JNUEE PHD Mathematical Sciences

Topic:- MATH897 JNUS21

1) A. Every uncountable subsets of $\mathbb{R} \setminus \mathbb{Z}$ is Lebesgue measurable.

B. Every uncountable subset of $\mathbb{R} \setminus \mathbb{Q}$ is Lebesgue measurable and has nonzero Lebesgue measure.

C. Every infinite subset of $\mathbb{Z}$ which contains only odd numbers is Lebesgue measurable.

D. $\{\frac{a}{2^b} \mid a \in \mathbb{Z}, b \in \mathbb{N}, b \text{ is odd}\}$ is Lebesgue measurable.

Which of the above statements is/are true? Choose the ***correct*** answer from the options given below:

[Question ID = 20203][Question Description = S1_MATH_897_PhD_Q001]

1. A and B only

   [Option ID = 146125]
2. C and D only

   [Option ID = 146126]
3. A and D only

   [Option ID = 146127]
4. C only

   [Option ID = 146128]

2) We define a relation $\sim$ on $\mathbb{R}^2$ as follows: For $a, b \in \mathbb{R}^2$, $a \sim b$ if $2a - 3b$ lies on the graph of a continuous function $f : \mathbb{R} \to \mathbb{R}$, which may depend on a and b.

Which of the following assertions is correct?

[Question ID = 20204][Question Description = S1_MATH_897_PhD_Q002]

1. $\sim$ is not reflexive.

   [Option ID = 146129]
2. $\sim$ is reflexive but neither symmetric nor transitive.

   [Option ID = 146130]
3. $\sim$ is an equivalence relation with finitely many equivalence classes.

   [Option ID = 146131]
4. $\sim$ is an equivalence relation with infinitely many equivalence classes.

   [Option ID = 146132]

3) Let $f : \mathbb{C} \to \mathbb{R}$ be any continuous function such that $f(e^{2i}z) = f(z)$ for all $z \in \mathbb{C}$, where $i = \sqrt{-1}$.
Consider the following statements:

A. The image of f is a compact set.

B. The image of f is an infinite countable set.

Which of the following is necessarily correct?

[Question ID = 20205][Question Description = S1_MATH_897_PhD_Q003]

1. A is true but B is false.

   [Option ID = 146133]
2. B is true but A is false.

   [Option ID = 146134]
3. Both A and B are true.

   [Option ID = 146135]
4. Both A and B are false.

   [Option ID = 146136]

4) $$\varrho(x) = \begin{cases} \sin 2x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ \frac{1}{|x|+1} & \text{if } x \in \mathbb{Q} \end{cases}$$

Consider the following statements:

A. ϱ is Lebesgue measurable.

B. ϱ is Lebesgue integrable.

C. $\varrho(E)$ is measurable for every nonempty bounded subset E of $\mathbb{R}$.

Which of the following is correct?

A.Q is Lebesgue measurable:

B.Q is Lebesgue measurable:

C. Q(E) is measruble for every nonempty bounded subset E or R.

Which of the following is correct?

[Question ID = 20206][Question Description = S1_MATH_897_PhD_Q004]

1. Both A and B are true but C is false.

   [Option ID = 146137]
2. Both A and B are false but C is true.

   [Option ID = 146138]
3. Both B and C are false but A is true.

   [Option ID = 146139]
4. Both A and C are true but B is false.

   [Option ID = 146140]

5)

Let $X=\mathbb{R}^2$, $Y=\mathbb{Q}\times\mathbb{Q}$, $Z-(\mathbb{R}\setminus\mathbb{Q})\times(\mathbb{R}\setminus\mathbb{Q})$ and $W=(\mathbb{R}\setminus\mathbb{Q})\times\mathbb{Q}$. Consider the following statements:

A. $X\setminus Y$ is connected.

B. $(X\setminus Y)\setminus W$ is connected.

C. Both $X\setminus Z$ and $X\setminus W$ are connected.

Which of the following is correct?

[Question ID = 20207][Question Description = S1_MATH_897_PhD_Q005]

1. Both A and B are true but C is false.

   [Option ID = 146141]
2. Both B and C are false but A is true.

   [Option ID = 146142]
3. Both A and C are true, but B is false.

   [Option ID = 146143]
4. All A, B and C are false.

   [Option ID = 146144]

6) Let $\mathcal{T}$ be the cofinite topology on $\mathbb{Z}$. Consider the following statements:

A. $(\mathbb{Z},\mathcal{T})$ is second countable.

B. $(\mathbb{Z},\mathcal{T})$ is separable.

C. $(\mathbb{Z},\mathcal{T})$ is compact.

D. $(\mathbb{Z},\mathcal{T})$ is path connected.

Which of the following is correct?

[Question ID = 20208][Question Description = S1_MATH_897_PhD_Q006]

1. A and B are true but C and D are false.

   [Option ID = 146145]
2. A, B and C are true but D is false.

   [Option ID = 146146]
3. C is true but A, B and D are false.

   [Option ID = 146147]
4. B and D are true but A and C are false.

   [Option ID = 146148]

7) Let $\mathcal{T}$ be the topology on $\mathbb{N}$ which has a subbasis $\{n, n+3 \mid n\in\mathbb{N}\}$. Consider the following statements:

A. $(\mathbb{N},\mathcal{T})$ is a Hausdorff space.

B. $(\mathbb{N},\mathcal{T})$ is a compact space.

C. $\mathcal{T}$ is countable.

Which of the following is correct?

[Question ID = 20209][Question Description = S1_MATH_897_PhD_Q007]

1. A is true but both B and C are false.

   [Option ID = 146149]
2. Both A and B are false but C is true.

   [Option ID = 146150]
3. Both A and C are false but B is true.

   [Option ID = 146151]
4. All A, B and C are false.

   [Option ID = 146152]

8)

Let $f:S^1\to\mathbb{R}$ be any continuous function, where S^1 is the unit circle centered at $(0,0)$ in $\mathbb{R}^2$ with the subspace topology inherited from the usual topology on $\mathbb{R}^2$. Consider the following statements:

A. f is not injective.

B. The image of f is a closed interval $[a,b]$ for some $a,b \in \mathbb{R}$ with $a < b$.

Which of the following is necessarily correct?

[Question ID = 20210][Question Description = S1_MATH_897_PhD_Q008]

1. A is true but B is false.

   [Option ID = 146153]
2. B is true but A is false.

   [Option ID = 146154]
3. Both A and B are true.

   [Option ID = 146155]
4. Both A and B are false.

   [Option ID = 146156]

9) Let $C_1 = \{]-2r, 2r[\mid r > 0\}$, $C_2 = \{[-2r, 2r] \mid r > 0\}$ and let $C_3 = \{\mathbb{R}, \emptyset\}$.

Which of the following assertions is correct?

[Question ID = 20211][Question Description = S1_MATH_897_PhD_Q009]

1. Given $\delta > 0$ there exists $\epsilon > 0$ such that $d(x,y) < \delta \Rightarrow d(f(x), f(y)) > \epsilon$

   [Option ID = 146157]
2. C_2 is a basis for a topology on $\mathbb{R}$ but C_1 is not.

   [Option ID = 146158]
3. $C_2 \cup \{\mathbb{R} \setminus X \mid X \in C_1\}$ is a basis for the usual topology on $\mathbb{R}$.

   [Option ID = 146159]
4. $C_1 \cup C_2 \cup C_3$ is a topology on $\mathbb{R}$.

   [Option ID = 146160]

10) Let (X, d) be a metric space and let $f : X \to X$ be any function. Which of the following is equivalent to the statement that "f is not uniformly continuous"?

[Question ID = 20212][Question Description = S1_MATH_897_PhD_Q010]

1. Given $\delta > 0$ there exists $\epsilon > 0$ such that $d(x,y) < \delta \Rightarrow d(f(x), f(y)) > \epsilon$

   [Option ID = 146161]
2. Given $\epsilon > 0$, there exists $\delta > 0$ and a pair of elements x, y in X, which depends on ϵ, such that $d(x,y) < \delta$ and $d(f(x), f(y)) > \epsilon$.

   [Option ID = 146162]
3. There exists $\delta > 0$ such that for every $\epsilon > 0$, $d(x,y) > \delta \Rightarrow d(f(x), f(y)) > \epsilon$.

   [Option ID = 146163]
4. There exists $\epsilon > 0$ such that for every $\delta > 0$, there exists a pair of elements x, y in X which depends on δ, with $d(x,y) < \delta$ and $d(f(x), f(y)) > \epsilon$.

   [Option ID = 146164]

11) Consider the following matrices

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 4 & 5 & 0 \end{pmatrix} \in M_3(\mathbb{Q}).$$

Which of the following is correct?

[Question ID = 20213][Question Description = S1_MATH_897_PhD_Q011]

1. A, B are similar in $M_3(\mathbb{Q})$.

   [Option ID = 146165]
2. A, B are similar in $M_3(\mathbb{R})$ but not in $M_3(\mathbb{Q})$.

