Mathematics and Statistics

Model Set - 1

Academic Year: 2020-2021 Marks: 80
Date: April 2021
Duration: 3h

1. The question paper is divided into four sections.
2. **Section A**: Q. No. 1 contains 8 multiple-choice type of questions carrying two marks each.
3. **Section A**: Q. No. 2 contains 4 very short answer type of questions carrying One mark each.
4. **Section B**: Q. No. 3 to Q. No. 14 contains Twelve short answer type of questions carrying Two marks each. **(Attempt any Eight)**.
5. **Section C**: Q. No.15 to Q. No. 26 contains Twelve short answer type of questions carrying Three marks each. **(Attempt any Eight)**.
6. **Section D**: Q.No. 27 to Q. No. 34 contains Five long answer type of questions carrying Four marks each. **(Attempt any Five)**.
7. Use of log table is allowed. Use of calculator is not allowed.
8. Figures to the right indicate full marks.
9. For each MCQ, correct answer must be written along with its alphabet. e.g., (a) ..... / (b) .... / (c) .... / (d) ..... Only first attempt will be considered for evaluation.
10. Use of graph paper is not necessary. Only rough sketch of graph is expected:
11. Start answers to each section on a new page.

Q. 1 | Select and write the most appropriate answer from the given alternatives for each sub-question:

1.i Which of the following statement is true

1. 3 + 7 = 4 or 3 – 7 = 4
2. If Pune is in Maharashtra, then Hyderabad is in Kerala
3. It is false that 12 is not divisible by 3
4. The square of any odd integer is even

1.ii

If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos 10\alpha \end{bmatrix}$, then A^{10} = _____

1. $\begin{bmatrix} \cos 10\,\alpha & -\sin 10\,\alpha \\ \sin 10\,\alpha & \cos 10\,\alpha \end{bmatrix}$

2. $\begin{bmatrix} \cos 10\,\alpha & \sin 10\,\alpha \\ -\sin 10\,\alpha & \cos 10\,\alpha \end{bmatrix}$

3. $\begin{bmatrix} \cos 10\,\alpha & \sin 10\,\alpha \\ -\sin 10\,\alpha & -\cos 10\,\alpha \end{bmatrix}$

4. $\begin{bmatrix} \cos 10\,\alpha & -\sin 10\,\alpha \\ -\sin 10\,\alpha & -\cos 10\,\alpha \end{bmatrix}$

1.iii Bernoulli distribution is a particular case of binomial distribution if n = _____

1. 4
2. 10
3. 2
4. 1

1.iv

If the p.m.f. of a d.r.v. X is P(X = x) = $\begin{cases} \frac{c}{x^3}, & \text{for } x = 1, 2, 3, \\ 0, & \text{otherwise} \end{cases}$ then E(X) = _____

1. $\frac{343}{297}$

2. $\frac{294}{251}$

3. $\frac{297}{294}$

4. $\frac{294}{297}$

1.v

The separate equations of the lines represented by $3x^2 - 2\sqrt{3}xy - 3y^2 = 0$ are _____

1. $x + \sqrt{3}y = 0$ and $\sqrt{3}x + y = 0$

2. $x - \sqrt{3}y = 0$ and $\sqrt{3}x - y = 0$

3. $x - \sqrt{3}y = 0$ and $\sqrt{3}x + y = 0$

4. $x + \sqrt{3}y = 0$ and $\sqrt{3}x - y = 0$

1.vi

Let $I_1 = \int_{e}^{e^2} \frac{1}{\log x} dx$ and $I_2 = \int_{1}^{2} \frac{e^x}{x} dx$ then

1. $I_1 = \frac{1}{3} I_2$

2. $I_1 + I_2 = 0$

3. $I_1 = 2I_2$

4. $I_1 = I_2$

1.vii

If $\int \frac{1}{x + x^5} dx = f(x) + c$, then $\int \frac{x^4}{x + x^5} dx =$ _____

1. f(x) − log x + c

2. f(x) + log x + c

3. log x − f(x) + c

4. $\frac{1}{5} x^5$ f(x) + c

1.viii If the foot of the perpendicular drawn from the origin to the plane is (4, −2, 5), then the equation of the plane is _____

1. 4x + y + 5z = 14
2. 4x − 2y − 5z = 45
3. x − 2y − 5z =10
4. 4x + y + 6z = 11

Q. 2 | Answer the following questions:

2.i

State the truth value of $\sqrt{3}$ is not an irrational number

Ans.

Let p : $\sqrt{3}$ is irrational number.

∴ Truth value of p is T.

∴ ~p : $\sqrt{3}$ is not irrational number.

∴ Truth value of ~p is F.

2.ii

Find the polar co-ordinates of point whose Cartesian co-ordinates are $\left(1\sqrt{3}\right)$

Ans.

(x, y) ≡ $\left(1\sqrt{3}\right)$ ....[Given]

Using x = r cos θ and y = r sin θ, where (r, θ) are the required polar co-ordinates, we get

1 = r cos θ, $\sqrt{3}$ = r sin θ

Now, r = $\sqrt{x^2+y^2}$

= $\sqrt{1+3}$

= 2

and tan θ = $\frac{\mathrm{r}\sin\theta}{\mathrm{r}\cos\theta}$

$= \dfrac{\sqrt{3}}{1}$

$= \sqrt{3}$

$= \tan \dfrac{\pi}{3}$

$\therefore \theta = n\pi + \dfrac{\pi}{3}, n \in Z$ .... $\left[\because \tan\theta = \tan\alpha \text{ implies } \theta = n\pi + \alpha, n \in Z\right]$

For polar co-ordinates, $0 \le \theta < 2\pi$

$\therefore \theta = \dfrac{\pi}{3}$ or $\theta = \pi + \dfrac{\pi}{3} = \dfrac{4\pi}{3}$

But the given point lies in the 1^{st} quadrant.

$\therefore \theta = \dfrac{\pi}{3}$

$\therefore$ The required polar co-ordinates are $\left(2, \dfrac{\pi}{3}\right)$.

2.iii Solve each of the following inequations graphically using XY-plane:

$4x - 18 \ge 0$

Ans.

