1. Two lines are given as $\frac{x-2}{3} = \frac{y-1}{3} = \frac{z-0}{2}$ and $\frac{x-1}{3} = \frac{y-2}{2} = \frac{z-1}{3}$ Then the shortest distance between the lines is:

A. $\frac{6}{\sqrt{43}}$
B. $\frac{11}{\sqrt{43}}$
C. $\frac{3}{\sqrt{43}}$
D. $\frac{5}{\sqrt{43}}$

Answer (B)

Sol.

Shortest distance $= \left| \frac{[\overrightarrow{a_2} - \overrightarrow{a_1} \quad \overrightarrow{b_1} \quad \overrightarrow{b_2}]}{|\overrightarrow{b_1} \times \overrightarrow{b_2}|} \right|$

From given data

$\overrightarrow{a_2} - \overrightarrow{a_1} = (2-1)\hat{\imath} + (1-2)\hat{\jmath} + (0-1)\hat{k} = \hat{\imath} - \hat{\jmath} - \hat{k}$

$\overrightarrow{b_1} = 3\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$

$\overrightarrow{b_2} = 3\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$

$$\overrightarrow{b_1} \times \overrightarrow{b_2} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & 3 & 2 \\ 3 & 2 & 3 \end{vmatrix} = 5\hat{\imath} - 3\hat{\jmath} - 3\hat{k}$$

Shortest Distance $= \frac{\begin{vmatrix} 1 & -1 & -1 \\ 3 & 3 & 2 \\ 3 & 2 & 3 \end{vmatrix}}{|5\hat{\imath} - 3\hat{\jmath} - 3\hat{k}|} = \frac{11}{\sqrt{43}}$

2. Tangent is drawn at a point on the parabola $y^2 = 24x$. It intersects the hyperbola $xy = 2$ at points A and B. Locus of mid point of AB is:

A. $y^2 = 3x$
B. $y^2 = -3x$
C. $y^2 = 6x$
D. $y^2 = -6x$

Answer (B)

Sol.

Let a point on $y^2 = 24x$ be $(6t^2, 12t)$

Equation of tangent $\equiv 12yt = 12(x + 6t^2)$

$\Rightarrow ty = x + 6t^2 \quad \cdots (1)$

Let midpoint of chord AB be (h, k)

Equation of chord bisect at this point is:

$\frac{xk + hy}{2} = hk \quad \cdots (2)$

Comparing (1) and (2) we get,

$\frac{t}{h/2} = \frac{-1}{k/2} = \frac{6t^2}{hk}$

$\Rightarrow t = \frac{-h}{k}$ and $-2h = 6(\frac{-h}{k})^2$

$\Rightarrow -2hk^2 = 6h^2$

$\Rightarrow k^2 = -3h$

The required locus is $y^2 = -3x$

3. $\lim_{t \to 0} [(1^{\frac{1}{\sin^2 t}} + 2^{\frac{1}{\sin^2 t}} + 3^{\frac{1}{\sin^2 t}} + \cdots + n^{\frac{1}{\sin^2 t}})^{\sin^2 t}$

A. 0
B. n
C. $\frac{n^2-n}{2}$
D. $n^2 + n$

Answer (B)

Sol.

$$\lim_{t \to 0} n\left[\left(\frac{1}{n}\right)^{cosec^2 t} + \left(\frac{2}{n}\right)^{cosec^2 t} + \left(\frac{3}{n}\right)^{cosec^2 t} + \cdots + \left(\frac{n}{n}\right)^{cosec^2 t}\right]^{\sin^2 t}$$

$\Rightarrow \quad n[0 + 0 + \cdots 1]$

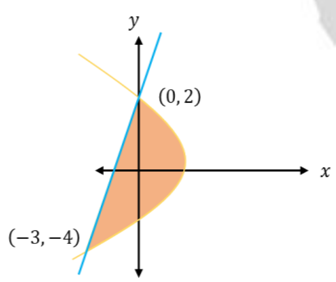
$\Rightarrow \quad n \times 1 = n$

4. The area enclosed between $y^2 = -4x + 4$ and $y = 2x + 2$ is:

A. 3
B. 6
C. 9
D. 12

Answer (C)

Sol.



