Mathematics and Statistics

Model Set - 2

Academic Year: 2020-2021 Marks: 80
Date: April 2021
Duration: 3h

1. The question paper is divided into four sections.
2. **Section A**: Q. No. 1 contains 8 multiple-choice type of questions carrying two marks each.
3. **Section A**: Q. No. 2 contains 4 very short answer type of questions carrying One mark each.
4. **Section B**: Q. No. 3 to Q. No. 14 contains Twelve short answer type of questions carrying Two marks each. **(Attempt any Eight)**.
5. **Section C**: Q. No.15 to Q. No. 26 contains Twelve short answer type of questions carrying Three marks each. **(Attempt any Eight)**.
6. **Section D**: Q.No. 27 to Q. No. 34 contains Five long answer type of questions carrying Four marks each. **(Attempt any Five)**.
7. Use of log table is allowed. Use of calculator is not allowed.
8. Figures to the right indicate full marks.
9. For each MCQ, correct answer must be written along with its alphabet.
   e.g., (a) ..... / (b) .... / (c) .... / (d) ..... Only first attempt will be considered for evaluation.
10. Use of graph paper is not necessary. Only rough sketch of graph is expected:
11. Start answers to each section on a new page.

Q. 1 | Select and write the most appropriate answer from the given alternatives for each sub-question:

1.i A biconditional statement is the conjunction of two ______ statements

1. Negative
2. Compound
3. Connective
4. Conditional

1.ii

If polar co-ordinates of a point are $\left(\frac{3}{4}, \frac{3\pi}{4}\right)$, then its Cartesian co-ordinate are ______

1. $\left(\frac{3}{4\sqrt{2}}, -\frac{3}{4\sqrt{2}}\right)$
2. $\left(\frac{3}{4\sqrt{2}}, \frac{3}{4\sqrt{2}}\right)$
3. $\left(-\frac{3}{4\sqrt{2}}, \frac{3}{4\sqrt{2}}\right)$
4. $\left(-\frac{3}{4\sqrt{2}}, -\frac{3}{4\sqrt{2}}\right)$

1.iii The feasible region is the set of point which satisfy.

1. The object functions
2. All the given constraints
3. Some of the given constraints
4. Only one constraint

1.iv

Select and write the correct alternative from the given option for the question

Differential equation of the function c + 4yx = 0 is

1. $xy + \frac{dy}{dx} = 0$
2. $x\frac{dy}{dx} + y = 0$
3. $\frac{dy}{dx} - 4xy = 0$
4. $x\frac{dy}{dx} + 1 = 0$

1.v

If $x = \cos^{-1}(t)$, $y = \sqrt{1 - t^2}$ then $\frac{dy}{dx}$ = ______

1. t

2. – t

3. $\dfrac{-1}{t}$

4. $\dfrac{1}{t}$

1.vi A ladder 5 m in length is resting against vertical wall. The bottom of the ladder is pulled along the ground, away from the wall at the rate of 1.5 m /sec. The length of the higher point of the when foot of the ladder is 4 m away from the wall decreases at the rate of ______

1. 1
2. 2
3. 2.5
4. 3

1.vii Select the correct option from the given alternatives:

If l, m, n are direction cosines of a line then $l\hat{i} + m\hat{j} + n\hat{k}$ is ______

1. null vector
2. the unit vector along the line
3. any vector along the line
4. a vector perpendicular to the line

1.viii Select the correct option from the given alternatives:

The 2 vectors $\hat{j} + \hat{k}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represents the two sides AB and AC respectively of a ΔABC. The length of the median through A is

1. $\dfrac{\sqrt{34}}{2}$

2. $\dfrac{\sqrt{48}}{2}$

3. $\sqrt{18}$

4. of the median through A is

Q. 2 | Answer the following questions:

2.i State the truth Value of $x^2 = 25$

Ans. '$x^2 = 25$' is an open sentence.

It is not a statement in logic.

2.ii

Evaluate: $\int_0^1 \frac{x}{\sqrt{e^x - 1}} dx$

Ans.

$$\int_0^1 \frac{x}{\sqrt{e^x - 1}} dx = \left[2\sqrt{e^x - 1}\right]_0^1 \qquad \text{......}\left[\because \int \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} + c\right]$$

$$= 2\left(\sqrt{e^1 1} - \sqrt{e^0 - 1}\right)$$

$$= 2\left(\sqrt{e - 1} - \sqrt{1 - 1}\right)$$

$$= 2\sqrt{e - 1}$$

2.iii An urn contains 5 red and 2 black balls. Two balls are drawn at random. X denotes number of black balls drawn. What are possible values of X?

Ans. A The urn contains 5 red and 2 black balls. If two balls are drawn from the urn, it contains either 0 or 1 or 2 black balls.

X can take values 0, 1, 2.

$\therefore$ X = {0, 1, 2}.

Ans. B X denotes the number of black balls drawn.

Sample space of the experiment is

S = {RR, BR, RB, BB}

The value of X corresponding to these outcomes are as follows:

X (RR) = 0

X(BR) = X(RB) = 1

X(BB) = 2

$\therefore$ Possible values of X are {0, 1, 2}.

2.iv The displacement of a particle at time t is given by $s = 2t^3 - 5t^2 + 4t - 3$. Find the velocity when $t = 2$ sec

Ans.

$s = 2t^3 - 5t^2 + 4t - 3$

$\therefore v = \dfrac{ds}{dt}$

$= 6t^2 - 10t + 4$

$v_{(t=2)} = 6(2)^2 - 10(2) + 4$

$= 24 - 20 + 4$

$= 8$ units/sec

Q. 3 | Attempt any Eight:

Write the converse and contrapositive of the following statements.
"If a function is differentiable then it is continuous"

Ans. Let p: A function is differentiable,

q: It is continuous.

$\therefore$ The symbolic form of the given statement is $p \rightarrow q$.

Converse: $q \rightarrow p$

i.e. If a function is continuous then it is differentiable

Contrapositive: $\sim q \rightarrow \sim p$

i.e. If a function is not continuous then it is not differentiable.

Q. 4

With usual notations, prove that $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2 + b^2 + c^2}{2abc}$

Ans.

