PSA
2019

1. Let $A = ((a_{ij}))$ be an $m \times n$ matrix with all non-zero real entries. Let B be obtained from A by replacing a_{11} by 0 and keeping all other entries unchanged. If r is the rank of A, then what is the set of possible values for the rank of B?

   (A) $\{r\}$ (B) $\{r-1, r, r+1\}$ (C) $\{r, r+1\}$ (D) $\{r-1, r\}$

2. What is the period of the function $g(x) = |\cos x| + |\sin x|$?

   (A) π (B) $\pi/2$ (C) 2π (D) $\pi/4$

3. If $3(\cos 100^\circ + i \sin 100^\circ)(\cos 110^\circ + i \sin 110^\circ) = x + iy$, where x and y are real numbers, then

   (A) $x = -\frac{3\sqrt{3}}{2}, \quad y = -\frac{3}{2}$.
   (B) $x = \frac{3\sqrt{3}}{2}, \quad y = \frac{3}{2}$.
   (C) $x = \frac{3\sqrt{3}}{2}, \quad y = -\frac{3}{2}$.
   (D) $x = -\frac{3\sqrt{3}}{2}, \quad y = \frac{3}{2}$.

ROUGH WORK

4. What is the number of 6 digit positive integers in which the sum of the digits is at least 52?

(A) 66 (B) 24 (C) 28 (D) 120

5. Let the sum

$$3 + 33 + 333 + \cdots + \underbrace{33\ldots3}_{200 \text{ times}}$$

be $\ldots zyx$ in the decimal system, i.e., x is the unit's digit, y the ten's digit, and so on. What is z?

(A) 0 (B) 9 (C) 7 (D) 3

6. How many times does the digit '2' appear in the set of integers $\{1, 2, ..., 1000\}$?

(A) 590 (B) 600 (C) 300 (D) 299

ROUGH WORK

7. Let t be a real number. Then the rank of $\begin{bmatrix} 0 & 1 & t \\ 2 & t & -1 \\ 2 & 2 & 0 \end{bmatrix}$ equals

(A) 2 if $t = -1$, and 3 if $t \neq -1$.

(B) 2 if $t = 1$, and 3 if $t \neq 1$.

(C) 2 if $t = \pm 1$, and 3 if $|t| \neq 1$.

(D) 3 for all t.

8. The number $\binom{200}{100}/4^{100}$ lies in

(A) $[\frac{3}{4}, 1)$ (B) $(0, \frac{1}{2})$ (C) $[1, \infty)$ (D) $[\frac{1}{2}, \frac{3}{4})$

9. Let $P(x) = x^4 + 4x^3 - 8x^2 - 1$. Which of the following is **false**?

(A) $P(x)$ has a real root in $(-4, 1)$

(B) $P(x)$ has a real root < -4

(C) $P(x)$ has a real root > 1

(D) $P(x)$ has at least two real roots.

ROUGH WORK

10. Two friends, one from Kolkata and one from Delhi, start driving towards each other at the same time. It is given that the distance between Kolkata and Delhi is 1455 km. One of them drives at a constant speed of 80 kmph (km per hour), while the other drives at a speed of 50 kmph during the first hour, 55 kmph during the second hour, 60 kmph during the third hour, and so on (i.e., his speeds over successive hours are in an arithmetic progression). How long will it take for them to meet each other?

   (A) 8 hrs 56 mins
   (B) 8 hrs 46 mins
   (C) 10 hrs 26 mins
   (D) 9 hrs 36 mins

11. How many positive divisors of $2^5\,5^3\,11^4$ are perfect squares?

   (A) 60 (B) 18 (C) 120 (D) 4

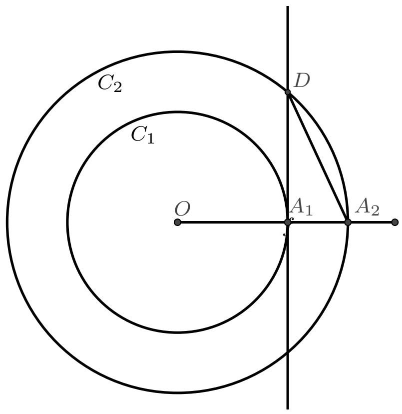
12. What is the set of numbers x in $(0, 2\pi)$ such that $\log\log(\sin x + \cos x)$ is well-defined?

   (A) $[\frac{\pi}{8}, \frac{3\pi}{8}]$ (B) $(0, \frac{\pi}{2})$ (C) $(0, \frac{\pi}{4}]$ (D) $(0, \pi) \cup (\frac{3\pi}{2}, 2\pi)$

ROUGH WORK

13. Let C_1 and C_2 be concentric circles with centre at O and radii r_1 and r_2 respectively. The line OA_2 intersects C_1 at A_1. The line A_1D is tangent to C_1 at A_1. What is the length of the line segment A_2D?

(A) $\sqrt{2r_1(r_2 - r_1)}$ (B) $\sqrt{r_2^2 - r_1^2}$ (C) r_2 (D) $\sqrt{2r_2(r_2 - r_1)}$



14. The reflection of the point $(1, 2)$ with respect to the line $x + 2y = 15$ is

(A) $(3, 6)$. (B) $(6, 3)$. (C) $(10, 5)$. (D) $(5, 10)$.

15. How many solutions does the equation $\cos^2 x + 3\sin x \cos x + 1 = 0$ have for $x \in [0, 2\pi)$?

(A) 1 (B) 3 (C) 4 (D) 2

ROUGH WORK

16. The functions $f, g : [0,1] \to [0,1]$ are given by $f(x) = \frac{1}{2}x(x+1)$ and $g(x) = \frac{1}{2}x^2(x+1)$. What is the area enclosed between the graphs of f^{-1} and g^{-1}?

