MATHEMATICS

1. An urn contains 6 black and 9 red balls. Four balls are drawn from the urn twice without replacement. The probability that first four balls are black & 2^{nd} four balls are red in colour is:

(1) $\frac{3}{765}$ (2) $\frac{6}{715}$ (3) $\frac{3}{715}$ (4) $\frac{6}{615}$

Ans. **(3)**

Sol. $\frac{{}^6C_4}{{}^{15}C_4} \times \frac{{}^9C_4}{{}^{11}C_4} = \frac{3}{715}$

2. A line x + y = 0 touches the circle $(x - \alpha)^2 + (y - \beta)^2 = 50$, α, β, > 0. The distance of origin from its points of contact is $4\sqrt{2}$. Find $\alpha^2 + \beta^2$.

Ans. 82

Sol. Point of contact is $(0 + 4\sqrt{2}\cos 135^\circ, 0 + 4\sqrt{2}\sin 135^\circ) = (-4, 4)$

$\left|\frac{\alpha+\beta}{\sqrt{2}}\right| = 5\sqrt{2}$ $\alpha + \beta = 10$ ….(i)

$(-4 - \alpha)^2 + (4 - \beta)^2 = 50$

$(\alpha + 4)^2 + (4 - 10 + \alpha)^2 = 50$

$(\alpha + 4)^2 + (\alpha - 6)^2 = 50$

$\alpha = 1, \beta = 9$

$\alpha^2 + \beta^2 = 82$

Also point of contact is (4, –4)

Satisfying this point of contact in the equation of circle we get

$(4 - \alpha)^2 + (-4 - \beta)^2 = 50$

$(4 - \alpha)^2 + (\beta + 4)^2 = 50$

$(4 - \alpha)^2 + (14 - \alpha)^2 = 50$

$\Rightarrow \alpha = 9, \beta = 1$

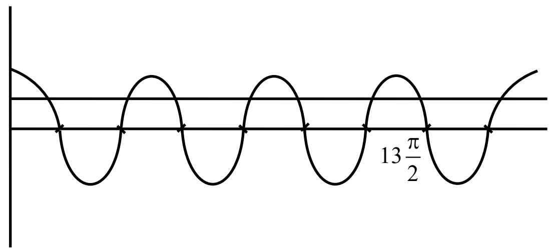
$\alpha^2 + \beta^2 = 82$

3. Let $2\tan^2 x - 5\sec x - 1 = 0$ has 7 solutions in $x \in \left[0, \frac{n\pi}{2}\right]$, then the minimum value of n is N find $\sum_{k=1}^{N} \frac{k}{2^k}$

(1) $2.\left(\frac{2^{13}-1}{2^{13}}\right) - \frac{13}{2^{13}}$ (2) $\left(\frac{2^{13}-1}{2^{13}}\right) - \frac{13}{2^{13}}$

(3) $2.\left(\frac{2^{13}-1}{2^{13}}\right) + \frac{13}{2^{14}}$ (4) $2.\left(\frac{2^{13}-1}{2^{13}}\right) - \frac{13}{2^{14}}$

Ans. **(1)**

Sol. 

$2\tan^2x - 5\sec x - 1 = 0$

$2\sec^2x - 5\sec x - 3 = 0$

$2\sec^2x - 6\sec x + \sec x - 3 = 0$

$(2\sec x + 1)(\sec x - 3) = 0$

$\sec x = 3, -\frac{1}{2}$

$\Rightarrow \sec x = 3$

$\Rightarrow \cos x = \frac{1}{3}$

For 7 solutions, n = 13 = N

so $\sum_{k=1}^{13}\frac{k}{2^k}$

let $S = \frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \ldots\ldots + \frac{13}{2^{13}}$

$\frac{1}{2}S = \frac{1}{2^2} + \frac{2}{2^3} + \ldots\ldots.. + \frac{13}{2^{14}}$

$\frac{1}{2}S = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots\ldots. \frac{1}{2^{13}} - \frac{13}{2^{14}}$

$$\frac{1}{2}S = \frac{1}{2}.\frac{\left(1-\frac{1}{2^{13}}\right)}{1-\frac{1}{2}} - \frac{13}{2^{14}}$$

$$S = 2.\left(\frac{2^{13}-1}{2^{13}}\right) - \frac{13}{2^{13}}$$

4. The vertices of a triangle are A(1, 2, 2), B(2, 1, 2) & C(2, 2, 1). The perpendicular distance of its orthocentre from the given sides are ℓ_1, ℓ_2 & ℓ_3. Find the value of $\ell_1^2 + \ell_2^2 + \ell_3^2$.

