PART : MATHEMATICS

1. The coefficient of x^{2012} in $(1-x)^{2008}(1+x+x^2)^{2007}$ is

(1) 0 (2) 1 (3) 2 (4) 3

Ans. (1)

Sol. $(1-x)^{2008}(1+x+x^2)^{2007} = (1-x)(1-x^3)^{2007}$

$= (1-x)\sum_{r=0}^{2007} {}^{2007}C_r\left(-x^3\right)^r$

$3r = 2012$ or $3r + 1 = 2012$

$r \notin W$ $r \notin W$

2. If the equation $2\tan^2\theta - 5\sec\theta = 1$ has 7 solution in $\theta \in [0, n\frac{\pi}{2}]$. for least value of $n \in N$ then value of

$\sum_{k=1}^{n}\frac{k}{2^n}$ is equal to

(1) $\frac{9}{2^9}$ (2) $\frac{91}{2^{13}}$ (3) $\frac{11}{2^{12}}$ (4) $\frac{7}{2^7}$

Ans. (2)

Sol. $2(\sec^2\theta-1) - 5\sec\theta - 1 = 0$

$2\sec^2\theta - 5\sec\theta - 3 = 0$

$(2\sec\theta + 1)(\sec\theta - 3) = 0$

$(\sec\theta = -\frac{1}{2})$ or $(\sec\theta = 3)$

↓

Rejected

the solution are in 1st and 4th quadrant only

the least value of n is 13 for which equation has 7 solutions in $\left[0, \frac{13\pi}{2}\right]$

now $\sum_{k=1}^{n}\frac{k}{2^n} = \sum_{k=1}^{13}\frac{k}{2^{13}} = \frac{91}{2^{13}}$

3. $\lim_{x\to 0}\frac{3+\alpha\sin x+\beta\cos x+\log_e(1+x)}{3\tan^2 x} = \frac{1}{3}$, then the value of $(2\alpha - \beta)$ is equal to

Ans. (1)

Sol. $\lim_{x\to 0}\frac{3+\alpha\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}....\right)+\beta\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\right)+\left(x-\frac{x^2}{2}+\frac{x^3}{3}....\right)}{3\tan^2 x} = \frac{1}{3}$

$\Rightarrow \lim_{x\to 0}\frac{(3+\beta)+(\alpha+1)x-\left(\frac{\beta+1}{2}\right)x^2...}{3\tan^2 x} = \frac{1}{3}$

$3 + \beta = 0$ and $\alpha + 1 = 0 \Rightarrow \alpha = -1$ and $\beta = -3$

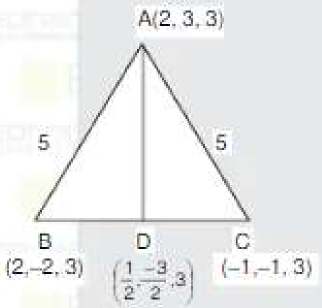
$$\lim_{x \to 0} \frac{x^2 + \frac{1}{2}x^3.....}{3\tan^2 x} = \frac{1}{3}$$

Hence $2\alpha - \beta = -2 + 3 = 1$

4. The position vector of the vertices A, B, C of a triangle are $2\hat{i} + 3\hat{j} + 3\hat{k}, 2\hat{i} - 2\hat{j} + 3\hat{k}, -\hat{i} - \hat{j} + 3\hat{k}$ respectively. Let ℓ denotes the length of the angle bisector AD of $\angle BAC$ Where 'D' is on the line segment BC, then $2\ell^2$ equals ?

Ans. (45)

Sol.



$$\ell^2 = \left(2 - \frac{1}{2}\right)^2 + \left(3 + \frac{3}{2}\right)^2 + 0$$

$$\ell^2 = \frac{9}{4} + \frac{81}{4}$$

$$\ell^2 = \frac{45}{2}$$

$$2\ell^2 = 45$$

5. $\int_0^{\pi} \frac{dx}{1 + \alpha^2 - 2\alpha \cos x}$ is equal (where $|\alpha| > 1$)

(1) $\frac{\pi}{1 - \alpha^2}$ (2) $\frac{\pi}{\alpha^2 - 1}$ (3) $\frac{\pi}{2(\alpha^2 - 1)}$ (4) $\frac{\pi}{2(1 - \alpha^2)}$

Ans. (2)

Sol. Let $I = \int_0^{\pi} \frac{dx}{1 + \alpha^2 - 2\alpha \cos x}$

$$I = \frac{1}{2\alpha} \int_0^{\pi} \frac{dx}{\frac{1 + \alpha^2}{2\alpha} - \cos x}$$

