JEE-Main-24-01-2023 (Memory Based) [Paper-1]

Mathematics

Question: The shortest distance between lines $\frac{x-2}{3}=\frac{y-1}{3}=\frac{z-0}{2}$ and $\frac{x-1}{3}=\frac{y-2}{2}=\frac{z-1}{3}$. is

Options:

(a) $\frac{6}{\sqrt{43}}$

(b) $\frac{11}{\sqrt{43}}$

(c) $\frac{3}{\sqrt{43}}$

(d) $\frac{5}{\sqrt{43}}$

Answer: (b)

Solution:

$a_1 \equiv (2,1,0)$

$a_2 \equiv (1,2,1)$

$\vec{a}_2-\vec{a}_1=-\hat{i}+\hat{j}+\hat{k}$

$\vec{b}_1=3\hat{i}+3\hat{j}+2\hat{k}$

$\vec{b}_2=3\hat{i}+2\hat{j}+3\hat{k}$

$$\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 3 & 2\\ 3 & 2 & 3\end{vmatrix}$$

$=5\hat{i}-3\hat{j}-3\hat{k}$

Shortest distance $=\left|\frac{(\vec{a}_2-\vec{a}_1)\cdot(\vec{b}_1\times\vec{b}_2)}{|\vec{b}_1\times\vec{b}_2|}\right|$

$$\left|\frac{-5-3-3}{\sqrt{25+9+9}}\right|=\frac{11}{\sqrt{43}}$$

Question: Tangent is drawn at a point on the parabola $y^2 = 24x$. It intersects the hyperbola $xy = 2$ at points A and B. Locus of midpoint of AB is

Options:

(a) $y^2 = 3x$

(b) $y^2 = -3x$

(c) $y^2 = 6x$

(d) $y^2 = -6x$

Answer: (b)

Solution:

Let a point on $y^2 = 24x$ be $\left(6t^2, 12t\right)$

Equation of tangent $\equiv 12yt = 12\left(x + 6t^2\right)$

$\Rightarrow ty = x + 6t^2 \qquad ....(1)$

Let mid point of AB be (h, k)

Equation of chord bisect at this point is:

$$\frac{xk + hy}{2} = hk \qquad ....(2)$$

Comparing (1) and (2) we get,

$$\frac{t}{\frac{h}{2}} = \frac{-1}{\frac{k}{2}} = \frac{6t^2}{hk} \qquad ....(3)$$

$$\Rightarrow t = -\frac{h}{k} \text{ and } -2h = 6\left(-\frac{h}{k}\right)^2$$

Required locus is $y^2 = -3x$

Question: If A and B are two non-zero matrices of order $n \times n$ and $A^2 + B = A^2B$, then

Options:

(a) $A^2 = I$ or $B = I$

(b) $AB = I$

(c) $A^2B = BA^2$

(d) None of these

Answer: (c)

Solution:

Subtract I on both sides

$A^2+B-A^2B-I=-I$

$A^2(I-B)+(B-I)=-I$

$(B-I)(-A^2+I)=-I$

$(B-I)(A^2-I)=I$

Inverse of each other

$(B-I)(A^2-I)=(A^2-I)(B-I)$

By expanding

$BA^2=A^2B$

Question: $\sim(\sim p\wedge q)\Rightarrow(\sim p\vee q)$ is equivalent to:

Options:

(a) $\sim p\vee q$

(b) $\sim p\wedge q$

(c) $p\wedge q$

(d) $p\vee q$

Answer: (a)

Solution:

$\sim(\sim p\wedge q)\Rightarrow(\sim p\vee q)$

$=\sim p\vee(\sim p\vee q)\wedge q\vee(\sim p\vee q)$

$=(\sim p\vee q)\wedge(q\vee\sim p)$

$=\sim p\vee q$

Question: Let $f(x)=\begin{cases} x^2\sin\frac{1}{x} & ; \quad x\neq 0 \\ 0 & ; \quad x=0 \end{cases}$, then

Options:

(a) f is continuous and f' is discontinuous at $x=0$

(b) f and f' both are continuous at $x=0$

(c) f and f' both are discontinuous at $x=0$

(d) f is discontinuous and f' is continuous at $x=0$

Answer: (a)

Solution:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin\frac{1}{x} = 0 = f(0)$$

$\therefore\ f(x)$ is continuous at $x=0$

$$f'(x) = 2x\sin\frac{1}{x} + x^2\left(\cos\frac{1}{x}\right)\left(\frac{-1}{x^2}\right)$$

$$f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

$\lim_{x\to 0} f'(x)$ does not exists

$\therefore\ f'(x)$ is discontinuous at $x=0$

Question: $\frac{8}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^{2023} x}{\sin^{2023} x + \cos^{2023} x} =$ _____.