Consider L.H.S. = $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$

$= \frac{1}{a}(\cos A) + \frac{1}{b}(\cos B) + \frac{1}{c}(\cos C)$

$= \frac{1}{a}\left(\frac{b^2 + c^2 - a^2}{2bc}\right) + \frac{1}{b}\left(\frac{a^2 + c^2 - b^2}{2ac}\right) + \frac{1}{c}\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$ ....[By consine rule]

$= \frac{b^2 + c^2 - a^2}{2abc} + \frac{a^2 + c^2 - b^2}{2abc} + \frac{a^2 + b^2 - c^2}{2abc}$

$= \frac{b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2}{2abc}$

$= \frac{a^2 + b^2 + c^2}{2abc}$

= R.H.S.

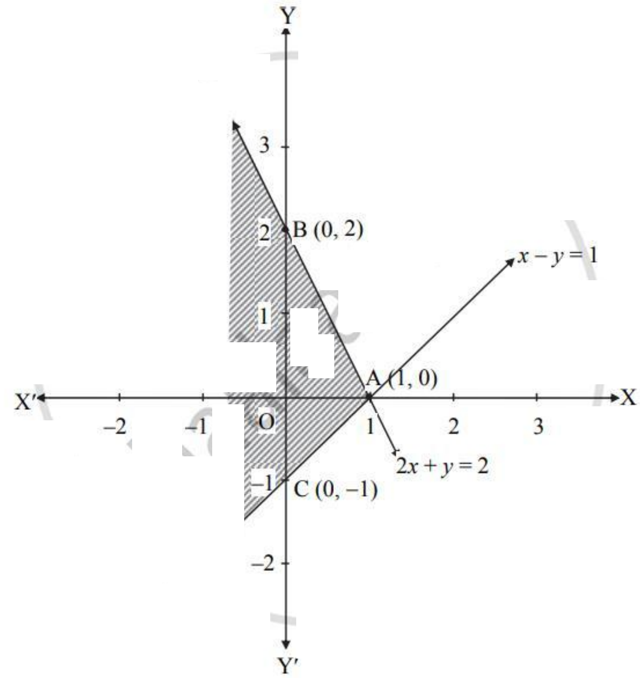
$\therefore \frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$

Q. 5 Find the graphical solution for the system of linear inequation $2x + y \le 2$, $x - y \le 1$

Ans. To find graphical solution, construct the table as follows:

Inequation	Equation	Double intercept form	Points (x, y)	Region
$2x + y \le 2$	$2x + y = 2$	$\frac{x}{1} + \frac{y}{2} = 1$	A(1, 0) B(0, 2)	$2(0) + 0 \le 2$ $\therefore 0 \le 2$ $\therefore$ origin side
$x - y \le 1$	$x - y = 1$	$\frac{x}{1} + \frac{y}{-1} = 1$	A(1, 0) C(0, −1)	$0 - 0 \le 1$ $\therefore 0 \le 1$ $\therefore$ origin side

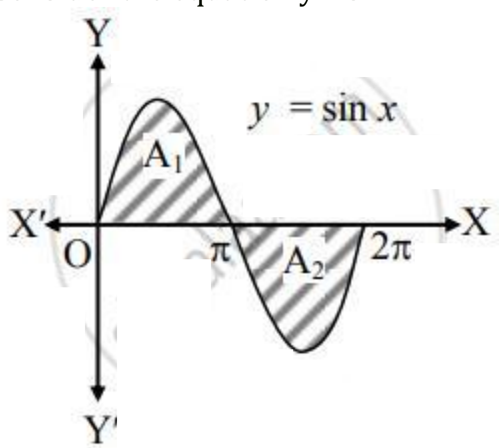
The shaded portion represents the graphical solution.



Q. 6 Find the area enclosed between the X-axis and the curve y = sin x for values of x between 0 to 2π

Ans. Let A be the required area.

Consider the equation y = sin x.



$$A_1 = \int_0^{a} \sin x \, dx$$

$= [-\cos x]_0^{\pi}$

$= -(\cos \pi - \cos 0)$

$= -(-1 - 1)$

$= 2$

$$A_2 = \int_{\pi}^{2\pi} \sin x \, dx$$

$= [-\cos x]_{\pi}^{2\pi}$

$= -[1 - (-1)]$

$= -2$

$\therefore A = A_1 + |A_2|$

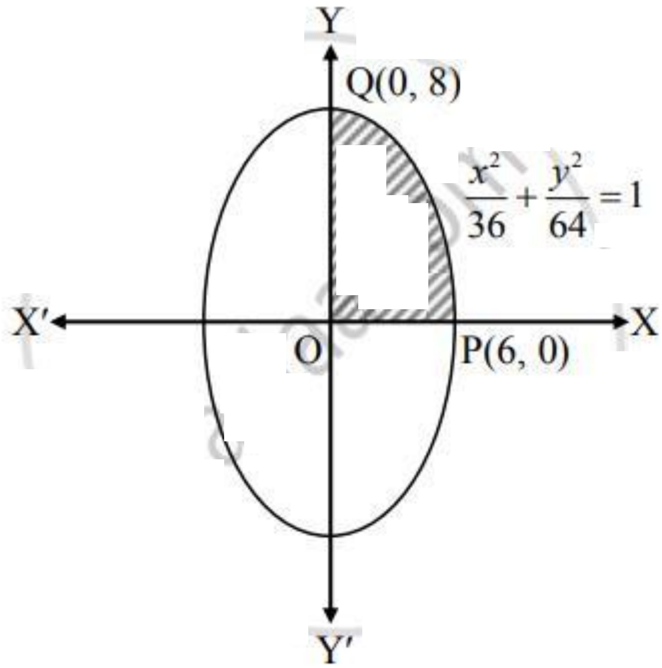
$= 2 + |(-2)|$

= 4 sq.units

Q. 7

Find the area of the ellipse $\frac{x^2}{36} + \frac{y^2}{64} = 1$, using integration

Ans. By the symmetry of the ellipse, required area of the ellipse is 4 times the area of the region OPQO.



For the region OPQO, the limits of integration are x = 0 and x = 6.