   [Option ID = 146166]
3. A, B are similar in $M_3(\mathbb{C})$ but not in $M_3(\mathbb{R})$.

   [Option ID = 146167]
4. A, B are not similar in $M_3(F)$ for any field F containing $\mathbb{Q}$.

   [Option ID = 146168]

12) Let $\mathbb{F}_2$ be the field of order 2 and $S = \left\{ A \in M_3(\mathbb{F}_2) : A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$.

What is the cardinality of the set S ?

[Question ID = 20645][Question Description = S1_MATH_897_PhD_Q012]

1. 0

   [Option ID = 146169]
2. 1

   [Option ID = 146170]
3. 2

   [Option ID = 146171]

4. 4

[Option ID = 146172]

13) Let $\mathbb{F}_q$ denote the finite field with q elements.

What is the number of finite degree separable extensions of $\mathbb{F}_q$ that are not normal?

[Question ID = 20646][Question Description = S1_MATH_897_PhD_Q013]

1. 0

[Option ID = 146173]

2. 1

[Option ID = 146174]

3. 2

[Option ID = 146175]

4. infinite

[Option ID = 146176]

14) Consider the inner product space $\mathbb{R}[X]$ of the polynomials over $\mathbb{R}$ with the inner product given by

$\langle f(X), g(X)\rangle := \int_{-1}^{1} f(X)g(X)dX$ for $f(X), g(X) \in \mathbb{R}[X]$.

Write

$f_0(X) = X,$

$f_1(X) = 1 - X + X^2 + \frac{5}{3}X^3,$

$f_2(X) = 1 - X + X^2 - X^3,$

$f_3(X) = 1 + \frac{3}{5}X + X^2 - X^3.$

Which of the above polynomials are orthogonal to $f_0(X)$?

[Question ID = 20647][Question Description = S1_MATH_897_PhD_Q014]

1. Only $f_1(X), f_2(X)$

[Option ID = 146177]

2. Only $f_2(X), f_3(X)$

[Option ID = 146178]

3. Only $f_1(X), f_3(X)$

[Option ID = 146179]

4. All of $f_1(X), f_2(X), f_3(X)$

[Option ID = 146180]

15) Let V be a finite dimensional vector space over a field F and $B : V \times V \to F$ any symmetric bilinear form on V.

For a subspace W, let $W^{\perp} := \{v \in V : B(v, w) = 0 \text{ for all } w \in W\}$.

Which of the following is necessarily correct?

[Question ID = 20648][Question Description = S1_MATH_897_PhD_Q015]

1. $\dim W^{\perp} = \dim V - \dim W$

[Option ID = 146181]

2. $\dim W^{\perp} > \dim V - \dim W$

[Option ID = 146182]

3. $\dim W^{\perp} \leq \dim V - \dim W$

[Option ID = 146183]

4. $\dim W^{\perp} \geq \dim V - \dim W$

[Option ID = 146184]

16) Let T be a linear operator on the two dimensional vector space $\mathbb{C}^2$ over $\mathbb{C}$ which preserves the bilinear form B on $\mathbb{C}^2$ defined by $B((x_1, x_2), (y_1, y_2)) = x_1y_2 - x_2y_1$. Let S be a linear operator on $\mathbb{C}^2$ which preserves the quadratic form Q on $\mathbb{C}^2$ defined by $Q((x_1, x_2)) = x_1^2 - x_2^2$. Which of the following is correct?

[Question ID = 20649][Question Description = S1_MATH_897_PhD_Q016]

1. $\det(T) = 1$ and $\det(S) = 1$.

[Option ID = 146185]

2. $\det(T) = \pm 1$ and $\det(S) = 1$.

[Option ID = 146186]

3. $\det(T) = 1$ and $\det(S) = \pm 1$.

[Option ID = 146187]

4. $\det(T) = \pm 1$ and $\det(S) = \pm 1$.

[Option ID = 146188]

17) Let $L = \mathbb{Q}(2^{\frac{1}{4}})$ and $K = \mathbb{Q}(2^{\frac{1}{5}})$ be field extensions of $\mathbb{Q}$.

For any field extension K/F, write $Aut(K/F)$ for the group of F-automorphisms of K. Then,

[Question ID = 20650][Question Description = S1_MATH_897_PhD_Q017]

1. $|Aut(L/\mathbb{Q})| = 1$ and $|Aut(K/\mathbb{Q})| = 1$.

[Option ID = 146189]

2. $|Aut(L/\mathbb{Q})| = 2$ and $|Aut(K/\mathbb{Q})| = 5$.

[Option ID = 146190]

3. $|Aut(L/\mathbb{Q})| = 2$ and $|Aut(K/\mathbb{Q})| = 1$.

[Option ID = 146191]

4. $|Aut(L/\mathbb{Q})| = 4$ and $|Aut(K/\mathbb{Q})| = 1$.

[Option ID = 146192]

18) Let L and K be the splitting fields of the polynomials $(X^2-2)(X^2-3)(X^2-5)$ and $(X^2-2)(X^2-3)(X^2-6)$ over $\mathbb{Q}$, respectively. For any field extension K/F write $Aut(K/F)$ for the group of F-automorphisms of K. Then,

[Question ID = 20651][Question Description = S1_MATH_897_PhD_Q018]

1. $|Aut(L/\mathbb{Q})| = 8$ and $|Aut(K/\mathbb{Q})| = 8$.

[Option ID = 146193]

2. $|Aut(L/\mathbb{Q})| = 8$ and $|Aut(K/\mathbb{Q})| = 4$.

[Option ID = 146194]

3. $|Aut(L/\mathbb{Q})| = 4$ and $|Aut(K/\mathbb{Q})| = 8$.

[Option ID = 146195]

4. $|Aut(L/\mathbb{Q})| = 4$ and $|Aut(K/\mathbb{Q})| = 4$.

[Option ID = 146196]

19) For any positive integer n, let $\Phi_n(X)$ denote the n-th cyclotomic polynomial, i.e. $\Phi_n(X) := \Pi_\zeta(X-\zeta)$ where the product varies over all the n-th primitive roots of unity. Which of the following is correct?

[Question ID = 20652][Question Description = S1_MATH_897_PhD_Q019]

1. $\Phi_6(X) = X^6 - 1$ and $\Phi_8(X) = X^8 - 1$.

[Option ID = 146197]

2. $\Phi_6(X) = X^3 + 1$ and $\Phi_8(X) = X^4 + 1$.

[Option ID = 146198]

3. $\Phi_6(X) = X^2 - X + 1$ and $\Phi_8(X) = X^4 + 1$.

[Option ID = 146199]

4. $\Phi_6(X) = X^2 + X + 1$ and $\Phi_8(X) = X^4 + 1$.

[Option ID = 146200]

20) Let $\mathbb{F}_3$ be a field containing 3 elements and $\mathbb{F}_3(t)$ the quotient field of the polynomial ring $\mathbb{F}_3[t]$.

Let $S := \{\phi : \mathbb{F}_3(t) \to \mathbb{F}_3(t) \mid \phi \text{ is a field homorphism } \}$.

What is the cardinality of the set S?

[Question ID = 20653][Question Description = S1_MATH_897_PhD_Q020]

1. 1 [Option ID = 146201]
2. 2 [Option ID = 146202]
3. greater than 2 but finite [Option ID = 146203]
4. infinte [Option ID = 146204]

21) Consider $\ell^2 = \{(x_n)_{n=1}^{\infty} \mid x_n \in \mathbb{C} \text{ and } \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$. Endow ℓ^2 with $||\cdot||_2$-norm defined by

$||(x_n)_{n=1}^{\infty}||_2 = \left[\sum_{n=1}^{\infty} |x_n|^2\right]^{\frac{1}{2}}$ for all $(x_n)_{n=1}^{\infty} \in \ell^2$.

Let $X = \left\{\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \mid f_{ij} \in \ell^2 \text{ for all } i,j = 1,2\right\}$. Define $||\cdot||: X \to \mathbb{R}$ by

$\left|\left|\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}\right|\right| = max\,\{||f_{ij}||_2 \mid i,j = 1,2\}$ Then which of the following statements is correct?

[Question ID = 20654][Question Description = S1_MATH_897_PhD_Q021]

1. $||\cdot||$ is not a norm on X.

[Option ID = 146205]

2. $||\cdot||$ is a norm on X but $(X, ||\cdot||)$ is not a Banach space.

[Option ID = 146206]

3. $(X, ||\cdot||)$ is Banach space but not a Hilbert space.

[Option ID = 146207]

4. $(X, ||\cdot||)$ is a Hilbert space.

[Option ID = 146208]

22) Let V denote the vector space of all polynomials in one variable with real coefficients over $\mathbb{R}$ of degree less than or equal to **10**. Then which of the following functions $||\cdot||: V \to \mathbb{R}$ is not a norm on V?

[Question ID = 20655][Question Description = S1_MATH_897_PhD_Q022]

1. $||p|| = \int_0^1 |p(x)|dx$ for all $p \in V$.

