The following notations are used in the question paper:

$\mathbb{R}$ is the set of real numbers,

$\mathbb{C}$ is the set of complex numbers,

$\mathbb{Z}$ is the set of integers,

$\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ for all $n = 1, 2, 3, \ldots$ and $r = 0, 1, \ldots, n$.

1. For a real number x,

$$x^3 - 7x + 6 > 0$$

if and only if

(A) $x > 2$.

(B) $-3 < x < 1$.

(C) $x < -3$ or $1 < x < 2$.

(D) $-3 < x < 1$ or $x > 2$.

2. Define a polynomial $f(x)$ by

$$f(x) = \begin{vmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{vmatrix}$$

for all $x \in \mathbb{R}$, where the right hand side above is a determinant. Then the roots of $f(x)$ are of the form

(A) α, $\beta \pm i\gamma$ where $\alpha, \beta, \gamma \in \mathbb{R}$, $\gamma \neq 0$ and i is a square root of -1.

(B) α, α, β where $\alpha, \beta \in \mathbb{R}$ are distinct.

(C) α, β, γ where $\alpha, \beta, \gamma \in \mathbb{R}$ are all distinct.

(D) α, α, α for some $\alpha \in \mathbb{R}$.

3. Let S be the set of those real numbers x for which the identity

$$\sum_{n=2}^{\infty} \cos^n x = (1 + \cos x) \cot^2 x$$

is valid, and the quantities on both sides are finite. Then

(A) S is the empty set.

(B) $S = \{x \in \mathbb{R} : x \neq n\pi \text{ for all } n \in \mathbb{Z}\}$.

(C) $S = \{x \in \mathbb{R} : x \neq 2n\pi \text{ for all } n \in \mathbb{Z}\}$.

(D) $S = \{x \in \mathbb{R} : x \neq (2n+1)\pi \text{ for all } n \in \mathbb{Z}\}$.

4. The number of consecutive zeroes adjacent to the digit in the unit's place of 401^{50} is

(A) 3. (B) 4. (C) 49. (D) 50.

5. Consider a right angled triangle $\triangle ABC$ whose hypotenuse AC is of length 1. The bisector of $\angle ACB$ intersects AB at D. If BC is of length x, then what is the length of CD?

(A) $\sqrt{\frac{2x^2}{1+x}}$ (B) $\frac{1}{\sqrt{2+2x}}$

(C) $\sqrt{\frac{x}{1+x}}$ (D) $\frac{x}{\sqrt{1-x^2}}$

6. Consider a triangle with vertices $(0,0)$, $(1,2)$ and $(-4,2)$. Let A be the area of the triangle and B be the area of the circumcircle of the triangle. Then $\frac{B}{A}$ equals

(A) $\frac{\pi}{2}$. (B) $\frac{5\pi}{4}$. (C) $\frac{3}{\sqrt{2}}\pi$. (D) 2π.

7. Let f, g be continuous functions from $[0, \infty)$ to itself,

$$h(x) = \int_{2^x}^{3^x} f(t)\, dt\,, x > 0\,,$$

and

$$F(x) = \int_0^{h(x)} g(t)\, dt\,, x > 0\,.$$

If F' is the derivative of F, then for $x > 0$,

(A) $F'(x) = g(h(x))$.

(B) $F'(x) = g(h(x))\,[f(3^x) - f(2^x)]$.

(C) $F'(x) = g(h(x))\,[x3^{x-1}f(3^x) - x2^{x-1}f(2^x)]$.

(D) $F'(x) = g(h(x))\,[3^x f(3^x)\ln 3 - 2^x f(2^x)\ln 2]$.

8. How many numbers formed by rearranging the digits of 234578 are divisible by 55?

(A) 0 (B) 12 (C) 36 (D) 72

9. Let

$$S = \left\{ \left(\theta \sin \frac{\pi\theta}{1+\theta}, \frac{1}{\theta} \cos \frac{\pi\theta}{1+\theta} \right) : \theta \in \mathbb{R},\ \theta > 0 \right\}$$

and

$$T = \left\{ (x, y) : x \in \mathbb{R},\ y \in \mathbb{R},\ xy = \frac{1}{2} \right\}.$$

How many elements does $S \cap T$ have?

(A) 0 (B) 1 (C) 2 (D) 3

10. The limit

$$\lim_{n\to\infty} n^{-\frac{3}{2}} \left((n+1)^{(n+1)}(n+2)^{(n+2)}\dots(2n)^{(2n)}\right)^{\frac{1}{n^2}}$$

equals

(A) 0. (B) 1. (C) $e^{-\frac{1}{4}}$. (D) $4e^{-\frac{3}{4}}$.