Consider the line whose equation is $4x - 18 \ge 0$ i.e. $x = \dfrac{18}{4} = \dfrac{9}{2} = 4.5$

This represents a line parallel to Y-axis passing3through the point (4.5,0)

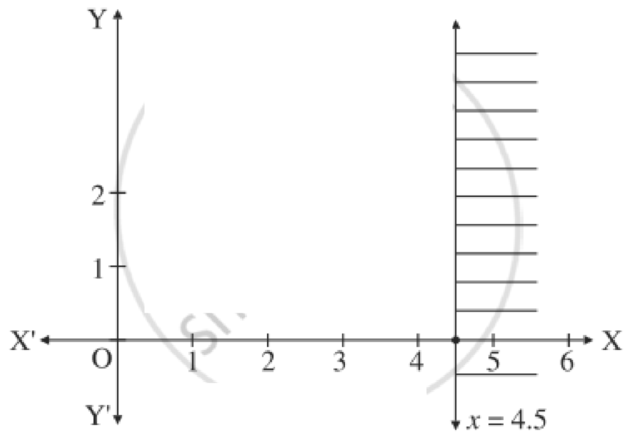
Draw the line x = 4.5

To find the solution set, we have to check the position of the origin (0, 0).

When x = 0, 4x – 18 = 4 x 0 – 18 = – 18 > 0

∴ the coordinates of the origin does not satisfy thegiven inequality.

∴ the solution set consists of the line x = 4.5 and the non-origin side of the line which is shaded in the graph.



2.iv The displacement of a particle at time t is given by $s = 2t^3 - 5t^2 + 4t - 3$. Find the velocity when $t = 2$ sec

Ans.

$$s = 2t^3 - 5t^2 + 4t - 3$$

$$\therefore v = \frac{ds}{dt}$$

$$= 6t^2 - 10t + 4$$

$$v_{(t=2)} = 6(2)^2 - 10(2) + 4$$

$$= 24 - 20 + 4$$

$$= 8 \text{ units/sec}$$

Q. 3 | Attempt any Eight:

Write the converse and contrapositive of the following statements.
"If a function is differentiable then it is continuous"

Ans. Let p: A function is differentiable,

q: It is continuous.

$\therefore$ The symbolic form of the given statement is $p \rightarrow q$.

Converse: $q \rightarrow p$

i.e. If a function is continuous then it is differentiable

Contrapositive: $\sim q \rightarrow \sim p$

i.e. If a function is not continuous then it is not differentiable.

Q. 4

Find the principal solutions of tan x = $-\sqrt{3}$

Ans.

tan x = $-\sqrt{3}$

$\therefore$ tan x = $-\tan\left(\frac{\pi}{3}\right)$

$\therefore$ tan x = $\tan\left(\pi - \frac{\pi}{3}\right)$

= $\tan\left(\frac{2\pi}{3}\right)$ and tan x = $\tan\left(2\pi - \frac{\pi}{3}\right)$

= $\tan\left(\frac{5\pi}{3}\right)$

such that $0 \le \frac{2\pi}{3} < 2\pi$ and $0 \le \frac{5\pi}{3} < 2\pi$

$\therefore$ The required principal solutions are x = $\frac{2\pi}{3}$ and x = $\frac{5\pi}{3}$.

Q. 5 Find the combined equation of the following pair of lines passing through (2, 3) and parallel to the coordinate axes.

Ans. Equations of the coordinate axes are x = 0 and y = 0

$\therefore$ The equations of the lines passing through (2, 3) and parallel to the coordinate axes are x = 2 and y = 3.

i.e. x − 2 = 0 and y − 3 = 0

$\therefore$ Their combined equation of these lines is

(x − 2)(y − 3) = 0

i.e., xy − 3x − 2y + 6 = 0

Q. 6 Find k, if the sum of the slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$ is twice their product.

Ans. Comparing the equation $x^2 + kxy - 3y^2 = 0$ with $ax^2 + 2hxy - by^2 = 0$,

We get, a = 1, 2h = k, b = −3.

Let m_1 and m_2 be the slopes of the lines represented by $x^2 + kxy - 3y^2 = 0$

$$\therefore m_1 + m_2 = \frac{-2h}{b} = -\frac{k}{-3} = \frac{k}{3}$$

$$\text{And } m_1 m_2 = \frac{a}{b} = \frac{1}{-3} = -\frac{1}{3}$$

$$\text{Now, } m_1 + m_2 = 2(m_1 m_2) \quad \text{.......(Given)}$$

$$\therefore \frac{k}{3} = 2\left(-\frac{1}{3}\right)$$

$$\therefore k = -2.$$

Q. 7 Find the differential equation of family of all ellipse whose major axis is twice the minor axis

Ans. Let the equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{......(i)}$$

Since the major axis is twice the minor axis,

$2a = 2(2b)$

$\therefore a = 2b$ ......(ii)

Substituting (ii) in (i), we get

$$\frac{x^2}{(2b)^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \frac{x^2}{4b^2} + \frac{y^2}{b^2} = 1$$

$$\therefore x^2 + 4y^2 = 4b^2$$

Differentiating w.r.t. x, we get

$$2x + 8y\frac{dy}{dx} = 0$$

$\therefore x + 4y\dfrac{dy}{dx} = 0$, where is the required differential equation.

Q. 8 Solve the differential equation $\sec^2 y \tan x \, dy + \sec^2 x \tan y \, dx = 0$

Ans. $\sec^2 y \tan x \, dy + \sec^2 x \tan y \, dx = 0$

Dividing both sides by tan x tan y, we get

$$\frac{\sec^2 y \tan x}{\tan x \tan y} \, dy + \frac{\sec^2 x \tan y}{\tan x \tan y} \, dx = 0$$

$$\therefore \frac{\sec^2 x}{\tan x} \, dx + \frac{\sec^2 y}{\tan y} \, dy = 0$$

Integrating on both sides, we get

$$\int \frac{\sec^2 x}{\tan x} \, dx + \int \frac{\sec^2 y}{\tan y} \, dy = 0$$

$\therefore \log |\tan x| + \log |\tan y| = \log |c|$

$\therefore \log |\tan x . \tan y| = \log |c|$

$\therefore \tan x \tan y = c$

Q. 9 Find the derivative of the inverse of function $y = 2x^3 - 6x$ and calculate its value at $x = -2$

Ans. $y = 2x^3 - 6x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(2x^3 - 6x\right)$$

$= 2(3x^2) - 6$

$= 6x^2 - 6$

$= 6(x^2 - 1)$

$$\therefore \left(\frac{dy}{dx}\right)_{x=-2} = 6\left[(-2)^2 - 1\right]$$

$= 6(3)$

$= 18$

$$\therefore \left(\frac{dx}{dy}\right)_{x=-2} = \frac{1}{\left(\frac{dy}{dx}\right)_{x=-2}}$$

$$= \frac{1}{18}$$

Q. 10 A car is moving in such a way that the distance it covers, is given by the equation $s = 4t^2 + 3t$, where s is in meters and t is in seconds. What would be the velocity and the acceleration of the car at time t = 20 seconds?