Let $a = \frac{1 + \alpha^2}{2\alpha}$ then $|a| > 1$

$$I = \frac{1}{2\alpha} \int_0^{\pi} \frac{dx}{a - \cos x}$$ where

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$$= \frac{\quad}{2\alpha}\int_0 \frac{\quad}{a - \frac{1-\tan^2 x/2}{1+\tan^2 x/2}}$$

$$= \frac{1}{2\alpha}\int_0^{\pi} \frac{\sec^2 x/2}{(a-1)+(a+1)\tan^2\frac{x}{2}}\,dx$$

Let $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$= \frac{1}{\alpha}\int_0^{\infty} \frac{dt}{(a-1)+(a+1)t^2}$$

$$= \frac{1}{\alpha(a+1)}\int_0^{\infty} \frac{dt}{\left(\frac{a-1}{a+1}\right)+t^2}$$

$$= \frac{1}{\alpha(a+1)}\int_0^{\infty} \frac{dt}{\left(\frac{a-1}{a+1}\right)+t^2}$$

$$= \frac{1}{\alpha(a+1)}\frac{1}{\sqrt{\frac{a-1}{a+1}}}\tan^{-1}\left(\frac{t}{\sqrt{\frac{a-1}{a+1}}}\right)\Bigg|_0^{\infty}$$

$$= \frac{1}{\alpha\sqrt{a^2-1}}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{\pi}{2\alpha\sqrt{a^2-1}} = \frac{\pi}{2\alpha\sqrt{\left(\frac{1+\alpha^2}{2\alpha}\right)^2-1}} = \frac{\pi}{\sqrt{(1-\alpha^2)^2}} = \frac{\pi}{|1-\alpha^2|} = \frac{\pi}{\alpha^2-1}$$

6. Number of solution of equation $\tan^{-1}x + \tan^{-1}2x = \pi/4$ is

(1) 0 (2) 1 (3) 2 (4) 3

Ans. **(2)**

Sol. $\tan^{-1}x + \tan^{-1}2x = \pi/4$

$\tan^{-1}2x = \tan^{-1}1 - \tan^{-1}x$

$2x = \frac{1-x}{1+x}$

$3x = 1-2x^2$

$2x^2 + 3x - 1 = 0$

$x = \frac{-3 \pm \sqrt{9+8}}{4}$

$x = \frac{-3 \pm \sqrt{17}}{4}$

$x = \frac{-3 + \sqrt{17}}{4}$ is accepted

Number of solution is 1.

7. 20^{th} term from the end of the progression $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots\ldots, -129\frac{1}{4}$ is ______.

(1) –120 (2) –115 (3) –125 (4) –110

Ans. **(2)**

Sol. $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, \ldots\ldots, -129\frac{1}{4}$

are in A.P. with common difference

$d = 19\frac{1}{4} - 20 = -\frac{3}{4}$

20^{th} term from end $= -129\frac{1}{4} + (20-1)\left(\frac{3}{4}\right)$

$= -129\frac{1}{4} + 19.\frac{3}{4}$

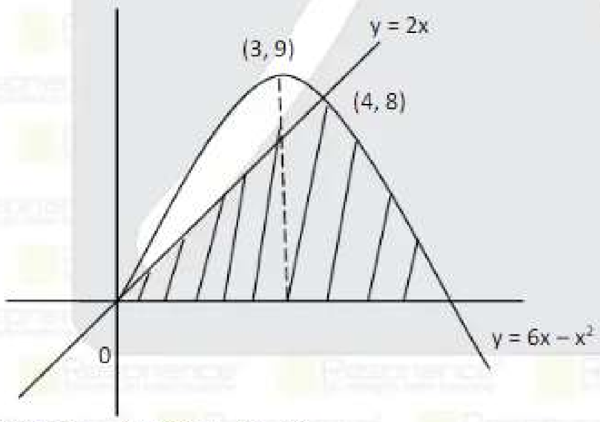
$= \frac{-517}{4} + \frac{57}{4}$

$= \frac{-460}{4} = -115$

8. A is the area of region $0 \le y \le \min(2x; 6x - x^2)$ then find 12A

Ans. **(304)**

Sol.



Solving $y = 2x$ and $y = 6x - x^2$

$2x = 6x - x^2$

$x^2 - 4x = 0$

$y = x = 0, x = 4, y = 8$

$A = \frac{1}{2} 4.8 + \int_4^6 (6x - x^2)dx$

$= 16 + \left(3x^2 - \frac{x^3}{3}\right)_4^6$

$A = 16 + 3(36 - 16) - \frac{1}{3}(216 - 64)$

$A = 16 + 60 - \frac{152}{3}$

$A = 16 + \frac{28}{3}$

12A = 304 Ans.