Given equation of the ellipse is $\frac{x^2}{36}+\frac{y^2}{64}$ = 1

$\therefore \frac{y^2}{64}=1-\frac{x^2}{36}$

$\therefore$ y^2 = $64\left(1-\frac{x^2}{36}\right)$

$\therefore$ y^2 = $\frac{64}{36}(36-x^2)$

$\therefore$ y = $\pm\frac{8}{6}\sqrt{36-x^2}$

$\therefore$ y = $\frac{4}{3}\sqrt{36-x^2}$ ......[$\because$ In first quadrant, y > 0]

$\therefore$ Required area = 4(area of the region OPQO)

$= 4 \int_0^6 y \, dx$

$= 4 \int_0^6 \frac{4}{3} \sqrt{36 - x^2} \, dx$

$= \frac{16}{3} \left[\frac{x}{2} \sqrt{36 - x^2} + \frac{36}{2} \sin^{-1}\left(\frac{x}{6}\right) \right]_0^6$

$= \frac{16}{3} \left[\frac{6}{2} \sqrt{36 - 36} + \frac{36}{2} \sin^{-1}(1) - \left\{ 0 + \frac{36}{2} \sin^{-1}(0) \right\} \right]$

$= \frac{16}{3} \left(0 + \frac{36}{2} \cdot \frac{\pi}{2} - 0 \right)$

= 48π sq.units

Q. 8

Let the p.m.f. of r.v. X be P(x) = ${}^4C_x \left(\frac{5}{9}\right)^x \left(\frac{4}{9}\right)^{4-x}$, x = 0, 1, 2, 3, 4. Find E(X) and Var(X)

Ans.

Here X ~ B$\left(4, \frac{5}{9}\right)$

i.e., n = 4, p = $\frac{5}{9}$ and q = $\frac{4}{9}$

∴ E(X) = np

$= 4 \times \frac{5}{9}$

$= \frac{20}{9}$

= 2.22

and

V(X) = npq

$= 4 \times \frac{5}{9} \times \frac{4}{9}$

$= \frac{80}{81}$

$= 0.9876$

Q. 9 Find the value of h, if the measure of the angle between the lines $3x^2 + 2hxy + 2y^2 = 0$ is 45°.

Ans. Given equation of the lines is $3x^2 + 2hxy + 2y^2 = 0$

Comparing with $ax^2 + 2hxy + by^2 = 0$,

We get a = 3, b = 2

Given that 45° is the acute angle between the lines.

$$\therefore \tan(45°) = \left|\frac{2\sqrt{h^2 - ab}}{a + b}\right|$$

$$\therefore 1 = \left|\frac{2\sqrt{h^2 - (3)(2)}}{3 + 2}\right|$$

$$\therefore 1 = \left|\frac{2\sqrt{h^2 - 6}}{5}\right|$$

$$\therefore \left(\frac{5}{2}\right)^2 = h^2 - 6$$

$$\therefore h^2 = \frac{25}{4} + 6$$

$$\therefore h^2 = \frac{49}{4}$$

$$\therefore h = \pm\frac{7}{2}$$

Q. 10

Evaluate: $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{(1+\cos x)^2} dx$

Ans.

Let I = $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{(1+\cos x)^2} dx$

Put $\tan\left(\frac{x}{2}\right)$ = t

$\therefore$ x = 2tan^{-1}t

$\therefore$ dx = $\frac{2\text{dt}}{1+\text{t}^2}$, sin x $\frac{2\text{t}}{1+\text{t}^2}$ and x = $\frac{1-\text{t}^2}{1+\text{t}^2}$

When x = 0, t = 0 and when x = $\frac{\pi}{2}$, t = 1

$$\therefore \text{I} = \int_0^1 \frac{\left(\frac{2\text{t}}{1+\text{t}^2}\right)^2}{\left(1+\frac{1-\text{t}^2}{1+\text{t}^2}\right)^2} \cdot \frac{2\text{dt}}{1+\text{t}^2}$$

$$\therefore \text{I} = \int_0^1 \frac{\left(\frac{2\text{t}}{1+\text{t}^2}\right)^2}{\left(1+\frac{1-\text{t}^2}{1+\text{t}^2}\right)^2} \cdot \frac{2\text{dt}}{1+\text{t}^2}$$

$$= \int_0^1 \frac{\frac{4\text{t}^2}{\left(1+\text{t}^2\right)^2}}{\frac{4}{\left(1+\text{t}^2\right)^2}} \cdot \frac{2\text{dt}}{1+\text{t}^2}$$

$$= 2\int_0^1 \frac{\text{t}^2}{1+\text{t}^2} \, \text{dt}$$

$$= 2\int_0^1 \left(\frac{1+\text{t}^2-1}{1+\text{t}^2}\right) \text{dt}$$

$$= 2\int_0^1 \left(1+\frac{1}{1+\text{t}^2}\right) \text{dt}$$

$= 2\left[t - \tan^{-1} t\right]_0^1$

$= 2[(1 - \tan^{-1}1) - (0 - \tan^{-1}0)]$

$= 2\left(1 - \frac{\pi}{4}\right)$

$= \frac{4 - \pi}{2}$

Q. 11 Water is being poured at the rate of 36 m^3/sec in to a cylindrical vessel of base radius 3 meters. Find the rate at which water level is rising

Ans. Let h be the height of water level, r be the radius of the base and V be the volume of the cylindrical vessel.

Then, r = 3 metres, $\frac{dV}{dT}$

= 36 m^3/sec

$V = \pi r^2 h$

$= \pi(3)^2 h$

$= 9\pi h$

Differentiating w.r.t.t, we get

$\frac{dV}{dt} = 9\pi \cdot \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \left(\frac{dV}{dt}\right) \times \frac{1}{9\pi}$

$= \frac{36}{9\pi}$

$= \frac{4}{\pi}$ m/sec

Thus, water level is rising at the rate of $\frac{4}{\pi}$ m/sec.

Q. 12

$\int [\text{cosec}(\log x)][1 - \cot(\log x)]\ dx$

Ans.

Let I = $\int [\text{cosec}(\log x)][1 - \cot(\log x)]\ dx$

Put $\log_e x$ = t

∴ x = e^t

∴ dx = $e^t \cdot dt$

∴ I = $\int \text{cosec}\ t(1 - \cot t)\ e^t\ dt$

= $\int e^t(\text{cosec}\ t - \text{cosec}\ t \cdot \cot t)\ dt$

Put f(t) = cosec t

∴ f'(t) = −cosec t.cot t

∴ I = $\int e^t\ [f(t) + f'(t)]\ dt$

= $e^t \cdot f(t) + c = e^t \text{cosec}\ t + c$

∴ I = $x\ \text{cosec}\ (\log x) + c$

Q. 13

If $\bar{a}, \bar{b}$ and $\bar{c}$ are position vectors of the points A, B, C respectively and $5\bar{a} - 3\bar{b} - 2\bar{c} = \bar{0}$, then find the ratio in which the point C divides the line segement BA

Ans.