Answer: 2.00

Solution:

Given, $\frac{8}{\pi}\int_0^{\frac{\pi}{2}} \frac{\sin^{2023} x}{\sin^{2023} x + \cos^{2023} x}$

$$\int_0^{\frac{\pi}{2}} \frac{\sin^{2023} x}{\sin^{2023} x + \cos^{2023} x}$$

Take $x = \frac{\pi}{2} - x$

$$I = \int_0^{\frac{\pi}{2}} \frac{\cos^{2023} x}{\cos^{2023} x + \sin^{2023} x}$$

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^{2023} x + \cos^{2023} x}{\sin^{2023} x + \cos^{2023} x}dx$$

$$2I = (x)_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}$$

$$\frac{8}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sin^{2023}x}{\sin^{2023}x+\cos^{2023}x}=\frac{8}{\pi}\times\frac{\pi}{4}=2$$

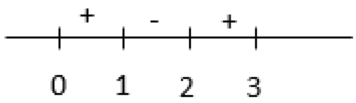
Question: $12\int_0^3\left|x^2-3x+2\right|dx$ is equal to

Answer: 22.00

Solution:

$$\int_0^3\left|x^2-3x+2\right|dx$$

$$=\int_0^3\left|(x-1)(x-2)\right|dx$$



$$=\int_0^1\left(x^2-3x+2\right)dx-\int_1^2\left(x^2-3x+2\right)dx+\int_2^3\left(x^2-3x+2\right)dx$$

$$=\frac{x^3}{3}-\frac{3x^2}{2}+2x\Big|_0^1-\frac{x^3}{3}-\frac{3x^2}{2}+2x\Big|_1^2+\frac{x^3}{3}-\frac{3x^2}{2}+2x\Big|_2^3$$

$$=\left(\frac{1}{3}-\frac{3}{2}+2\right)-\left\{\left(\frac{8}{3}-6+4\right)-\left(\frac{1}{3}-\frac{3}{2}+2\right)\right\}+\left\{\left(9-\frac{27}{2}+6\right)-\left(\frac{8}{3}-6+4\right)\right\}$$

$$=\left(\frac{5}{6}\right)-\left\{\frac{2}{3}-\frac{5}{6}\right\}+\left\{\frac{3}{2}-\frac{2}{3}\right\}$$

$$=\frac{10}{6}-\frac{8}{9}+\frac{9}{6}$$

$$=\frac{11}{6}$$

$$12\int_0^3\left|x^2-3x+2\right|dx=12\times\frac{11}{6}=22$$

Question: Out of 12 subjects, there are 5 languages & 7 others. 5 has to be chosen such that 2 language subjects are chosen.

Answer: 546.00

Solution:

Case-1:

If no language is selected from given 5 particular languages

$\Rightarrow {}^{7}C_{5}$

Case-2

If 1 language is chosen from the given 5

$\Rightarrow {}^{7}C_{4}.{}^{5}C_{1}$

Case-3

If 2 languages are chosen from the given 5

$\Rightarrow {}^{7}C_{3}.{}^{5}C_{2}$

$\therefore$ Total ways $= {}^{7}C_{5} + {}^{7}C_{4}.{}^{5}C_{1} + {}^{7}C_{3}.{}^{5}C_{2}$

$= 21 + 175 + 350$

$= 546$

Question: $\lim_{t\to 0}\left[1^{\frac{1}{\sin^2 t}} + 2^{\frac{1}{\sin^2 t}} + 3^{\frac{1}{\sin^2 t}} + ... + n^{\frac{1}{\sin^2 t}}\right]^{\sin^2 t} =$ _____.

