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JEE Main 2023 (Memory based)

1st February 2023 - Shift 2

Answer & Solutions

MATHEMATICS

1. If the term independent of x in the expansion of $\left(x^{\frac{2}{3}}+\frac{\alpha}{x^3}\right)^{22}$ is 7315, then $|\alpha|$ is:

A. 1
B. 2
C. 0
D. 3

Answer (A)

Solution:

$T_{r+1} = {}^{22}C_r\left(x^{\frac{2}{3}}\right)^{22-r}\left(\frac{\alpha}{x^3}\right)^r$

$\Rightarrow \frac{2(22-r)}{3} - 3r = 0$

$\Rightarrow 44 - 2r - 9r = 0$

$\Rightarrow r = 4$

$\therefore T_5 = {}^{22}C_4\alpha^4 = 7315$

$\Rightarrow \alpha^4 = \frac{7315}{7315} = 1$

$\Rightarrow |\alpha| = 1$

2. The value of $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x+\frac{\pi}{4}}{2-\cos 2x}\,dx$ is:

A. $\frac{3\pi^2}{\sqrt{6}}$
B. $\sqrt{3}\pi^2$
C. $\frac{\pi^2}{6\sqrt{3}}$
D. $\frac{6\pi^2}{\sqrt{3}}$

Answer (C)

Solution:

$I = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x\,dx}{2-\cos 2x} + \frac{\pi}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{2-\cos 2x}$

$$= 0 + \frac{\pi}{4} \cdot 2 \int_0^{\frac{\pi}{4}} \frac{dx}{2-\cos 2x}$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{4}} \frac{\sec^2 x dx}{1+3\tan^2 x}$$

Now, $\tan x = t$

$$= \frac{\pi}{2} \int_0^1 \frac{dt}{1+3t^2}$$

$$= \frac{\pi}{2} \left[\frac{\tan^{-1}(\sqrt{3}t)}{\sqrt{3}}\right]_0^1$$

$$= \frac{\pi}{2\sqrt{3}} \left(\frac{\pi}{3}\right) = \frac{\pi^2}{6\sqrt{3}}$$

3. The area determined by $xy < 8$, $y < x^2$ and $y > 1$ is:

A. $4 \ln 2 - \frac{14}{3}$

B. $4 \ln 2 + \frac{20}{3}$

C. $8 \ln 4 - \frac{14}{3}$

D. $8 \ln 4 - \frac{20}{3}$

Answer (C)

Solution:

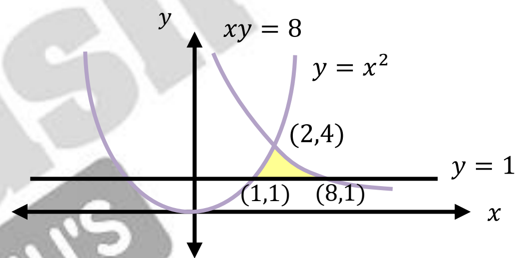
Area $= \int_1^2 (x^2 - 1)\, dx + \int_2^8 \left(\frac{8}{x} - 1\right) dx$

$$= \frac{x^3}{3} - x\Big|_1^2 + (8 \ln x - x)|_2^8$$

$$= \left(\frac{8}{3} - 2\right) - \left(\frac{1}{3} - 1\right) + (8 \ln 8 - 8) - (8 \ln 2 - 2)$$

$$= \frac{4}{3} + 8 \ln 4 - 6$$

$$= 8 \ln 4 - \frac{14}{3}$$



4. If $f(x) + f\left(\frac{1}{1-x}\right) = 1 - x$, then $f(2)$ equals:

A. $\frac{1}{4}$

B. $-\frac{5}{4}$

C. $\frac{3}{4}$

D. $-\frac{3}{4}$

Answer (B)

Solution:

$f(x) + f\left(\frac{1}{1-x}\right) = 1 - x \cdots (i)$

Put $x = 2$ in (i)

$f(2) + f(-1) = -1 \cdots (ii)$

Put $x = -1$ in (i)

$f(-1) + f\left(\frac{1}{2}\right) = 2 \cdots (iii)$

Put $x = \frac{1}{2}$ in (i)

$f\left(\frac{1}{2}\right) + f(2) = \frac{1}{2} \cdots (iv)$

$(ii) + (iv) - (iii)$ gives,

$f(2) + f(-1) + f\left(\frac{1}{2}\right) + f(2) - f(-1) - f\left(\frac{1}{2}\right) = -1 + \frac{1}{2} - 2$