Required area $= \int_{-4}^{2} \left(\frac{4-y^2}{4} - \frac{y-2}{2}\right) dy$

$= \left[2y - \frac{y^3}{12} - \frac{y^2}{4}\right]_{-4}^{2}$

$= \left(4 - \frac{8}{12} - 1\right) - \left(-8 + \frac{16}{3} - 4\right)$

$= \left(3 - \frac{2}{3}\right) + 12 - \frac{16}{3}$

$= 9$

5. $\sum_{r=0}^{22}\ {}^{22}C_r\ {}^{23}C_r$ is equal to:

A. ${}^{44}C_{22}$
B. ${}^{45}C_{23}$
C. ${}^{45}C_{24}$
D. ${}^{44}C_{23}$

Answer (B)

Sol.

$$\sum_{r=0}^{22}\ {}^{22}C_r\ {}^{23}\sum_{r=0}^{22}\ {}^{22}C_r\ \ {}^{23}C_r\ = \sum_{r=0}^{22}\ {}^{22}C_r\ {}^{23}\sum_{r=0}^{22}\ {}^{22}C_r\ \ {}^{23}C_{23-r}$$

$$= {}^{22}C_0\ {}^{23}C_{23} + {}^{22}C_1\ {}^{23}C_{22} + \cdots + {}^{22}C_{21}\ {}^{23}C_2 + {}^{22}C_{22}\ {}^{23}C_1$$

$$(1+x)^{22} = {}^{22}C_0 + {}^{22}C_1\,x + \cdots + {}^{22}C_{21}\,x^{21} + {}^{22}C_{22}\,x^{22}$$

$$(1+x)^{23} = {}^{23}C_0 + {}^{23}C_1\,x + \cdots + {}^{23}C_{22}\,x^{22} + {}^{23}C_{23}\,x^{23}$$

coefficient of x^{23} in $(1+x)^{22}(1+x)^{23} = \sum_{r=0}^{22}\ {}^{22}C_r\ {}^{23}\sum_{r=0}^{22}\ {}^{22}C_r\ \ {}^{23}C_{23-r}$

$$= {}^{45}C_{23}$$

6. $\sim(\sim p \wedge q) \Rightarrow (\sim p \vee q)$ is equivalent to:

A. $\sim p \vee q$
B. $\sim p \wedge q$
C. $p \wedge q$
D. $p \vee q$

Answer (A)

Sol.

$\sim(\sim p \wedge q) \Rightarrow (\sim p \vee q)$ $\qquad$ $[p \to q \Leftrightarrow \sim p \vee q]$

$= (\sim p \wedge q) \vee (\sim p \vee q)$

$= (\sim p \vee (\sim p \vee q)) \wedge (q \vee (\sim p \vee q))$

$= (\sim p \vee q) \wedge (q \vee \sim p)$

$= \sim p \vee q$

7. There are 12 languages. One can choose at most 2 from 5 particular languages. The number of ways in which one can select 5 languages is:

A. 540
B. 535
C. 546
D. 525

Answer (C)

Sol.

Case -1: If no language is selected from given 5 particular languages

$\Rightarrow\ {}^{7}C_5$

Case -2: If 1 language is chosen from the given 5 and 4 from other 7 languages

$\Rightarrow\ {}^{7}C_4\ {}^{5}C_1$

Case -3: If 2 languages are chosen from the given 5 and 3 from other 7 languages

$\Rightarrow\ {}^7C_3\ {}^5C_2$

$\therefore$ Total ways $= {}^7C_5 + {}^7C_4\ {}^5C_1 + {}^7C_3\ {}^5C_2$

$= 21 + 175 + 350$

$= 546$

8. The solution of differential equation $\frac{dy}{dx} + \frac{y}{x^2} = \frac{1}{x^3}$ is:

A. $y = \left(1 + \frac{1}{x}\right) + ce^{\frac{1}{x}}$

B. $y = \left(1 - \frac{1}{x}\right) + ce^{\frac{1}{x}}$

C. $y = \left(x + \frac{1}{x}\right) + ce^{\frac{1}{x}}$

D. $y = \left(x - \frac{1}{x}\right) + ce^{\frac{1}{x}}$

Answer (A)

Sol.