Ans. Let v be the velocity and a be the acceleration of the car.

Distance travelled by the car is given by

$\therefore$ Velocity $= v = \frac{ds}{dt}$

$= \frac{d}{dt}\left(4t^2 + 3t\right)$

$= 8t + 3$ ....(i)

Acceleration $= a = \frac{dv}{dt}$

$= \frac{d}{dt}(8t + 3)$

= 8 ....(ii)

Velocity of the car at t = 20 seconds is

$V_{(t=20)} = 8(20) + 3$ ....[From (i)]

$= 163$ m/sec.

Acceleration of the car at t = 20 seconds is

$a_{(t=20)} = 8$ m/sec^2 ....[From (ii)]

Q. 11

$$\int e^{3\log x}\left(x^4+1\right)^{-1}dx$$

Ans.

Let I = $\int e^{3\log x}\left(x^4+1\right)^{-1}dx$

$$= \int \frac{e^{\log\left(x^3\right)}}{x^4+1}\,dx$$

$$= \int \frac{x^3}{x^4+1}\,dx$$

Put $x^4 + 1 = t$

Differentiating w.r.t. x, we get

$4x^3dx = dt$

$4x^3dx = dt$

$\therefore x^3dx = \frac{1}{4}\,dt$

$\therefore I = \frac{1}{4}\int \frac{dt}{t}$

$= \frac{1}{4}\log|t| + c$

$\therefore I = \frac{1}{4}\log\left|x^4+1\right| + c$

Q. 12

Reduce the equation $\bar{r} \cdot \left(3\hat{i} + 4\hat{j} + 12\hat{k}\right)$ = 8 to normal form

Ans.

Equation of plane is $\bar{r} \cdot \left(3\hat{i} + 4\hat{j} + 12\hat{k}\right)$ = 8

This is of the form,

$\bar{r} \cdot \bar{n}$ = 8, where $\bar{n} = 3\hat{i} + 4\hat{j} + 12\hat{k}$

Now, $\left|\bar{n}\right| = \sqrt{3^2 + 4^2 + 12^2}$

= $\sqrt{9 + 16 + 144}$

= 13

The equation $\bar{r} \cdot \bar{n}$ = 8 can be written as

$$\bar{r} \cdot \frac{\bar{n}}{\left|\bar{n}\right|} = \frac{8}{\left|\bar{n}\right|}$$

i.e., $\bar{r} \cdot \left(\frac{3}{13}\hat{i} + \frac{4}{13}\hat{j} + \frac{12}{13}\hat{k}\right) = \frac{8}{13}$,

which is the normal form of the plane.

Q. 13 Find the Cartesian equation of the line passing through A(1, 2, 3) and B(2, 3, 4)

Ans. The Cartesian equation of the line passing through A(x_1, y_1, z_1) and B(x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

∴ The Cartesian equation of line is

$$\frac{x - 1}{2 - 1} = \frac{y - 2}{3 - 2} = \frac{z - 3}{4 - 3}$$

∴ x – 1 = y – 2 = z – 3

Q. 14 If aba ̄,b ̄ and cc ̄ are position vectors of the points A, B, C respectively and abc5a ̄-3b ̄-2c ̄=0 ̄, then find the ratio in which the point C divides the line segement BA

Ans.

$$5\bar{a} - 3\bar{b} - 2\bar{c} = 0$$

$$\therefore 2\bar{c} = 5\bar{a} - 3\bar{b}$$

$$\therefore \bar{c} = \frac{5\bar{a} - 3\bar{b}}{2}$$

$$\therefore \bar{c} = \frac{5\bar{a} - 3\bar{b}}{5 - 3}$$

∴ The point C divides the line segment BA externally in ratio 5:3.

Q. 15 | Attempt any Eight:

Write the converse, inverse, and contrapositive of the following statement.
"If it snows, then they do not drive the car"

Ans. Let **p:** It snows.
q: They do not drive the car.

∴ The given statement is $p \rightarrow q$.

Its converse is $q \rightarrow p$.
If they do not drive the car, then it snows.

Its inverse is $\sim p \rightarrow \sim q$.
If it does not snow, then they drive the car.

Its contrapositive is $\sim q \rightarrow \sim p$.
If they drive the car, then it does not snow.

Q. 16

In ΔABC, if $\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$, then show that the triangle is a right angled

Ans. In ΔABC by cosine rule, we get

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{a^2 + c^2 - b^2}{2ac}, \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca} \quad \text{.......[Given]}$$

$$\therefore \frac{2\left(b^2 + c - a^2\right)}{2abc} + \frac{a^2 + c^2 - b^2}{2abc} + \frac{2\left(a^2 + b^2 - c^2\right)}{2abc} = \frac{2a^2 + 2b^2}{2abc}$$

$$\therefore 2b^2 + 2c^2 - 2a^2 + a^2 + c^2 - b^2 + 2a^2 + 2b^2 - 2c^2 = 2a^2 + 2b^2$$

$$\therefore b^2 - a^2 + c^2 = 0$$

$$\therefore a^2 = b^2 + c^2$$

Hence, ΔABC is a right angled triangle.

