PART : MATHEMATICS

1. If $g(x)=\sqrt{x}-1$ and $fog(x) = x + \sqrt{x}-1$ then f(0) is

(1) 0 (2) 1 (3) 2 (4) –1

Ans. (2)

Sol. $f(g(x)) = x + \sqrt{x} - 1$

$f(\sqrt{x}-1) = x + \sqrt{x} - 1$.....(1)

$\sqrt{x}-1 = 0 \Rightarrow x=1$

So put x = 1 in equation (1)

$f(0) = 1+\sqrt{1}-1 = 1$

2. Number of 7 digit numbers which contain only digits 1, 2, 3 and 4 and sum of digits of number is always 12, is

(1) 420 (2) 413 (3) 252 (4) 392

Ans. (2)

Sol. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 12$ where $x_i \in \{1, 2, 3, 4\}$

Number of solution = coefficient of x^{12} in $(x^1 + x^2 + x^3 + x^4)^7$

$= \text{coefficient of } x^5 \text{ in } (1 + x + x^2 + x^3)^7$

$= \text{coefficient of } x^5 \text{ in } \frac{(1-x^4)^7}{(1-x)^7}$

$= \text{coefficient of } x^5 \text{ in } (1 - 7x^4)(1-x)^{-7}$

$= \text{coefficient of } x^5 \text{ in } (1 - 7x^4)\sum_{r=0}^{\infty} {}^{7+r-1}C_r x^r$

$= {}^{11}C_5 - 7 \times {}^7C_1$

$= 462-49 = 413$

Alt. $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 = 12$

Case-I	1	1	1	1	1	3	4
Case-II	1	1	1	1	2	2	4
Case-III	1	1	2	2	2	2	2
Case-IV	1	1	1	2	2	2	3
Case-V	1	1	1	1	2	3	3

$\frac{7!}{5!} + \frac{7!}{4!\,2!} + \frac{7!}{2!\,5!} + \frac{7!}{3!\,3!} + \frac{7!}{4!\,2!}$

$= 42 + 105 \times 2 + \frac{7\times6}{2} + \frac{7\times6\times5\times4}{3\times2}$

$= 42 + 210 + 21 + 140$

$= 413$

3. Let $\frac{dy}{dx} = y + 7$, $y_1(x)$ & $y_2(x)$ are solutions of $y_1(0) = 0$, $y_2(0) = 1$, then number of points of intersection of curve $y_1(x)$ & $y_2(x)$ is

(1) 2 points (2) more than 2 points (3) No point (4) one point

Ans. (3)

Sol. $\frac{dy}{y+7} = dx$

$\ell n(y+7) = x + c$

$\Rightarrow y + 7 = e^{x+c}$

$\Rightarrow y = \lambda e^x - 7$

Let $y_1(x) = \lambda_1 e^x - 7$ & $y_2(x) = \lambda_2 e^x - 7$

As $y_1(0) = 0 \Rightarrow \lambda = 7$ & $y_2(0) = 1 \Rightarrow \lambda_2 = 8$

$\Rightarrow y_1(x) = 7e^x - 7$

$y_2(x) = 8e^x - 7$

Hence it's point of intersection by solving

$7e^x - 7 = 8e^x - 7$

$\Rightarrow e^x = 0$ $\Rightarrow$ No solution

4. The fractional part of $\frac{4^{2022}}{15}$ is equal to

(1) $\frac{1}{15}$ (2) $\frac{2}{15}$ (3) $\frac{8}{15}$ (4) $\frac{4}{15}$

Ans. (1)

Sol. $\frac{4^{2022}}{15} = \frac{16^{1011}}{15} = \frac{(15+1)^{1011}}{15} = \left(\frac{15\lambda + 1}{15}\right)$

Hence fractional part is $\frac{1}{15}$

5. Number of symmetric matrices of order 3×3 by using digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 is

(1) 10^6 (2) 9^6 (3) 10^9 (4) 10^5

Ans. (1)

Sol. $\begin{bmatrix} a & \alpha & \beta \\ \alpha & b & \gamma \\ \beta & \gamma & c \end{bmatrix}$