[Option ID = 146209]

2. $||p|| = sup\{|p(x)| \mid 0 \le x \le 1\}$, for all $p \in V$.

[Option ID = 146210]

3. $||p|| = sup\{|p(x)| \mid 0 \le x \le 1\} + |p(0)| + |p(1)|$, for all $p \in V$.

[Option ID = 146211]

4. $||p|| = \left[\int_0^1 |p(x)|dx\right]^{\frac{1}{2}}$, for all $p \in V$.

[Option ID = 146212]

23) Consider $C[0,1] = \{f: [0,1] \to \mathbb{R} \mid \text{f is continuous}\}$. Endow $C[0,1]$ with the $||\cdot||_\infty$-norm defined by $||f||_\infty = sup\{|f(x)| \mid 0 \le x \le 1\}$, for all $f \in C[0,1]$. Define $T: C[0,1] \to C[0,1]$ by $(Tf)(t) = \int_0^t f(s)ds$ for all $f \in C[0,1], t \in [0,1]$. Then which of the following statements is correct?

[Question ID = 20656][Question Description = S1_MATH_897_PhD_Q023]

1. $||T^{2021}|| = 2021$

[Option ID = 146213]

2. $||T^{2021}|| = \frac{1}{2021}$

[Option ID = 146214]

3. $||T^{2021}|| = \frac{1}{2021!}$

[Option ID = 146215]

4. T^{2021} is an unbounded operator.

[Option ID = 146216]

24) Let S be a non-empty countable set. For every positive integer n, define $C_{S,n} = \{h: S \to \mathbb{R} \mid \sum_{s \in S} |h(s)|^n < \infty\}$. Also, for every positive integer n and $f: S \to \mathbb{R}$ define $||f||_n = [\sum_{s \in S} |f(s)|^n]^{\frac{1}{n}}$. Then which of the following statements is correct?

[Question ID = 20657][Question Description = S1_MATH_897_PhD_Q024]

1. $(C_{S,2021}, ||\cdot||_{2021})$ is a Banach space.

[Option ID = 146217]

2. $(C_{S,2021}, ||\cdot||_{2021})$ is a Banach space if and only if S is a finite set.

[Option ID = 146218]

3. $(C_{S,2021}, ||\cdot||_{2021})$ is a Hilbert space.

[Option ID = 146219]

4. $(C_{S,2021}, ||\cdot||_{2020})$ is a Banach space.

[Option ID = 146220]

25) Consider $\ell^{50} = \{(x_n)_{n=1}^\infty \mid x_n \in \mathbb{C} \text{ and } \sum_{n=1}^\infty |x_n|^{50} < \infty\}$. Endow ℓ^{50} with $||\cdot||_{50}$-norm defined by $||(x_n)_{n=1}^\infty||_{50} = [\sum_{n=1}^\infty |x_n|^{50}]^{\frac{1}{50}}$ for all $(x_n)_{n=1}^\infty \in \ell^{50}$. Fix a bounded sequence $(y_n)_{n=1}^\infty$ of complex numbers. Define $T: \ell^{50} \to \ell^{50}$ by $T((x_n)_{n=1}^\infty) = (x_n y_n)_{n=1}^\infty$ for all $(x_n)_{n=1}^\infty \in \ell^{50}$. Then which of the following statements is correct?

[Question ID = 20658][Question Description = S1_MATH_897_PhD_Q025]

1. T is not continuous.

[Option ID = 146221]

2. $||T|| > sup\{|y_n| \mid \text{for all positive integers } n\}$.

[Option ID = 146222]

3. $||T|| < sup\{|y_n| \mid \text{for all positive integers } n\}$.

[Option ID = 146223]

4. $||T|| = sup\{|y_n| \mid \text{for all positive integers } n\}$.

[Option ID = 146224]

26) Let X be a normal, second countable topological space. Consider a family $\mathfrak{F}$ of continuous functions from X to $[0,1]$ that separates points and closed sets in X. Then which of the following statements is correct?

[Question ID = 20695][Question Description = S1_MATH_897_PhD_Q026]

1. No such family $\mathfrak{F}$ exists.

[Option ID = 146513]

2. Any such family $\mathfrak{F}$ is necessarily finite.

[Option ID = 146514]

3. There exists such a family $\mathfrak{F}$ which is countable.

[Option ID = 146515]

4. Any such family $\mathfrak{F}$ is necessarily uncountable.

[Option ID = 146516]

27) Let X be the disjoint union of open intervals I_1 and I_2 in $\mathbb{R}$. Endow $\mathbb{R}$ with the usual topology and X with the subspace topology. Then the one point compactification of X

[Question ID = 20659][Question Description = S1_MATH_897_PhD_Q027]

1. does not exist.

   [Option ID = 146225]
2. is homeomorphic to a circle.

   [Option ID = 146226]
3. is homeomorphic to the disjoint union of two circles.

   [Option ID = 146227]
4. is homeomorphic to $\{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 1\} \cup \{(x,y) \in \mathbb{R}^2 \mid (x+1)^2 + y^2 = 1\}$ equipped with the usual topology on $\mathbb{R}^2$.

   [Option ID = 146228]

28) Consider the set of natural numbers $\mathbb{N}$ (equipped with the discrete topology) as a subset of its Stone-Čech compactification denoted by $\beta\mathbb{N}$. If A and B are non-empty disjoint subsets of $\mathbb{N}$, then their closures in $\beta\mathbb{N}$

[Question ID = 20660][Question Description = S1_MATH_897_PhD_Q028]

1. are disjoint.

   [Option ID = 146229]
2. contain exactly one point in common.

   [Option ID = 146230]
3. contain countably infinitely many points in common.

   [Option ID = 146231]
4. contain uncountably many points in common.

   [Option ID = 146232]

29) Suppose X is a normal topological space containing an infinite discrete closed subset A. Then which of the following statements is correct?

[Question ID = 20661][Question Description = S1_MATH_897_PhD_Q029]

1. Every continuous function $f: X \to \mathbb{R}$ is constant.

   [Option ID = 146233]
2. Every continuous function $f: X \to \mathbb{R}$ is bounded.

   [Option ID = 146234]
3. Every continuous function $f: A \to \mathbb{R}$ is bounded.

   [Option ID = 146235]
4. There exists an unbounded continuous function $f: X \to \mathbb{R}$.

   [Option ID = 146236]

30) Let $X = \{(x,y) \in \mathbb{R}^2 \mid 2x^2 + 3y^2 = 1\}$. Endow $\mathbb{R}^2$ with the discrete topology. Then which of the following statements is correct?

[Question ID = 20662][Question Description = S1_MATH_897_PhD_Q030]

1. X is a compact subset of $\mathbb{R}^2$ in this topology.

   [Option ID = 146237]
2. X is a connected subset of $\mathbb{R}^2$ in this topology.

   [Option ID = 146238]
3. X is an open subset of $\mathbb{R}^2$ in this topology.

   [Option ID = 146239]
4. X is neither open nor closed subset of $\mathbb{R}^2$ in this topology.

   [Option ID = 146240]

31) Let $c > a > 0$ be fixed. The set of complex numbers z satisfying $0 < |z-a| + |z+a| \leq 2c$ is

[Question ID = 20663][Question Description = S1_MATH_897_PhD_Q031]

1. neither open nor closed.

   [Option ID = 146241]
2. closed but not bounded.

   [Option ID = 146242]
3. open.

   [Option ID = 146243]
4. compact.

   [Option ID = 146244]

32) The radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2n)!}{n!n!} z^n$$

is

[Question ID = 20664][Question Description = S1_MATH_897_PhD_Q032]

1. 1

   [Option ID = 146245]
2. 1/2

   [Option ID = 146246]
3. 1/4

[Option ID = 146247]
4. 1/e

[Option ID = 146248]

33) Which of the following inequalities is true for all $z \in \mathbb{C}$?

[Question ID = 20532][Question Description = S1_MATH_897_PhD_Q033]

1. $|\sin z| \le 1$

[Option ID = 146249]

2. $|\sin z| \le e^{|z|}$

[Option ID = 146250]

3. $|\sin z| \le |e^{z}|$

[Option ID = 146251]

4. $|\sin z| \le |z|$

[Option ID = 146252]

34) Let w and z be complex numbers. What are the necessary and sufficient conditions for the equality

$$|z+w| = |z| - |w|$$

to hold?

[Question ID = 20533][Question Description = S1_MATH_897_PhD_Q034]

1. $w = 0$ or $\frac{z}{w} \le 0$

[Option ID = 146253]

2. $w = 0$ or $\frac{z}{w} \le -1$

[Option ID = 146254]

3. $w = 0$

[Option ID = 146255]

4. $w = z = 0$

[Option ID = 146256]

35) Given below are two statements

Statement A:

A branch of the square root function can be defined on $\mathbb{C}\backslash\{z \in \mathbb{C} | z \text{ is real and } z \ge 0\}$.