Q. 17 Maximize $z = 10x + 25y$ subject to $x + y \leq 5$, $0 \leq x \leq 3$, $0 \leq y \leq 3$

Ans. To draw the feasible region, construct table as follows:

Inequality	$x \leq 3$	$y \leq 3$	$x + y \leq 5$
Corresponding equation (of line)	$x = 3$	$y = 3$	$x + y = 5$
Intersection of line with X-axis	(3, 0)	–	(5, 0)
Intersection of line with Y-axis	–	(0, 3)	(0, 5)
Region	Origin side	Origin side	Origin side

$x \geq 0$, $y \geq 0$ represent 1st quadrant.

Shaded portion OABCD is the feasible region,

whose vertices are O(0, 0), A(3, 0), B, C and D(0, 3).

B is the point of intersection of the lines $x = 3$ and $x + y = 5$.

Substituting $x = 3$ in $x + y = 5$, we get

$y = 2$

$\therefore B \equiv (3, 2)$

C is the point of intersection of the lines $y = 3$ and $x + y = 5$.

Substituting $y = 3$ in $x + y = 5$, we get

$x = 2$

$\therefore C \equiv (2, 3)$

Here, the objective function is $Z = 10x + 25y$

$\therefore$ Z at O(0, 0) = 10(0) + 25(0) = 0

Z at A(3, 0) = 10(3) + 25(0) = 30

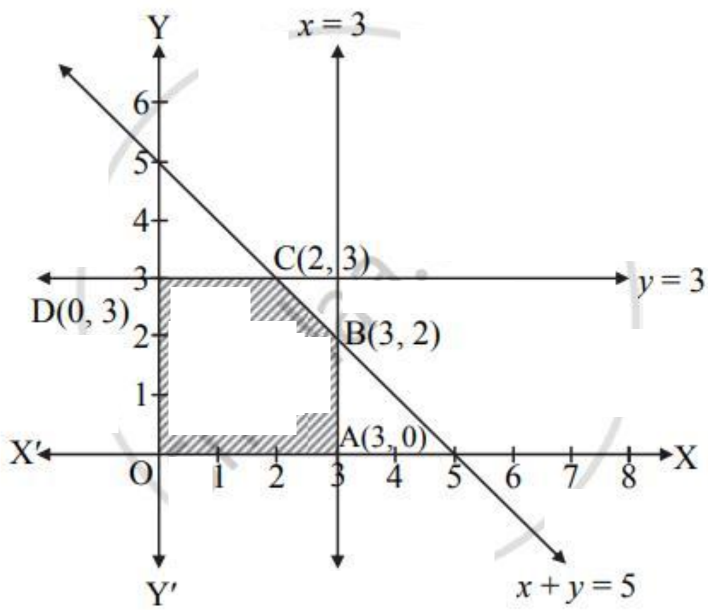
Z at B(3, 2) = 10(3) + 25(2) = 30 + 50 = 80

Z at C(2, 3) = 10(2) + 25(3) = 20 + 75 = 95

Z at D(0, 3) = 10(0) + 25(3) = 75

$\therefore$ Z has maximum value 95 at C(2, 3).

$\therefore$ Z has maximum value 95 when x = 2 and y = 3.



Q. 18 The probability that certain kind of component will survive a check test is 0.6. Find the probability that exactly 2 of the next 4 tested components survive

Ans. Let X denote the number of tested components survive.

P(component survive the check test) = p = 0.6 ....[Given]

$\therefore q = 1 - p$

$= 1 - 0.6$

$= 0.4$

Given, n = 4

$\therefore$ X ~ B (4, 0.6)

The p.m.f. of X is given by

$$P(X = x) = {}^4C_x(0.6)^x(0.4)^{4-x}, x = 0, 1, ..., 4$$

$\therefore$ P(exactly 2 components tested survive)

= P(X = 2)

= ${}^4C_2$ $(0.6)^2$ $(0.4)^2$

= 6(0.36)(0.16)

= 0.3456

Q. 19 A random variable X has the following probability distribution :

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2 + k$

Determine :
(i) k
(ii) P(X < 3)
(iii) P(X > 4)

Ans. (A) (i) Since P(x) is a probability distribution of x,

$$\Sigma_{x=0}^{7} P(x) = 1$$

$\therefore$ P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) = 1

$\therefore$ 0 + k + 2k + 2k + 3k + k^2 + $2k^2$ + $7k^2$ + k = 1

$\therefore$ $10k^2 + 9k - 1 = 0$

$\therefore$ $10k^2 + 10k - k - 1 = 0$

$\therefore$ 10k (k + 1) − 1 (k + 1) = 0

$\therefore$ (k + 1) (10k − 1) = 0

$\therefore$ 10k − 1 = 0

$\therefore$ 10k − 1 = 0 ....(k ≠ - 1)

$$\therefore k = \frac{1}{10}$$

(ii) P(X < 3) = P(0) + P(1) + P(2)

= 0 + k + 2k

= 3k

= `3 (1/ 10)

= 3/10`.

(ii) $P(0 < X < 3) = + P(1) + P(2)$

= k + 2k

= 3k

= 3`(1/ 10)

= 3/ 10`.

Ans. (B)

i. The table gives a probability distribution and therefore $\sum_{i=1}^{8} P_i = 1$

$\therefore 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\therefore 10k^2 + 9k - 1 = 0$

$\therefore 10k^2 + 10k - k - 1 = 0$

$\therefore 10k(k + 1) - 1(k + 1) = 0$

$\therefore (10k - 1)(k + 1) = 0$

$\therefore k = \frac{1}{10}$ or $k = -1$

But k cannot be negative

$\therefore k = \frac{1}{10}$

ii. $P(X < 3)$

$= P(X = 0 \text{ or } X = 1 \text{ or } X = 2)$

$= P(X = 0) + P(X = 1) + P(X = 2)$

$= 0 + k + 2k$

$= 3k$

$= \frac{3}{10}$

iii. $P(X > 4)$

= P(X = 5 or X = 6 or X = 7)

= P(X = 5) + P(X = 6) + P(X = 7)

= $k^2 + 2k^2 + 7k^2 + k$

= $10k^2 + k$

$$= 10\left(\frac{1}{10}\right)^2 + \frac{1}{10}$$

$$= \frac{1}{10} + \frac{1}{10}$$

$$= \frac{1}{5}$$

Q. 20

If $y = \log\left[\sqrt{\frac{1-\cos\left(\frac{3x}{2}\right)}{1+\cos\left(\frac{3x}{2}\right)}}\right]$, find $\frac{dy}{dx}$

Ans.