$a, b, c, \alpha, \beta, \gamma$ each has 10 options

$= 10 \times 10 \times 10 \times 10 \times 10 \times 10 = 10^6$

6. Sum of series $2.2^2 - 3^2 + 2.4^2 - 5^2 + 2.6^2 + \text{--------}$ upto 20 terms, is equal to.

(1) 1510 (2) 1300 (3) 1310 (4) 1410

Ans. (3)

Sol. $2.2^2 - 3^2 + 2.4^2 - 5^2 + 2.6^2 - 7^2 + \text{-----} + 2.(20)^2 - (21)^2$

$3(2^2 + 4^2 + 6^2 + \text{----} + (20)^2) - (2^2 + 3^2 + 4^2 + \text{----} + (21)^2)$

$12(1^2 + 2^2 + 3^2 + \text{----} + (10)^2) - (2^2 + 3^2 + 4^2 + \text{----} + 21^2)$

$$= 12\left(\frac{10\times11\times21}{6}\right) - \left(\frac{21\times22\times43}{6} - 1\right) = 2(2310) - (7\times11\times43 - 1) = 4620 - 3310 = 1310$$

7. Consider the following statements

S-1 $\lim\limits_{n\to\infty} \frac{1}{n^2}(2+4+6+......+2n) = 1$

S-2 $\lim\limits_{n\to\infty} \frac{1}{n^{26}}(1^{25}+2^{25}+3^{25}+......+n^{25}) = \frac{1}{16}$

then

(1) Statement-1 and Statement-2 both are correct

(2) Statement-1 is correct and Statement-2 is incorrect

(3) Statement-1 is incorrect and Statement-2 is correct

(4) Statement-1 and Statement-2 both are incorrect

Ans. (2)

Sol. S-1 $\lim\limits_{n\to\infty} \frac{1}{n^2}(2+4+6+......+2n)$

$\lim\limits_{n\to\infty} \frac{n(n+1)}{n^2} = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$

So Statement-1 is correct

S-2 $\lim\limits_{n\to\infty} \frac{1}{n}\left(\frac{1^{25}+2^{25}+......+n^{25}}{n^{25}}\right)$

$= \lim\limits_{n\to\infty} \sum\limits_{r=1}^{n}\left(\frac{r}{n}\right)^{25}.\frac{1}{n}$

$= \int_0^1 x^{25}dx = \left.\frac{x^{26}}{26}\right|_0^1 = \frac{1}{26}$

8. The value of $\int_0^\infty \frac{6dx}{e^{3x}+6e^{2x}+11e^x+6}$ is equal to

(1) $\ell n\frac{512}{27}$ (2) $\ell n\frac{256}{27}$ (3) $\ell n\frac{64}{27}$ (4) $\ell n\frac{32}{27}$

Ans. (4)

Sol. Let $e^x = t \Rightarrow e^x\,dx = et$

$$I = \int_1^{\infty} \frac{6}{t(t+1)(t+2)(t+3)}dt$$

$$= = 6\int_1^{\infty}\left\{\frac{A}{t}+\frac{B}{t+1}+\frac{C}{t+2}+\frac{D}{t+3}\right\}dt$$

$$= 6\int_1^{\infty}\left(\frac{1}{6t}+\frac{1}{(-2)(t+1)}+\frac{1}{2(t+2)}-\frac{1}{6(t+3)}\right)dt$$

$$= \left[\ell nt - 3\ell n(t+1) + 3\ell n(t+2) - \ell n(t+3)\right]_1^{\infty}$$

$$= \left[\ell n\frac{t(t+2)^3}{(t+1)^3(t+3)}\right]_1^{\infty}$$

$$= 0 - \ell n\frac{3^3}{2^3.4} = \ell n\frac{32}{27}$$

9. $3f(x) + 2f\left(\frac{1}{x}\right) = \frac{1}{x} - 10$ for $x \in R - \{0\}$, then find $\left|f(3) + f'\left(\frac{1}{4}\right)\right| = ?$

Ans. (13)