Statement B:

A branch of logarithm can be defined on $\mathbb{C}\backslash\{z \in \mathbb{C} | z \text{ is real and } z \ge 0\}$.

Choose the correct answer from the options given below.

[Question ID = 20534][Question Description = S1_MATH_897_PhD_Q035]

1. Both Statements A and B are true.

[Option ID = 146257]

2. Both Statements A and B are false.

[Option ID = 146258]

3. Statement A is true but Statement B is false.

[Option ID = 146259]

4. Statement A is false but statement B is true.

[Option ID = 146260]

36) Let $\mathbb{D} = \{z \in \mathbb{C} | \ |z| < 1\}$ be the open unit disk. Suppose that $f, g : \mathbb{D} \to \mathbb{C}$ are analytic and satisfy

$f(\frac{1}{n}) = \frac{1}{n^2}$ and $g(1 - \frac{1}{n}) = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$.

Which of the following two statements is/are necessarily true?

Statement A:

$f(z) = z^2$ for all $z \in \mathbb{D}$.

Statement B:

$g(z) = z$ for all $z \in \mathbb{D}$.

[Question ID = 20535][Question Description = S1_MATH_897_PhD_Q036]

1. Both Statements A and B

[Option ID = 146261]

2. Statement A only

[Option ID = 146262]

3. Statement B only

[Option ID = 146263]

4. Neither Statement A nor Statement B

[Option ID = 146264]

37) If $f : \mathbb{C} \to \mathbb{C}$ is a nonconstant entire function, which of the following is necessarily correct?

If f is a nonconstant entire function, which of the following is necessarily correct?

[Question ID = 20536][Question Description = S1_MATH_897_PhD_Q037]

1. $\lim_{|z|\to\infty} |f(z)| = \infty$.

   [Option ID = 146265]
2. $|f^{(n)}(0)| \leq \frac{n!}{100^n}$ for all sufficiently large n.

   [Option ID = 146266]
3. $|f^{(n)}(0)| \leq 100^n$ for all sufficiently large n.

   [Option ID = 146267]
4. f has infinitely many zeros.

   [Option ID = 146268]

38) The integral $\int_0^{\infty} \frac{\sin 2x}{x}\, dx$

[Question ID = 20537][Question Description = S1_MATH_897_PhD_Q038]

1. does not converge.

   [Option ID = 146269]
2. converges and equals 0.

   [Option ID = 146270]
3. converges and equals $\pi/2$.

   [Option ID = 146271]
4. converges and equals 2.

   [Option ID = 146272]

39) Suppose that f, g: $\mathbb{C}\to\mathbb{C}$ are entire functions such that

Re f(z) + Im f(z) $\neq 0$ for any $z \in \mathbb{C}$

and

$(\text{Re } g(z))^2 + (\text{Im } g(z))^2 \neq 0$ for any $z \in \mathbb{C}$.

Then

[Question ID = 20538][Question Description = S1_MATH_897_PhD_Q039]

1. f must be a constant and g must be a constant.

   [Option ID = 146273]
2. f must be a constant but g need not be a constant.

   [Option ID = 146274]
3. f need not be a constant but g must be a constant.

   [Option ID = 146275]
4. f need not be a constant and g need not be a constant.

   [Option ID = 146276]

40) If f is analytic in the disk $\{z \in \mathbb{C} \mid |z| < 3\}$, then $\sup_{|z|=2} \left|f(z) - \frac{1}{z}\right|$

[Question ID = 20696][Question Description = S1_MATH_897_PhD_Q040]

1. can be at most 1/2. [Option ID = 146517]
2. is at least 1/2. [Option ID = 146518]
3. can take any positive real value for suitably chosen f.

   [Option ID = 146519]
4. can take any positive real value less than or equal to 2 but cannot be more than 2. [Option ID = 146520]

41) Let $E = \{p \in \mathbb{Q} | 2 < p^2 < 3\}$.

A. E is closed in $\mathbb{Q}$.

B. E is bounded in $\mathbb{Q}$.

C. E is compact.

Choose the ***correct*** answer from the options given below.

[Question ID = 20539][Question Description = S1_MATH_897_PhD_Q041]

1. A, B and C are true.

   [Option ID = 146277]
2. B is true, but A and C are false.

   [Option ID = 146278]
3. A and B are true, but C is false.

   [Option ID = 146279]
4. A, B and C are false.

   [Option ID = 146280]

42) For $z = a + ib, w = c + id$, we say $z \prec w$ if either $a < c$ or $a = c, b \leq d$.

A. $(\mathbb{C}, \prec)$ is an ordered field.

B. $(\mathbb{C}, \prec)$ has least upper bound property.

Choose the ***correct*** answer from the options given below:

[Question ID = 20540][Question Description = S1_MATH_897_PhD_Q042]

1. Both A and B are true.

   [Option ID = 146281]
2. A is true but B is false.

   [Option ID = 146282]
3. A is false but B is true.

   [Option ID = 146283]
4. Both A and B are false.

   [Option ID = 146284]

43) Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined as

$$\phi(x) = \begin{cases} \pi & \text{if } x \leq 0 \\ \pi + \exp\left(\frac{-2}{5x}\right) & \text{if } x > 0. \end{cases}$$

Let the nth order derivative of ϕ at x be denoted by $\phi^{(n)}(x)$, if it exists.

A. ϕ is infinitely differentiable at every point in $\mathbb{R} \setminus \{0\}$.

B. ϕ is infinitely differentiable at 0, and for each natural number n, $\phi^{(n)}(0) = 0$.

Choose the ***correct*** answer from the options given below:

[Question ID = 20541][Question Description = S1_MATH_897_PhD_Q043]

1. A and B are true.

   [Option ID = 146285]
2. A is true, but B is false.

   [Option ID = 146286]
3. B is true, but A is false.

   [Option ID = 146287]
4. A and B are false.

   [Option ID = 146288]

44) Let $x, y, z \in \mathbb{R}^n$ be linearly independent vectors. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, let $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$.

A. For every $m \in \mathbb{N}$, there exist $a_1^{(m)}, a_2^{(m)}, a_3^{(m)} \in \mathbb{R}$ with $|a_1^{(m)}| + |a_2^{(m)}| + |a_3^{(m)}| = 1$ such that $\|a_1^{(m)}x + a_2^{(m)}y + a_3^{(m)}z\| < \frac{1}{m}$.

B. There is a $C > 0$ such that if $a_1, a_2, a_3 \in \mathbb{R}$ and $|a_1| + |a_2| + |a_3| = 1$, then $\|a_1x + a_2y + a_3z\| \geq C$.

Choose the ***correct*** answer from the options given below:

[Question ID = 20542][Question Description = S1_MATH_897_PhD_Q044]

1. Both A and B are true.

   [Option ID = 146289]
2. A is false and B is true.

   [Option ID = 146290]
3. A is true and B is false.

   [Option ID = 146291]
4. Both A and B are false.

   [Option ID = 146292]

45) Define $f : \mathbb{R}^2 \to \mathbb{R}$ as

$$f(x,y) = \begin{cases} 0 & \text{if } (x,y) = (0,0) \\ (x^2+y^2)\sin\frac{1}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0). \end{cases}$$

A. f is differentiable at $(0,0)$.

B. The directional derivatives of f are continuous at $(0,0)$.

Choose the ***correct*** answer from the options given below:

[Question ID = 20543][Question Description = S1_MATH_897_PhD_Q045]

1. Both A and B are true. [Option ID = 146293]
2. A is false but B is true. [Option ID = 146294]
3. A is true but B is false. [Option ID = 146295]
4. Both A and B are false. [Option ID = 146296]

46) Let $f : \mathbb{M}_n(\mathbb{R}) \to \mathbb{M}_n(\mathbb{R})$ be defined as $f(A) = 2A + 5A^t$, where A^t denotes the transpose of A. Let $f'(A)$ denote the total derivative of f at A, if it exists.

Choose the ***correct*** answer from the options given below:

[Question ID = 20544][Question Description = S1_MATH_897_PhD_Q046]

1. f is not continuous on $\mathbb{M}_n(\mathbb{R})$.

[Option ID = 146297]

2. f is continuous but not differentiable on $\mathbb{M}_n(\mathbb{R})$.

[Option ID = 146298]

3. f is differentiable at every $A \in \mathbb{M}_n(\mathbb{R})$ and $f'(A)(X) = 2A + 5A^t$.

[Option ID = 146299]

4. f is differentiable at every $A \in \mathbb{M}_n(\mathbb{R})$ and $f'(A)(X) = 2X + 5X^t$.

[Option ID = 146300]

47) Let $U \subseteq \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^n$ be a continuously differentiable injective function such that $\det f'(x) \neq 0$ for each x.