9. α and β are the roots of $x^2 - x - 1 = 0$ and $S_n = 2023\,\alpha^n + 2024\beta^n$ then

(1) $S_{12} = S_{11} + S_{10}$ (2) $2S_{12} = S_{11} + S_{10}$ (3) $S_{11} = 2S_{10} + S_{12}$ (4) $S_{10} = S_{11} + S_{12}$

Ans. (1)

Sol. $x^2 - x - 1 =$

$S_n - S_{n-1} - S_{n-2} = 0$

$S_n = S_{n-1} + S_{n-2}$

$S_{12} = S_{11} + S_{10}$

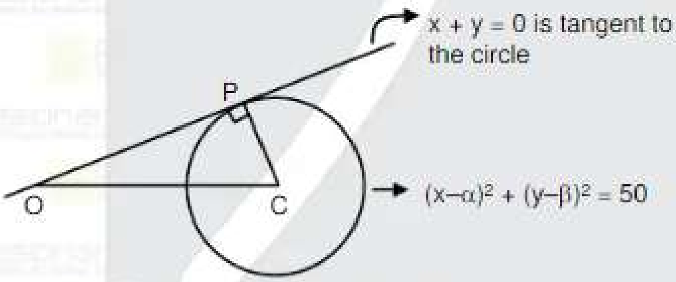
n = 12

10. If circle $(x-\alpha)^2 + (y-\beta)^2 = 50$ and line $x + y = 0$ intersect at only one point P distance of P from origin is $4\sqrt{2}$ then $\alpha^2 + \beta^2$ is

(1) 81 (2) 82 (3) 85 (4) 169

Ans. **(2)**

Sol.



$OP = 4\sqrt{2}$ $CP = \sqrt{50} = 5\sqrt{2}$

$(OC)^2 = \alpha^2 + \beta^2 = (OP)^2 + (PC)^2$

$= 16 \times 2 + 50 = 82$

11. Two finite set A and B have m and n elements respectively. If subset of A is 56 more than that of B then the distance between A (m,n) and B(−2,−3) is

(1) 8 (2) 10 (3) 11 (4) 15

Ans. **(2)**

Sol. $2^m - 2^n = 56$

we known that 64 − 8 = 56

m = 6, n = 3

A (6,3) , B(−2,−3)

$AB = \sqrt{64 + 36} = 10$

12. The value of $\int \frac{(x^8 - x^2)}{(x^{12} + 3x^6 + 1)\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)} dx$

(1) $\ell n\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right| + C$

(2) $\frac{1}{2}\ell n\left|\tan^{-1}\left(x^3 - \frac{1}{x^3}\right)\right| + C$

(3) $\frac{1}{3}\ell n\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right| + C$

(4) $\frac{1}{2}\ell n\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right| + C$

Ans. **(3)**

Sol. $\tan^{-1}\left(x^3 + \frac{1}{x^3}\right) = t$

$$\frac{1}{1 + \left(x^3 + \frac{1}{x^3}\right)^2} \times 3\left(x^2 - \frac{1}{x^4}\right)dx = dt$$

$$\int \frac{1}{3}\frac{dt}{t} = \frac{1}{3}\ell n|t| + C$$

$$= \frac{1}{3}\ell n\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right| + C$$

13. If mean and standard deviation of 15 observation are 12 and 3 respectively. But an error is found that 10 is written in place of 12. If the correct mean is μ and the correct variance σ^2 then find the value of $15(\mu + \mu^2 + \sigma^2)$

Ans. **(2521)**

Sol. Since $\frac{x_1 + x_2 + x_3 ..... + x_{15}}{15} = 12 \Rightarrow \sum_{i=1}^{15} x_i = 180$

but when data's are corrected the new mean is

$$\Rightarrow \mu = \frac{\sum_{i=1}^{15} x_i - 10 + 12}{15} = \frac{180 - 10 + 12}{15} = \frac{182}{15}$$

also given SD $= \sqrt{\frac{\sum x_i^2}{15} - (12)^2}$

$$\Rightarrow 9 + 144 = \frac{\sum x_i^2}{15}$$

$$\Rightarrow \sum_{i=1}^{15} x_i^2 = 153 \times 15 = 2295$$

new variance $\sigma^2 = \frac{\sum x_i^2 - 100 + 144}{15} - (\mu)^2$

$$\sigma^2 + \mu^2 = \frac{2295 + 44}{15}$$

$\sigma^2 + \mu^2 = \frac{2339}{15}$

$\Rightarrow 15(\mu + \mu^2 + \sigma^2) = 15\left(\frac{182}{15} + \frac{2339}{15}\right)$