$y = \log\left[\sqrt{\frac{1-\cos\left(\frac{3x}{2}\right)}{1+\cos\left(\frac{3x}{2}\right)}}\right]$, find $\frac{dy}{dx}$

$$= \log\left[\sqrt{\frac{2\sin^2\left(\frac{3x}{4}\right)}{2\cos^2\left(\frac{3x}{4}\right)}}\right]$$

$$= \log\left[\sqrt{\tan^2\left(\frac{3x}{4}\right)}\right]$$

$$= \log\left[\tan\left(\frac{3x}{4}\right)\right]$$

Differentiating w. r. t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log\left(\tan\left(\frac{3x}{4}\right)\right)\right]$$

$$= \frac{1}{\tan\left(\frac{3x}{4}\right)} \cdot \frac{d}{d}x\left[\tan\left(\frac{3x}{4}\right)\right]$$

$$= \cot\left(\frac{3x}{4}\right) \cdot \sec^2\left(\frac{3x}{4}\right) \cdot \frac{d}{dx}\left(\frac{3x}{4}\right)$$

$$= \frac{\cos\left(\frac{3x}{4}\right)}{\sin\left(\frac{3x}{4}\right)} \cdot \frac{1}{\cos^2\left(\frac{3x}{4}\right)} \cdot \frac{3}{4}$$

$$= \frac{3}{2\left[2\sin\left(\frac{3x}{4}\right)\cos\left(\frac{3x}{4}\right)\right]}$$

$$= \frac{3}{2\sin\left(\frac{3x}{2}\right)}$$

$$= \frac{3}{2}\operatorname{cosc}\left(\frac{3x}{2}\right)$$

Q.21 Find the values of x, for which the function $f(x) = x^3 + 12x^2 + 36x + 6$ is monotonically decreasing

Ans. $f(x) = x^3 + 12x^2 + 36x + 6$

$\therefore f'(x) = 3x^2 + 24x + 36$

$= 3(x^2 + 8x + 12)$

$= 3(x + 2)(x + 6)$

f(x) is monotonically decreasing, if $f'(x) < 0$

$\therefore 3(x + 2)(x + 6) < 0$

$\therefore (x + 2)(x + 6) < 0$

$ab < 0 \Leftrightarrow a > 0$ and $b < 0$ or $a < 0$ and $b > 0$

$\therefore$ Either $x + 2 > 0$ and $x + 6 < 0$

or

$x + 2 < 0$ and $x + 6 > 0$

Case I: $x + 2 > 0$ and $x + 6 < 0$

$\therefore x > -2$ and $x < -6$,

which is not possible.

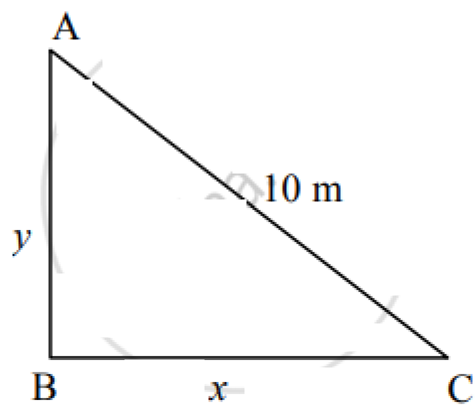
Case II: $x + 2 < 0$ and $x + 6 > 0$

$\therefore x < -2$ and $x > -6$

Thus, f(x) is monotonically decreasing for $x \in (-6, -2)$.

Q. 22 A ladder 10 meter long is leaning against a vertical wall. If the bottom of the ladder is pulled horizontally away from the wall at the rate of 1.2 meters per seconds, find how fast the top of the ladder is sliding down the wall when the bottom is 6 meters away from the wall

Ans. Let AC be the ladder. BC = x be the distance of the bottom of the ladder from the wall and AB = y be the distance of the top of the ladder from the floor.



Then, $\frac{dx}{dt} = 1.2$ m/sec, AC = 10 m, BC = 6 m ....[Given]

By Pythagoras theorem, we get

$x^2 + y^2 = AC^2$

$\therefore y^2 = AC^2 - x^2$

$\therefore y^2 = (10)^2 - x^2$ ....(i)

Differentiating w.r.t. t, we get

$$2y\frac{dy}{dt} = -2x\frac{dx}{dt}$$

$$\therefore \frac{dy}{dt} = \frac{-x}{y} \cdot \frac{dx}{dt}$$

$$= \frac{-6(1.2)}{y} \quad \text{.......(ii)}$$

Substituting x = 6 in (i), we get

$y^2 = (10)^2 - (6)^2$

= 100 – 36

= 64

∴ y = 8

Substituting y = 8 in (ii), we get

$$\frac{dy}{dt} = \frac{(-6)(1.2)}{8}$$

= – 0.9 metre/sec

Thus, the top of the ladder is sliding down at the rate of 0.9 meters/sec.

Q. 23

$$\int e^{\sin^{-1}x}\left[\frac{x+\sqrt{1-x^2}}{\sqrt{1-x^2}}\right]dx$$

Ans.

f'(x) = $x-\frac{3}{x^3}$, f(1) = $\frac{11}{2}$ ....[Given]

f(x) = $\int$ f'(x) dx

= $\int\left(x-\frac{3}{x^3}\right)dx$

= $\int x$ d$x-3\int x^{-3}$dx

= $\frac{x^2}{2}-3\left(\frac{x^{-2}}{2}\right)+$ c

∴f(x) = $\frac{x^2}{2}+\frac{3}{2x^2}+$ c

∴ f(1) = $\frac{(1)^2}{2}+\frac{3}{2(1)^2+\text{c}}$

∴ $\frac{11}{2}=\frac{1}{2}+\frac{3}{2}+$ c

∴ $\frac{11}{2}$ =2 + c

∴ c = $\frac{7}{2}$

Substituting c = $\frac{7}{2}$ in (i),, w get

f(x) = $\frac{x^2}{2}+\frac{3}{2x^2}+\frac{7}{2}$

Q. 24

If f'(x) = $x-\frac{3}{x^3}$, f(1) = $\frac{11}{2}$ find f(x)

