1. How many distinct straight lines can one form that are given by an equation $ax + by = 0$, where a and b are numbers from the set $\{0, 1, 2, 3, 4, 5, 6, 7\}$?

   (A) 63.

   (B) 57.

   (C) 37.

   (D) 49.

2. Which of the following sequences is not monotone?

   (A) $a_n = n^2 - 3n, \quad n \geq 1.$

   (B) $a_n = 3n^2 - n, \quad n \geq 1.$

   (C) $a_n = \log(\frac{3}{4})^n, \quad n \geq 1.$

   (D) $a_n = 7n - n^2, \quad n \geq 1.$

3. Let $A = (1, -1)$, $B = (-2, 0)$, $C = (1, 2)$ and D be the vertices of a parallelogram in the X-Y plane listed clockwise. Then the point D is:

   (A) $(4, 1)$.

   (B) $(-2, -3)$.

   (C) $(3, 0)$.

   (D) $(-2, 1)$.

ROUGH WORK

4. Let f be the function on the set of real numbers defined by

$$f(x) = \begin{vmatrix} e^x & \sin x & \cos x \\ e^{2x} & \sin 2x & \cos 2x \\ e^{3x} & \sin 3x & \cos 3x \end{vmatrix}.$$

Then $f'(0)$ is

(A) 1. (B) 0. (C) -1. (D) 2.

5. The period of the function f given by $f(x) = \sin(\frac{x}{3} - \frac{\pi}{2})$, where x is real, is:

(A) 6π. (B) $9\frac{\pi}{2}$. (C) 2π. (D) $13\frac{\pi}{2}$.

6. Let

$$X = \frac{1}{2}\begin{pmatrix} \sqrt{3} & 1 \\ -1 & \sqrt{3} \end{pmatrix} \text{ and } A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}.$$

Let X^t denote the transpose of the matrix X. If $B = XAX^t$, then $X^tB^{105}X$ equals

(A) I. (B) A. (C) X^tAX. (D) B.

ROUGH WORK

7. Suppose

$$a_j(n) = \sum_{k=1}^{n} k^j, \qquad j = 0, 1, 2, 3, \quad n = 1, 2, \ldots.$$

Then $\lim_{n\to\infty} \frac{a_1(n)a_2(n)}{a_0(n)a_3(n)}$ equals

(A) 0. (B) $\frac{1}{3}$. (C) $\frac{2}{3}$. (D) 1.

8. Let S be the set of all 3×3 real matrices $A = ((a_{ij}))$ such that the matrix $((a_{ij}^3))$ has rank one. Let R be the set

$$R = \{\text{rank}(A) : A \in S\}.$$

Then R is equal to

(A) $\{1\}$. (B) $\{1, 2\}$. (C) $\{1, 3\}$. (D) $\{1, 2, 3\}$.

9. Consider the circles

$$S_1 : x^2 + y^2 + 4x + 2y - 4 = 0,$$
$$S_2 : x^2 + y^2 - 4x - 4y + 4 = 0.$$

Then the number of common tangents of S_1 and S_2 is

(A) 1. (B) 2. (C) 3. (D) 4.

ROUGH WORK

10. Let A be the 3×3 matrix

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

Then the determinant of the matrix $A^{17} + A^{10} - I$ is

(A) 1. (B) 2. (C) -1. (D) 0.

11. Let m and n be nonzero integers. Define

$$A_{m,n} = \left\{ x \in \mathbb{R} : n^2x^3 + 2020x^2 + mx = 0 \right\}.$$

Then the number of pairs (m, n) for which $A_{m,n}$ has exactly two points is

(A) 0. (B) 10. (C) 16. (D) ∞.

12. The set of all real solutions of the inequality $2|x| > |x - 1|$ is

(A) $\left\{x : x < -\frac{1}{3}\right\} \cup \{x : x > 1\}$.

(B) $\left\{x : x > \frac{1}{3}\right\}$.

(C) $\{x : -1 < x < 1\}$.

(D) $\{x : x < -1\} \cup \left\{x : x > \frac{1}{3}\right\}$.

ROUGH WORK

13. Let S and T be two non-empty sets and $f : S \to T$ be a function such that for all subsets A and B of S, we have $f(A \cap B) = f(A) \cap f(B)$. Then

(A) There exists an $A \subseteq S$ such that $f^{-1}(f(A)) \neq A$.

(B) There exist disjoint subsets A and B of S such that $f(A) \cap f(B) \neq \phi$.

(C) f is injective.

(D) None of the above statements is true.

14. Let S be the set of all 3×3 matrices A such that among the 9 entries of A, there are exactly three 0's, exactly three 1's and exactly three 2's. The number of matrices in S that have trace divisible by 3 is

(A) 580. (B) 600. (C) 150. (D) 120.