(1) 1 (2) $\frac{1}{2}$ (3) $\frac{1}{3}$ (4) $\frac{1}{4}$

Ans. **(2)**

Sol. ΔABC is equilateral

$\therefore$ orthocentre & centroid will be same $\left(\frac{5}{3}, \frac{5}{3}, \frac{5}{3}\right)$

midpoint of AB is $\left(\frac{3}{2}, \frac{3}{2}, 2\right)$

$\Rightarrow \ell_1 = \sqrt{\frac{1}{36} + \frac{1}{36} + \frac{1}{9}}$

$= \frac{1}{\sqrt{6}} = \ell_2 = \ell_3$

5. Let two sets A and B having 'm' & 'n' elements respectively such that difference of the number of subsets of A and that of B is 56, then (m, n) is

(1) (8, 3) (2) (8, 5) (3) (6, 3) (4) (7, 4)

Ans. **(3)**

Sol. $2^m - 2^n = 56; m > n$

$\Rightarrow 2^n(2^{m-n} - 1) = 8(2^3 - 1)$

$\Rightarrow m = 6, n = 3$

6. If 'A' is a square matrix of order '2' such that roots of the equation $\det(A - \lambda I) = 0$ are 1 and –3, then sum of diagonal elements of matrix 'A^2' is

(1) 2 (2) –3 (3) 9 (4) 10

Ans. **(4)**

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\therefore |A - \lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - (a + d)\lambda + ad - bc = 0$

Sum of the roots = $a + d = 2$

Product of roots = $ad - bc = -3$

Now $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{bmatrix}$

$\therefore tr(A^2) = a^2 + d^2 + 2bc = (a + d)^2 - 2(ad - bc) = 4 + 6 = 10$

7. Let $\tan^{-1}x + \tan^{-1}2x = \frac{\pi}{4}$; $x > 0$, then number of positive values of x is/are

(1) 0 (2) 1 (3) 2 (4) 3

Ans. **(2)**

Sol. $\tan^{-1}\left(\frac{x+2x}{1-2x^2}\right) = \frac{\pi}{4}$

$\Rightarrow \frac{3x}{1-2x^2} = 1 \qquad \Rightarrow x = \frac{-3\pm\sqrt{17}}{4}$

$\therefore x = \frac{\sqrt{17}-3}{4}$

8. Find the coefficient of x^{2012} in $(1 - x)^{2008}.(1 + x + x^2)^{2007}$

Ans. **(0)**

Sol. Coefficient of x^{2012} in $(1 - x^3)^{2007}.(1 - x)$

Coefficient of x^{2012} in $(1 - x^3)^{2007} - x(1 - x^3)^{2007}$

Coefficient of x^{2012} in ${}^{2007}C_{r_1}\left(-x^3\right)^{r_1} + {}^{2007}C_{r_2}\left(-1\right)^{r_2} x^{3r_2+1}$

$3r_1 = 2012$ Which is not possible for any $r_1 \in w$

and $3r_2 + 1 = 2012$ also not possible for any $r_2 \in w$

$\therefore$ no term containing x^{2012}

$\therefore$ Coefficient of x^{2012} is 0

9. An ellipse is passing through focii of hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ and product of their eccentricities is 1, then the length of chord of ellipse passing through (0, 2) and parallel to x-axis is

(1) $\frac{5\sqrt{5}}{3}$ (2) $\frac{3}{5\sqrt{5}}$ (3) $\frac{10\sqrt{5}}{3}$ (4) $\frac{20\sqrt{5}}{3}$

Ans. **(3)**

Sol. $\frac{x^2}{16}-\frac{y^2}{9}=1 \qquad \Rightarrow e_H = \frac{5}{4} \qquad \Rightarrow e_E = \frac{4}{5}$

Ellipse is passing through (±5, 0)