Ans.

f'(x) = $x-\frac{3}{x^3}$, f(1) = $\frac{11}{2}$ ....[Given]

f(x) = $\int f'(x)\,dx$

= $\int\left(x-\frac{3}{x^3}\right)dx$

= $\int x\,dx-3\int x^{-3}dx$

= $\frac{x^2}{2}-3\left(\frac{x^{-2}}{2}\right)+c$

$\therefore$f(x) = $\frac{x^2}{2}+\frac{3}{2x^2}+c$

$\therefore$ f(1) = $\frac{(1)^2}{2}+\frac{3}{2(1)^2+c}$

$\therefore \frac{11}{2}=\frac{1}{2}+\frac{3}{2}+c$

$\therefore \frac{11}{2}$ =2 + c

$\therefore$ c = $\frac{7}{2}$

Substituting c = $\frac{7}{2}$ in (i),, w get

f(x) = $\frac{x^2}{2}+\frac{3}{2x^2}+\frac{7}{2}$

Q. 25

If A(5, 1, p), B(1, q, p) and C(1, −2, 3) are vertices of triangle and G$\left(r,-\frac{4}{3},\frac{1}{3}\right)$ is its centroid then find the values of p, q and r

Ans.

Let $\bar{a}, \bar{b}, \bar{c}$ be the position vectors of points A, B, C respectively of ΔABC and $\bar{g}$ be the position vector of its centroid G.

$\therefore \bar{a} = 5\hat{i} + \hat{j} + p\hat{k},$

$\bar{b} = \hat{i} + q\hat{j} + p\hat{k},$

$\bar{c} = \hat{i} - 2\hat{j} + 3\hat{k}$

and

$\bar{g} = r\hat{i} - \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$

$\therefore$ By using centroid formula,

$\bar{g} = \frac{\bar{a} + \bar{b} + \bar{c}}{3}$

$\therefore 3\bar{g} = \bar{a} + \bar{b} + \bar{c}$

$$\therefore 3\left(r\hat{i} - \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}\right) = \left(5\hat{i} + \hat{j} + p\hat{k}\right) + \left(\hat{i} + q\hat{j} + p\hat{k}\right) + \left(\hat{i} - 2j\widehat{+ 3\text{hat}}\boldsymbol{k}\right)$$

$$\therefore 3r\hat{i} - 4\hat{j} + \hat{k} = 7\hat{i} + (q - 1)\hat{j} + (2p + 3)\hat{k}$$

$\therefore$ By equality of vectors, we get

$3r = 7, -4 = q - 1$ and $1 = 2p + 3$

$\therefore r = \frac{7}{3}, q = -3$ and $p = -1$

Q. 26 Show that the points A(2, −1, 0) B(−3, 0, 4), C(−1, −1, 4) and D(0, − 5, 2) are non coplanar

Ans.

Let $\bar{a}, \bar{b}, \bar{c}, \bar{c}, \bar{d}$ be the position vectors of points A, B, C, D respectively.

$\therefore\ \bar{a} = 2\hat{i} - \hat{j}, \bar{b} = -3\hat{i} + 4\hat{k}, \bar{c} = -\hat{i} - \hat{j} + 4\hat{k}, \bar{d} = -5\hat{j} + 2\hat{k}$

$\therefore\ \overline{AB} = \bar{b} - \bar{a}$

= $\left(-3\hat{i} + 4\hat{k}\right) - \left(2\hat{i} - \hat{j}\right)$

= $-5\hat{i} + \hat{j} + 4\hat{k}$

$\overline{AC} = \bar{c} - \bar{a}$

= $\left(-\hat{i} - \hat{j} + 4\hat{k}\right) - \left(2\hat{i} - \hat{j}\right)$

= $-3\hat{i} + 4\hat{k}$

$\overline{AD} = \bar{d} - \bar{a}$

= $\overline{AD} = \bar{d} - \bar{a}$

= $\left(-5\hat{j} + 2\hat{k}\right) - \left(2\hat{i} - \hat{j}\right)$

= $-2\hat{i} - 4\hat{j} + 2\hat{k}$

Points A, B, C, D are non-coplanar if $\overline{AB}, \overline{AC}$ and $\overline{AD}$ are non-coplanar.

$$\overline{AD} \cdot \left(\overline{AC} \times \overline{AD}\right) = \begin{vmatrix} -5 & 1 & 4 \\ -3 & 0 & 4 \\ -2 & -4 & 2 \end{vmatrix}$$

= – 5(0 + 16) – 1(– 6 + 8) + 4(12 – 0)

= – 5(16) – 1(2) + 4(12)

= – 80 – 2 + 48

= – 34 ≠ 0

∴ The points A, B, C, D are non-coplanar.

Q. 27 | Attempt any Five:

If three numbers are added, their sum is 2. If 2 times the second number is subtracted from the sum of first and third numbers, we get 8. If three times the first number is added to the sum of second and third numbers, we get 4. Find the numbers using matrices.

Ans. Let the three numbers be x, y, z.

According to the first condition,

x + y + z = 2

According to the second condition,

(x + z) − 2y = 8

i.e. x − 2y + z = 8

According to the third condition,

3x + y + z = 4

Matrix form of the given system of equations is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 4 \end{bmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & 0 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}$$

Applying $R_2 \rightarrow \left(-\frac{1}{3}\right) R_2$ and $R_3 \rightarrow \left(-\frac{1}{2}\right) R_3$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

Applying $R_3 \rightarrow R_3 - R_2$,

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

Hence, the original matrix is reduced to an upper triangular matrix

∴ By equality of matrices, we get

$x + y + z = 2$ ....(i)

$y = -2$ ....(ii)

$z = 3$ ....(iii)

Putting $y = -2$ and $z = 3$ in equation (i), we get

$x - 2 + 3 = 2$

∴ $x = 1$

Hence, 1, −2 and 3 are the required numbers.

Q. 28

Prove that $2\tan^{-1}\left(\frac{1}{8}\right) + \tan^{-1}\left(\frac{1}{7}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \frac{\pi}{4}$

Ans.

