PSB
2019

GROUP A

1. Let $f(x) = x^3 - 3x + k$, where k is a real number. For what values of k will $f(x)$ have three distinct real roots?

2. Let A and B be 4×4 matrices. Suppose that A has eigenvalues x_1, x_2, x_3, x_4 and B has eigenvalues $1/x_1, 1/x_2, 1/x_3, 1/x_4$, where each $x_i > 1$.

   (a) Prove that $A + B$ has at least one eigenvalue greater than 2.

   (b) Prove that $A - B$ has at least one eigenvalue greater than 0.

   (c) Give an example of A and B so that 1 is not an eigenvalue of AB.

3. Elections are to be scheduled on any seven days in April and May. In how many ways can the seven days be chosen such that elections are not scheduled on two consecutive days?

GROUP B

4. Let X and Y be independent and identically distributed random variables with mean $\mu > 0$ and taking values in $\{0, 1, 2, \dots\}$. Suppose, for all $m \geq 0$,

$$\mathrm{P}(X = k \mid X + Y = m) = \frac{1}{m+1}, \quad k = 0, 1, \dots, m.$$

   Find the distribution of X in terms of μ.

5. Suppose $X_1, X_2, \ldots, X_n$ are independent random variables such that

$$\mathrm{P}(X_i = 1) = p_i = 1 - \mathrm{P}(X_i = 0),$$

where $p_1, p_2, \ldots, p_n \in (0,1)$ are all distinct and unknown. Consider $X = \sum_{i=1}^{n} X_i$ and another random variable Y which is distributed as Binomial$(n, \bar{p})$, where $\bar{p} = \frac{1}{n} \sum_{i=1}^{n} p_i$. Between X and Y, which is a better estimator of $\sum_{i=1}^{n} p_i$ in terms of their respective mean squared errors?

6. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from Uniform$(0, \theta)$ for some unknown $\theta > 0$. Let Y_n be the minimum of $X_1, X_2, \ldots, X_n$.

(a) Suppose F_n is the cumulative distribution function (c.d.f.) of nY_n. Show that for any real x, $F_n(x)$ converges to $F(x)$, where F is the c.d.f. of an exponential distribution with mean θ.

(b) Find $\lim_{n \to \infty} \mathrm{P}(n[Y_n] = k)$ for $k = 0, 1, 2, \ldots$, where $[x]$ denotes the largest integer less than or equal to x.

7. Suppose an SRSWOR of size n has been drawn from a population labelled $1, 2, \ldots, N$, where the population size N is unknown.

(a) Find the maximum likelihood estimator $\widehat{N}$ of N.

(b) Find the probability mass function of $\widehat{N}$.

(c) Show that $\frac{n+1}{n}\widehat{N} - 1$ is an unbiased estimator of N.

8. Suppose $\{(x_i, y_i, z_i) : i = 1, 2, \ldots, n\}$ is a set of trivariate observations on three variables: X, Y, and Z, where $z_i = 0$ for $i = 1, 2, \ldots, n-1$ and $z_n = 1$. Suppose the least squares linear regression equation of Y on X based on the first $n-1$ observations is

$$y = \widehat{\alpha}_0 + \widehat{\alpha}_1 x$$

and the least squares linear regression equation of Y on X and Z based on all n observations is

$$y = \widehat{\beta}_0 + \widehat{\beta}_1 x + \widehat{\beta}_2 z.$$

Show that $\widehat{\alpha}_1 = \widehat{\beta}_1$.

9. Let Z be a random variable with probability density function

$$f(z) = \frac{1}{2} e^{-|z-\mu|}, z \in \mathbb{R}$$

with parameter $\mu \in \mathbb{R}$. Suppose we observe $X = \max(0, Z)$.

(a) Find the constant c such that the test that "rejects when $X > c$" has size 0.05 for the null hypothesis $H_0 : \mu = 0$.

(b) Find the power of this test against the alternative hypothesis $H_1 : \mu = 2$.