Sol. $3f(x) + 2f\left(\frac{1}{x}\right) = \frac{1}{x} - 10$

Replace x be $\frac{1}{x}$

$3f\left(\frac{1}{x}\right) + 2f(x) = x - 10$

Now $9f(x) + 6f\left(\frac{1}{x}\right) = \frac{3}{x} - 30$

$4f(x) + 6f\left(\frac{1}{x}\right) = 2x - 20$

$5f(x) = \frac{3}{x} - 2x - 10$

$f(x) = \frac{3}{5x} - \frac{2x}{5} - 2$

$f'(x) = -\frac{3}{5x^2} - \frac{2}{5}$

$f(3) = \frac{1}{5} - \frac{6}{5} - 2 = -3$

$f'\left(\frac{1}{4}\right) = \frac{-48}{5} - \frac{2}{5} = -10$

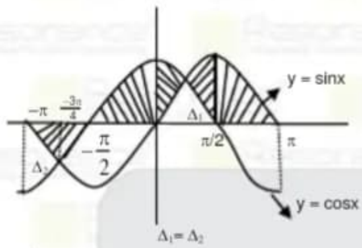
$\left|f(3) + f'\left(\frac{1}{4}\right)\right| = 13$

10. Find the area bounded by curve $y = \max(\sin x, \cos x)$, $x \in (-\pi, \pi)$ and x-axis.

(1) $2\sqrt{2}(\sqrt{2}+1)$ (2) $2\sqrt{2}$ (3) $2\sqrt{2}+\sqrt{3}$ (4) 4

Ans. (4)

Sol.



$$\int_0^{\pi/2} \sin x . dx = 1$$

Required area = $2 + 1 + 1 + \Delta_1 - \Delta_2 = 4$

11. $S_1, S_2, \ldots\ldots, S_{10}$ be sum of 12 terms of 10 A.P's whose first terms are 1, 2, 3, ... , 10 and common differences are 1, 3, 5, 7, ..., 19 respectively, then $\sum_{r=1}^{10} S_r$ is.....

Ans. (7260)

Sol. $1 \le r \le 10$

$S_r = \frac{12}{2}[2r + (12-1)(2r-1)] = 6(2r + 22r - 11)$

$= 6(24r - 11)$

$\sum_{r=1}^{10} S_r = 6\sum_{r=1}^{10}(24r - 11) = 6 \times \frac{10}{2}[24 - 11 + 240 - 11]$

$= 30\,[242] = 7260$

12. $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$

$\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$

$\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$

& $\vec{d}.\vec{a} = 24$ & $\vec{d} \times \vec{b} = \vec{c} \times \vec{b}$ then $|\vec{d}|^2$ is ....

Ans. (413)

Sol. $\vec{a}. \times \vec{b} - \vec{c} \times \vec{b} = \vec{0} \Rightarrow (\vec{d} - \vec{c}) \times \vec{b} = \vec{0}$

So $\vec{d} - \vec{c} = \lambda\vec{b} \Rightarrow \vec{d} = \vec{c} + \lambda\vec{b}$

$\vec{a}.\vec{d} = \vec{a}.\vec{c} + \lambda\vec{a}.\vec{b}$

$24 = (2 - 4 + 8) + \lambda\,(3 - 8 + 14)$

$24 = 6 + 9\lambda \Rightarrow 9\lambda = 18 \Rightarrow \lambda = 2$

So $\vec{d} = \vec{c} + 2\vec{b}$

$(2\hat{i} - 4\hat{j} + 4\hat{k}) + 6\hat{i} - 4\hat{j} + 14\hat{k}$

$\vec{d} = 8\hat{i} - 5\hat{j} + 18\hat{k}$

$|\vec{d}|^2 = 64 + 25 + 324 = 413$

13. If $\sin^{-1}\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right) - \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{\pi}{4}$ then value of $\sin(x^2+x+5)\frac{\pi}{2} + \cos(x^2+x+5)\pi$ is equal to.

(1) 2 (2) 0 (3) 1 (4) –1

Ans. **(2)**

Sol. $\sin^{-1}\left(\frac{x+1}{\sqrt{x^2+2x+2}}\right) - \sin^{-1}\left(\frac{x}{\sqrt{x^2+1}}\right) = \frac{\pi}{4}$

$\tan^{-1}(x+1) - \tan^{-1}x = \frac{\pi}{4} \Rightarrow \tan^{-1}\left(\frac{(x+1)-x}{1+(x+1)x}\right) = \frac{\pi}{4}$

$\left(\frac{1}{1+x^2+x}\right) = \frac{\pi}{4} \Rightarrow \quad \frac{1}{1+x^2+x} = 1$

$x^2+x+1 = 1 \quad \Rightarrow \quad x^2+x+5 = 5$

Now $\sin\frac{5\pi}{2} + \cos 5\pi = 1 - 1 = 0$

14. For the given observations in table

x	1	3	5	7	9
f	4	24	28	α	8

Mean = 5, M.D. = m, Variance = σ^2

Find value of $\frac{3\alpha}{m+\sigma^2}$.

