1. Let A be a 2×2 nonzero real matrix. Which of the following is true?

   (A) A has a nonzero eigenvalue.

   (B) A^2 has at least one positive entry.

   (C) trace (A^2) is positive.

   (D) All entries of A^2 cannot be negative.

2. Let A be a 3×3 real matrix with zero diagonal entries. If $1 + i$ is an eigenvalue of A, the determinant of A equals

   (A) 4. (B) -4. (C) 2. (D) -2.

3. Let A be an $n \times n$ matrix and let b be an $n \times 1$ vector such that $Ax = b$ has a unique solution. Let A' denote the transpose of A. Then which of the following statements is **false**?

   (A) $A'x = 0$ has a unique solution.

   (B) $A'x = c$ has a unique solution for any non-zero c.

   (C) $Ax = c$ has a solution for any c.

   (D) $A^2x = c$ is inconsistent for some vector c.

ROUGH WORK

4. Let A and B be $n \times n$ matrices. Assuming all the inverses exist,

$$(A^{-1} - B^{-1})^{-1}$$

equals

(A) $(I - AB^{-1})^{-1}B$.

(B) $A(B - A)^{-1}B$.

(C) $B(B - A)^{-1}A$.

(D) $B(A - B)^{-1}A$.

5. Let f be a function defined on $(-\pi, \pi)$ as

$$f(x) = (|\sin x| + |\cos x|) \cdot \sin x.$$

Then f is differentiable at

(A) all points.

(B) all points except at $x = -\pi/2, \pi/2$.

(C) all points except at $x = 0$.

(D) all points except at $x = 0, -\pi/2, \pi/2$.

6. The equation of the tangent to the curve $y = \sin^2(\pi x^3/6)$ at $x = 1$ is

(A) $y = \frac{1}{4} + \frac{\sqrt{3}\pi}{4}(x - 1)$.

(B) $y = \frac{\sqrt{3}\pi}{4}x + \frac{1-\sqrt{3}\pi}{4}$.

(C) $y = \frac{\sqrt{3}\pi}{4}x - \frac{1-\sqrt{3}\pi}{4}$.

(D) $y = \frac{1}{4} - \frac{\sqrt{3}\pi}{4}(x - 1)$.

ROUGH WORK

7. Let f be a function defined from $(0, \infty)$ to $\mathbb{R}$ such that

$$\lim_{x\to\infty} f(x) = 1 \text{ and } f(x+1) = f(x) \quad \text{for all } x.$$

Then f is

(A) continuous and bounded.

(B) continuous but not necessarily bounded.

(C) bounded but not necessarily continuous.

(D) neither necessarily continuous nor necessarily bounded.

8. The value of $\lim_{x\to\infty} \left(\log x\right)^{1/x}$

(A) is e. (B) is 0. (C) is 1. (D) does not exist.

9. The number of real solutions of the equation,

$$x^7 + 5x^5 + x^3 - 3x^2 + 3x - 7 = 0$$

is

(A) 5. (B) 7. (C) 3. (D) 1.

ROUGH WORK

10. Let x be a real number. Then

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\cos^{2n}(m!\pi x)\right)$$

(A) does not exist for any x.

(B) exists for all x.

(C) exists if and only if x is irrational.

(D) exists if and only if x is rational.

11. Let $\{a_n\}_{n\geq 1}$ be a sequence such that $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots$. Suppose the subsequence $\{a_{2n}\}_{n\geq 1}$ is bounded. Then

(A) $\{a_{2n}\}_{n\geq 1}$ is always convergent but $\{a_{2n+1}\}_{n\geq 1}$ need not be convergent.

(B) both $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are always convergent and have the same limit.

(C) $\{a_{3n}\}_{n\geq 1}$ is not necessarily convergent.

(D) both $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are always convergent but may have different limits.

12. Let $\{a_n\}_{n\geq 1}$ be a sequence of positive numbers such that $a_{n+1} \leq a_n$ for all n, and $\lim_{n\to\infty} a_n = a$. Let $p_n(x)$ be the polynomial

$$p_n(x) = x^2 + a_n x + 1,$$

and suppose $p_n(x)$ has no real roots for every n. Let α and β be the roots of the polynomial $p(x) = x^2 + ax + 1$. Then

(A) $\alpha = \beta$, α and β are not real.

(B) $\alpha = \beta$, α and β are real.