Answer: n

Solution:

$$\lim_{t\to 0}\left(\left(\frac{1}{n}\right)^{\operatorname{cosec}^2 t} + \left(\frac{2}{n}\right)^{\operatorname{cosec}^2 t} + ... + \left(\frac{n}{n}\right)^{\operatorname{cosec}^2 t}\right)^{\sin^2 t}$$

$$= \lim_{t\to 0}(n)(0 + 0 + ... + 1)$$

$$= \lim_{t\to 0}(n)\times 1$$

$$= n$$

Question: $\sum_{r=0}^{2023} r^2 \times {}^{2023}C_r = 2023 \times 2^{2022}\alpha$ then $\alpha = ?$

Answer: 1012.00

Solution:

$$\sum_{r=0}^{2023} r^2\, {}^{2023}C_r = (2023)2^{2022}\alpha$$

$$2023\sum_{r=1}^{2023} r\, {}^{2022}C_{r-1} = 2023\left[\sum (r-1)\, {}^{2022}C_{r-1} + \sum {}^{2022}C_{r-1}\right]$$

$$2023\left[2022 \cdot 2^{2021} + 2^{2022}\right] = 2023 \cdot 2^{2022}\alpha$$

$$2^{2022}[1011 + 1] = 2^{2022}\alpha$$

$\alpha = 1012$

Question: $\sum_{r=0}^{22} {}^{22}C_r \times {}^{23}C_r = ?$

Answer: ${}^{45}C_{23}$

Solution:

$$\sum_{r=0}^{22} {}^{22}C_r \times {}^{23}C_r = \sum_{r=0}^{22} {}^{22}C_r \, {}^{23}C_{23-r}$$

$$= {}^{22}C_0 \, {}^{23}C_{23} + {}^{22}C_1 \, {}^{23}C_{22} + \ldots + {}^{22}C_{21} \, {}^{23}C_2 + {}^{22}C_{22} \, {}^{23}C_1$$

$$(1+x)^{22} = {}^{22}C_0 + {}^{22}C_1 x + {}^{22}C_2 x + {}^{22}C_3 x^3 + \ldots + {}^{22}C_{21} x^{21} + {}^{22}C_{22} x^{22}$$

$$(1+x)^{23} = {}^{23}C_0 + {}^{23}C_1 x + {}^{23}C_2 x^2 + {}^{23}C_3 x^3 + \ldots + {}^{23}C_{22} x^{22} + {}^{23}C_{23} x^{23}$$

Coefficient of x^{23} in $(1+x)^{22}(1+x)^{23}$

$$= \sum_{r=0}^{22} {}^{22}C_r \, {}^{23}C_{23-r}$$

$$= {}^{45}C_{23}$$

Question: $1,1,1,3,3,4,4,2,2$ form 9 digit number in which even digits are at even places.

Answer: 60.00

Solution:

4 even places 4, 4, 2, 2

$$\left(\frac{4!}{2!2!}\right)$$

5 odd places 1, 1, 1, 3, 3

$$\left(\frac{5!}{3!2!}\right)$$

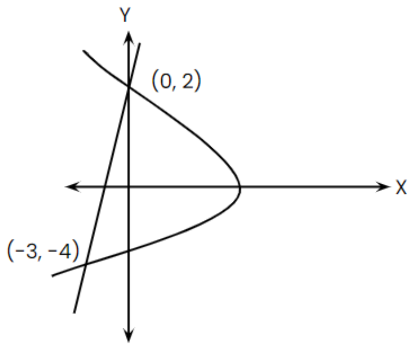
$$\left(\frac{4!}{2!2!}\right) \times \left(\frac{5!}{3!2!}\right)$$

$$= 60$$

Question: Find the area under the curve between parabola $y^2 + 4x = 4$ and $2x = y + 2$ is

Answer: 9.00

Solution:



Required area $= \int_{-4}^{2}\left(\frac{4-y^2}{4}-\frac{y-2}{2}\right)dy$

$=\left[2y-\frac{y^3}{12}-\frac{y^2}{4}\right]_{-4}^{2}$

$=\left(4-\frac{8}{12}-1\right)-\left(-8+\frac{16}{3}-4\right)$

$=\left(3-\frac{2}{3}\right)+12-\frac{16}{3}$

$=9$

Question: $\tan^{-1}\left(\frac{1+\sqrt{3}}{3+\sqrt{3}}\right)+\sec^{-1}\sqrt{\frac{8+4\sqrt{3}}{6+3\sqrt{3}}}=?$