$\therefore$ ellipse : $\frac{x^2}{25}+\frac{y^2}{9}=1$

and points of chord : $\left(\pm\frac{5\sqrt{5}}{3}, 2\right)$

$\therefore$ Length of chord = $\frac{10\sqrt{5}}{3}$

10. If α and $\frac{1}{\alpha}$ are two complex numbers which satisfy the equations $|z - z_0|^2 = 4$ and $|z - z_0|^2 = 16$ respectively, where $z_0 = 1 + i$, then the value of $5|\alpha|^2$ is

Ans. (1)

Sol. $|\alpha - z_0|^2 = 4$

$\Rightarrow (\alpha - z_0)(\bar{\alpha} - \bar{z}_0) = 4$

$|\alpha|^2 - \alpha\bar{z}_0 - \bar{\alpha}z_0 + |z_0|^2 = 4$

$|\alpha|^2 - \alpha\bar{z}_0 - \bar{\alpha}z_0 = 2 \quad \ldots(i)$

(ii) $\left|\frac{1}{\bar{\alpha}} - z_0\right|^2 = 16 \Rightarrow (1 - \bar{\alpha}z_0)(1 - \alpha\bar{z}_0) = 16|\alpha|^2$

$\Rightarrow 1 - \alpha\bar{z}_0 - \bar{\alpha}z_0 + |\alpha|^2 . 2 = 16|\alpha|^2 \quad \ldots.(ii)$

from (i) & (ii) $- 1 - |\alpha|^2 = 2 - 16|\alpha|^2 \Rightarrow 15|\alpha|^2 = 3 \Rightarrow 5|\alpha|^2 = 1$

11. Let $x^2 - x - 1 = 0$ has roots α and β such that $S_n = 2023\,\alpha^n + 2024\,\beta^n$, then

(1) $S_{12} = S_{11} - S_{10}$
(2) $S_{12} = S_{10} - S_{11}$
(3) $S_{12} = S_{10} + S_{11}$
(4) $S_{12} = -S_{10} - S_{11}$

Ans. **(3)**

Sol. $S_n = 2023\,\alpha^n + 2024\,\beta^n$

$\Rightarrow S_n - S_{n-1} - S_{n-2} = 0$

$\Rightarrow S_{12} = S_{11} + S_{10}$

12. For the series $20, 19\frac{1}{4}, 18\frac{1}{2}, \ldots\ldots., -129\frac{1}{4}$, the 20^{th} term from end is

(1) –115
(2) – 119
(3) –117
(4) –120

Ans. **(1)**

Sol. T_{20} for $a = -129\frac{1}{4} = -\frac{517}{4}$, $d = \frac{3}{4}$

$T_{20} = -\frac{517}{4} + 19 . \frac{3}{4} = -\frac{460}{4} = -115$

13. $$\int \frac{x^8 - x^2}{(x^{12} + 3x^6 + 1)\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)}dx =$$

(1) $\frac{1}{3}\ \ell n \tan^{-1}\left(x^3 + \frac{1}{x^3}\right) + C$
(2) $\ell n \tan^{-1}\left(x^3 + \frac{1}{x^3}\right) + C$
(3) $\tan^{-1}\left(x^3 + \frac{1}{x^3}\right) + C$
(4) None of these

Ans. **(1)**

Sol. $\int \frac{x^8 - x^2}{(x^{12}+3x^6+1)\tan^{-1}\left(x^3+\frac{1}{x^3}\right)} dx =$

Let $\tan^{-1}\left(x^3+\frac{1}{x^3}\right) = t$

$$\frac{1}{1+\left(x^3+\frac{1}{x^3}\right)^2}\left(3x^2-\frac{3}{x^4}\right) dx = dt$$

$$\frac{x^6}{x^{12}+2x^6+1+x^6} \times \frac{3x^6-3}{x^4} \ dx \ = dt$$

$$\frac{1}{3}\int\frac{1}{t}dt = \frac{1}{3}\ell n(t) + C$$

$$= \frac{1}{3}\ell n \tan^{-1}\left(x^3+\frac{1}{x^3}\right) + C$$

14. If $\lim_{x\to 0}\frac{\alpha \sin x + \beta \cos x + \ln(1-x) + 3}{3\tan^2 x} = \frac{1}{3}$ find $2\alpha - \beta$

(1) 10 (2) 7 (3) 8 (4) 5

Ans. **(4)**

Sol. $\lim_{x\to 0}\frac{\alpha \sin x + \beta \cos x + \ln(1-x) + 3}{\frac{3\tan^2 x}{x^2}} = \frac{1}{3}$