$5\bar{a} - 3\bar{b} - 2\bar{c} = 0$

$\therefore 2\bar{c} = 5\bar{a} - 3\bar{b}$

$\therefore \bar{c} = \dfrac{5\bar{a} - 3\bar{b}}{2}$

$\therefore \bar{c} = \dfrac{5\bar{a} - 3\bar{b}}{5 - 3}$

$\therefore$ The point C divides the line segment BA externally in ratio 5:3.

Q. 14 If a line has the direction ratios 4, −12, 18, then find its direction cosines

Ans. Direction ratios of the line are a = 4, b = −12, c = 18.

Let l, m, n be the direction cosines of the line.

Then $l = \dfrac{a}{\sqrt{a^2 + b^2 + c^2}}$

$= \dfrac{4}{\sqrt{4^2 + (-12)^2 + (18)^2}}$

$= \dfrac{4}{\sqrt{16 + 144 + 324}}$

$= \dfrac{4}{22}$

$= \dfrac{2}{11}$

$m = \dfrac{b}{\sqrt{a^2 + b^2 + c^2}}$

$$= \frac{-12}{\sqrt{16 + 144 + 324}}$$

$$= \frac{-12}{22}$$

$$= \frac{-6}{11}$$

and

$$n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{18}{\sqrt{4^2 + (-12)^2 + (18)^2}}$$

$$= \frac{18}{\sqrt{16 + 144 + 324}}$$

$$= \frac{18}{22}$$

$$= \frac{9}{11}$$

Hence, the direction cosines of the line are $\frac{2}{11}, \frac{-6}{11}, \frac{9}{11}$.

Q. 15 | Attempt any Eight:

Q. 15 (A) Write the following statements in symbolic form
Milk is white if and only if the sky is not blue

Q. 15 (B) Write the following statements in symbolic form
If Kutab – Minar is in Delhi then Taj - Mahal is in Agra

Q. 15 (C) Write the following statements in symbolic form
Even though it is not cloudy, it is still raining

Ans. (A) Let p: Milk is white.

q: Sky is blue.

The given statement in symbolic form is $p \leftrightarrow \sim q$.

Ans. (B) Let p: If Kutub – Minar is in Delhi.

q: Taj – Mahal Is in Agra.

The given statement in symbolic form is $p \to q$.

Ans. (C) Let p: It is cloudy.

q: It is still raining.

$\therefore$ The symbolic form of the given statement is $\sim p \wedge q$

Q. 16 Three chairs and two tables cost ₹ 1850. Five chairs and three tables cost ₹2850. Find the cost of four chairs and one table by using matrices

Ans. Let the cost of 1 chair and 1 table be ₹ x and ₹ y respectively.

According to the first condition,

$3x + 2y = 1850$

According to the second condition,

$5x + 3y = 2850$

Matrix form of the above system of equations is

$$\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1850 \\ 2850 \end{bmatrix}$$

Applying $R_2 \to 3R_2 - 5R_1$, we get

$$\begin{bmatrix} 3 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1850 \\ -700 \end{bmatrix}$$

$\therefore$ By equality of matrices, we get

$3x + 2y = 1850$ ....(i)

$-y = -700$

Substituting $y = 700$ in equation (i), we get

$3x + 2(700) = 1850$

$\therefore 3x = 450$

$\therefore x = 150$

$\therefore$ The cost of four chairs $= 4 \times 150 =$ ₹ 600

∴ The cost of four chairs and one table is ₹ 600 + ₹ 700 = ₹ 1300.

Q. 17

Transform $\begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ into an upper triangular matrix by using suitable row transformations

Ans.

Let A = $\begin{bmatrix} 1 & 2 & 4 \\ 3 & -1 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Applying $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$, we get

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -7 & -7 \\ 0 & 0 & -2 \end{bmatrix}$$

This is required upper triangular matrix.

Q. 18 If the angles A, B, C of ΔABC are in A.P. and its sides a, b, c are in G.P., then show that a^2, b^2, c^2 are in A.P.

Ans. A, B, C are in A.P.

∴ A + C = 2B

We know that A + B + C = 180°

∴ 2B + B = 180°

∴ 3B = 180°

∴ ∠B = 60° ....(i)

Also, it is given that sides a, b, c are in G.P.

∴ $ac = b^2$ ....(ii)

$\therefore ac = b^2$ ....(ii)

Consider, $\cos B = \dfrac{a^2 + c^2 - b^2}{2ac}$ ....[By cosine rule]

$\therefore \cos(60°) = \dfrac{a^2 + c^2 - b^2}{2b^2}$ ....[From (i) and (ii)]

$\therefore \dfrac{1}{2} = \dfrac{a^2 + c^2 - b^2}{2b^2}$

$\therefore b^2 = a^2 + c^2 - b^2$

$\therefore a^2 + c^2 = 2b^2$

$\therefore a^2, b^2, c^2$ are in A.P.

Q. 19

In ΔABC, prove that $\dfrac{\cos 2A}{a^2} - \dfrac{\cos 2c}{c^2} = \dfrac{1}{a^2} - \dfrac{1}{c^2}$

Ans.

Consider L.H.S. $= \dfrac{\cos 2A}{a^2} - \dfrac{\cos 2c}{c^2}$

$= \dfrac{1 - 2\sin^2 A}{a^2} - \dfrac{1 - 2\sin^2 C}{c^2}$

$= \dfrac{1}{a^2} - 2\dfrac{\sin^2 A}{a^2} - \dfrac{1}{c^2} + 2\dfrac{\sin^2 C}{c^2}$

$= \dfrac{1}{a^2} - 2k^2 - \dfrac{1}{c^2} + 2k^2$ ....[By since rule]

$= \dfrac{1}{a^2} - \dfrac{1}{c^2}$

= R.H.S.