$= 2521$

14. If $\frac{dy}{dx} = \frac{x+y-2}{x-y}$ and y (0) = 2 then y (2) is

(1) 0 (2) 2 (3) e (4) e^2

Ans. **(1)**

Sol. Put x = X + 1 & y = Y +1

$\frac{dY}{dX} = \frac{X+Y}{X-Y}$

$Y = vX$

$\frac{dY}{dX} = v + X\frac{dv}{dX}$

$v + X\frac{dv}{dX} = \frac{1+v}{1-v}$

$X\frac{dv}{dX} = \frac{1+v-v+v^2}{1-v}$

$\frac{1-v}{1+v^2}dv = \frac{dX}{X}$

$\tan^{-1} v - \frac{1}{2}\ell n\left(1+v^2\right) = \ell n(x-1) + c$

$\tan^{-1}\left(\frac{y-1}{x-1}\right) - \frac{1}{2}\ell n\left(1+\left(\frac{y-1}{x-1}\right)^2\right) = \ell n(x-1) + c$

x = 0, y = 2

$-\frac{\pi}{4} - \frac{1}{2}\ell n2 = \ell n1 + c$

$c = -\frac{\pi}{4} - \frac{1}{2}\ell n2$

hence at x = 2 $\Rightarrow \tan^{-1}(y-1) - \frac{1}{2}\ell n\left(1+(y-1)^2\right) = c = -\frac{\pi}{4} - \frac{1}{2}\ell n2$

y = 0

15. A is a 2 × 2 matrix, I is 2 × 2 identity matrix. |A – xI| = 0 has the roots – 1, 3. Then find the sum of diagonal elements of A^2.

Ans. **(10)**

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A - xI| = 0 \Rightarrow \begin{bmatrix} a-x & b \\ c & d-x \end{bmatrix} = 0$

$(a - x)(d - x) - bc = 0$

$x^2 - (a + d)x + ad - bc = 0$

has roots -1 & 3

$a + d = 2 \qquad ad - bc = -3$

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

Sum of diagonal element of A^2

$= a^2 + d^2 + 2bc$

$= (a + d)^2 + 2(bc - ad)$

$= 4 + 6 = 10$

16. For $x \in (0, 3)$, $g(x) = 3f\left(\frac{x}{3}\right) + f(3 - x)$ and $f''(x) > 0 \; \forall x \in (0, 3)$

If $g(x)$ is increasing in $(\alpha, 3)$ and decreasing in $(0, \alpha)$ then find 'α'

(1) $\frac{1}{2}$ (2) $\frac{9}{4}$ (3) $\frac{3}{2}$ (4) $\frac{1}{4}$

Ans. **(2)**

Sol. Since $f''(x) > 0 \Rightarrow f'(x)$ is increasing in $(0, 3)$.

Now $g'(x) = f'\left(\frac{x}{3}\right) - f'(3 - x)$

For $g(x)$ to be increasing

$$g'(x) = f'\left(\frac{x}{3}\right) - f'(3 - x) > 0$$

$\Rightarrow f'\left(\frac{x}{3}\right) > f'(3 - x)$

$\Rightarrow \frac{x}{3} > 3 - x \quad \Rightarrow x > 9 - 3x \;\Rightarrow x > \frac{9}{4}$

For $g(x)$ to be decreasing

$g'(x) < 0 \qquad \Rightarrow x < \frac{9}{4}$

$\alpha = \frac{9}{4}$

17. Let $f(x) = \int_0^x g(t)\,\ell n\left(\frac{1-t}{1+t}\right)dt$. If g is odd continuous function and $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(f(x) + \frac{x^2\cos x}{1+e^x}\right)dx = \frac{\pi^2}{\alpha^2} - \alpha$, then value of α is ______

(1) 1 (2) 2 (3) 3 (4) 4

Ans. **(2)**

Sol. $f(x) = \int_0^x g(t)\,\ell n\left(\frac{1-t}{1+t}\right)dt$ is odd function.