Given equation is linear differential equation

$\therefore I.F = e^{\int \frac{1}{x^2} dx} = e^{-\left(\frac{1}{x}\right)}$

$\Rightarrow \int d(ye^{-\frac{1}{x}}) = \int \frac{e^{-\frac{1}{x}}}{x^3} dx$

$\Rightarrow ye^{-\frac{1}{x}} = \frac{1}{x} e^{-\frac{1}{x}} + e^{-\frac{1}{x}} + c$

$\Rightarrow y = (\frac{1}{x} + 1) + ce^{\frac{1}{x}}$

9. The equation $x^2 - [x] + x + 3 = x[x]^2$ has (where [.] represents greatest integer function)

A. No solution

B. 1 solution in $(-\infty, 1)$

C. 2 solution in $(-\infty, \infty)$

D. 1 solution in $(-\infty, \infty)$

Answer (D)

Sol.

$x^2 - [x] + x + 3 = x[x]^2$

$x^2 + \{x\} + 3 = x[x]^2$

Case 1: $x < 0$

L.H.S> 0, R.H.S< 0

$\therefore$ No solution

Case 2: $x \in [0,1)$

L.H.S≥ 3, R.H.S$= 0$

No solution

Case 3: $x \in [1,2)$

L.H.S$\in [4,8)$, R.H.S$\in [1,2)$

No solution

Case 4: $x \in [2,3)$

L.H.S$\in [7,13)$, R.H.S$\in [8,12)$

$\therefore\ x^2 + x - 2 + 3 = 4x$

$\Rightarrow x^2 - 3x + 1 = 0$

$\Rightarrow x = \frac{3+\sqrt{5}}{2}$ one solution

Case 5: $x \in [3,4)$

L.H.S$\in [12,20)$, R.H.S$\in [27,36)$

Similarly, For $x > 4$, R.H.S > L.H.S Always

$\therefore$ one solution in $[1,2)$ or only one solution in $(-\infty, \infty)$

10. The function $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is:

A. Continuous but non- differentiable at $x = 0$
B. Discontinuous at $x = 0$
C. $f'(x)$ is differentiable but not continuous
D. $f'(x)$ is continuous but non- differentiable

Answer (D)

Sol.

$f(x) = x^2\, sin\,(\frac{1}{x})$

At $x = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0)$

$f(x)$ is continuous at $x = 0$

LHD at $x = 0$ is $\lim_{h\to 0} \frac{f(0-h)-f(0)}{-h}$

$\lim_{h\to 0} \frac{-h^2 sin(\frac{1}{h})}{-h} = 0$

RHD at $x = 0$ is $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = 0$

$\therefore$ $f(x)$ is continuous as well as differentiable hence Option A and B are wrong.

A function cannot be differentiable unless continuous hence Option C is wrong.

11. The distance of the point $P(1,2,3)$ from the plane containing the points A(1, 4, 5), B(2, 3, 4) and C(3, 2, 1) is equal to:

A. $5\sqrt{2}$
B. $3\sqrt{2}$
C. 1
D. $\sqrt{2}$

Answer (D)

Sol.

