CSM – 52/18
Mathematics
Paper – I

Time : 3 hours

Full Marks : 300

The figures in the right-hand margin indicate marks.

Candidates should attempt Q. No. 1 from Section – A and Q. No. 5 from Section – B which are compulsory and ***three*** *of the remaining questions, selecting at least* ***one*** *from each Section.*

SECTION – A

1. Answer any **five** of the following :

   (a) Show that $W = \{(x, y, z) \in \mathbb{R}^3 : x + 2y - z = 0, 2x - y + 3z = 0\}$ is a subspace of $\mathbb{R}^3$. Find a basis and dimension of W. 12

   (b) The order of each subgroup of a finite group is a divisor of the order of the group. 12

   (c) Prove that, for any positive integer n, the ring Z_n of all integers modulo n is an integral domain if and only if n is prime integer. 12

(d) Check whether the matrix $\begin{pmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{pmatrix}$ is diagonalisable. If so diagonalise it and also find the diagonalising matrix. 12

(e) Prove that the plane $ax + by + cz = 0$ cuts the cone $yz + zx + xy = 0$ in two perpendicular lines if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$. 12

(f) A variable sphere passes through the point $(0, 0, \pm c)$ and cuts the straight lines $y = x \tan\alpha$, $z = c$ and $y = -x \tan\alpha$, $z = -c$ in the points p, p′ respectively. If p . p′ = 2a, where a is a constant, then show that the centre of the sphere lies on the circle $z = 0$, $x^2 + y^2 = (a^2 - c^2)\operatorname{cosec}^2 2\alpha$. 12

2. (a) Prove that the set S of all real numbers of the form $a + b\sqrt{2}$, where a, b are integers, is an integral domain with respect to usual addition and multiplication. 15

(b) Show that the intersection of two normal subgroups of a group G is also a normal subgroup of G. 15

(c) Prove that the set $\mathbb{Z}_5$ of integers modulo 5 forms a field under multiplication modulo 5. 15

(d) Let $R' = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$ where $\mathbb{R}$ is a ring and $f : R' \to \mathbb{R}$ be defined by $f\left(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \right) = a$ for all $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \in R'$. Show that f is an isomorphism. 15

3. (a) Let V be the real vector space spanned by the rows of the matrix

$$A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}.$$

Find a basis for V. If (x_1, x_2, x_3, x_4) is in V, what are its coordinates relative to the ordered basis of V ? 15

(b) The matrix of a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered bases {(0, 1, 1), (1, 0,1), (1, 1, 0)} of $\mathbb{R}^3$ and {(1, 0), (1, 1)} of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. Find T. 15

(c) Show that, the eigenvalues of a real skew symmetric matrix are either zero or pure imaginary ? 15

(d) Let $A = \begin{pmatrix} 1 & -3 & 2 \\ -3 & 7 & -5 \\ 2 & -5 & 8 \end{pmatrix}$. Find a non-singular matrix P such that $D = P^T A P$ is diagonal and also find the signature of A. 15

4. (a) Show that only one tangent plane can be drawn to the sphere $x^2 + y^2 + z^2 - 2x + 6y + 2z + 8 = 0$ through the straight line $3x - 4y - 8 = 0 = y - 3z + 2$. Also, find the plane. 15

(b) If the two pair of lines $x^2 - 2pxy - y^2 = 0$ and $x^2 - 2qxy - y^2 = 0$ be such that each pair bisects the angle between the other pair then prove that $pq = -1$. 15

(c) If a sphere touches the planes $2x + 3y - 6z + 14 = 0$ and $2x + 3y - 6z + 42 = 0$ and if its centre lies on the straight line $2x + z = 0$, $y = 0$, find the sphere. 15

(d) A plane meets the coordinate axes at A, B, C such that the centroid of the triangle ABC is the point (α, β, γ). Show that the equation of the plane is $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3$. 15

SECTION – B

5. Answer any **five** questions of the following :

(a) Prove that every Cauchy sequence of real numbers converges. 12

(b) Examine the convergence of the series :

$$x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \ldots\ldots\ldots (x > 0).$$ 12

(c) The smaller segment of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cut off by the chord $\frac{x}{a} + \frac{y}{b} = 1$ revolves completely about this chord, find the volume of the solid spindle thus generated. 12

(d) Using contour integration method, prove that :

$$\int_0^{\infty} \frac{\cos mx}{x^2 + 1} dx = \frac{\pi}{2} e^{-m}, m > 0.$$ 12

(e) Find the Laurent's series of the following functions : 12

(i) $\frac{e^z}{(z-2)^3}$ about the point $z = 2$.

(ii) $\frac{e^{z^2}}{z^3}$ about the point $z = 0$.

(f) Find the area of the surface of the cylinder $x^2 + y^2 = 4a^2$ above the xy-plane and bound by the planes $y = 0$, $z = a$ and $y = z$. 12

6. (a) Using Cauchy's residue theorem, evaluate

$$\int_0^{\infty} \frac{1}{x^4 + 1} dx.$$ 15

(b) If $u = e^x(x\cos y - y\sin y)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z. 15

(c) Prove that $\sum_{n=1}^{\infty}\{nxe^{-nx^2} - (n-1)xe^{-(n-1)x^2}\}$ converges on $0 \leq x \leq 1$ but not uniformly. 15

(d) Prove that every bounded sequence has a convergent subsequence. 15

7. (a) Apply Lagrange's Mean Value theorem to the function $\log_e(1+x)$ to show that $0 < [\log_e(1+x)]^{-1} - x^{-1} < 1$, for all $x > 0$. 15

(b) Evaluate $\iint_E xe^{x^2-y^2}dxdy$, where E is the closed region bounded by the lines $y = x$, $y = x - 1$, $y = 0$, $y = 1$. 15

(c) Show that :

$$\iint_S (yzdydz + zxdzdx + xydxdy) = \frac{3}{8},$$

where S is the outer surface of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant. 15

(d) Test the convergence of $\int_0^{\pi/2} \frac{\sin x}{x^p} dx$. 15

8. (a) If $r = |\vec{r}|$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find f(r) such that $\vec{\nabla} f = \frac{\vec{r}}{r^5}$ and f(1) = 0. 15

(b) Verify the divergence theorem for the vector function $\vec{F} = 4xz\hat{i} + y^2\hat{j} + yz\hat{k}$ taken over the cube bounded by x = 0, x = 1 ; y = 0, y = 1 ; z = 0, z = 1. 15

(c) Evaluate by Green's theorem $\oint_C \{(x^2 - \cosh y)dx + (y + \sin x)dy\}$, where C is the rectangle with vertices (0, 0), (π, 0), (π, 1) and (1, 0). 15

(d) Verify Stoke's theorem for $\vec{F} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. 15