Now $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(f(x)+\frac{x^2\cos x}{1+e^x}\right)dx=\int_{0}^{\frac{\pi}{2}}\left(\frac{(x^2\cos x)(1+e^x)}{1+e^x}\right)dx$

$=\int_0^{\frac{\pi}{2}} x^2\cos x dx=\left(x^2\sin x\right)_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}} x\sin x dx$

$=\frac{\pi^2}{4}+2(x\cos x)_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}\cos x dx$

$=\frac{\pi^2}{4}-2\times 1=\frac{\pi^2}{4}-2 \Rightarrow \alpha=2$

18. Let R be the interior region between the lines $3x-y+1=0$ and $x+2y-5=0$ containing the origin. The set of all values of a for which the points $(a^2, a+1)$ lies in R is

(1) $(-\infty,-1)\cup(3,\infty)$ (2) $(-3, 0)\cup\left(\frac{1}{3},1\right)$ (3) $(-\infty,-1)\cup\left(0,\frac{1}{3}\right)$ (4) $(-\infty,-2)\cup\left(0,\frac{1}{3}\right)$

Ans. (2)

Sol. $(a^2, a+1)$ and $(0,0)$ lies on the same side of $3x-y+1=0$ and $x+2y-5=0$

$\Rightarrow 3a^2-a-1+1>0$ and $a^2+2a+2-5<0$

$a(3a-1)>0$ and $(a+3)(a-1)<0$

$a\in(-3, 0)\cup\left(\frac{1}{3},1\right)$

19. If $\begin{vmatrix} 1 & \frac{3}{2} & \alpha+\frac{3}{2} \\ 1 & \frac{1}{3} & \alpha+\frac{1}{3} \\ 2\alpha+3 & 3\alpha+1 & 0 \end{vmatrix}=0$ then α lies in the interval

(1) $(0, 3)$ (2) $(-3, 0)$ (3) $(-2, 1)$ (4) $(-2, 0)$

Ans. (2)

Sol. $C_3\rightarrow C_3-(\alpha C_1+C_2)$

$\begin{vmatrix} 1 & \frac{3}{2} & 0 \\ 1 & \frac{1}{3} & 0 \\ 2\alpha+3 & 3\alpha+1 & -(2\alpha^2+6\alpha+1) \end{vmatrix}=0$

$\Rightarrow(2\alpha^2+6\alpha+1)=0$

$\Rightarrow \alpha=\frac{-3+\sqrt{7}}{2},\frac{-3-\sqrt{7}}{2}$

20. If $f(x) = 6x - x^2$; $x \in [0,2]$ and $g(x) = \begin{cases} \min\{g(t); & 0 \le t \le x\}, 0 \le x < 1 \\ 3 + x; & x \in [1,2] \end{cases}$, then numbers of points where g(x) is not differentiable is :

(1) 1 (2) 0 (3) 2 (4) 3

Ans. **(1)**

Sol. $f(x) = 6x - x^2$

$g(x) = \begin{cases} 0; & 0 \le x < 1 \\ 3 + x & 1 \le x \le 2 \end{cases}$

g(x) is not differentiable at x = 1

21. Three lines $2x - y - 3 = 0$, $6x + 3y + 4 = 0$, $\alpha x + 2y + 4 = 0$ does not form triangle then find $\left[\sum \alpha^2\right]$ (where [.] denotes the greatest integer functions)

Ans. **(32)**

Sol. Triangle will not form if either at least two lines are parallel or lines are concurrent.

If two line are parallel

$\frac{2}{\alpha} = \frac{-1}{2} \Rightarrow \alpha = -4$

$\frac{6}{\alpha} = \frac{3}{2} \Rightarrow \alpha = 4$

If lines are concurrent

$\begin{vmatrix} 2 & -1 & -3 \\ 6 & 3 & 4 \\ \alpha & 2 & 4 \end{vmatrix} = 0$

$2(12 - 8) + 1(24 - 4\alpha) - 3(12 - 3\alpha) = 0$

$8 + 24 - 4\alpha - 36 + 9\alpha = 0$

$\alpha = \frac{4}{5}$

$\sum \alpha^2 = 16 + 16 + \frac{16}{25}$

$\left[\sum \alpha^2\right] = 32$

22. Let $S_1 = \frac{(4!)!}{4!^{(3!)}}$ and $S_2 = \frac{(5!)!}{(5!)^{(4!)}}$ then

(1) $S_1 \in N$ and $S_2 \notin N$ (2) $S_1 \in N$ and $S_2 \in N$

(3) $S_1 \notin N$ and $S_2 \in N$ (4) $S_1 \notin N$ and $S_2 \notin N$

Ans. **(2)**

Sol. divide 24 different objects into 6 persons of 4 each

Number of ways of making groups = $\frac{24!}{(4!)^6} = I_1$

$S_1 \in N$

divide 120 different objects into 24 persons of 5 each

$\frac{5!}{(5!)^{24}} = I_2$

Hence $S_2 \in N$