Ans. **(8.00)**

Sol. $\frac{4+72+28\times5+7\alpha+72}{56+8+\alpha} = 5 \Rightarrow \alpha = 16$

$\sum f_i = 80$

M.D. = $\frac{\sum f_i|x_i-5|}{\sum f_i} = \frac{4+4+24\times2+0+16\times2+8\times4}{80} = \frac{16}{10}$

$\sigma^2 = \frac{\sum f_i(x_i-5)^2}{\sum f_i} = \frac{4+16+24\times4+0+16\times4+8\times16}{80} = \frac{44}{10}$

$\frac{3\alpha}{m+\sigma^2} = \frac{3\times16}{\frac{16}{10}+\frac{44}{10}} = \frac{3\times16\times10}{60} = 8$

15. If $f(x) = x - 2\sin x\cos x + \frac{1}{3}\sin 3x$, $x \in [0, \pi]$, then maximum value of f(x) is

(1) $\frac{5\pi}{6} - \frac{\sqrt{3}}{2} + \frac{1}{3}$ (2) $\frac{5\pi}{6} + \frac{\sqrt{3}}{2} + \frac{1}{3}$ (3) 0 (4) π

Ans. **(2)**

Sol. $f(x) = x - 2\sin x\cos x + \frac{1}{3}\sin 3x$

$f'(x) = 1 - 2\cos 2x + \cos 3x$

$f''(x) = 4\sin 2x - 3\sin 3x$

$f'(x) = 0$

$1 - 2\cos 2x + \cos 3x = 0$

$1 - 2(2\cos^2 x - 1) + 4\cos^3 x - 3\cos x = 0$

$4\cos^3 x - 4\cos^3 x - 3\cos x + 3 = 0$

$4\cos^3 x\,(\cos x - 1) - 3(\cos x - 1) = 0$

$(2\cos x + \sqrt{3})\,(2\cos x - \sqrt{3})\,(\cos x - 1) = 0$

$\cos x = -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}, 1$

$x = \frac{5\pi}{6}, \frac{\pi}{6}, 0$

$f''\left(\frac{5\pi}{6}\right) = 4\sin\frac{5\pi}{3} - 3\sin\frac{5\pi}{2} = -2\sqrt{3} - 3 < 0$

$f''\left(\frac{\pi}{6}\right) = 4\sin\frac{\pi}{3} - 3\sin\frac{\pi}{2} = 2\sqrt{3} - 3 > 0$

$f''(0) = 0$

So $x = \frac{5\pi}{6}$ is local maxima point

Maximum value of f(x) is $f\left(\frac{5\pi}{6}\right) = \frac{5\pi}{6} + \frac{\sqrt{3}}{2} + \frac{1}{3}$

16. In the expansion of $\left(\sqrt{x} - \frac{6}{x^{3/2}}\right)^n$, the constant term is A and sum of coefficients of other terms is 649. If the coefficent of x^{-n} is λA then λ = ?

Ans. **(36)**

Sol. General term $= T_{r+1} = {}^nC_r\left(\sqrt{x}\right)^{n-r}\left(-\frac{6}{x^{3/2}}\right)^r = {}^nC_r\,(-6)^r.\ x^{\frac{n}{2}-\frac{x}{2}-\frac{3r}{2}} = {}^nC_r\,(-6)^r.\ x^{\frac{n}{2}-2r}$

For constant term $\frac{n}{2} - 2r = 0$

$r = \frac{n}{4}$

Now ${}^nC_{\frac{n}{4}}\,(-6)^{\frac{n}{4}} = A$ .....(1)

Sum of coefficient of other terms = 649

$\left(1 - \frac{6}{1}\right)^n - A = 649$

$(-5)^n - A = 649$

$A = (-5)^n - 649$ .....(2)