Q. 20 The probability that a person who undergoes a kidney operation will be recovered is 0.5. Find the probability that out of 6 patients who undergo similar operation none will recover

Ans. Let X denote the number of patients recovered.

P(patient recovers) = p = 0.5

$\therefore$ q = 1 – p = 1 – 0.5 = 0.5

Given, n = 6

$\therefore$ X ~ B(6, 0.5)

The p.m.f. of X is given by

$P(X = x) = {}^6C_x\,(0.5)^x\,(0.5)^{6-x}$, x = 0, 1, …, 6

P(none will recover) = P(X = 0)

$= {}^6C_0\,(0.5)^0\,(0.5)^6$

= `1/2^6

$= \dfrac{1}{64}$

Q. 21

Prove that: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. Hence find $\int_0^{\frac{\pi}{2}} \sin^2 x\,dx$

Ans.

Consider R.H.S : $\int_0^a f(a-x)\,dx$

Let I = $\int_0^a f(a-x)\,dx$

Put a – x = t

$\therefore$ – dx = dt

$\therefore$ – dx = dt

When x = 0, t = a – 0 = a

and when x = a, t = a – a = 0

$$\therefore \text{I} = \int_4^0 \text{f(t)}(-\text{dt})$$

$$= -\int_a^0 \text{f(t) dt}$$

$$= \int_0^a \text{f(t) dt} \quad \ldots\ldots \left[\because \int_a^b \text{f}(x)\,\text{d}x = -\int_b^a \text{f}(x)\,\text{d}x\right]$$

$$= \int_0^a \text{f}(x)\,\text{d}x \quad \ldots\ldots \left[\because \int_a^b \text{f}(x)\,\text{d}x = \int_a^b \text{f(t) dt}\right]$$

= L.H.S.

$$\therefore \int_0^a \text{f}(x)\,\text{d}x = \int_0^a \text{f}(\text{a} - x)\,\text{d}x$$

$$\text{Let I} = \int_0^{\frac{\pi}{2}} \sin^2 x\,\text{d}x \quad \ldots\ldots\text{(i)}$$

$$= \int_0^{\frac{\pi}{2}} \sin^2\left(\frac{\pi}{2} - x\right)\text{d}x \quad \ldots\ldots \left[\because \int_0^a \text{f}(x)\,\text{d}x = \int_0^a \text{f}(\text{a} - x)\,\text{d}x\right]$$

$$\therefore \text{I} = \int_0^{\frac{\pi}{2}} \cos^2\ \text{d}x \quad \ldots\ldots\text{(ii)}$$

Adding (i) and (ii), we get

2I = $\int_0^{\frac{\pi}{2}} \sin^2 x \, \mathrm{d}x + \int_0^{\frac{\pi}{2}} \cos^2 x \, \mathrm{d}x$

= $\int_0^{\frac{\pi}{2}} (\sin^2 x + \cos^2 x) \, \mathrm{d}x$

∴ 2I = $\int_0^{\frac{\pi}{2}} 1 \cdot \mathrm{d}x$

∴ I = $\frac{1}{2} [x]_0^{\frac{\pi}{2}}$

∴ I = $\frac{1}{2} \left(\frac{\pi}{2} - 0 \right)$

∴ I = $\frac{\pi}{4}$

Q. 22

Verify y = log x + c is the solution of differential equation $x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}$ = 0

Ans. y = log x + c

Differentiating w.r.t. x, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

∴ $x \frac{\mathrm{d}y}{\mathrm{d}x}$ = 1

Again, differentiating w.r.t. x, we get

$x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} \times \mathbf{1}$ = 0

∴ $x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}$ = 0

∴ y = log x + c is the solution of $x \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}$ = 0

Q. 23 Find the differential equation by eliminating arbitrary constants from the relation $x^2 + y^2 = 2ax$

Ans. $x^2 + y^2 = 2ax$ ......(i)

Here, a is an arbitrary constant.

Differentiating (i) w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} = 2a$$

$$\therefore 2x + 2y\frac{dy}{dx} = \frac{x^2 + y^2}{x} \quad \text{.....[From (i)]}$$

$$\therefore 2x^2 + 2xy\frac{dy}{dx} = x^2 + y^2$$

$$\therefore 2xy\frac{dy}{dx} = y^2 - x^2$$

Q. 24

If $\log_5\left(\frac{x^4 + y^4}{x^4 - y^4}\right) = 2$, show that $\frac{dy}{dx} = \frac{12x^2}{13y^3}$

Ans.

$$\log_5\left(\frac{x^4 + y^4}{x^4 - y^4}\right) = 2$$

$$\therefore \frac{x^4 + y^4}{x^4 - y^4} = 5^2 = 25$$

$$\therefore x^4 + y^4 = 25x^4 - 25y^4$$

$$\therefore -24x^4 + 26y^4 = 0$$

$$\therefore -12x^4 + 13y^4 = 0$$

Differentiating w. r. t. x, we get

$\therefore -12(4x^3) + 13\left(4y^3\frac{dy}{dx}\right) = 0$

$\therefore -12x^3 + 13y^3\frac{dy}{dx} = 0$

$\therefore 13y^3\frac{dy}{dx} = 12x^3$

$\therefore \frac{dy}{dx} = \frac{12x^3}{13y^3}$

Q. 25

Find the vector equation of the line passing through the point having position vector $-\hat{i} - \hat{j} + 2\hat{k}$ and parallel to the line $\bar{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \mu\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$, μ is a parameter

Ans.

Let $\bar{a}$ be the position vector of the point

$\therefore \bar{a} = -\hat{i} - \hat{j} + 2\hat{k}$

Equation of given line is $\bar{r} = \left(\hat{i} + 2\hat{j} + 3\hat{k}\right) + \mu\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$

$\therefore$ Direction ratios of the line are 3, 2, 1.

Let $\bar{b}$ be the vector parallel to this line.

$\therefore \bar{b} = 3\hat{i} + 2\hat{j} + \hat{k}$

The vector equation of a line passing through a point with position vector $\bar{a}$ and parallel to $\bar{b}$ is $\bar{r} = \bar{a} + \lambda\bar{b}$.

$\therefore$ Vector equation of the line is $\bar{r} = \left(-\hat{i} - \hat{j} + 2\hat{k}\right) + \lambda\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$

Q. 26

Find the equation of the plane passing through the point (7, 8, 6) and parallel to the plane $\bar{r} \cdot \left(6\hat{i} + 8\hat{j} + 7\hat{k}\right) = 0$

Ans. The plane passes through the point A(7, 8, 6).