A. f is an open map

B. $f(U)$ is open but f may not be an open map.

C. $f^{-1} : f(U) \to U$ is differentiable.

Choose the ***correct*** answer from the options given below:

[Question ID = 20545][Question Description = S1_MATH_897_PhD_Q047]

1. A and C are true.

[Option ID = 146301]

2. B and C are true.

[Option ID = 146302]

3. B is true, but C can be false.

[Option ID = 146303]

4. A, B and C can be false.

[Option ID = 146304]

48) Let $\{r_1, r_2, \ldots\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$. For each $n \in \mathbb{N}$, define $f_n : [0,1] \to \mathbb{R}$ as

$$f_n(t) = \begin{cases} 1 & \text{if } t \in \{r_1, r_2, \ldots, r_n\}, \\ 0 & \text{otherwise .} \end{cases}$$

A. (f_n) is pointwise convergent

B. (f_n) is uniformly convergent

C. The pointwise limit of (f_n), if it exists, is Riemann integrable.

Choose the ***correct*** answer from the options given below:

[Question ID = 20546][Question Description = S1_MATH_897_PhD_Q048]

1. A, B and C are true.

[Option ID = 146305]

2. A is true, but B and C are false.

[Option ID = 146306]

3. A and C are true, but B is false.

[Option ID = 146307]

4. A, B and C are false.

[Option ID = 146308]

49) Let $f : [0,1] \to [0,1]$ be defined as

$$f(t) = \begin{cases} 1 & \text{if } t = 0 \\ \frac{1}{q} & \text{if } t = \frac{p}{q} \text{ with } p, q \in \mathbb{N},\ \gcd(p,q) = 1, \\ 0 & \text{if } t \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

A. f is Riemann integrable on $[0,1]$.

B. f is continuous at every irrational point.

C. f is discontinuous at every rational point.

Choose the ***correct*** answer from the options given below:

[Question ID = 20547][Question Description = S1_MATH_897_PhD_Q049]

1. A, B and C are true. [Option ID = 146309]
2. A is false, but B and C are true. [Option ID = 146310]
3. B is false, but A and C are true. [Option ID = 146311]
4. C is false, but A and B are true. [Option ID = 146312]

50) A. $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ and $\{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$ are homeomorphic.

B. $\{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$ and $\{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$ are homeomorphic.

B. $\{(x,y) \in \mathbb{R}^2 \mid xy = 0\}$ and $\{(x,y) \in \mathbb{R}^2 \mid xy = 1\}$

C. The open interval $]-1,1[$ and the set $\{(x,y) \in \mathbb{R}^2 \mid y = x^2\}$ are homeomorphic.

Choose the ***correct*** answer from the options given below:

[Question ID = 20548][Question Description = S1_MATH_897_PhD_Q050]

1. A, B and C are true.

   [Option ID = 146313]
2. A is true, but B and C are false.

   [Option ID = 146314]
3. B is true, but A and C are false.

   [Option ID = 146315]
4. C is true, but A and B are false.

   [Option ID = 146316]

51) Let $X = (\mathbb{C}^2, \|\cdot\|_1)$, where $\|(a,b)\|_1 = |a| + |b|$ for $a, b \in \mathbb{C}$. Let $X_0 = \{(a,0) : a \in \mathbb{C}\}$.
Define $f_0 : X_0 \to \mathbb{C}$ by $f_0(a,0) = a$ for all $a \in \mathbb{C}$. Let $\theta \in \mathbb{R} \setminus \{0\}$ be fixed. Define $f_1, f_2 : X \to \mathbb{C}$ by

$$f_1(a,b) = a + e^{i\theta} b,\ a, b \in \mathbb{C},$$
$$f_2(a,b) = a - e^{i\theta} b,\ a, b \in \mathbb{C}.$$

Which of the following statements is correct?

[Question ID = 20549][Question Description = S1_MATH_897_PhD_Q051]

1. Only f_1 is a Hahn-Banach extension of f_0.

   [Option ID = 146317]
2. Only f_2 is a Hahn-Banach extension of f_0.

   [Option ID = 146318]
3. Neither of f_1 and f_2 are Hahn-Banach extensions of f_0.

   [Option ID = 146319]
4. Both f_1 and f_2 are Hahn-Banach extensions of f_0.

   [Option ID = 146320]

52) For $n \in \mathbb{N}$, define $L_n = \{(x_k) \in \ell^2 : \sum_{k=1}^{n} x_k = 0\}$. Let $e_1 = (1, 0, 0, \ldots,)$.

What is the distance between e_1 and L_5?

[Question ID = 20665][Question Description = S1_MATH_897_PhD_Q052]

1. $\sqrt{5}$

   [Option ID = 146321]
2. $\dfrac{1}{\sqrt{5}}$

   [Option ID = 146322]
3. 0

   [Option ID = 146323]
4. 5

   [Option ID = 146324]

53) Let $(X, \|\cdot\|)$ be a finite dimensional complex normed linear space. Let $T : X \to X$ be a non-zero linear map. Define $\|x\|_T := \|Tx\|,\ x \in X$.

Consider the following statements:

[Question ID = 20666][Question Description = S1_MATH_897_PhD_Q053]

1. A and B only.

   [Option ID = 146325]
2. C and D only.

   [Option ID = 146326]
3. All of the statements are correct.

   [Option ID = 146327]
4. None of the statements is correct.

   [Option ID = 146328]

54) Define $f : \ell^2 \to \mathbb{C}$ by $f((x_n)) = \sum_{n=1}^{\infty} \frac{1}{n} x_n,\ (x_n) \in \ell^2$.

What is the value of $\|f\|$?

[Question ID = 20667][Question Description = S1_MATH_897_PhD_Q054]

1. π

2

[Option ID = 146329]
2. $\frac{\pi}{6}$

[Option ID = 146330]
3. $\frac{\sqrt{\pi}}{6}$

[Option ID = 146331]
4. $\frac{\pi}{\sqrt{6}}$

[Option ID = 146332]

55) Let $C = \{(x_n) \in \ell^2 : |x_n| \leq \frac{1}{n^{\frac{3}{2}}}, \text{ for all } n \in \mathbb{N}\}$ Then C is

[Question ID = 20668][Question Description = S1_MATH_897_PhD_Q055]
1. closed and bounded but not compact.

[Option ID = 146333]
2. compact.

[Option ID = 146334]
3. bounded but not closed.

[Option ID = 146335]
4. closed but not bounded.

[Option ID = 146336]

56) The sum of the series $\sum_{n=0}^{\infty} \frac{\cos(2^n x)}{3^n} \ (x \in \mathbb{R})$

[Question ID = 20669][Question Description = S1_MATH_897_PhD_Q056]
1. converges uniformly to a bounded continuous function on $\mathbb{R}$.

[Option ID = 146337]
2. converges uniformly to an unbounded continuous function on $\mathbb{R}$.

[Option ID = 146338]
3. does not converge uniformly but converges pointwise to a continuous function.

[Option ID = 146339]
4. does not converge uniformly but converges pointwise to a discontinuous function.

[Option ID = 146340]

57) Define $f : [0,1] \to \mathbb{R}$ by $f(x) = \begin{cases} \sin(\frac{1}{x}), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$

Which of the following statements is correct?

[Question ID = 20670][Question Description = S1_MATH_897_PhD_Q057]
1. The graph of f is a connected subset of $\mathbb{R}^2$ and f is not continuous.

[Option ID = 146341]
2. The graph of f is a disconnected subset of $\mathbb{R}^2$ and f is not continuous.

[Option ID = 146342]
3. The graph of f is a disconnected subset of $\mathbb{R}^2$ and f is continuous.

[Option ID = 146343]
4. The graph of f is a connected subset of $\mathbb{R}^2$ and f is continuous.

[Option ID = 146344]

58) Let (X, d) be a compact metric space and $f : X \to X$ be an isometry, that is, $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$.

Which of the following statements is necessarily true?

[Question ID = 20671][Question Description = S1_MATH_897_PhD_Q058]
1. f must be surjective but need not be injective.

[Option ID = 146345]
2. f must be injective but need not be surjective.

[Option ID = 146346]
3. f is neither injective nor surjective.

[Option ID = 146347]
4. f is both injective and surjective.

[Option ID = 146348]

59) Let $0 < K < \infty$ and $0 < \alpha \leq 1$. Define

$L_{K,\alpha} = \left\{ f \in C([0,1], \mathbb{R}) : |f(x) - f(y)| \leq K|x-y|^{\alpha}, \ x, y \in [0,1] \right\}.$

Then, the family $L_{K,\alpha}$ is

[Question ID = 20672][Question Description = S1_MATH_897_PhD_Q059]
1. equicontinuous but not totally bounded.

   [Option ID = 146349]
2. not equicontinuous but totally bounded.

   [Option ID = 146350]
3. neither equicontinuous nor totally bounded.

   [Option ID = 146351]
4. both equicontinuous and totally bounded.

   [Option ID = 146352]

60)

Let (X, d) be a metric space such that every sequence (x_n) in X with $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ converges in X. Consider the following statements;

A. X is complete.

B. Every bounded sequence in X must have a convergent subsequence.

C. There exists a Cauchy sequence without any convergent subsequence.

Which of the above statements is/are correct?