11. Suppose x and y are positive integers. If $4x+3y$ and $2x+4y$ are divided by 7, then the respective remainders are 2 and 5. If $11x+5y$ is divided by 7, then the remainder equals

(A) 0. (B) 1. (C) 2. (D) 3.

12. The value of

$$\sum_{k=0}^{202}(-1)^k \binom{202}{k} \cos\left(\frac{k\pi}{3}\right)$$

equals

(A) $\sin\left(\frac{202}{3}\pi\right)$. (B) $-\sin\left(\frac{202}{3}\pi\right)$.
(C) $\cos\left(\frac{202}{3}\pi\right)$. (D) $\cos^{202}\left(\frac{\pi}{3}\right)$.

13. For real numbers $a, b, c, d, a', b', c', d'$, consider the system of equations

$$ax^2 + ay^2 + bx + cy + d = 0\,,$$
$$a'x^2 + a'y^2 + b'x + c'y + d' = 0\,.$$

If S denotes the set of all real solutions (x, y) of the above system of equations, then the number of elements in S can never be

(A) 0. (B) 1. (C) 2. (D) 3.

14. The limit

$$\lim_{x\to 0} \frac{1}{x}\left(\cos(x) + \cos\left(\frac{1}{x}\right) - \cos(x)\cos\left(\frac{1}{x}\right) - 1\right)$$

(A) equals 0.
(B) equals $\frac{1}{2}$.
(C) equals 1.
(D) does not exist.

15. Let n be a positive integer having 27 divisors including 1 and n, which are denoted by $d_1, \ldots, d_{27}$. Then the product of $d_1, d_2, \ldots, d_{27}$ equals

(A) n^{13}.
(B) n^{14}.
(C) $n^{\frac{27}{2}}$.
(D) $27n$.

16. Suppose $F : \mathbb{R} \to \mathbb{R}$ is a continuous function which has exactly one local maximum. Then which of the following is true?

(A) F cannot have a local minimum.
(B) F must have exactly one local minimum.
(C) F must have at least two local minima.
(D) F must have either a global maximum or a local minimum.

17. Suppose $z \in \mathbb{C}$ is such that the imaginary part of z is non-zero and $z^{25} = 1$. Then

$$\sum_{k=0}^{2023} z^k$$

equals

(A) 0.
(B) 1.
(C) $-1 - z^{24}$.
(D) $-z^{24}$.

18. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable one-to-one function. If $f(2) = 2$, $f(3) = -8$ and

$$\int_2^3 f(x)\,dx = -3\,,$$

then

$$\int_{-8}^{2} f^{-1}(x)\,dx$$

equals

(A) -25. (B) 25. (C) -31. (D) 31.

19. If $f : [0, \infty) \to \mathbb{R}$ is a continuous function such that

$$f(x) + \ln 2 \int_0^x f(t)\,dt = 1\,, x \geq 0\,,$$

then for all $x \geq 0$,

(A) $f(x) = e^x \ln 2$. (B) $f(x) = e^{-x} \ln 2$.
(C) $f(x) = 2^x$. (D) $f(x) = \left(\frac{1}{2}\right)^x$.

20. If $[x]$ denotes the largest integer less than or equal to x, then

$$\left[(9 + \sqrt{80})^{20}\right]$$

equals

(A) $(9 + \sqrt{80})^{20} - (9 - \sqrt{80})^{20}$.
(B) $(9 + \sqrt{80})^{20} + (9 - \sqrt{80})^{20} - 20$.
(C) $(9 + \sqrt{80})^{20} + (9 - \sqrt{80})^{20} - 1$.
(D) $(9 - \sqrt{80})^{20}$.

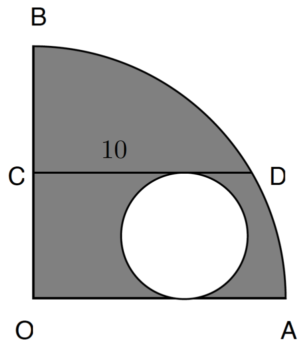
21. The limit

$$\lim_{n\to\infty}\left(2^{-2^{n+1}}+2^{-2^{n-1}}\right)^{2^{-n}}$$

equals

(A) 1. (B) $\frac{1}{\sqrt{2}}$. (C) 0. (D) $\frac{1}{4}$.

22. In the following figure, OAB is a quarter-circle. The unshaded region is a circle to which OA and CD are tangents.



If CD is of length 10 and is parallel to OA, then the area of the shaded region in the above figure equals

(A) 25π. (B) 50π. (C) 75π. (D) 100π.

23. Three left brackets and three right brackets have to be arranged in such a way that if the brackets are serially counted from the left, then the number of right brackets counted is always less than or equal to the number of left brackets counted. In how many ways can this be done?