$\beta + 3 = 0 \Rightarrow \beta = -3$

$$\lim_{x\to 0}\frac{\alpha\cos x - \beta \sin x - \frac{1}{1-x}}{2x}$$

$\alpha - 1 = 0 \Rightarrow \alpha = 1$

so $2\alpha - \beta = 5$

15. Let a_1, a_2, a_3 ……. a_{15} are 15 observations having mean and variance as 12 & 9 respectively. One of the observation which was 12, misread as 10. The correct mean and variance are μ and σ^2 respectively, then $15(\mu + \mu^2 + \sigma^2)$

(1) 2521 (2) 2522 (3) 2518 (4) 2621

Ans. **(1)**

Sol. old mean $12 = \frac{\Sigma x_i}{n} \Rightarrow 12 = \frac{a_1 + a_2 + ...a_{14} + 10}{15}$

$$\sum_{i=1}^{14} a_i = 170$$

old variance = 9 ⇒ $9 + (12)^2 = \frac{a_1^2 + a_2^2 + ... + a_{14}^2 + 10^2}{15}$

$$\sum_{i=1}^{14} a_i^2 = 2195$$

new mean $(\mu) = \frac{\sum_{i=1}^{14} a_i + 12}{15} = \frac{170 + 12}{15} = \frac{182}{15}$

new variance (σ^2)

$\sigma^2 + \mu^2 = \frac{\sum_{i=1}^{14} a_i^2 + 12^2}{15} = \frac{2339}{15}$

$\sigma^2 + \mu^2 + \mu = \frac{2339}{15} + \frac{182}{15} = \frac{2521}{15}$

$15(\sigma^2 + \mu^2 + \mu) = 2521$

16. Values of α for which $\begin{vmatrix} 1 & \frac{3}{2} & \alpha + \frac{3}{2} \\ 1 & \frac{1}{3} & \alpha + \frac{1}{3} \\ 2\alpha + 3 & 3\alpha + 1 & 0 \end{vmatrix} = 0$ lies in the interval

(1) (0, 3) (2) (–3, 0) (3) (–2, 1) (4) (–2, 0)

Ans. **(2)**

Sol. $C_3 \rightarrow C_3 - (\alpha C_1 + C_2)$

$$\begin{vmatrix} 1 & \frac{3}{2} & 0 \\ 1 & \frac{1}{3} & 0 \\ 2\alpha + 3 & 3\alpha + 1 & -(2\alpha^2 + 6\alpha + 1) \end{vmatrix} = 0$$

$\Rightarrow \frac{7}{6}(2\alpha^2 + 6\alpha + 1) = 0$

$\Rightarrow \alpha = \frac{-3 + \sqrt{7}}{2}, \frac{-3 - \sqrt{7}}{2}$

17. Let $g(x) = 3f\left(\frac{x}{3}\right) + f(3 - x)$, where $f''(x) > 0$ and $x \in (0, 3)$, g(x) is decreasing in $x \in (0, \alpha)$ and increasing in $(\alpha, 3)$, then 8α is

Ans. **(18)**

Sol. $g'(x) = 3 \cdot \frac{1}{3} f'\left(\frac{x}{3}\right) - f'(3 - x) = f'\left(\frac{x}{3}\right) - f'(3 - x)$