${}^nC_{\frac{n}{4}}\,(-6)^{\frac{n}{4}} = (-5)^n - 649$

here n will be multiple of 4

Let n = 4 = ${}^nC_1\,(-6)^1 = (-5)^4 - 649$ which is true

So value of n is 4

for coefficient of x^{-4}, $\frac{4}{2} - 2r = -4 \Rightarrow 2r = 6 \Rightarrow r = 3$

coefficient of $x^{-4} = {}^4C_3\,(-6)^3 = 4(-216) = \lambda A \Rightarrow \lambda = -\frac{864}{A}$

and $A = (-5)^4 - 649 = 625 - 649 = -24$ $\qquad \therefore \lambda = \frac{864}{24} = 36$

17. A biased coin is tossed and the probability of getting head is 3 times the probability of getting tail and this process is done till either 1 head or 3 continuous tail occurs, then the mean of probability distribution is

(1) $\frac{21}{8}$ (2) $\frac{7}{16}$ (3) $\frac{21}{16}$ (4) $\frac{21}{4}$

Ans. **(2)**

Sol. clearly $P(H) = \frac{3}{4}$, $P(T) = \frac{1}{4}$

Since coin is tossed till either 1 head or 3 continuous tail ssssoccurs. So process will be end in the maximum 3 thrown so probability distribution is

x_i	1	2	3
p_i	$\frac{3}{4}$	$\frac{1}{4}.\frac{3}{4}$	$\frac{1}{4}.\frac{1}{4}.\frac{3}{4}+\frac{1}{4}.\frac{1}{4}.\frac{1}{4}$

Hence mean $= \sum p_i x_i = 1\left(\frac{3}{4}\right)+2\left(\frac{3}{16}\right)+3\left(\frac{1}{16}\right)=\frac{12+6+3}{16}=\frac{21}{16}$

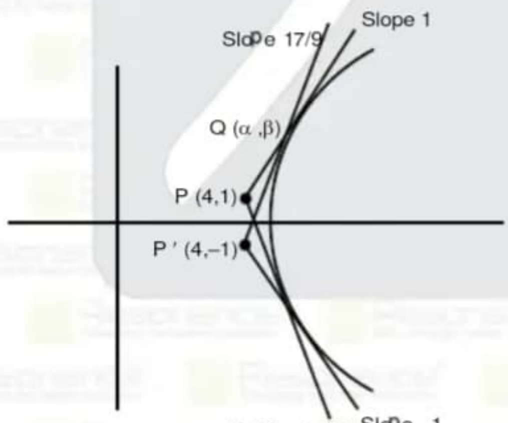
18. If tangents are drawn to the curve $\frac{x^2}{25}-\frac{y^2}{16}=1$ from point P(4, 1) with slope a and b and tangents are drawn from point Q(α, β) to given curve with slope |a|, |b|, then value of $\frac{PQ^2}{\alpha\beta}$ is

(1) $\frac{162}{425}$ (2) $\frac{162}{325}$ (3) $\frac{81}{325}$ (4) $\frac{81}{425}$

Ans. **(2)**

Sol. Let equation of tangent to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ is

$y = mx \pm \sqrt{25m^2-16}$



Passes through P(4, 1) so

$1 = 4m \pm \sqrt{25m^2-16}$

$(1-4m)^2 = 25m^2 - 16$

$9m^2 + 8m - 17 = 0$ → $a = 1$, $b = -17/9$

$m = 1, -\frac{17}{9}$

Equation of PA → $y - 1 = 1.(x - 4) \Rightarrow y = x - 3$ ....(1)

Let P′ is image of P in x-axis

Equation of P′B → $y + 1 = \frac{17}{9}(x - 4) \Rightarrow 9y = 17x - 77$ ....(2)

Solving (1) & (2) $Q\left(\frac{25}{4}, \frac{13}{4}\right)$

$PQ^2 = \frac{81}{16} + \frac{81}{16} = \frac{81}{8}$

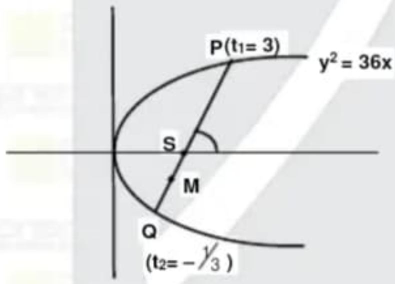
So $\frac{PQ^2}{\alpha\beta} = \frac{81 \times 4 \times 4}{8 \times 25 \times 13} = \frac{162}{25 \times 13} = \frac{162}{325}$

19. Let PQ be a focal chord of the parabola $y^2 = 36x$ of length 100, making an acute angle with the positive direction of x axis and M be the point on the line segment PQ such that PM : MQ = 3 : 1, then which of the following points does not lie on the line passing through 'M' and perpendicular to the line PQ.