[Question ID = 20673][Question Description = S1_MATH_897_PhD_Q060]
1. Only A is correct.

   [Option ID = 146353]
2. Only B and C are correct.

   [Option ID = 146354]
3. All are correct.

   [Option ID = 146355]
4. None of the statements is correct.

   [Option ID = 146356]

61) What is the largest order of an element in the group S_8 of permutations of eight symbols?

[Question ID = 20674][Question Description = S1_MATH_897_PhD_Q061]
1. 8 [Option ID = 146357]
2. 10 [Option ID = 146358]
3. 15 [Option ID = 146359]
4. 20 [Option ID = 146360]

62) A finite group G has two subgroups H and K such that the order of H is 8 and the index of K is 9. If K is normal in G then which of the following statements is true?

[Question ID = 20675][Question Description = S1_MATH_897_PhD_Q062]
1. H is contained in K

   [Option ID = 146361]
2. K is contained in H

   [Option ID = 146362]
3. $H \cap K = \{1\}$

   [Option ID = 146363]
4. $HK = G$

   [Option ID = 146364]

63) Which of the following three statements is/are true for the dihedral group D_n whose order is $2n$?

A. The center of D_n is the trivial group $\{1\}$.

B. The center of D_n has exactly two elements.

C. There is exactly one normal subgroup of D_n.

Choose the ***correct*** answer from the options given below:

[Question ID = 20676][Question Description = S1_MATH_897_PhD_Q063]
1. A only

   [Option ID = 146365]
2. B only

   [Option ID = 146366]
3. C only

   [Option ID = 146367]
4. None of A, B and C

   [Option ID = 146368]

64) If A is a 2×2 matrix with real entries and $2i$ is a characteristic root of A. Then A^{-1} is given by

[Question ID = 20677][Question Description = S1_MATH_897_PhD_Q064]
1. $\frac{1}{2}A$

   [Option ID = 146369]

2. $-\frac{1}{2}A$

[Option ID = 146370]

3. $\frac{1}{4}A$

[Option ID = 146371]

4. $-\frac{1}{4}A$

[Option ID = 146372]

65) Let F be the field having exactly 8 elements. Then the number of subspaces of dimension 1 in F^3 is

[Question ID = 20678][Question Description = S1_MATH_897_PhD_Q065]

1. 3 [Option ID = 146373]
2. 64 [Option ID = 146374]
3. 73 [Option ID = 146375]
4. 511 [Option ID = 146376]

66) Suppose H is a subgroup of a finite group G. Which of the following is true about the ac on of G on the set $\frac{G}{H}$ of lecosets given by $x \star gH = xgH$ for $x, g \in G$?

[Question ID = 20679][Question Description = S1_MATH_897_PhD_Q066]

1. The number of orbits depends upon the index $[G: H]$ of H in G

[Option ID = 146377]

2. The size of the orbit that contains gH depends upon the choice of g

[Option ID = 146378]

3. The size of the stabilizer group at the point gH depends upon the choice of g

[Option ID = 146379]

4. The stabilizer group at the point gH depends upon the of choice of g

[Option ID = 146380]

67) Which of $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$ is/are diagonalizable over the field of complex numbers?

[Question ID = 20680][Question Description = S1_MATH_897_PhD_Q067]

1. A only

[Option ID = 146381]

2. B only

[Option ID = 146382]

3. Both A and B

[Option ID = 146383]

4. Neither A nor B

[Option ID = 146384]

68) Set $u = (1,1,1), v = (1,2,3), w = (0,1,1),\ z = (0,1,2)$. Let $S = \{u, v, w\}$ and $T = \{u, v, z\}$. Suppose $f: S \to T$ and $g: T \to S$ are given by $f(u) = z, f(v) = u, f(w) = v$ and $g(u) = v, g(v) = w, g(z) = u$. Then which of f and g can be extended to a linear transformation from $\mathbb{R}^3$ o itself?

[Question ID = 20681][Question Description = S1_MATH_897_PhD_Q068]

1. f but not g

[Option ID = 146385]

2. g but not f

[Option ID = 146386]

3. Both f and g

[Option ID = 146387]

4. Neither f nor g

[Option ID = 146388]

69) Which of the following are sizes of the conjugacy classes of the permutation group S_4 on four symbols?

[Question ID = 20682][Question Description = S1_MATH_897_PhD_Q069]

1. 1, 1, 1, 3, 4, 6, 8 [Option ID = 146389]
2. 1, 4, 5, 6, 8 [Option ID = 146390]
3. 1, 4, 4, 7, 8 [Option ID = 146391]
4. 1, 3, 6, 6, 8 [Option ID = 146392]

70) Suppose T is a linear operator of a finite dimensional complex vector space. If the semi-simple (diagonalizable) and nilpotent parts of T are respectively D and N so that $T = D + N$. Then the nilpotent part of $T^2 + 2T$ is given by

[Question ID = 20683][Question Description = S1_MATH_897_PhD_Q070]

1. $N^2 + 2N$

[Option ID = 146393]

2. N^2

[Option ID = 146394]

3. $N^2 + 2N + 2DN$

[Option ID = 146395]
4. $N^2 + 2DN$

[Option ID = 146396]

71) Let $G = < g >$ be a cyclic group of order 105. Which of the following is an **incorrect** statement?

[Question ID = 20684][Question Description = S1_MATH_897_PhD_Q071]

1. G has 8 distinct subgroups.

   [Option ID = 146397]
2. The number of generators of G is 48.

   [Option ID = 146398]
3. The number of generators of G in the list $g^{21}, \cdots, g^{30}$ is 5.

   [Option ID = 146399]
4. Order of the element g^{-100} in G is 21.

   [Option ID = 146400]

72) Let $G = [0,1[= \{x \in \mathbb{R} : 0 \le x < 1\}$. Define a binary operation on G as follows: For $x, y \in G$,

$$x * y = \begin{cases} x + y & \text{if} \quad x + y < 1 \\ x + y - 1 & \text{if} \quad x + y \ge 1. \end{cases}$$

With this operation, G is an abelian group. Consider the following statements:
A. Any finite subgroup of G is cyclic.
B. For integers $1 < m < n$, the subgroup of G generated by $\frac{m}{n}$ is of order n.
C. The subgroup of G generated by $x \in G$ is finite only if x is rational.
Then

[Question ID = 20569][Question Description = S1_MATH_897_PhD_Q072]

1. 1, 1, 1, 3, 4, 6, 8

   [Option ID = 146401]
2. 1, 4, 5, 6, 8

   [Option ID = 146402]
3. 1, 4, 4, 7, 8

   [Option ID = 146403]
4. 1, 3, 6, 6, 8

   [Option ID = 146404]

73) Let $|G| = p^3 q$ with odd primes p, q satisfying $p^2 \le q \le p^2 + p$. Consider the following statements:
A. G has a normal Sylow $p-$subgroup.
B. G has a normal Sylow $q-$subgroup.
Then

[Question ID = 20570][Question Description = S1_MATH_897_PhD_Q073]

1. A is true and B is false.

   [Option ID = 146405]
2. both A and B are true.

   [Option ID = 146406]
3. B is true and A is false.

   [Option ID = 146407]
4. both A and B are false.

   [Option ID = 146408]

74) Let $p > 3$ be a prime and $n = p^3 + p - 3$. Suppose that G is a subgroup of the symmetric group S_n with $|G| = p^3$. Let
$S = \{1 \le i \le n : \sigma(i) = i \quad \text{for all} \quad \sigma \in G\} \quad \text{and} \quad |S| = s.$
Then

[Question ID = 20571][Question Description = S1_MATH_897_PhD_Q074]

1. $s = 0.$

   [Option ID = 146409]
2. $s = 1.$

   [Option ID = 146410]
3. $p | s$ with $s > 0.$

   [Option ID = 146411]
4. $p \nmid s$ with $s > 1.$

   [Option ID = 146412]

75) Let G be a finite group and p be a prime such that p^3 divides $|G|$. Which of the following is a possible value of the number of elements of order p in G?

[Question ID = 20572][Question Description = S1_MATH_897_PhD_Q075]

1. $p - 1$

   [Option ID = 146413]
2. $p + 1$

[Option ID = 146414]
3. p^2

[Option ID = 146415]
4. $p^2 + p$

[Option ID = 146416]

76) Let d be a squarefree integer. Recall that $\mathbb{Z}[\sqrt{d}] = \{m + n\sqrt{d} : m, n \in \mathbb{Z}\}$ is a ring under standard addition and multiplication. For $x = m + n\sqrt{d} \in \mathbb{Z}[\sqrt{d}]$, let $N(x) = |m^2 - dn^2|$. Consider the following statements about $\mathbb{Z}[\sqrt{d}]$:

A. $N(x) = N(y)$ for associates $x, y \in \mathbb{Z}[\sqrt{d}]$.

B. $2 + \sqrt{-3}$ is a unit in $\mathbb{Z}[\sqrt{-3}]$.

C. $N(x) \geq 4$ for $x \in \mathbb{Z}[\sqrt{5}], x \neq 0$ and x not an unit.

Then

[Question ID = 20573][Question Description = S1_MATH_897_PhD_Q076]
1. A and C are true and B is false.