g(x) is decreasing $g'(x) < 0$

$f'\left(\frac{x}{3}\right) < f'(3 - x)$

$\because f''(x) > 0 \Rightarrow f'(x)$ is increasing

$\frac{x}{3} < 3 - x$

$\frac{4x}{3} < 3 \Rightarrow x < \frac{9}{4}$

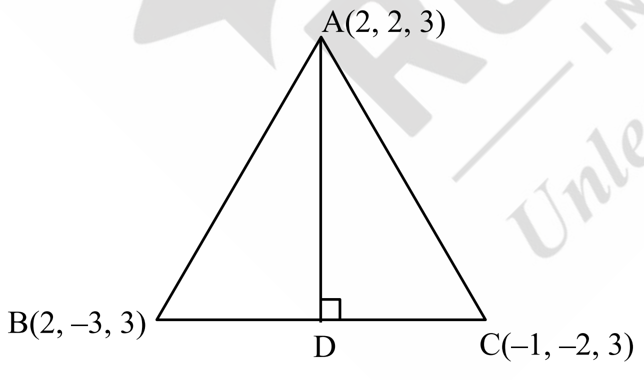
$\alpha = \frac{9}{4}$

$\therefore 8\alpha = 18$

18. Let ΔABC have vertices A(2, 2, 3), B(2, –3, 3) C(–1, –2, 3) and length of internal angle bisector of angle A is ℓ, then the value of $2\ell^2$ is

Ans. **(45)**

Sol.



$\overrightarrow{AB} = -5j$

$\overrightarrow{AC} = -3i - 4j$

$\left|\overrightarrow{AB}\right| = \left|\overrightarrow{AC}\right|$

$D\left(\frac{-1+2}{2}, \frac{-2-3}{2}, \frac{3+3}{2}\right)$

$D\left(\frac{1}{2},\frac{-5}{2},3\right)$

$AD = \sqrt{\left(\frac{1}{2}-2\right)^2+\left(2+\frac{5}{2}\right)^2+(3-3)^2}$

$\ell = \sqrt{\frac{9}{4}+\frac{81}{4}}$

$2\ell^2 = 45$

$= \sqrt{\frac{90}{4}} = \sqrt{\frac{45}{2}}$

19. Let $S_1 = \frac{\lfloor 4!}{(4!)^{3!}}$ and $S_2 = \frac{\lfloor 5!}{(5!)^{4!}}$, then

(1) $S_1 \in N$ and $S_2 \notin N$
(2) $S_1 \in N$ and $S_2 \in N$
(3) $S_1 \notin N$ and $S_2 \in N$
(2) $S_1 \notin N$ and $S_2 \notin N$

Ans. **(2)**

Sol. Make 6 groups of 4 each

$24 \rightarrow (4, 4, 4, 4, 4, 4, 4)$

Number of ways of making groups $= \frac{24!}{(4!)^6 . 6!} = I_1$

$\frac{(24)!}{(4!)^6} = \frac{\lfloor 4!}{(4!)^{3!}} = (6! I_1)$

$S_1 \in N$

$(5!) \rightarrow (5, 5, 5, 5 \ldots\ldots, 5(24 \text{ times}))$

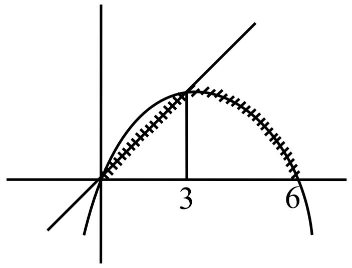
$\frac{\lfloor 5!}{(5!)^{24} . 24!} = I_2 \Rightarrow S_2 = (24!)\, I_2$

Hence $S_2 \in N$

20. Let area bounded by $y = \min.(3x, 6x - x^2)$; $y \geq 0$ is A, then 2A is

Ans. **(63)**

Sol.



$2x = 6x - x^2$

$A = \frac{1}{2} \times 3 \times 9 + \int_3^6 \sqrt{6x - x^2}\,dx$

$A = \frac{27}{2} + \int_3^6 \sqrt{9 - (x-3)^2}\,dx$

$A = \frac{27}{2} + \left(\frac{x-3}{2}\sqrt{9-(x-3)^2} + \frac{9}{2}\sin^{-1}\left(\frac{x-3}{3}\right)\right)_4^6$

$A = \frac{27}{2} + \frac{9\pi}{4}$

$12A = 162 + 27\pi$

21. Let $(x^2 - 4)dy = y(y-3)dx$ satisfying $y(4) = \frac{3}{2}$ then $y(10)$ is equal to

(1) $\frac{3}{1-8^{\frac{1}{4}}}$ (2) $\frac{3}{1+8^{\frac{1}{4}}}$ (3) $\frac{3}{1+2^{\frac{1}{4}}}$ (4) $\frac{3}{1-2^{\frac{1}{4}}}$