L.H.S. $= 2\tan^{-1}\left(\frac{1}{8}\right) + \tan^{-1}\left(\frac{1}{7}\right) + 2\tan^{-1}\left(\frac{1}{5}\right)$

$= 2\left[\tan^{-1}\left(\frac{1}{8}\right) + \tan^{-1}\left(\frac{1}{5}\right)\right] + \tan^{-1}\left(\frac{1}{7}\right)$

$= 2\left[\tan^{-1}\left(\frac{\frac{1}{8}+\frac{1}{5}}{1-\frac{1}{8}\times\frac{1}{5}}\right)\right] + \tan^{-1}\left(\frac{1}{7}\right)$

$= 2\left[\tan^{-1}\left(\frac{\frac{13}{40}}{\frac{39}{40}}\right)\right] + \tan^{-1}\left(\frac{1}{7}\right)$

$= 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{3}}{1-\frac{1}{3}\times\frac{1}{3}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right)$

$= \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right)$

$= \tan^{-1}(1)$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$

Q. 29 Find the area of the region lying between the parabolas $4y^2 = 9x$ and $3x^2 = 16y$

Ans. Given equations of the parabolas are

$4y^2 = 9x$ ....(i)

and $3x^2 = 16y$

$\therefore y = \dfrac{3x^2}{16}$ ......(ii)

From (i), we get

$y^2 = \dfrac{9}{4}x$

$\therefore y = \dfrac{3}{2}\sqrt{x}$ ....(iii) ......[$\because$ In first quadrant, $y > 0$]

Find the points of intersection of $4y^2 = 9x$ and $3x^2 = 16y$.

Substituting (ii) in (i), we get

$4\left(\dfrac{3x^2}{16}\right)^2 = 9x$

$\therefore x^4 = 64x$

$\therefore x^4 - 64x = 0$

$\therefore x(x^3 - 64) = 0$

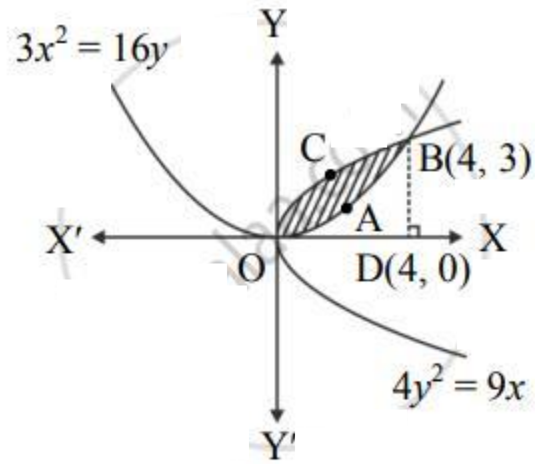
$\therefore x = 0$ or $x^3 = 64 = 4^3$

$\therefore x = 0$ or $x = 4$

When $x = 0$, $y = 0$ and when $x = 4$, $y = 3$

$\therefore$ The points of intersection are O(0, 0) and B(4, 3).

Draw BD ⊥ OX.



Required area = area of the region OABCO

= area of the region ODBCO – area of the region ODBAO

= area under the parabola $4y^2 = 9x$ – area under the parabola $3x^2 = 16y$

$$= \int_0^4 \frac{3}{2}\sqrt{x}\,dx - \int_0^4 \frac{3x^2}{16}\,dx$$

$$= \frac{3}{2}\int_0^4 x^{\frac{1}{2}}\,dx - \frac{3}{16}\int_0^4 x^2\,dx$$

$$= \frac{3}{2}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 - \frac{3}{16}\left[\frac{x^3}{3}\right]_0^4$$

$$= \left[(4)^{\frac{3}{2}} - 0\right] - \frac{1}{16}\left[(4)^3 - 0\right]$$

$$= 8 - \frac{1}{16}(64)$$

$= 8 - 4$

$= 4$ sq.units

Q. 30

Evaluate: $\int_0^1 \left(\frac{1}{1+x^2}\right)\sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

Ans.

Let I = $\int_0^1 \left(\frac{1}{1+x^2}\right)\sin^{-1}\left(\frac{2x}{1+x^2}\right) \mathrm{d}x$

Put x = tan θ

∴ dx = $\sec^2\theta$ dθ

When x = 0, θ = 0 and when x = 1, θ = $\frac{\pi}{4}$

∴ I = $\int_0^{\frac{\pi}{4}} \left(\frac{1}{1+\tan^2\theta}\right)\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\,\mathrm{d}\theta$

= $\int_0^{\frac{\pi}{4}} \left(\frac{1}{\sec^2\theta}\right)\sin^{-1}(\sin 2\theta)\sec^2\theta\,\mathrm{d}\theta$

= $\int_0^{\frac{\pi}{4}} 2\theta\,\mathrm{d}\theta$

= $2\left[\frac{\theta^2}{2}\right]_0^{\frac{\pi}{4}}$

= $\left(\frac{\pi}{4}\right)^2 - 0$

∴ I = $\frac{\pi^2}{16}$

Q. 31 The rate of growth of bacteria is proportional to the number present. If initially, there were 1000 bacteria and the number doubles

in 1 hour, find the number of bacteria after $\frac{5}{2}$ hours $\left(\text{Given } \sqrt{2} = 1.414\right)$

Ans. Let 'x' be the number of bacteria present at time 't'.

∴ $\frac{\mathrm{d}x}{\mathrm{dt}} \propto x$

∴ $\frac{\mathrm{d}x}{\mathrm{dt}}$ = kx, where k is the constant of proportionality.

$\therefore \frac{dx}{x} = kdt$

Integrating on both sides, we get

$$\int \frac{dx}{x} = k \int dt$$

$\therefore \log x = kt + c$ .....(i)

When t = 0, x = 1000

$\therefore \log (1000) = k(0) + c$

$\therefore c = \log (1000)$

$\therefore \log x = kt + \log (1000)$ ....(ii) ....[From (i)]

When t = 1, x = 2000

$\therefore \log (2000) = k(1) + \log (1000)$

$\therefore \log (2000) - \log (1000) = k$

$\therefore k = \log\left(\frac{2000}{1000}\right)$

$= \log 2$ .....(iii)

When $t = \frac{5}{2}$, we get

$\log x = \frac{5}{2}k + \log(1000)$ ....[From (ii)]

$\therefore \log x = \left(\frac{5}{2}\right) \log 2 + \log(1000)$ .....[From (iii)]

$= \log\left(2^{\frac{5}{2}}\right) + \log(1000)$

$= \log\left(4\sqrt{2}\right) + \log(1000)$

$= \log\left(4000\sqrt{2}\right)$

$= \log (4000 \times 1.414)$

$\therefore \log x = \log (5656)$

$\therefore x = 5656$

Thus, there will be 5656 bacteria after $\frac{5}{2}$ hours.

