frac{y-2}{a^2-b^2} = \frac{z}{1}$ and the plane $x - y + z = 0$ is (α, β, γ), then $\alpha + \beta + \gamma$ is equal to __________.

Official Ans. by NTA (10)

Allen Ans. (10)

Sol. $(a,-4a,-7) \perp$ to $(3,-1,2b)$

$a = 2b \qquad \ldots(i)$

$(a,-4a,-7) \perp$ to $(b,a,-2)$

$3a + 4a - 14b = 0$

$ab - 4a^2 + 14 = 0 \qquad \ldots(ii)$

From Equations (i) and (ii)

$2b^2 - 16b^2 + 14 = 0$

$b^2 = 1$

$a^2 = 4b^2 = 4$

$\frac{x+1}{5} = \frac{y-2}{3} = \frac{z}{1} = k$

$\alpha = 5k - 1, \beta = 3k + 2, \gamma = k$

As (α, β, γ) satisfies $x - y + z = 0$

$5k - 1 - (3k + 2) + k = 0$

$k = 1$

$\therefore\ \alpha + \beta + \gamma = 9k + 1 = 10$

4. Let $a_1, a_2, a_3, \ldots$ be an A.P. If $\sum_{r=1}^{\infty} \frac{a_r}{2^r} = 4$, then $4a_2$ is equal to ________.

Official Ans. by NTA (16)

Allen Ans. (16)

Sol. $S = \frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \ldots.$

$\frac{S}{2} = \qquad \frac{a_1}{2^2} + \frac{a_2}{2^3} + \ldots$

$\frac{S}{2} = \frac{a_1}{2} + d\left(\frac{1}{2^2} + \frac{1}{2^3} + \ldots\right)$

$\frac{S}{2} = \frac{a_1}{2} + d\left(\frac{\frac{1}{4}}{1-\frac{1}{2}}\right)$

$\therefore S = a_1 + d = a_2 = 4$

Or $\quad 4a_2 = 16$

5. Let the ratio of the fifth term from the beginning to the fifth term from the end in the binomial expansion of $\left(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}}\right)^n$, in the increasing powers of $\frac{1}{\sqrt[4]{3}}$ be $\sqrt[4]{6} : 1$. If the sixth term from the beginning is $\frac{\alpha}{\sqrt[4]{3}}$, then α is equal to _______.

Official Ans. by NTA (84)

Allen Ans. (84)

Sol. $\frac{T_5}{T_{n-3}} = \frac{{}^nC_4(2^{1/4})^{n-4}(3^{-1/4})^4}{{}^nC_{n-4}(2^{1/4})^4(3^{-1/4})^{n-4}} = \frac{\sqrt[4]{6}}{1}$

$\Rightarrow 2^{\frac{n-8}{4}}3^{\frac{n-8}{4}} = 6^{1/4}$

$\Rightarrow 6^{n-8} = 6$

$\Rightarrow n-8=1 \Rightarrow n=9$

$T_6 = {}^9C_5\,(2^{1/4})^4(3^{-1/4})^5 = \frac{84}{\sqrt[4]{3}}$

$\therefore \alpha = 84$

6. The number of matrices of order 3 × 3, whose entries are either 0 or 1 and the sum of all the entries is a prime number, is______.

Official Ans. by NTA (282)

Allen Ans. (282)

Sol. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} a_{ij} \in \{0,1\}$

$\sum a_{ij} = 2,3,5,7$

Total matrix = ${}^9C_2 + {}^9C_3 + {}^9C_5 + {}^9C_7$

= 282

7. Let p and p + 2 be prime numbers and let

$$\Delta = \begin{vmatrix} p! & (p+1)! & (p+2)! \\ (p+1)! & (p+2)! & (p+3)! \\ (p+2)! & (p+3)! & (p+4)! \end{vmatrix}$$

Then the sum of the maximum values of α and β, such that p^α and $(p+2)^\beta$ divide Δ, is _________.