(A) 3 (B) 4 (C) 5 (D) 6

24. The polynomial $x^{10} + x^5 + 1$ is divisible by

(A) $x^2 + x + 1$. (B) $x^2 - x + 1$.
(C) $x^2 + 1$. (D) $x^5 - 1$.

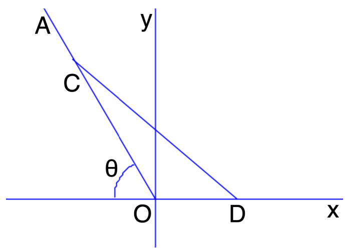
25. Suppose $a, b, c \in \mathbb{R}$ and

$$f(x) = ax^2 + bx + c\,, x \in \mathbb{R}\,.$$

If $0 \le f(x) \le (x-1)^2$ for all x, and $f(3) = 2$, then

(A) $a = \frac{1}{2}$, $b = -1$, $c = \frac{1}{2}$. (B) $a = \frac{1}{3}$, $b = -\frac{1}{3}$, $c = 0$.
(C) $a = \frac{2}{3}$, $b = -\frac{5}{3}$, $c = 1$. (D) $a = \frac{3}{4}$, $b = -2$, $c = \frac{5}{4}$.

26. As in the following figure, the straight line OA lies in the second quadrant of the (x, y)-plane and makes an angle θ with the negative half of the x-axis, where $0 < \theta < \frac{\pi}{2}$.

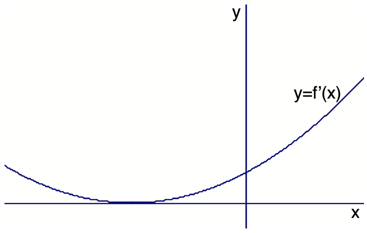


The line segment CD of length 1 slides on the (x, y)-plane in such a way that C is always on OA and D on the positive side of the x-axis. The locus of the mid-point of CD is

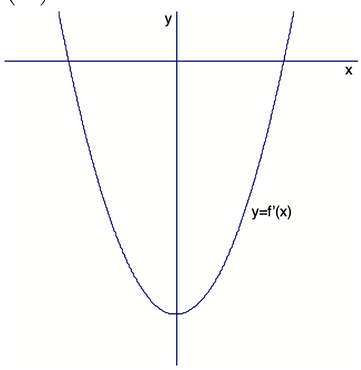
(A) $x^2 + 4xy \cot\theta + y^2 (1 + 4\cot^2\theta) = \frac{1}{4}$.
(B) $x^2 + y^2 = \frac{1}{4} + \cot^2\theta$.
(C) $x^2 + 4xy \cot\theta + y^2 = \frac{1}{4}$.
(D) $x^2 + y^2 (1 + 4\cot^2\theta) = \frac{1}{4}$.

27. Suppose that $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c, d are real numbers with $a \neq 0$. The equation $f(x) = 0$ has exactly two distinct real solutions. If $f'(x)$ is the derivative of $f(x)$, then which of the following is a possible graph of $f'(x)$?

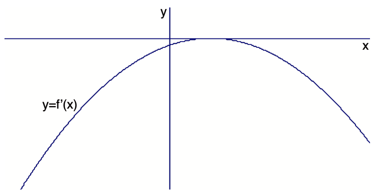
(A)



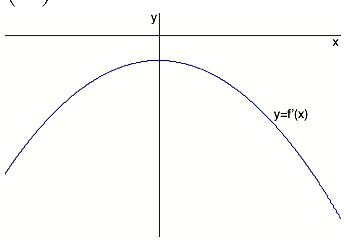
(B)



(C)



(D)



28. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f\,(a + ib) = e^a\,(\cos b + i \sin b)\ \ , a, b \in \mathbb{R}\,,$$

where i is a square root of -1. Then

(A) f is one-to-one and onto.

(B) f is one-to-one but not onto.

(C) f is onto but not one-to-one.

(D) f is neither one-to-one nor onto.

29. Suppose $f : \mathbb{Z} \to \mathbb{Z}$ is a non-decreasing function. Consider the following two cases:

$$\text{Case 1. } f(0) = 2\,, f(10) = 8\,,$$
$$\text{Case 2. } f(0) = -2\,, f(10) = 12\,.$$

In which of the above cases it is necessarily true that there exists an n with $f(n) = n$?

(A) In both cases.

(B) In neither case.

(C) In Case 1. but not necessarily in Case 2.

(D) In Case 2. but not necessarily in Case 1.

30. How many functions $f : \{1, 2, \ldots, 10\} \to \{1, \ldots, 2000\}$, which satisfy

$$f(i+1) - f(i) \geq 20\,, \text{ for all } 1 \leq i \leq 9\,,$$

are there?

(A) $10!\binom{1829}{10}$

(B) $11!\binom{1830}{11}$

(C) $\binom{1829}{10}$

(D) $\binom{1830}{11}$