[Option ID = 146417]
2. B and C are true and A is false.

[Option ID = 146418]
3. A, B, C are all true.

[Option ID = 146419]
4. A and B are true and C is false.

[Option ID = 146420]

77) The number of distinct ideals of $\mathbb{Z}/360\mathbb{Z}$, the ring of integers modulo 360, is

[Question ID = 20574][Question Description = S1_MATH_897_PhD_Q077]
1. 360

[Option ID = 146421]
2. 24

[Option ID = 146422]
3. 30

[Option ID = 146423]
4. 8

[Option ID = 146424]

78) Let R be a commutative ring with identity. Consider the following statements about R:

A. If $x \in R$ is nilpotent, then $1 + x^2$ is a unit.

B. There are nilpotent elements $x, y \in R$ such that $x + y$ is not nilpotent.

C. $\{1 - x : x^3 \text{ is nilpotent in } R\}$ is a subgroup of the unit group of R.

Then

[Question ID = 20575][Question Description = S1_MATH_897_PhD_Q078]
1. B and C are true and A is false.

[Option ID = 146425]
2. A and B are true and C is false.

[Option ID = 146426]
3. A, B, C are all true.

[Option ID = 146427]
4. A and C are true and B is false.

[Option ID = 146428]

79) Let $f(x) = x^n + x + p \in \mathbb{Q}[x]$ where $p > 5$ is a prime and $n > p$. Consider the following statements:

A. All the roots of $f(x)$ over $\mathbb{C}$ have absolute value > 1.

B. $f(x)$ is irreducible over $\mathbb{Q}$.

Then

[Question ID = 20576][Question Description = S1_MATH_897_PhD_Q079]
1. both A and B are true.

[Option ID = 146429]
2. A is true and B is false.

[Option ID = 146430]
3. both A and B are false.

[Option ID = 146431]
4. B is true and A is false.

[Option ID = 146432]

80) Consider the following statements in $\mathbb{Z}[x]$. Here $f(x), g(x) \in \mathbb{Z}[x]$.

A. $3x - 15$ is irreducible in $\mathbb{Z}[x]$.

B. $f(x+3)$ is irreducible iff $f(x-15)$ is irreducible.

C. Let $\deg(f(x)) = n^3 > 8$ and $f(0) \neq 0$. Then $x^{n^3} f(\frac{1}{x})$ is irreducible iff $f(x)$ is irreducible.

D. If $f(x^2)$ is irreducible, then $f(x)$ is irreducible.

E. If the monic irreducible polynomials $f(x)$ and $g(x)$ have a common root $\alpha \in \mathbb{C}$, then $f(x) = g(x)$.

The number of **correct** statements above is

[Question ID = 20577][Question Description = S1_MATH_897_PhD_Q080]

1. 2

   [Option ID = 146433]
2. 3

   [Option ID = 146434]
3. 4

   [Option ID = 146435]
4. 5

   [Option ID = 146436]

81) Let $\sigma(m)$ denote the sum of positive divisors of m and $\varphi(m)$ denote the number of positive integers $\leq m$ and coprime to m. For $m \in \mathbb{N}$, let $f(m) = \frac{\sigma(m)}{\varphi(m)}$. Consider the following statements:

A. $f(p^r q^s) = f(p^r) f(q^s)$ for distinct primes p, q and, $r, s \in \mathbb{N}$.

B. $f(p^r) < f(p^s)$ for a prime p and positive integers $r < s$.

C. $f(p^r) \leq f(q^s)$ for primes $2 < p < q$ and, $r, s \in \mathbb{N}$.

Then

[Question ID = 20578][Question Description = S1_MATH_897_PhD_Q081]

1. A and B are true and C is false.

   [Option ID = 146437]
2. A and C are true and B is false.

   [Option ID = 146438]
3. B and C are true and A is false.

   [Option ID = 146439]
4. A, B, C are all true.

   [Option ID = 146440]

82) If

$$\left(\binom{n}{0}+\binom{n}{1}\right)\left(\binom{n}{1}+\binom{n}{2}\right)\cdots\left(\binom{n}{n-1}+\binom{n}{n}\right) = \kappa\binom{n}{0}\binom{n}{1}\cdots\binom{n}{n-1}$$

then κ equals

[Question ID = 20579][Question Description = S1_MATH_897_PhD_Q082]

1. $\frac{n^n}{n!}$

   [Option ID = 146441]
2. $\frac{(n+1)^n}{n!}$

   [Option ID = 146442]
3. $\frac{(n+1)^n}{nn!}$

   [Option ID = 146443]
4. $\frac{(n+1)^{n+1}}{n!}$

   [Option ID = 146444]

83) Let $s(n)$ be the sum of decimal digits of a positive integer n. Then the number of solutions of the equation $n + s(n) = 2021$ with $n \geq 2000$ is

[Question ID = 20580][Question Description = S1_MATH_897_PhD_Q083]

1. 1

   [Option ID = 146445]
2. 3

   [Option ID = 146446]
3. 2

   [Option ID = 146447]
4. 21

   [Option ID = 146448]

84) Let $n = 2021^2$. For any permutation $\sigma = (a_1, a_2, \cdots, a_n) \in S_n$, let

$$\mathcal{P}(\sigma) = \prod_{j=1}^{n}(j - a_j) = (1 - a_1)(2 - a_2)\cdots(n - a_n).$$

Then

[Question ID = 20581][Question Description = S1_MATH_897_PhD_Q084]

1. $\mathcal{P}(\sigma) = 0$ for all $\sigma \in S_n$.

   [Option ID = 146449]
2. $\mathcal{P}(\sigma)$ is even for all $\sigma \in S_n$.

   [Option ID = 146450]
3. $\mathcal{P}(\sigma)$ is odd for half of all $\sigma \in S_n$.

   [Option ID = 146451]
4. there exist $\sigma_1, \sigma_2 \in S_n$ for which $\mathcal{P}(\sigma_1)$ is even and $\mathcal{P}(\sigma_2)$ is odd.

   [Option ID = 146452]

85) Let $n \geq 2021$ be odd and $1 \leq a_1 < a_2 < \cdots < a_k \leq n$ be k integers. The least value of k for which there is always a pair (i, j) with $1 \leq i < j \leq k$ such that $a_j - a_i = a_1$ is

[Question ID = 20582][Question Description = S1_MATH_897_PhD_Q085]

1. $\frac{n-1}{2}$

   [Option ID = 146453]
2. $\frac{n+1}{2}$

   [Option ID = 146454]
3. $\frac{n+3}{2}$

   [Option ID = 146455]
4. $\frac{n-3}{2}$

   [Option ID = 146456]

86) Consider the fields $F = \mathbb{Q}[\sqrt{2}]$ and $L = \mathbb{Q}[\sqrt{-2}]$. Then F and L are

[Question ID = 20697][Question Description = S1_MATH_897_PhD_Q086]

1. isomorphic as rings [Option ID = 146521]
2. isomorphic as vector spaces over $\mathbb{Q}$ but not as rings

   [Option ID = 146522]
3. isomorphic as groups but not as vector spaces over $\mathbb{Q}$

   [Option ID = 146523]
4. not isomorphic as groups [Option ID = 146524]

87) The extension degree of the splitting field of $f(x) = x^8 + 1$ over $\mathbb{Q}$ is

[Question ID = 20583][Question Description = S1_MATH_897_PhD_Q087]

1. 8!

   [Option ID = 146457]
2. 24

   [Option ID = 146458]
3. 16

   [Option ID = 146459]
4. 8

   [Option ID = 146460]

88) Let $K = F[\alpha]$ be a finite extension of F with $[K:F] = n$. The F-linear map $T: K \to K$ is given by $x \mapsto \alpha x$ for all $x \in K$. Which of the following statements is/are true?

[Question ID = 20584][Question Description = S1_MATH_897_PhD_Q088]

1. A only

   [Option ID = 146461]
2. C only

   [Option ID = 146462]
3. A and B only

   [Option ID = 146463]
4. None of A, B and C

   [Option ID = 146464]

89) Let F be a field and let $f(x), g(x), h(x) \in F[x]$ be non-constant monic polynomials. Suppose $f(x)$ divides $g(x)$ and $f(x)$ divides $h(x)$ in $F[x]$. Consider the following statements.