$\therefore x_1 = 7, y_1 = 8, z_1 = 6$

Since the required plane is parallel to the plane $\bar{r} \cdot \left(6\hat{i} + 8\hat{j} + 7\hat{k}\right) = 0$

Direction ratios of normal vector will be a = 6, b = 8, c = 7.

Equation of a plane in Cartesian form is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\therefore 6(x - 7) + 8(y - 8) + 7(z - 6) = 0$

$\therefore 6x - 42 + 8y - 64 + 7z - 42 = 0$

$\therefore 6x + 8y + 7z = 42 + 42 + 64$

$\therefore 6x + 8y + 7z = 148$

Q. 27 | Attempt any Five:

The following is the c.d.f. of r.v. X:

X	−3	−2	−1	0	1	2	3	4
F(X)	0.1	0.3	0.5	0.65	0.75	0.85	0.9	1

Find p.m.f. of X.

i. $P(-1 \le X \le 2)$

ii. $P(X \le 3 / X > 0)$.

Ans. From the given table

$F(-3)= 0.1, F(-2) = 0.3, F(-1) = 0.5$

$F(0) = 0.65, f(1) = 0.75, F(2) = 0.85$

$F(3) = 0.9, F(4) = 1$

$P(X =-3) = F(-3) = 0.1$

$P(X= -2)= F(-2) - F(-3) = 0.3 - 0.1 = 0.2$

$P(X= -1) = F(-1) - F(-2) = 0.5 - 0.3 = 0.2$

$P(X = 0) = F(0) - F(-1) = 0.65 - 0.5 = 0.15$

$P(X = 1) = F(1) - F(0) = 0.75 - 0.65 = 0.1$

P(X = 2) = F(2) − F(1) = 0.85 − 0.75 = 0.1

P(X = 3) = F(3) − F(2) = 0.9 − 0.85 = 0.05

P(X = 4) = F(4) − F(3) = 1 − 0.9 = 0.1

∴ The probability distribution of X is as follows:

X = x	−3	−2	−1	0	1	2	3	4
P(X = x)	0.1	0.2	0.2	0.15	0.1	0.1	00.5	0.1

i. P(−1 ≤ X ≤ 2)

= P(X = −1 or X = 0 or X = 1 or X = 2)

= P(X = −1) + P(X = 0) + P(X = 1) + P(X = 2)

= 0.2 + 0.15 + 0.1 + 0.1

= 0.55

ii. P(X ≤ 3 / X > 0)

$$= \frac{P(X = 1 \text{ or } X = 2 \text{ or } X + 3)}{P(X = 1 \text{ or } X = 2 \text{ or } X = 3 \text{ or } X = 4)} \quad \ldots\ldots \left[\begin{array}{c}\text{Using conditional probability} \\ P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}\end{array}\right]$$

$$= \frac{P(X = 1) + P(X = 2) + P(X = 3)}{P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)}$$

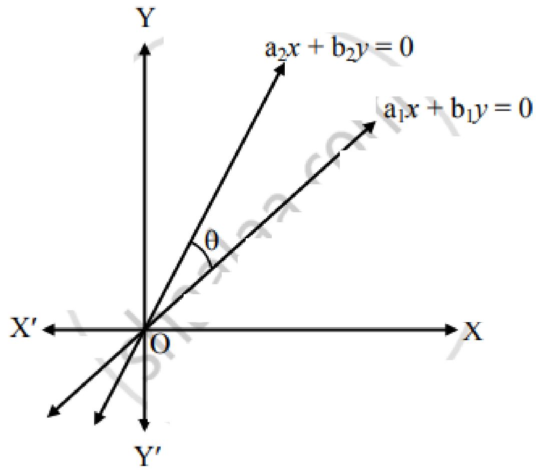
$$= \frac{0.1 + 0.1 + 0.05}{0.1 + 0.1 + 0.05 + 0.1}$$

$$= \frac{0.25}{0.35}$$

$$= \frac{5}{7}$$

Q. 28 Show that the combined equation of pair of lines passing through the origin is a homogeneous equation of degree 2 in x and y. Hence find the combined equation of the lines 2x + 3y = 0 and x − 2y = 0

Ans.



Let $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$ be a pair of lines passing through the origin.

∴ Their combined equation is $(a_1x + b_1y)(a_2x + b_2y) = 0$

∴ $a_1a_2x_2 + a_1b_2xy + b_1a_2xy + b_1b_2y^2 = 0$

∴ $(a_1a_2)x^2 + (a_1b_2 + a_2b_1)xy + (b_1b_2)y^2 = 0$

In this if we put $a_1a_2 = a$, $a_1b_2 + a_2b_1 = 2h$, $b_1b_2 = b$, We get $ax^2 + 2hxy + by^2 = 0$ which is a homogeneous equation of degree 2 in x and y.

Now, on comparing $2x + 3y = 0$ and $x - 2y = 0$ with $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$,

we get $a_1 = 2$, $b_1 = 3$, $a_2 = 1$ and $b_2 = -2$

Substituting in equation (i), we get

$2(1)x^2 + [2(-2) + 1(3)]xy + 3(-2)y^2 = 0$

i.e., $2x^2 - xy - 6y^2 = 0$,

Which is the required combined equation.