Direction ratio of $\overrightarrow{AB} = < 1, -1, -1 >$

Direction ratio of $\overrightarrow{AC} = < 2, -2, -4 >$

Vector normal to $\overrightarrow{AB}$ and $\overrightarrow{AC} = (1,1,0)$

Equation of plane $(x-1, y-4, z-5) \cdot (1,1,0) = 0$

$\Rightarrow x + y - 5 = 0$

Distance from $(1,2,3) = \left|\frac{1+2-5}{\sqrt{(1^2+1^2)}}\right| = \sqrt{2}$

12. If $\left(1+\sqrt{3}i\right)^{200} = 2^{199}(p+iq)$, then $p+q+q^2$ and $p-q+q^2$ are roots of the equation:

A. $x^2-4x+1=0$
B. $x^2-4x-1=0$
C. $x^2+4x+1=0$
D. $x^2+4x-1=0$

Answer (A)

Sol.

$(1+\sqrt{3}i)^{200} = 2^{199}(p+iq)$

$\Rightarrow 2^{200}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{200} = 2^{199}(p+iq)$

$\Rightarrow 2^{200}\left(cos\left(\frac{\pi}{3}\right)+i\ sin\left(\frac{\pi}{3}\right)\right)^{200} = 2^{199}(p+iq)$

$\Rightarrow 2^{200}(e^{i\frac{\pi}{3}})^{200} = 2^{199}(p+iq)$

$\Rightarrow 2^{200}\left(cos\left(\frac{200\pi}{3}\right)+i\ sin\left(\frac{200\pi}{3}\right)\right) = 2^{199}(p+iq)$

$\Rightarrow 2^{200}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) = 2^{199}(p+iq)$

$\Rightarrow 2^{199}(-1+\sqrt{3}i) = 2^{199}(p+iq)$
$\Rightarrow p=-1, q=\sqrt{3}$
$p+q+q^2 = 2+\sqrt{3}$
$p-q+q^2 = 2-\sqrt{3}$
So, the equation whose roots are $2+\sqrt{3}$ and $2-\sqrt{3}$ is
$x^2-(2+\sqrt{3}+(2-\sqrt{3})x+(2+\sqrt{3})(2-\sqrt{3})=0$
$\Rightarrow x^2-4x+1=0$

13. $\sum_{r=0}^{2023} r^2 \cdot {}^{2023}C_r = \alpha \cdot 2023 \cdot 2^{2022}$ then α is equal to:

A. 1012
B. 1011
C. 1020
D. 1022

Answer (A)

Sol.

$\sum_{r=0}^{2023} r^2 \cdot {}^{2023}C_r = \sum_{r=0}^{2023}(r^2-r)\cdot {}^{2023}C_r + \sum_{r=0}^{2023} r \cdot {}^{2023}C_r$

$= 2023 \cdot 2022 \cdot \sum_{r=2}^{2023} {}^{2021}C_{r-2} + 2023\sum_{r=1}^{2023} {}^{2022}C_{r-1}$

$= 2023 \cdot 2022 \cdot 2^{2021} + 2023 \cdot 2^{2022}$

$= 2^{2022} \cdot 2023 \cdot 1012$

Then $\alpha = 1012$

14. If $y^2 + ln\,(cos^2 x) = y, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ then

A. $y''(0) = 0$
B. $|y''(0)| = 2$
C. $y'(0) = 3$
D. $y'(0) = -3$

Answer (B)

Sol.

Differentiating both sides we have
$2yy' + \frac{1}{cos^2 x} \cdot (2\,cos\,x) \cdot (-\,sin\,x) = y'$
$2yy' - 2\,tan\,x = y' \cdots (i)$

Differentiating both sides again we have
$\Rightarrow 2(y')^2 - 2yy'' - 2\,sec^2\,x = y'' \cdots (ii)$
From (i) substituting $x = 0$
$y'(0) = 0$
and substituting $x = 0$ and $y'(0) = 0$ in (ii)
$y''(0) = -2$
$|y''(0)| = 2$

15. if $\vec{v}.\vec{w} = 2, \vec{u} \times \vec{w} = \vec{v} + \alpha\vec{u}, \vec{u} = 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k}, \vec{v} = \hat{\imath} + 2\hat{\jmath} - 4\hat{k}$ then $\vec{u} \cdot \vec{w}$

A. $\frac{28}{12}$
B. $\frac{12}{29}$
C. $-\frac{29}{12}$
D. $\frac{29}{12}$

Answer (D)

Sol.