(C) $\alpha \neq \beta$, α and β are real.

(D) $\alpha \neq \beta$, α and β are not real.

ROUGH WORK

13. Consider the set of all functions from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$, where $n > m$. If a function is chosen from this set at random, what is the probability that it will be strictly increasing?

(A) $\binom{n}{m}/m^n$. (B) $\binom{n}{m}/n^m$. (C) $\binom{m+n-1}{m}/m^n$. (D) $\binom{m+n-1}{m-1}/n^m$.

14. A flag is to be designed with 5 vertical stripes using some or all of the four colours: green, maroon, red and yellow. In how many ways can this be done so that no two adjacent stripes have the same colour?

(A) 576. (B) 120. (C) 324. (D) 432.

15. Suppose $x_1, \ldots, x_6$ are real numbers which satisfy

$$x_i = \prod_{j \neq i} x_j, \quad \text{for all } i = 1, \ldots, 6.$$

How many choices of $(x_1, \ldots, x_6)$ are possible?

(A) Infinitely many. (B) 2. (C) 3. (D) 1.

ROUGH WORK

16. Suppose X is a random variable with $P(X > x) = 1/x^2$, for all $x > 1$. The variance of $Y = 1/X^2$ is

(A) 1/4. (B) 1/12. (C) 1. (D) 1/2.

17. Let $X \sim N(0, \sigma^2)$, where $\sigma > 0$, and

$$Y = \begin{cases} -1 & \text{if } X \leq -1, \\ X & \text{if } X \in (-1, 1), \\ 1 & \text{if } X \geq 1. \end{cases}$$

Which of the following statements is correct?

(A) $\text{Var}(Y) = \text{Var}(X)$.

(B) $\text{Var}(Y) < \text{Var}(X)$.

(C) $\text{Var}(Y) > \text{Var}(X)$.

(D) $\text{Var}(Y) \geq \text{Var}(X)$ if $\sigma \geq 1$, and $\text{Var}(Y) < \text{Var}(X)$ if $\sigma < 1$.

18. If a fair coin is tossed 5 times, what is the probability of obtaining at least 3 consecutive heads?

(A) 1/8. (B) 5/16. (C) 1/4. (D) 3/16.

ROUGH WORK

19. Let X and Y be random variables with mean λ. Define

$$Z = \begin{cases} \min(X,Y) & \text{with probability } \frac{1}{2}\,, \\ \max(X,Y) & \text{with probability } \frac{1}{2}\,. \end{cases}$$

What is $\mathrm{E}(Z)$?

(A) λ. (B) $4\lambda/3$. (C) λ^2. (D) $\sqrt{3}\lambda/2$.

20. A finite population has $N(\geq 10)$ units marked $\{U_1, \ldots, U_N\}$. The following sampling scheme was used to obtain a sample s. One unit is selected at random: if this is the i-th unit, then the sample is $s = \{U_{i-1}, U_i, U_{i+1}\}$, provided $i \notin \{1, N\}$. If $i = 1$ then $s = \{U_1, U_2\}$ and if $i = N$ then $s = \{U_{N-1}, U_N\}$. The probability of selecting U_2 in s is

(A) $\frac{2}{N}$. (B) $\frac{3}{N}$. (C) $\frac{1}{(N-2)} + \frac{2}{N}$. (D) $\frac{3}{(N-2)}$.

21. Suppose $X_1, \ldots, X_n$ are i.i.d. observations from a distribution assuming values $-1, 1$ and 0 with probabilities p, p and $1-2p$, respectively, where $0 < p < \frac{1}{2}$. Define $Z_n = \prod_{i=1}^n X_i$ and $a_n = P(Z_n = 1)$, $b_n = P(Z_n = -1)$, $c_n = P(Z_n = 0)$. Then as $n \to \infty$,

(A) $a_n \to \frac{1}{4}$, $b_n \to \frac{1}{2}$, $c_n \to \frac{1}{4}$.

(B) $a_n \to \frac{1}{3}$, $b_n \to \frac{1}{3}$, $c_n \to \frac{1}{3}$.

(C) $a_n \to 0$, $b_n \to 0$, $c_n \to 1$.

(D) $a_n \to p$, $b_n \to p$, $c_n \to 1-2p$.