Q. 29

If $y = \cos(m \cos^{-1}x)$, then show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$

Ans. $y = \cos(m \cos^{-1}x)$ ....(i)

Differentiating w. r. t. x, we get

$$\frac{dy}{dx} = -\sin\left(m\cos^{-1}x\right) \cdot \frac{d}{dx}\left(m\cos^{-1}x\right)$$

$$\therefore \frac{dy}{dx} = -\sin\left(m\cos^{-1}x\right)\left[\frac{m}{\sqrt{1-x^2}}\right]$$

$$\therefore \frac{dy}{dx} = \sin\left(m\cos^{-1}x\right)\left[\frac{m}{\sqrt{1-x^2}}\right]$$

.......(ii)

$\therefore \sqrt{1-x^2}\frac{dy}{dx}$ = m sin(m $\cos^{-1}$x)

Squaring on both sides, we get

$(1-x^2)\left(\frac{dy}{dx}\right)^2 = m^2 \sin^2(m\cos^{-1}x)$

Again, differentiating w. r. t. x, we get

$$(1-x^2)\cdot 2\left(\frac{dy}{dx}\right)\cdot\frac{d}{dx}\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)^2(-2x) = m^2\left[2\sin\left(m\cos^{-1}x\right)\right]\cdot\frac{d}{dx}\left[\sin\left(m\cos^{-1}x\right)\right]$$

$\therefore 2\left(1-x^2\right)\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = -2m^2y\frac{dy}{dx}$ ....[From (i) and (ii)]

Dividing both sides by 2 $\frac{dy}{dx}$, we get

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -m^2y$$

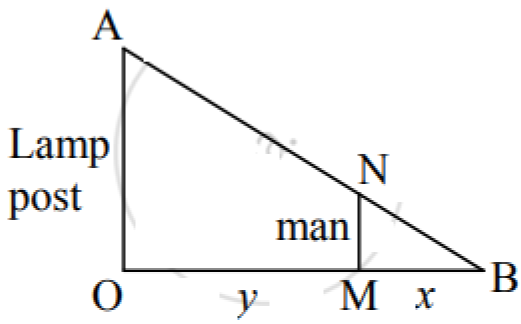
$$\therefore \left(1-x^2\right)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0$$

Q. 30 A man of height 180 cm is moving away from a lamp post at the rate of 1.2 meters per second. If the height of the lamp post is 4.5 meters, find the rate at which

(i) his shadow is lengthening

(ii) the tip of the shadow is moving

Ans. Let OA be the lamp post, MN be the man, MB = x be the length of the shadow and OM = y be the distance of the man from the lamp post at time t.



Then,

$\frac{dy}{dt}$ = 1.2 m/sec, MN = 180 cm = 1.8 m, OA = 4.5 m ....[Given]

(i) $\Delta NMB \sim \Delta AOB$

$\therefore \frac{MB}{MN} = \frac{OB}{OA}$

$\therefore \frac{x}{1.8} = \frac{x+y}{4.5}$

$\therefore$ 4.5x = 1.8x + 1.8y

$\therefore$ 2.7x = 1.8y

$\therefore$ x = $\frac{1.8y}{2.7}$

= $\frac{2y}{3}$

Differentiating w.r.t. t, we get

$\frac{dx}{dt} = \frac{2}{3} \times \frac{dy}{dt}$

= $\frac{2}{3} \times 1.2$

= 0.8 m/sec

(ii) B is the tip of the shadow and it is at a distance of (x + y) from the lamp post.

$$\frac{d}{dt}(x+y) = \frac{dx}{dt} + \frac{dy}{dt}$$

$\therefore \frac{d}{dt}(x+y)$ = 0.8 + 1.2

= 2 m/sec

Thus, the shadow is lengthening at the rate of 0.8 m/sec and its tip is moving at the rate of 2 m/sec.

Q. 31

$$\int \frac{(2\log x+3)}{x(3\log x+2)\left[(\log x)^2+1\right]}\,dx$$

Ans.

Let I = $\int \frac{(2\log x+3)}{x(3\log x+2)\left[(\log x)^2+1\right]}\,dx$

Put log x = t

$\therefore \frac{1}{x}\,dx = dt$

$\therefore$ I = $\int \frac{2t+3}{(3t+2)\left(t^2+1\right)}\,dt$

Let $\frac{2+3}{(3t+2)\left(t^2+1\right)} = \frac{A}{3t+2} + \frac{Bt+C}{t^2+1}$

$\therefore$ $2t + 3 = A(t^2 + 1) + (Bt + C)(3t + 2)$ ....(i)

Putting $t = -\frac{2}{3}$ in (i), we get

$$2\left(\frac{-2}{3}\right) + 3 = A\left[\left(\frac{-2}{3}\right)^2 + 1\right]$$

$$\therefore \frac{-4}{3} + 3 = A\left(\frac{4}{9} + 1\right)$$

$$\therefore \frac{5}{3} = A\left(\frac{13}{9}\right)$$

$$\therefore A = \frac{15}{13}$$

Putting $t = 0$ in (i), we get

$3 = A(1) + C(2)$

$$\therefore 3 = \frac{15}{13} + 2C$$

$$\therefore 3 - \frac{15}{13} = 2C$$

$$\therefore \frac{24}{13} = 2C$$

$$\therefore C = \frac{12}{13}$$

Putting $t = 1$ in (i), we get

$2 + 3 = A(1 + 1) + (B + C)(3 + 2)$

$\therefore$ $5 = 2A + 5(B + C)$

$$\therefore 5 = 2\left(\frac{15}{13}\right) + 5\left(B + \frac{12}{13}\right)$$

$$\therefore 5 = \frac{30}{13} + 5B + \frac{60}{13}$$

$$\therefore 5B = 5 - \frac{30}{13} - \frac{60}{13}$$

$$\therefore 5B = -\frac{25}{13}$$

$$\therefore B = \frac{-5}{13}$$

$$\therefore \frac{2t+3}{(3t+2)\left(t^2+1\right)} = \frac{\frac{15}{13}}{3t+2} + \frac{-\frac{5}{13}t + \frac{12}{13}}{t^2+1}$$

$$\therefore I = \int\left(\frac{\frac{15}{13}}{3t+2} + \frac{\frac{-5}{13}t + \frac{12}{13}}{t^2+1}\right) dt$$

$$= \frac{15}{13}\int \frac{1}{3t+2}\,dt - \frac{5}{13}\int \frac{t}{t^2+1}\,dt + \frac{12}{13}\int \frac{1}{t^2+1}\,dt$$

$$= \frac{15}{13}\int \frac{1}{3t+2}\,dt - \frac{5}{13}\cdot\frac{1}{2}\int \frac{2t}{t^2+1}\,dt + \frac{12}{13}\int \frac{1}{t^2+1}\,dt$$

$$= \frac{15}{13}\cdot\frac{\log|3t+2|}{3} - \frac{5}{26}\log\left|t^2+1\right| + \frac{12}{13}\tan^{-1} t + c$$

$$\therefore I = \frac{5}{13}\log|3\log x\ 2| - \frac{5}{26}\log\left|(\log x)^2+1\right| + \frac{12}{13}\tan^{-1}(\log x) + c$$

Q. 32

$$\int \frac{3x+4}{\sqrt{2x^2+2x+1}}\,dx$$

Ans.