(A) 1/8 (B) 1/4 (C) 5/12 (D) 7/24

17. If $f(a) = 2, f'(a) = 1, g(a) = -1$ and $g'(a) = 2$, then what is

$$\lim_{x \to a} \frac{g(x)f(a) - f(x)g(a)}{x - a} \quad ?$$

(A) 5 (B) 3 (C) -3 (D) -5

18. Draw one observation N at random from the set $\{1, 2, \ldots, 100\}$. What is the probability that the last digit of N^2 is 1?

(A) 1/20 (B) 1/50 (C) 1/10 (D) 1/5

ROUGH WORK

19. Let X be the number of tosses of a fair coin required to get the first head. If $Y \mid X = n$ is distributed as Binomial$(n, \frac{1}{2})$, then what is $\mathrm{P}(Y = 1)$?

(A) 4/9 (B) 1/4 (C) 1/3 (D) 5/9

20. Suppose X is distributed uniformly on $(-1, 1)$. For $i = 0, 1, 2, 3$, let $p_i = \mathrm{P}\left(X^2 \in (\frac{i}{4}, \frac{i+1}{4})\right)$. For which value of i is p_i the largest?

(A) 3 (B) 1 (C) 0 (D) 2

21. In a simulation experiment, two independent observations X_1 and X_2 are generated from the Poisson distribution with mean 1. The experiment is said to be successful if $X_1 + X_2$ is odd. What is the expected value of the number of experiments required to obtain the first success?

(A) $2(1 + e^{-2})$
(B) $2/(1 - e^{-2})$
(C) $2/(1 - e^{-4})$
(D) $2(1 + e^{-4})$

ROUGH WORK

22. A shopkeeper has 12 bulbs of which 3 are defective. She sells the bulbs by selecting them at random one at a time. What is the probability that the seventh bulb sold is the last defective one?

(A) 3/44 (B) 9/44 (C) 13/44 (D) 7/44

23. The chances of Smitha getting admitted to colleges A and B are 60% and 40% respectively. Assume that colleges admit students independently of each other. If Smitha is told that she has been admitted to at least one college, what is the probability that she got admitted to college A?

(A) 3/5 (B) 15/19 (C) 10/13 (D) 5/7

24. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from an exponential distribution with mean λ. If $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are, respectively, the maximum likelihood estimators of the mean and the median of the underlying distribution, then

(A) $\hat{\lambda}_1 < \hat{\lambda}_2$.

(B) $\hat{\lambda}_1 = \hat{\lambda}_2$.

(C) $\hat{\lambda}_1 < \hat{\lambda}_2$ and $\hat{\lambda}_1 > \hat{\lambda}_2$ are both possible.

(D) $\hat{\lambda}_1 > \hat{\lambda}_2$.

ROUGH WORK

25. Suppose $Y, X_1, X_2, \ldots, X_n$ are i.i.d. $N(\mu, 1)$ random variables, and $I_n = (\bar{X}_n - a_n, \bar{X}_n + a_n)$ is a 95% confidence interval for μ. Then $\mathrm{P}(Y \in I_n)$

(A) converges to 1 as $n \to \infty$.

(B) is greater than 0.95 for all $n \geq 1$.

(C) is less than 0.95 for all $n \geq 1$.

(D) equals 0.95 for all $n \geq 1$.

26. Suppose $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ is an i.i.d. sample from $N_2(0, 0, 1, 1, \rho)$ where $|\rho| \leq 1$. Let $(i_1, \ldots, i_n)$ be a random permutation of $\{1, 2, \ldots, n\}$. Define $T_n = \frac{1}{n} \sum_{j=1}^{n} X_j Y_{i_j}$. What is $\mathrm{E}(T_n)$?

(A) $1/n$ (B) 0 (C) ρ/n (D) ρ

27. Suppose X is a $N(\mu, \sigma^2)$ random variable, and $Y = \Phi(X)$, where Φ is the cumulative distribution function of a standard normal random variable. What is $\mathrm{E}(Y)$?

(A) $\Phi(\mu/\sqrt{2 + \sigma^2})$

(B) $\Phi(\mu/\sqrt{1 + \sigma^2})$

(C) $\Phi(\mu/\sigma)$

(D) $\Phi(\mu/\sqrt{4 + \sigma^2})$

ROUGH WORK

28. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f_\lambda(x) = \begin{cases} (\lambda+1)\, x^\lambda & 0 \le x \le 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > -1$. What is the maximum likelihood estimator of λ?

(A) $1 + n/(\sum_{i=1}^{n} \log X_i)$

(B) $-1 - n/(\sum_{i=1}^{n} \log X_i)$

(C) $-1 - \frac{1}{n} \sum_{i=1}^{n} \log X_i$

(D) $1 - n/(\sum_{i=1}^{n} \log X_i)$

29. Suppose the joint distribution of (X_1, X_2) is $N_2(0, 0, 2, 2, -0.5)$. What is the value of $\mathrm{P}(2X_1 + X_2 \le 2\sqrt{2})$? Here Φ denotes the cumulative distribution function of a standard normal random variable.

(A) $\Phi(2/\sqrt{7})$ (B) $\Phi(2/\sqrt{3})$ (C) $\Phi(2/\sqrt{5})$ (D) $\Phi(1)$

30. Suppose the joint probability density function of (X, Y) is

$$f(x) = \begin{cases} e^{-x} & 0 \le y \le x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

What is $\mathrm{E}(X)$?

(A) 2 (B) 1 (C) 6 (D) 1/2

ROUGH WORK