ROUGH WORK

22. Suppose X_1, X_2 and X_3 are i.i.d. positive valued random variables. Define $Y_i = \frac{X_i}{X_1+X_2+X_3}$, $i = 1, 2, 3$. The correlation between Y_1 and Y_3 is

(A) 0. (B) $-1/6$. (C) $-1/3$. (D) $-1/2$.

23. Assume (y_i, x_i) satisfies the linear regression model,

$$y_i = \beta x_i + \epsilon_i, \quad \text{for } i = 1, \ldots, n,$$

where, $\beta \in \mathbb{R}$ is unknown, $\{x_i : 1 \leq i \leq n\}$ are fixed constants and $\{\epsilon_i : 1 \leq i \leq n\}$ are i.i.d. errors with mean zero and variance $\sigma^2 \in (0, \infty)$. Let $\widehat{\beta}$ be the least squares estimate of β and $\widehat{y}_i = \widehat{\beta} x_i$ be the predicted value of y_i. For each $n \geq 1$, define

$$a_n = \frac{1}{\sigma^2} \sum_{i=1}^{n} \text{Cov}(y_i, \widehat{y}_i).$$

Then,

(A) $a_n = 1$. (B) $a_n \in (0, 1)$. (C) $a_n = n$. (D) $a_n = 0$.

24. Let X and Y be two random variables with $\text{E}(X|Y = y) = y^2$, where Y follows $N(\theta, \theta^2)$, with $\theta \in \mathbb{R}$. Then $\text{E}(X)$ equals

(A) θ. (B) θ^2. (C) $2\theta^2$. (D) $\theta + \theta^2$.

ROUGH WORK

25. Suppose X is a random variable with finite variance. Define $X_1 = X$, $X_2 = \alpha X_1,\ X_3 = \alpha X_2, \ldots, X_n = \alpha X_{n-1}$, for $0 < \alpha < 1$. Then $\mathrm{Corr}(X_1, X_n)$ is

(A) α^n. (B) 1. (C) 0. (D) α^{n-1}.

26. Let X be a random variable with $P(X = 2) = P(X = -2) = 1/6$ and $P(X = 1) = P(X = -1) = 1/3$. Define $Y = 6X^2 + 3$. Then

(A) $\mathrm{Var}(X - Y) < \mathrm{Var}(X)$.

(B) $\mathrm{Var}(X - Y) < \mathrm{Var}(X + Y)$.

(C) $\mathrm{Var}(X + Y) < \mathrm{Var}(X)$.

(D) $\mathrm{Var}(X - Y) = \mathrm{Var}(X + Y)$.

27. Suppose X is a random variable on $\{0, 1, 2, \ldots\}$ with unknown p.m.f. $p(x)$. To test the hypothesis $H_0 : X \sim \text{Poisson}(1/2)$ against $H_1 : p(x) = 2^{-(x+1)}$ for all $x \in \{0, 1, 2, \ldots\}$, we reject H_0 if $x > 2$. The probability of type-II error for this test is

(A) $\frac{1}{4}$. (B) $1 - \frac{13}{8}e^{-1/2}$. (C) $1 - \frac{3}{2}e^{-1/2}$. (D) $\frac{7}{8}$.

ROUGH WORK

28. Let X be a random variable with

$$P_\theta(X = -1) = \frac{(1-\theta)}{2}, \; P_\theta(X = 0) = \frac{1}{2}, \text{ and } P_\theta(X = 1) = \frac{\theta}{2}$$

for $0 < \theta < 1$. In a random sample of size 20, the observed frequencies of $-1, 0$ and 1 are $6, 4$ and 10, respectively. The maximum likelihood estimate of θ is

(A) 1/5. (B) 4/5. (C) 5/8. (D) 1/4.

29. Two judges evaluate n individuals, with (R_i, S_i) the ranks assigned to the i-th individual by the two judges. Suppose there are no ties and $S_i = R_i + 1$, for $i = 1, \ldots, (n-1)$, and $S_i = 1$ if $R_i = n$. If the Spearman's rank correlation between the two evaluations is 0, what is the value of n?

(A) 7. (B) 11. (C) 4. (D) 5.

30. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with variance 2. Then for all x,

$$\lim_{n\to\infty} P\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (-1)^i X_i \leq x\right)$$

equals

(A) $\Phi(x\sqrt{2})$. (B) $\Phi\left(x/\sqrt{2}\right)$. (C) $\Phi(x)$. (D) $\Phi(2x)$.

ROUGH WORK