A. Any root of $h(x) + g(x)$ is a root of $f(x)$.

B. Any root of $f(x)$ is a root of $h(x)+g(x)$.

C. Any common root of $h(x)$ and $g(x)$ is a root $f(x)$. Choose the **correct** answer from the options given below:

[Question ID = 20585][Question Description = S1_MATH_897_PhD_Q089]

1. B is true but A and C are false

   [Option ID = 146465]
2. A and B are both true if and only if h(x) and g(x) have a common root in F

   [Option ID = 146466]
3. C is true but A and B are false

   [Option ID = 146467]
4. Insufficient information to determine whether or not A, B and C are true

   [Option ID = 146468]

90) Let F be the field with 729 elements. Then the number of subfields of F that are different from F is

[Question ID = 20586][Question Description = S1_MATH_897_PhD_Q090]

1. 3

   [Option ID = 146469]
2. 2

   [Option ID = 146470]
3. 1

   [Option ID = 146471]
4. 0

   [Option ID = 146472]

91) Consider the ring $R = \mathbb{Z}[\sqrt{10}]$. Which of the following is correct?

[Question ID = 20685][Question Description = S1_MATH_897_PhD_Q091]

1. R is a UFD but not a PID

   [Option ID = 146473]
2. R is a PID but not a Euclidean domain

   [Option ID = 146474]
3. R is a Euclidean domain

   [Option ID = 146475]
4. R is not a UFD

   [Option ID = 146476]

92) Consider the ring $R = \mathbb{Z}\left[\frac{1}{2}\right]$. Which of the following statements is/are correct?

A. R is an integral domain.

B. All ideals in R are principal.

C. R is generated by 1 and $\frac{1}{2}$ as a group.

Choose the **correct** answer from the options given below:

[Question ID = 20686][Question Description = S1_MATH_897_PhD_Q092]

1. B only

   [Option ID = 146477]
2. A and B only

   [Option ID = 146478]
3. A and C only

   [Option ID = 146479]
4. All are true

   [Option ID = 146480]

93)

Let $q = p^{28}$ for a prime p and let $\mathbb{F}_q$ be the field with q elements. Consider the map φ on $\mathbb{F}_q$ given by $x \mapsto x^{p^7}$. Which of the following statements is/are true?

[Question ID = 20687][Question Description = S1_MATH_897_PhD_Q093]

1. A only

   [Option ID = 146481]
2. B only

   [Option ID = 146482]
3. A and B only

   [Option ID = 146483]
4. All of A, B and C

   [Option ID = 146484]

94) Which of the following statements is not correct?

[Question ID = 20688][Question Description = S1_MATH_897_PhD_Q094]

1. Consider $\mathbb{R}$ and $\mathbb{R}^3$ as vector spaces over $\mathbb{Q}$. Then, every basis of $\mathbb{R}$ over $\mathbb{Q}$ has the same cardinality as a basis of $\mathbb{R}^3$ over $\mathbb{Q}$.

   [Option ID = 146485]

2. The set $S := \{(x, y, z) \in \mathbb{R}^3 : \max\{|x|, |y|, |z|\} \leq 1\}$ and the open interval $I = \{x \in \mathbb{R} : 0 < x < 1$ have same cardinality.

   [Option ID = 146486]

3. If E denotes the set of even natural numbers, then the set of all functions from E into the group $\mathbb{Z}_2$ containing two elements is a countable set.

   [Option ID = 146487]

4. There exists a surjective map from the set $X = \{x \in \mathbb{R} : -1 < x < 0 \text{ or } 0 < x < 1\}$ onto the set $Y = \{w \in \mathbb{C} : p(w) = 0 \text{ for some polynomial } p(t) \in \mathbb{Z}[t]\}$.

   [Option ID = 146488]

95) Consider the sets

$S = \{(x, y) \in \mathbb{R}^2 : y = x\};$
$T = \{(x, y) \in \mathbb{R}^2 : y = -x\};$
$U = \{(x, y) \in \mathbb{R}^2 : y = 0\}$ and
$W = \{(x, y) \in \mathbb{R}^2 : x = 0\}.$

Which of the following statements is **not** correct?

[Question ID = 20689][Question Description = S1_MATH_897_PhD_Q095]

1. $S \cup T$ is an equivalence relation on $\mathbb{R}$.

   [Option ID = 146489]

2. $S \cup U \cup W$ is an equivalence relation on $\mathbb{R}$.

   [Option ID = 146490]

3. $U \cup W$ is a symmetric relation on $\mathbb{R}$.

   [Option ID = 146491]

4. $S \cup W$ is not a symmetric relation on $\mathbb{R}$.

   [Option ID = 146492]

96) Let V be a 5 dimensional vector space over the field $\mathbb{Z}_5$ and let $\mathcal{Z} := \{T \in L(V) : T \circ S = S \circ T \text{ for all } S \in L(V)\}$, where $L(V)$ denotes the space of all linear maps from V into itself. Consider the following statements:

A. $\mathcal{Z}$ is a subspace of $L(V)$.

B. $\mathcal{Z}$ has only one element.

C. $\mathcal{Z}$ has precisely 5 elements.

D. $\mathcal{Z}$ has precisely 5^5 elements.

Which of the above statements is/are correct?

[Question ID = 20690][Question Description = S1_MATH_897_PhD_Q096]

1. A only

   [Option ID = 146493]

2. A and C only

   [Option ID = 146494]

3. A and B only

   [Option ID = 146495]

4. A and D only

   [Option ID = 146496]

97) Let V, W, Y and Z be finite dimensional vector spaces over the real field and let

$T : V \to W, S : W \to Z, P : V \to Y \text{ and } Q : Y \to Z$

be linear maps such that T and Q are injective, S and P are surjective and $S \circ T = Q \circ P$.

Consider the following statements:

A. dim(W) = dim (Y).

B. dim (Z) > dim (V).

Which of the above two statements is/are always true?

[Question ID = 20691][Question Description = S1_MATH_897_PhD_Q097]

1. A only

   [Option ID = 146497]

2. B only

   [Option ID = 146498]

3. Both A and B

[Option ID = 146499]
4. Neither A nor B

[Option ID = 146500]

98) Let X be an infinite set and Y be a finite set. Consider the following statements related to X and Y:

A. If $f: X \to Y$ is a surjective map, then $|f^{-1}(y)| = |X|$ for some y in Y.

B. If $f, g: X \to Y$ are two surjective maps such that $f^{-1}(y) = g^{-1}(y)$ for every y in Y, then $f = g$.

C. There exists an injective map from X into $\mathbb{Z}$ and a surjective map from $\mathbb{Q}$ onto Y.

D. If $f: X \to \mathbb{Z}$, $g: \mathbb{Z} \to Y$ and $h: Y \to X$ are three maps such that f is surjective, then has $h \circ g \circ f$ finite range.

Which of the above statements is/are necessarily correct?

[Question ID = 20692][Question Description = S1_MATH_897_PhD_Q098]
1. B and C only

[Option ID = 146501]
2. B and D only

[Option ID = 146502]
3. A and D only

[Option ID = 146503]
4. D only

[Option ID = 146504]

99)

Let d_1 and d_2 be two metrics on $\mathbb{C}^2$ given by $\begin{array}{l} d_1((z_1, z_2), (w_1, w_2)) = |z_1 - z_2| + |w_1 - w_2| \\ d_2((z_1, z_2), (w_1, w_2)) = \sqrt{|z_1 - z_2|^2 + |w_1 - w_2|^2} \end{array}$ for all $(z_1, z_2), (w_1, w_2) \in \mathbb{C}^2$. Consider the following statements:

A. The set $\{(z_1, z_2) \in \mathbb{C}^2 : d_2((z_1, z_2), (1, 1)) \leq 1\}$ is complete with respect to the metric d_1.

B. The set $\{(z, w) \in \mathbb{C}^2 : |z| + |w| \leq 5\}$ is complete with respect to the metric d_2.

C. There exists a linear isometry from $(\mathbb{C}^2, d_1)$ into $(\mathbb{C}^2, d_2)$.

Which of the above statements is/are correct?

[Question ID = 20693][Question Description = S1_MATH_897_PhD_Q099]
1. All are correct.

[Option ID = 146505]
2. A and B only

[Option ID = 146506]
3. C only

[Option ID = 146507]
4. B and C only

[Option ID = 146508]

100) For $p = 2, 3$, consider the metric $d_p : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ given by

$$d_p((x_1, x_2, x_3), (y_1, y_2, y_3)) = [\sum_{i=1}^{3} |x_i - y_i|^p]^{\frac{1}{p}}.$$

Let $K = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_i| \leq i \text{ for all } 1 \leq i \leq 3\} \subset \mathbb{R}^3$. Consider the following statements:

A. K is complete with respect to the metric d_3.

B. Every sequence in K has a convergent subsequence with respect to the metric d_2.

C. K is bounded with respect to the metric d_3.

D. If a subset S of $\mathbb{R}^3$ is bounded with respect to the metric d_3, then S is compact with respect to the metric d_2.

Which of the above statements is/are correct?

[Question ID = 20694][Question Description = S1_MATH_897_PhD_Q100]
1. All are correct.

[Option ID = 146509]
2. A, B and C only

[Option ID = 146510]
3. B and C only

[Option ID = 146511]
4. A and C only

[Option ID = 146512]