(1) (3, 33) (2) (9, 25) (3) (15, 17) (4) (18, 12)

Ans. (4)

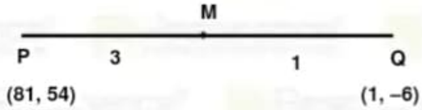
Sol.



Length $9\left(t_1 + \frac{1}{t_1}\right)^2 = 100$

$t_1 + \frac{1}{t_1} = \pm\frac{10}{3}$

$t_1 = 3$ or $\frac{1}{3}$



M = (21,9)

Required line $y - 9 = -\frac{80}{60}(x - 21)$ $\Rightarrow 4x + 3y = 111$

20. Let the tangent and normal at the point $(3\sqrt{3}, 1)$ on the ellipse $\frac{x^2}{36}+\frac{y^2}{4}=1$ meet the y–axis at the points A and B respectively. Let the circle 'C' be drawn taking AB as diameter and the line $x = 2\sqrt{5}$ intersects circle at points 'P' and 'Q'. If the tangents at the points 'P' & 'Q' on the circle intersects at the point (α, β), then $10(\alpha^2 - \beta^2)$ is equal to

Ans. **(608)**

Sol. $\frac{x^2}{36}+\frac{y^2}{4}=1$

Tangent $\Rightarrow \frac{x.3\sqrt{3}}{36}+\frac{y.1}{4}=1$

A (0, 4) ($\because$ it meets y axis at A)

Normal $\frac{x.36}{3\sqrt{3}}-\frac{y.4}{1}=36-4=32$

A(0, 4) and B(0,–8) are end points of diameter

$(x-0)(x-0)+(y-4)(y+8)=0$

$x^2+y^2+4y-32=0$

Equation of C.O.C from (α, β) with respect to circle

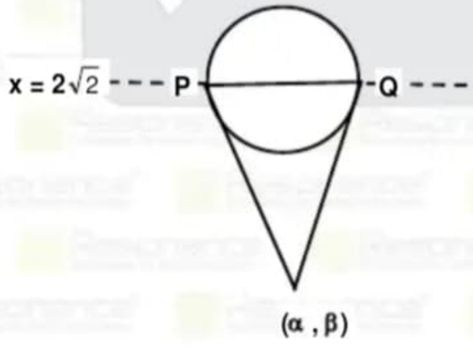
$\alpha x+\beta y+2(y+\beta)-32=0$

$\Rightarrow \alpha x+(\beta+2)y+2\beta-32=0$

Is same as $x = 2\sqrt{5}$

So $\beta+2=0$ & $\frac{32-2\beta}{\alpha}=2\sqrt{5} \Rightarrow \alpha=\frac{18}{\sqrt{5}}$

So, $10(\alpha^2-\beta^2) = 10\left(\frac{324}{5}-4\right)=608$



21. Let $\omega = Z\bar{Z} + K_1 Z + K_2 iZ + \lambda$, locus of Z is given by Re(ω)= 0 is a circle in first quadrant with r = 1 & touching y= 1 & y –axis. If again locus of Z is Im(ω)=0 intersects circle at A & B then $(AB)^2$

Ans. 4

Sol. $w = x^2 + y^2 + K_1 (x + iy) + k_2 (ix - y) + \lambda$

$c : x^2 + y^2 + k_1 x - K_2 y + \lambda = 0$ .... (i)

$(x - 1)^2 + (y - 2)^2 = 1$

$x^2 + y^2 - 2x - 4y + 4 = 0$ .... (ii)

compare (i) & (ii) $k_1 = -2$, $k_2 = 4$ and $\lambda = 4$

also Im(ω)=0 $\Rightarrow$ $k_1 y + k_2 x = 0$

$\Rightarrow$ $4x - 2y = 0$

$\Rightarrow$ $2x - y = 0$....... (iii)

Centre of circle lies of line $2x - y = 0$

Hence AB is a diameter

$(AB)^2 = (2)^2 = 4$