Answer: $\frac{\pi}{3}$

Solution:

$$\tan^{-1}\left(\frac{1+\sqrt{3}}{\sqrt{3}\left(1+\sqrt{3}\right)}\right)+\sec^{-1}\sqrt{\frac{4\left(2+\sqrt{3}\right)}{3\left(2+\sqrt{3}\right)}}$$

$$\tan^{-1}\frac{1}{\sqrt{3}}+\sec^{-1}\frac{2}{\sqrt{3}}=\frac{\pi}{6}+\frac{\pi}{6}=\frac{\pi}{3}$$

Question: $\left(1-\sqrt{3}i\right)^{100}=2^{99}\left(p+qi\right)$. $\left(p-q+q^2\right),\left(p+q+q^2\right)$, are roots of the equation

Answer:

Solution:

$\left(1-\sqrt{3}i\right)^{100}=2^{99}\left(p+iq\right)$

LHS = $\left(2\left[\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right]\right)^{100}$

$=2^{100}\left[\cos\left(\frac{100\pi}{3}\right)-i\sin\left(\frac{100\pi}{3}\right)\right]$

$=200\left(-\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

$=2^{100}\left(\frac{-1}{2}+\frac{\sqrt{3}i}{2}\right)$

$=2^{99}\left(-1+\sqrt{3}i\right)$

$=2^{99}\left(p+qi\right)$

$p=-1\ \&\ q=\sqrt{3}$

Roots are $p-q+q^2\ \ \&\ \ p+q+q^2$

$2-\sqrt{3}\ \&\ 2+\sqrt{3}$

Equation is $x^2-4x+1=0$

Question: Circle with centre (2, 0) is inside $\frac{x^2}{36}+\frac{y^2}{16}=1$. If circle pass through $\left(1,\alpha\right)$, then $10\alpha^2=$

Answer: 118.00

Solution:

Normal to ellipse at $P\left(6\cos\theta,4\sin\theta\right)$

$6x\sec\theta-4y\,\mathrm{cosec}\,\theta=20$

$\left(2,0\right)$ satisfies if

$12\sec\theta=20\Rightarrow\cos\theta=\frac{3}{5},\ \sin\theta=\frac{4}{5}$

$P=\left(\frac{18}{5},\frac{16}{5}\right)$

Circle: $\left(x-2\right)^2+y^2=\left(\frac{18}{5}-2\right)^2+\left(\frac{16}{5}\right)^2$

$(1,\infty)$ satisfies

$$1+\alpha^2=\frac{64}{25}+\frac{256}{25}=\frac{320}{25}$$

$$1+\alpha^2=\frac{64}{5}$$

$$10+10\alpha^2=128$$

$$\Rightarrow 10\alpha^2=118$$

Question: If $x^3dy+(xy-1)dx=0$ and $y\left(\frac{1}{2}\right)=(3-e)$, then $y(1)=?$

Answer: 1.00

Solution:

Given, $x^3dy+(xy-1)dx=0$ and $y\left(\frac{1}{2}\right)=(3-e)$

$$\frac{dy}{dx}+\frac{xy-1}{x^3}=0$$

$$\frac{dy}{dx}+\frac{y}{x^2}=\frac{1}{x^3}$$

$$\text{IF }=e^{\frac{-1}{x}}$$

$$ye^{\frac{-1}{x}}=\int e^{\frac{-1}{x}}\frac{1}{x^3}dx$$

$$=\int e^{\frac{-1}{x}}\left(\frac{1}{x}\right)d\left(\frac{-1}{x}\right)$$

$$=e^{\frac{-1}{x}}\left[\frac{1}{x}+1\right]+c$$

$$y=\frac{1}{x}+1+ce^{\frac{1}{x}}$$

$$3-e=3+ce^2$$

$$x=\frac{-1}{e}$$

$$y(1)=2-1=1$$