If $\vec{u} \times \vec{w} = \vec{v} + \alpha\vec{u}$

$\vec{v}.\vec{w} + \alpha\vec{u} \cdot \vec{w} = \vec{0}$ (taking dot product with $\vec{v}$)

$\alpha(\vec{u} \cdot \vec{w}) = -2 \cdots (i)$

$0 = \vec{u} \cdot \vec{v} + \alpha|\vec{u}|^2$

$\Rightarrow 29\alpha + 24 = 0$

$\Rightarrow \alpha = -\frac{24}{29}$

From (i)

$\vec{u} \cdot \vec{w} = \frac{-2}{\left(-\frac{24}{29}\right)} = \frac{58}{24} = \frac{29}{12}$

16. If $R = \{(a, b): g.c.d(a, b) = 1, a, b \in Z\}$. Then relation R is:

A. Reflexive
B. Symmetric
C. Transitive
D. None of the above

Answer (B)

Sol.

Reflexive relation:-
$(a, a) \in R\ \forall a \in A$
Let $a = 5, g.c.d(5,5) = 5 \neq 1$
$\Rightarrow R$ is not reflexive relation

Symmetric Relation:-
If $(a, b) \in R \Rightarrow (b, a) \in R$
If $g.c.d(a, b) = 1 \Rightarrow g.c.d(b, a) = 1$

Transitive Relation:-
If (a, b) and $(c, c) \in R \Rightarrow (a, c) \in R$
$(2,3)$ and $(3,4) \in R$ but $(2,4) \notin R$ (because $g.c.d(2,4) = 2$)
∴ R is symmetric relation

17. The sum of all the values of $\boldsymbol{x}$ satisfying $\boldsymbol{cos^{-1} x - 2\, sin^{-1} x = cos^{-1} 2x}$ is:

A. 0
B. 1
C. $\frac{1}{2}$
D. $-\frac{1}{2}$

Answer (A)

Sol.

$cos^{-1} x - 2\, sin^{-1} x = cos^{-1} 2x$

$\Rightarrow \frac{\pi}{2} - 3\, sin^{-1} x = cos^{-1} 2x$

$\Rightarrow cos\left(\frac{\pi}{2} - 3\, sin^{-1} x\right) = cos\,(cos^{-1} 2x)$

$\Rightarrow sin\,(3\, sin^{-1} x) = cos\,(cos^{-1} 2x)$

$\Rightarrow 3x - 4x^3 = 2x$

$\Rightarrow 4x^3 = x$

$\Rightarrow x = 0, \pm\frac{1}{2}$

Sum of values of $x = 0$

18. The value of $\boldsymbol{12\int_0^3 |x^2 - 3x + 2| dx}$ is equal to:

Answer (22)

Sol.

$I = \int_0^3 |x^2 - 3x + 2| dx = \int_0^3 |(x-1)(x-2)| dx \int_0^3 |x^2 - 3x + 2| dx = \int_0^3 |(x-1)(x-2)| dx$
$= \int_0^1 (x-1)(x-2) dx - \int_1^2 (x-1)(x-2) dx + \int_2^3 (x-1)(x-2) dx$
$= \int_0^1 (x-1)(x-2) dx - \int_1^2 (x-1)(x-2) dx + \int_2^3 (x-1)(x-2) dx$
$= \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^1 - \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_1^2 + \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_2^3$
$= \left(\frac{1}{3} - \frac{3}{2} + 2\right) - \left\{\left(\frac{8}{3} - 6 + 4\right) - \left(\frac{1}{3} - \frac{3}{2} + 2\right)\right\} + \left\{\left(9 - \frac{27}{2} + 6\right) - \left(\frac{8}{3} - 6 + 4\right)\right\}$
$= \frac{11}{6}$
Hence $12\int_0^3 |x^2 - 3x + 2| dx = \int_0^3 |x^2 - 3x + 2| dx = 12I = 12 \times \frac{11}{6} = 22$