Official Ans. by NTA (4)

Allen Ans. (4)

Sol. $\Delta = \begin{vmatrix} P! & (P+1)! & (P+2)! \\ (P+1)! & (P+2)! & (P+3)! \\ (P+2)! & (P+3)! & (P+4)! \end{vmatrix}$

$$\Delta = P!(P+1)!(P+2)!\begin{vmatrix} 1 & 1 & 1 \\ P+1 & P+2 & P+3 \\ (P+2)(P+1) & (P+3)(P+2) & (P+4)(P+3) \end{vmatrix}$$

$\Delta = 2P!(P+1)!(P+2)!$

Which is divisible by $P^\alpha \& (P+2)^\beta$

$\therefore \alpha = 3, \beta = 1$

Ans. 4

8. If $\frac{1}{2\times3\times4} + \frac{1}{3\times4\times5} + \frac{1}{4\times5\times6} + \ldots + \frac{1}{100\times101\times102} = \frac{k}{101}$, then 34 k is equal to _____.

Official Ans. by NTA (286)

Allen Ans. (286)

Sol. $\frac{1}{2.3.4} + \frac{1}{3.4.5} + ... + \frac{1}{100.101.102} = \frac{k}{101}$

$\frac{4-2}{2.3.4} + \frac{5-3}{3.4.5} + ... + \frac{102-100}{100.101.102} = \frac{2k}{101}$

$\frac{1}{2.3} - \frac{1}{3.4} + \frac{1}{3.4} - \frac{1}{4.5} + ... + \frac{1}{100.101} - \frac{1}{101.102} = \frac{2k}{101}$

$\frac{1}{2.3} - \frac{1}{101.102} = \frac{2k}{101}$

$\therefore 2k = \frac{101}{6} - \frac{1}{102}$

$\therefore 34k = 286$

9. Let S = {4, 6, 9} and T = {9, 10, 11, …, 1000}. If $A = \{a_1 + a_2 + \ldots + a_k : k \in N, a_1, a_2, a_3, \ldots, a_k \in S\}$, then the sum of all the elements in the set T – A is equal to ______.

Official Ans. by NTA (11)

Allen Ans. (11)

Sol. S={4,6,9} T = {9,10,11….1000}

$A\{a_1 + a_2 + ... + a_k : K \in N\}$ & $a_i \in S$

Here by the definition of set 'A'

$A = \{a : a = 4x + 6y + 9z\}$

Except the element 11, every element of set T is of of the form 4x + 6y + 9z for some x, y, z $\in W$

$\therefore$ T – A = {11}

Ans. 11

10. Let the mirror image of a circle $c_1 : x^2 + y^2 - 2x - 6y + \alpha = 0$ in line y = x + 1 be $c_2 : 5x^2 + 5y^2 + 10gx + 10fy + 38 = 0$. If r is the radius of circle c_2, then $\alpha + 6r^2$ is equal to ________

Official Ans. by NTA (12)

Allen Ans. (12)

Sol. Image of centre $c_1 \equiv (1,3)$ in x – y + 1 = 0 is given by

$$\frac{x_1 - 1}{1} = \frac{y_1 - 3}{-1} = \frac{-2(1-3+1)}{1^2+1^2}$$

$\Rightarrow x_1 = 2, y_1 = 2$

$\therefore$ Centre of circle $c_2 \equiv (2,2)$

$\therefore$ Equation of c_2 be $x^2 + y^2 - 4x - 4y + \frac{38}{5} = 0$

Now radius of c_2 is $\sqrt{4 + 4 - \frac{38}{5}} = \sqrt{\frac{2}{5}} = r$

$(\text{radius of } c_1)^2 = (\text{radius of } c_2)^2$

$\Rightarrow 10 - \alpha = \frac{2}{5} \Rightarrow \alpha = \frac{48}{5}$

$\therefore \alpha + 6r^2 = \frac{48}{5} + \frac{12}{5} = 12$