$\Rightarrow 2f(2) = -1 + \frac{1}{2} - 2$

$\Rightarrow 2f(2) = -\frac{5}{2}$

$\Rightarrow f(2) = -\frac{5}{4}$

5. If $f(x) = x^x, x > 0$ then $f''(2) + f'(2)$ is:

A. $10 + 12\ln 2 + 4(\ln 2)^2$

B. $10 + 4(\ln 2)^2$

C. $10 + 12\ln 2$

D. $2^{\ln 2} + (\ln 2)^2$

Answer (A)

Solution:

$f(x) = x^x$

$f'(x) = x^x(1 + \ln x)$

$\therefore f'(2) = 4(1 + \ln 2)$

$f''(x) = \frac{x^x}{x} + x^x(1 + \ln x)^2$

$\Rightarrow f''(2) = 2 + 4(1 + \ln 2)^2$

$\Rightarrow f''(2) + f'(2) = 4 + 4\ln 2 + 6 + 8\ln 2 + 4(\ln 2)^2$

$\Rightarrow f''(2) + f'(2) = 10 + 12\ln 2 + 4(\ln 2)^2$

6. Which of the following is a tautology?

A. $p \to (\sim p \wedge q)$

B. $p \to (p \vee q)$

C. $p \to (\sim p \vee q)$

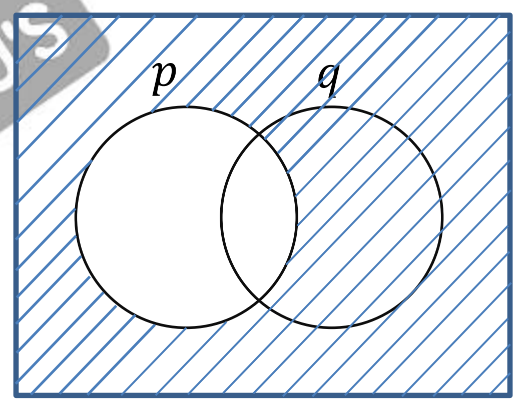
D. $p \to (\sim p \wedge \sim q)$

Answer (B)

Solution:

$a.\ \ p \to (\sim p \wedge q)$

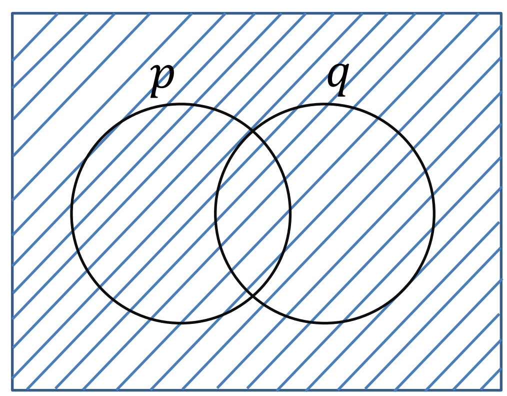
$\cong (\sim p) \vee (\sim p \wedge q)$



$b.\ \ p \to (p \vee q)$

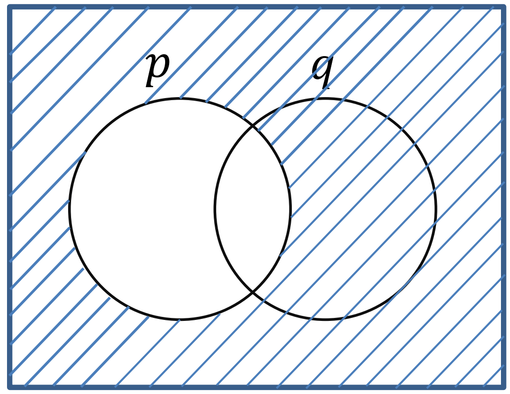
$\cong (\sim p) \vee (p \vee q)$

It can be inferred from the diagram it represents tautology.



c. $p \to (\sim p \vee q)$

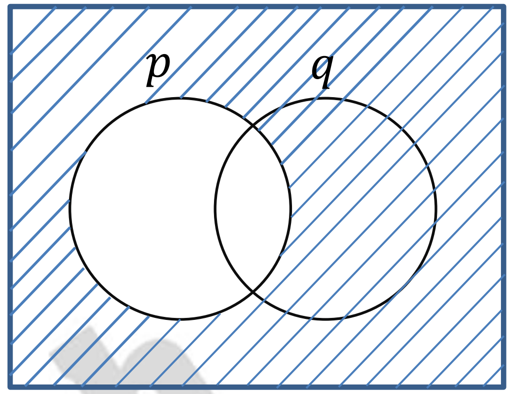
$\cong (\sim p) \vee (\sim p \vee q)$



d. $p \to (\sim p \wedge \sim q)$

$\cong (\sim p) \vee (\sim p \wedge \sim q)$

$\cong (\sim p) \vee (p \vee q)'$



7. If the system of equations

$\alpha x + y + z = 1,$

$x + \alpha y + z = 1,$

$x + y + \alpha z = \beta$ has infinitely many solutions, then:

A. $\alpha = 1,\ \beta = 1$

B. $\alpha = 1,\ \beta = -1$

C. $\alpha = -1,\ \beta = -1$

D. $\alpha = -1,\ \beta = 1$

Answer (A)

Solution:

For infinite solutions

$\Delta = \Delta_x = \Delta_y = \Delta_z = 0$

$$\Rightarrow \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & 1 \\ \beta & 1 & \alpha \end{vmatrix} = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & 1 & 1 \\ 1 & \beta & \alpha \end{vmatrix} = \begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \beta \end{vmatrix} = 0$$

Clearly $\alpha = 1 = \beta$ makes all the equations identical i.e., three coincidence planes.

$\therefore \alpha = 1 = \beta$

8. If $A = \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$, then which of the following is true?

A. $A^{30} = A^{25}$

B. $A^{30} + A^{25} + A = I$

C. $A^{30} - A^{25} + A = I$

D. $A^{30} = A^{25} + A$

Answer (C)

Solution:

$$A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

(a) $A^{30} = (A^3)^{10} = (-I)^{10} = I$

$A^{25} = (A^3)^8 \cdot A = (-I) \cdot A = A$

$\Rightarrow A^{30} \neq A^{25}$

(b) $A^{30} + A^{25} + A = I + A + A = I + 2A \neq I$

(c) $A^{30} - A^{25} + A = I - A + A = I$

(d) $A^{30} - A^{25} - A = I - A - A = I - 2A \neq 0$

9. 2 unbiased die are thrown independently. A is the event such that the number on the first die is less than second die. B is the event, such that number on the first die is even and number on the second die is odd. C is the event such that first die shows odd number and second die shows even number. Then:

A. $n\big((A \cup B) \cap C\big) = 6$

B. A and B are mutually exclusive events

C. A and B are independent events

D. $n(A) = 18$, $n(B) = 6$, $n(C) = 6$

Answer (A)

Solution:

$A = \{(1,2),\ (1,3),\ (1,4),\ (1,5),\ (1,6), (2,3),\ (2,4),\ (2,5),\ (2,6)(3,4),\ (3,5),\ (3,6)(4,5),\ (4,6), (5,6)\}$

$n(A) = 15$

$B = \{(2,1),\ (2,3),\ (2,5),\ (4,1),\ (4,3),\ (4,5),\ (6,1),\ (6,3),\ (6\ 5)\}$

$n(B) = 9$

$C = \{(1,2), (1,4),\ (1,6),\ (3,2),\ (3,4),\ (3,6),\ (5,2),\ (5,4),\ (5,6)\}$

$n(C) = 9$

$\big((A \cup B) \cap C\big) = \{(1,2), (1,4), (1,6), (3,4), (3,6), (5,6)\}$

$\Rightarrow n\big((A \cup B) \cap C\big) = 6$

$A \cap B = \{(2,3),\ (2,5),\ (4,5)\}$

$P(A \cap B) = \frac{3}{36} = \frac{1}{12}$

$P(A)=\frac{15}{36},\ P(B)=\frac{9}{36},\ P(C)=\frac{9}{36}$

$P(A)\cdot P(B)=\frac{15}{36}\cdot\frac{9}{36}=\frac{5}{48}$

$\Rightarrow A$ and B are not independent events

10. If $\frac{dy}{dx}=\frac{x^2+3y^2}{3x^2+y^2},\ y(1)=0$, then:

A. $\frac{2x^2}{(x-y)^2}=\ln|x-y|+\frac{2x}{x-y}$

B. $\frac{2x}{(x-y)^2}=\ln|x-y|+1$

C. $\frac{2x^2}{(x-y)^2}=\ln|x-y|+\frac{y}{x-y}$

D. $\frac{2x}{(x-y)^2}=\ln|x-y|+\frac{y}{x-y}$

Answer (A)

Solution:

$\frac{dy}{dx}=\frac{x^2+3y^2}{3x^2+y^2}$

Let $y=vx$

$\Rightarrow \frac{dy}{dx}=v+x\frac{dv}{dx}$

$\Rightarrow v+x\frac{dv}{dx}=\frac{1+3v^2}{3+v^2}$

$\Rightarrow x\frac{dv}{dx}=\frac{1+3v^2}{3+v^2}-v=\frac{-v^3+3v^2-3v+1}{v^2+3}$

$\Rightarrow \frac{(v^2+3)}{-v^3+3v^2-3v+1}dv=\frac{1}{x}dx$

$\Rightarrow \int\frac{(v^2+3)}{(1-v)^3}dv=\frac{1}{x}dx$

$\Rightarrow \int\frac{1}{(1-v)}dv-\int\frac{2}{(1-v)^2}dv+\int\frac{4}{(1-v)^3}dv=\int\frac{1}{x}dx$

$\Rightarrow -\ln|1-v|-\frac{2}{(1-v)}+\frac{2}{(1-v)^2}=\ln|x|+C$

$\because y(1)=0\Rightarrow v(1)=0\Rightarrow C=0$

$\therefore \frac{2}{\left(1-\frac{y}{x}\right)^2}=\ln\left|1-\frac{y}{x}\right|+\frac{2}{1-\frac{y}{x}}+\ln x$

$\Rightarrow \frac{2x^2}{(x-y)^2}=\ln|x-y|+\frac{2x}{x-y}$

11. $\vec{a}=\hat{\imath}-\hat{\jmath}+\hat{k},\vec{b}=2\hat{\imath}-3\hat{\jmath}+\hat{k},\vec{c}=4\hat{\imath}+5\hat{\jmath}-\hat{k}$. If $\vec{r}.\vec{b}=0$ and $\vec{r}\times\vec{a}=\vec{b}\times\vec{c}$, then $\vec{r}$ is equal to:

A. $-12\hat{\imath}-8\hat{\jmath}+\hat{k}$

B. $-12\hat{\imath}-\frac{23}{3}\hat{\jmath}+\hat{k}$

C. $12\hat{\imath}+\frac{23}{3}\hat{\jmath}+\hat{k}$

D. $12\hat{\imath}+8\hat{\jmath}+\hat{k}$

Answer (B)

Solution:

$\vec{r}\times\vec{a}=\vec{b}\times\vec{c}$

$\vec{b} \times (\vec{r} \times \vec{a}) = \vec{b} \times (\vec{b} \times \vec{c})$
$\Rightarrow (\vec{b}.\vec{a})\vec{r} - (\vec{b}.\vec{r})\vec{a} = (\vec{b}.\vec{c})\vec{b} - (\vec{b}.\vec{b})\vec{c}$
$\Rightarrow 6\vec{r} = -8(2\hat{\imath} - 3\hat{\jmath} + \hat{k}) - 14(4\hat{\imath} + 5\hat{\jmath} - \hat{k})$
$\Rightarrow 6\vec{r} = -72\hat{\imath} - 46\hat{\jmath} + 6\hat{k}$
$\Rightarrow \vec{r} = -12\hat{\imath} - \frac{23}{3}\hat{\jmath} + \hat{k}$

12. If $2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$, $x \in (0,\ 1)$ has:

A. 2 solutions for $x < \frac{1}{2}$

B. 2 solutions for $x > \frac{1}{2}$

C. 1 solutions for $x < \frac{1}{2}$

D. 1 solutions for $x > \frac{1}{2}$

Answer (C)

Solution:

$2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta$ we get,

$2\tan^{-1}\left(\tan\left(\frac{\pi}{4} - \theta\right)\right) = \cos^{-1}\cos 2\theta$

$\Rightarrow 2\left(\frac{\pi}{4} - \theta\right) = 2\theta$

$\Rightarrow \theta = \frac{\pi}{8}$

$\Rightarrow \tan^{-1} x = \frac{\pi}{8}$

$\Rightarrow x = \tan\frac{\pi}{8} = \sqrt{2} - 1$

$\therefore x = \sqrt{2} - 1$ only 1 solution for $x < \frac{1}{2}$

13. Number of non-negative integral solutions of $x + y + z = 21$ if $x \geq 1,\ y \geq 3,\ z \geq 6$ are __________.

Answer (78)

Solution:

$\because x + y + z = 21\ \ [x \geq 1,\ y \geq 3,\ z \geq 6]$
$\Rightarrow (x-1) + (y-3) + (z-6) = 11$
$\Rightarrow x_1 + y_1 + z_1 = 11$
Where, $x_1 \geq 0,\ y_1 \geq 0,\ z_1 \geq 0$
Total ${}^{11+3-1}C_{3-1}$ solutions
${}^{13}C_2 = \frac{13!}{2!11!} = 6 \times 13 = 78$