Q. 32 If y = f(u) is a differentiable function of u and u = g(x) is a differentiable function of x such that the composite function y = f[g(x)] is a

differentiable function of x, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$. Hence find $\frac{dy}{dx}$ if $y = \sin^2 x$

Ans. Let δx be a small increment in the value of x.

Since u is a function of x, there should be a corresponding increment δu in the value of u.

Also y is a function of u.

∴ There should be a corresponding increment δy in the value of y.

Consider, $\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \times \frac{\delta u}{\delta x}$

Taking $\lim\limits_{\delta x \to 0}$ on both sides, we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x}$$

As δx → 0, δu → 0 ....[u is a continuous function of x]

$$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta u \to 0} \frac{\delta y}{\delta u} \times \lim_{\delta x \to 0} \frac{\delta u}{\delta x} \quad \text{........(i)}$$

y is a differentiable function of u and u is a differentiable function of x.

$\therefore \lim\limits_{\delta u \to 0} \frac{\delta y}{\delta u} = \frac{dy}{du}$ exists and is finite.

From (i), we get

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{du} \times \frac{du}{dx} \quad \text{........(ii)}$$

Here, R.H.S. of (ii) exists and is finite.

Hence, L.H.S. of (ii) should also exists and be finite.

$\therefore \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$ exists and is finite.

$\therefore$ Equation (ii) becomes

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$y = \sin^2 x$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^2 x)$$

$$= 2 \sin x \cdot \frac{d}{dx}(\sin x)$$

$= 2 \sin x \cos x$

Q. 33

Show that the lines $\frac{x+1}{-10} = \frac{y+3}{-1} = \frac{z-4}{1}$ and $\frac{x+10}{-1} = \frac{y+1}{-3} = \frac{z-1}{4}$ intersect each other.also find the coordinates of the point of intersection

Ans.

The variable point on the line $\frac{x+1}{-10} = \frac{y+3}{-1} = \frac{z-4}{1}$ is $\frac{x+10}{-1} = \frac{y+1}{-3} = \frac{z-1}{4}$ = λ

∴ x + 1 = − 10λ, y + 3 = − λ, z − 4 = λ

∴ x = − 10λ −1, y = − λ − 3, z = λ + 4 ....(i)

Also, the variable point on the line

$\frac{x+10}{-1} = \frac{y+1}{-3} = \frac{z-1}{4}$ is

$\frac{x+10}{-1} = \frac{y+1}{-3} = \frac{z-1}{4}$ = μ

∴ x + 10 = −μ, y + 1 = −3μ, z − 1 = 4μ

∴ x = − μ − 10, y = − 3μ − 1, z = 4μ + 1 ....(ii)

Given lines intersect each other if there exist some values of λ and μ for which

−10λ − 1 = − μ − 10, − λ − 3

= − 3μ −1 and λ + 4

= 4μ + 1

∴ 10λ − μ = 9 ....(iiii)

λ − 3μ = − 2 ....(iv)

λ − 4μ = − 3 ....(v)

Subtracting equation (iv) from (v), we get

λ − 4μ = −3

λ − 3μ = −2

Subtracting equation (iv) from (v), we get

$\lambda - 4\mu = -3$

$\lambda - 3\mu = -2$

$\underline{-\quad + \quad = \quad +}$

$\underline{\mathbf{-\mu = -1}}$

$\therefore \mu = 1$

Substituting $\mu = 1$ in (iv), we get

$\lambda - 3(1) = -2$

$\therefore \lambda = -2 + 3$

$\therefore \lambda = 1$

Since the values of λ and μ exist, the given lines intersect each other. To find the point of intersection, substituting the value of $\lambda = 1$ in equation (i), we get

$x = -10 - 1, y = -1 - 3, z = 1 + 4$

$\therefore x = -11, y = -4, z = 5$

$\therefore$ Point of intersection of the lines is (x, y, z) i.e., (−11, − 4, 5).

Q. 34

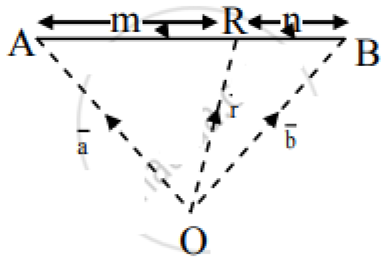
Let $A\left(\bar{a}\right)$ and $B\left(\bar{b}\right)$ are any two points in the space and $R\left(\bar{r}\right)$ be a point on the line segment AB dividing it internally in the ratio m : n, then prove that

$$\bar{r} = \frac{m\bar{b} + n\bar{a}}{m + n}$$

Ans.

R is a point on the line segment AB(A-R-B) and $\overline{AR}$ and $\overline{RB}$ are in the same direction.

Point R divides AB internally in the ratio m : n



$$\therefore \frac{AR}{RB} = \frac{m}{n}$$

$\therefore$ n(AR) = m(RB)

As n $\left(\overline{AR}\right)$ and m $\left(\overline{RB}\right)$ have same direction and magnitude,

$$n\left(\overline{AR}\right) = m\left(\overline{RB}\right)$$

$$\therefore n\left(\overline{OR} - \overline{OA}\right) = m\left(\overline{OB} - \overline{OR}\right)$$

$$\therefore n\left(\bar{r} - \bar{a}\right) = m\left(\bar{b} - \bar{r}\right)$$

$$\therefore n\bar{r} - n\bar{a} = m\bar{b} - m\bar{r}$$

$$\therefore m\bar{r} + n\bar{r} = m\bar{b} + n\bar{a}$$

$$\therefore (m + n)\bar{r} = m\bar{b} + n\bar{a}$$

$$\therefore \bar{r} = \frac{m\bar{b} + n\bar{a}}{m + n}$$