Let I = $\int \frac{3x+4}{\sqrt{2x^2+2x+1}}\,dx$

Let 3x + 4 = A$\frac{d}{dx}\left(2x^2+2x+1\right)+B$

∴ 3x + 4 = A(4x + 2) + B

∴ 3x + 4 = 4Ax + 2A + B

By equating the coefficients on both sides, we get

4A = 3 and 2A + B = 4

∴ A = $\frac{3}{4}$ and $2\left(\frac{3}{4}\right)+B=4$

∴ B = $\frac{5}{2}$

∴ 3x + 4 = $\frac{3}{4}(4x+2)+\frac{5}{2}$

∴ I = $\int \frac{\frac{3}{4}(4x+2)+\frac{5}{2}}{\sqrt{2x^2+2x+1}}\,dx$

= $\frac{3}{4}\int \frac{4x+2}{\sqrt{2x^2+2x+1}}\,dx+\frac{5}{2}\int \frac{1}{\sqrt{2x^2+2x+1}}\,dx$

= $I_1 + I_2$ ....(i)

$I_1 = \frac{3}{4}\int \frac{4x+2}{\sqrt{2x^2+2x+1}}\,dx$

Put $2x^2 + 2x + 1 = t$

∴ (4x + 2) dx = dt

∴ $I_1 = \frac{3}{4}\int \frac{dt}{\sqrt{t}}$

$= \frac{3}{4}\left(\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right) + c_1$

$= \frac{3}{2}\sqrt{t} + c_1$

$\therefore I_1 = \frac{3}{2}\sqrt{2x^2 + 2x + 1} + c_1$ ....(ii)

$I_2 = \frac{5}{2}\int \frac{1}{\sqrt{2x^2 + 2x + 1}}\, dx$

$= \frac{5}{2}\int \frac{1}{\sqrt{2\left(x^2 + x + \frac{1}{2}\right)}}\, dx$

$\left(\frac{1}{2} \text{ coefficient of } x\right)^2 = \left(\frac{1}{2} \times 1\right)^2$

$= \frac{1}{4}$

$\therefore I_2 = \frac{5}{2\sqrt{2}}\int \frac{1}{\sqrt{x^2 + x + \frac{1}{4} - \frac{1}{4} + \frac{1}{2}}}\, dx$

$= \frac{5}{2\sqrt{2}}\int \frac{1}{\sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\, dx$

$= \frac{5}{2\sqrt{2}}\log\left|x + \frac{1}{2} + \sqrt{\left(x + \frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + c_2$

$\therefore I_2 = \frac{5}{2\sqrt{2}}\log\left|x + \frac{1}{2} + \sqrt{x^2 + x + \frac{1}{2}}\right| + c_2$ ....(iii)

From (i), (ii) and (iii), we get

I = $\frac{3}{2}\sqrt{2x^2+2x+1}+\frac{5}{2\sqrt{2}}\log\left|x+\frac{1}{2}+\sqrt{x^2+x+\frac{1}{2}}\right|$ + c,

where c = $c^1 + c^2$

Q. 33 A(– 2, 3, 4), B(1, 1, 2) and C(4, –1, 0) are three points. Find the Cartesian equations of the line AB and show that points A, B, C are collinear

Ans. We find the cartesian equations of the line AB.

The cartesian equations of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$$

Here, $(x_1, y_1, z_1) \equiv (-2, 3, 4)$ and $(x_2, y_2, z_2) \equiv (4, -1, 0)$

∴ The required cartesian equations of the line AB are

$$\frac{x-(-2)}{4-(-2)}=\frac{y-3}{-1-3}=\frac{z-4}{0-4}$$

$$\therefore \frac{x+2}{6}=\frac{y-3}{-4}=\frac{z-4}{-4}$$

$$\therefore \frac{x+2}{3}=\frac{y-3}{-2}=\frac{z-4}{-2}$$

C = (4, –1, 0)

For x = 4, $\frac{x+2}{3}=\frac{4+2}{3}$ = 2

For y = –1, $\frac{y-3}{-2}=\frac{-1-3}{-2}$ = 2

For z = 0, $\frac{z-4}{-2}=\frac{0-4}{-2}$ = 2

∴ Coordinates of C satisfy the equations of the line AB.

∴ C lies on the line passing through A and B.

Hence, A, B, C are collinear.

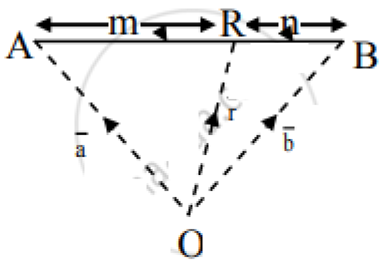
Q. 34

Let $A\left(\bar{a}\right)$ and $B\left(\bar{b}\right)$ are any two points in the space and $R\left(\bar{r}\right)$ be a point on the line segment AB dividing it internally in the ratio m : n, then prove that

$$\bar{r} = \frac{m\bar{b} + n\bar{a}}{m + n}$$

Ans. R is a point on the line segment AB(A-R-B) and ARAR¯ and RBRB¯ are in the same direction.

Point R divides AB internally in the ratio m : n



$$\therefore \frac{AR}{RB} = \frac{m}{n}$$

$\therefore$ n(AR) = m(RB)

As $n\left(\overline{AR}\right)$ and $m\left(\overline{RB}\right)$ have same direction and magnitude,

$$n\left(\overline{AR}\right) = m\left(\overline{RB}\right)$$

$$\therefore n\left(OR - \overline{OA}\right) = m\left(\overline{OB} - \overline{OR}\right)$$

$$\therefore n\left(\bar{r} - \bar{a}\right) = m\left(\bar{b} - \bar{r}\right)$$

$$\therefore n\bar{r} - n\bar{a} = m\bar{b} - m\bar{r}$$

$$\therefore m\bar{r} + n\bar{r} = m\bar{b} + n\bar{a}$$

$$\therefore (m + n)\bar{r} = m\bar{b} + n\bar{a}$$

$$\therefore \bar{r} = \frac{m\bar{b} + n\bar{a}}{m + n}$$