Ans. **(2)**

Sol. $= \frac{1}{3}\int \frac{y-(y-3)}{y(y-3)}dy = \frac{1}{4}\int \frac{(x+2)-(x-2)}{(x+2)(x-2)}dx$

$\frac{1}{3}(\ell n|y-3| - \ell n|y|) = \frac{1}{4}(\ell n|x-2| - \ell n|x+2|) + C$

$\frac{1}{3}\ell n\left|\frac{y-3}{y}\right| = \frac{1}{4}\left(\ell n\left|\frac{x-2}{x+2}\right|\right) + C$

$\frac{1}{3}\ell n\left|\frac{\frac{3}{2}-3}{\frac{3}{2}}\right| = \frac{1}{4}\ell n\left(\frac{4-2}{4+2}\right) + C$

$C = \frac{1}{4}\ell n 3$

$$\frac{1}{3}\ell n\left|\frac{y-3}{y}\right| = \frac{1}{4}\ell n\left|\frac{x-2}{x+2}\right| + \frac{1}{4}\ell n 3$$

$\Rightarrow x = 10$

$$\frac{1}{3}\ell n\left|\frac{y-3}{y}\right| = \frac{1}{4}\ell n\left|\frac{2}{3}\right| + \frac{1}{4}\ell n 3$$

$$\frac{1}{3}\ell n\left|\frac{y-3}{y}\right| = \frac{1}{4}\ell n 2$$

$$\ell n\left|\frac{y-3}{y}\right| = \ell n 2^{\frac{3}{4}}$$

$$\left|\frac{y-3}{y}\right| = 2^{\frac{3}{4}}$$

$$-y + 3 = 8^{\frac{1}{4}} y$$

$$y = \frac{3}{1+8^{\frac{1}{4}}}$$

22. Three lines $2x - y - 3 = 0$, $6x + 3y + 4 = 0$, $\alpha x + 2y + 4 = 0$ does not form triangle then find $[\Sigma\alpha^2]$ (where [.] denotes the greatest integer function)

Ans. **(32)**

Sol. If two lines are parallel

$$\frac{2}{\alpha} = \frac{-1}{2} \Rightarrow \alpha = -4$$

$$\frac{6}{\alpha} = \frac{3}{2} \Rightarrow \alpha = 4$$

If lines are concurrent

$$\begin{vmatrix} 2 & -1 & -3 \\ 6 & 3 & 4 \\ \alpha & 2 & 4 \end{vmatrix} = 0$$

$2(12 - 8) + 1(24 - 4\alpha) - 3(12 - 3\alpha) = 0$

$8 + 24 - 4\alpha - 36 + 9\alpha = 0$

$5\alpha = 4 \Rightarrow \alpha = \frac{4}{5}$

$\Sigma\alpha^2 = 16 + 16 + \frac{16}{25}$

$[\Sigma\alpha^2] = 32$

23. Let $f(x) = \int_0^x g(t)\ \log\left(\frac{1-t}{1+t}\right) dt$, (where g(x) is cont. odd function).

If $\int_{-\frac{\pi}{2}}^{\pi/2}\left(f(x)+\frac{x^2\cos x}{1+e^x}\right)dx = \left(\frac{\pi}{\alpha}\right)^2-\alpha$, then find α

Ans. $\alpha = 2$

Sol. $I = \int_0^{\pi/2}\left(f(x)+f(-x)+\frac{x^2\cos x}{1+e^x}+\frac{x^2\cos x}{1+e^{-x}}\right)dx = \int_0^{\pi/2}(f(x)+f(-x)+x^2\cos x)\,dx$ ….(i)

Now $f(-x) = \int_0^{-x} g(t)\ \log\left(\frac{1-t}{1+t}\right) dt$

$t = -p$

$= \int_0^x -g(-p)\log\left(\frac{1+p}{1-p}\right) dp = -f(x)$

$\therefore$ (i) becomes $I = \int_0^{\pi/2} x^2\cos x\,dx = x^2\sin x - 2\int x\ \sin x\,dx = (x^2\sin x - 2)\,(-x\cos x + \sin x))_0^{\pi/2}$

$= \frac{\pi^2}{4} - 2 \Rightarrow \frac{\pi^2}{4} - 2$