14. Total 6 digit numbers using the digits 4, 5, 9 which are divisible by 6 are __________.

Answer (81)

Solution:

We have,
For this, 4 will be fixed as unit place digit

Total number

Case I: $4's \rightarrow 6$ times 1

Case II: $4's \rightarrow 4$ times
$5's \rightarrow 1$ times $\frac{5!}{3!} = 20$
$9's \rightarrow 1$ times

Case III: $4's \rightarrow 3$ times
$5's \rightarrow 3$ times $\frac{5!}{2!3!} = 10$

Case IV: $4's \rightarrow 3$ times
$9's \rightarrow 3$ times $\frac{5!}{2!3!} = 10$

Case V: $4's \rightarrow 2$ times
$5's \rightarrow 2$ times $\frac{5!}{2!2!} = 30$
$9's \rightarrow 2$ times

Case VI: $4's \rightarrow 1$ times
$5's \rightarrow 1$ times $\frac{5!}{4!} = 5$
$9's \rightarrow 4$ times

Case VII: $4's \rightarrow 1$ times
$5's \rightarrow 4$ times $\frac{5!}{4!} = 5$
$9's \rightarrow 1$ times

Total numbers $= 81$

15. Let $3\ A.P.'s$ be

$S_1 = 2, 5, 8, 11, \cdots 394$

$S_2 = 1, 3, 5, 7, \cdots$

And $S_3 = 2,\ 7,\ 12,\ \cdots 397$
Then sum of common terms of these three $A.P.'s$ is _________.

Answer (2561)

Solution:

Common terms in $S_1,\ S_2,\ S_3$ are
$= 2,\ 17,\ 32,\ 47,\ \cdots$
S_2 has all odd numbers up to 397
Common terms in $S_1,\ S_2,\ S_3$ are
$= 17,\ 47,\ 77,\ \cdots,\ 377$
Sum of terms $= \frac{13}{2}(17 + 377)$
$= 2561$

16. Let $f(x) = |(x-3)(x-2)| - 3x + 2$ for $x \in [1,3]$. If M and m are absolute maximum & absolute minimum value of $f(x)$, then $|m| + |M|$ equals ______.

Answer (8)

Solution:

$$|(x-3)(x-2)| = \begin{cases} (x-2)(x-3), & x \in [1,2) \\ -(x-2)(x-3), & x \in [2,3] \end{cases}$$

$$f(x) = \begin{cases} x^2 - 5x + 6 - 3x + 2, & x \in [1,2) \\ -x^2 + 5x - 6 - 3x + 2, & x \in [2,3] \end{cases}$$

$f(x) = \begin{cases} x^2 - 8x + 8, & x \in [1,2) \\ -x^2 + 2x - 4, & x \in [2,3] \end{cases}$

$f'(x) = \begin{cases} 2x - 8, & x \in [1,2) \\ -2x + 2, & x \in [2,3] \end{cases}$

$\Rightarrow f'(x) = \begin{cases} < 0, & x \in [1,2) \\ < 0, & x \in [2,3] \end{cases}$

$\Rightarrow f'(x)$ is strictly decreasing in $[1,3]$

$f(1) = f(x)_{max} = M = 2 - 3 + 2 = 1$

$f(3) = f(x)_{min} = m = 0 - 9 + 2 = -7$

$\therefore |m| + |M| = |-7| + |1| = 8$

17. Let $X_1,\ X_2,\ X_3,\ \cdots,\ X_7$ is an A.P such that $X_1 < X_2 < X_3 \cdots < X_7,\ X_1 = 9,\ \sigma = 4$. The value of $\bar{X} + X_6$ is equal to __________.

Answer (34)

Solution:

Let the series be $a - 3d,\ a - 2d,\ a - d,\ a,\ a + d,\ a + 2d,\ a + 3d$

$a - 3d = 9$

Now if we shift the origin, the variance remains same

$\therefore$ for $-3d,\ -2d,\ -d,\ 0,\ d,\ 2d,\ 3d$

$\Rightarrow 16 = \frac{2}{7}(9d^2 + 4d^2 + d^2) - (\bar{X})^2$

$\Rightarrow 16 = \frac{2}{7}(14)d^2 - (0)^2$

$\Rightarrow d = 2$

$a - 3d = 9$

$\Rightarrow a = 15$

$\bar{X} = 15$

$X_6 = a + 2d = 19$

$\bar{X} + X_6 = 15 + 19 = 34$