15. Let $x_1, \ldots, x_n \in \mathbb{R}$ be distinct reals. Define the set

$$A = \Big\{ \big(f_1(t), \ldots, f_n(t)\big) : t \in \mathbb{R} \Big\},$$

where for $1 \leq k \leq n$

$$f_k(t) = \begin{cases} 1 & \text{if } x_k \leq t, \\ 0 & \text{otherwise.} \end{cases}$$

Then A contains

(A) exactly n distinct elements.

(B) exactly $(n+1)$ distinct elements.

(C) exactly 2^n distinct elements.

(D) infinitely many distinct elements.

ROUGH WORK

16. Let X, Y be independent and identically distributed random variables with

$$P(X = k) = (0.75)(0.25)^k, \qquad k = 0, 1, 2, \ldots.$$

Then $P(X > Y)$ is equal to

(A) 0.5 (B) 0.4 (C) 0.2 (D) 0.6

17. Let $X_1, X_2, \ldots$ be independent and identically distributed $\mathrm{Bin}(n, p)$ random variables. Define

$$S_k = X_1^2 + X_2^2 + \cdots + X_k^2, \qquad k = 1, 2, \ldots,$$

and let $\varepsilon > 0$. Then as $k \to \infty$,

(A) $P\left(\left|\frac{S_k}{k} - n^2p^2\right| > \varepsilon\right) \to 0$ for all $p \in (0, 1)$.

(B) $P\left(\left|\frac{S_k}{k} - np^2\right| > \varepsilon\right) \to 0$ for all $p \in (0, 1)$.

(C) $P\left(\left|\frac{S_k}{k} - \frac{n}{4}\right| > \varepsilon\right) \to 0$ when $p = 1/2$.

(D) $P\left(\left|\frac{S_k}{k} - \frac{n(n+1)}{4}\right| > \varepsilon\right) \to 0$ when $p = 1/2$.

18. Let X, Y have joint probability density function

$$f(x, y) = \begin{cases} \frac{1}{2} y e^{-xy} & \text{if } x > 0,\ \ 2 \le y \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbb{E}(XY)$ is given by

(A) 3. (B) 1. (C) 2. (D) 1.5

ROUGH WORK

19. Let X and Y be random variables with $V(X) = 9$ and $V(Y) = 4$. Then which of the following is true for non negative reals a and b?

(A) $V(aX + bY) = 0$ if and only if $a = b = 0$.

(B) $V(aX + bY)$ lies between $(3a - 2b)^2$ and $(3a + 2b)^2$.

(C) $V(aX + bY)$ lies between $3a$ and $2b$.

(D) $V(aX + bY)$ lies between $(3a)^2$ and $(2b)^2$.

20. The moment generating function of a continuous random variable X is given by $M(t) = e^{t(t+1)}$, $-\infty < t < \infty$. Let Φ denote the cumulative distribution function of a standard normal random variable. Then $P(X \geq 0)$ is

(A) $\Phi(0)$. (B) $\Phi(-\frac{1}{\sqrt{2}})$. (C) $\Phi(\frac{1}{\sqrt{2}})$. (D) $\Phi(\frac{1}{2})$.

21. A multiple choice test uses the following scoring procedure to discourage students from guessing. For each correct response the score is 5, for each incorrect response the score is 0 and for no response the score is 2. If there are 4 choices for each question, what is the minimum number of incorrect choices that the student must eliminate before it is advantageous to guess among the rest than leave the question unanswered?

(A) 0. (B) 1. (C) 2. (D) 3.

ROUGH WORK

22. Each of 100 laboratory rats has available plain water and a mixture of water and caffeine in their cages. After 24 hours, two measurements were recorded for each rat: the amount of caffeine consumed (X) and the blood pressure (Y). Based on the data, the correlation coefficient between X and Y was found to be 0.428. Which of the following conclusions is justified on the basis of the study?

   (A) The correlation between caffeine consumed and blood pressure in the population of rats is 0.428.

   (B) If rats stop drinking the mixture of water and caffeine, their blood presure will go down.

   (C) Rats with low blood pressure do not consume the mixture of water and caffeine as much as rats with high blood pressure.

   (D) About 18% of variation in blood pressure can be explained by a linear relationship between blood pressure and caffeine consumed.

23. Suppose X is distributed as $\text{Bin}(3, \frac{2}{3})$ and the conditional distribution of Y given X is $N(X, X^2 + 1)$. Then the variance of Y is

   (A) $\frac{19}{3}$. (B) $\frac{17}{3}$. (C) $\frac{13}{3}$. (D) $\frac{11}{3}$.

24. Suppose X and Y are independent observations from $N(0, 1)$. For $t > 0$, define $g(t) = P(|X - Y| > t \mid (X + Y) > t)$. Then

   (A) g is a strictly increasing function.

   (B) g is a strictly decreasing function.

   (C) g is a constant function.

   (D) g is not a monotonic function.

ROUGH WORK

25. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed $N(1, \sigma^2)$ random variables. Let $\widehat{\sigma^2}$ denote the maximum likelihood estimator of σ^2. Then

(A) $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$ and it is biased.

(B) $\widehat{\sigma^2} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ and it is unbiased.

(C) $\widehat{\sigma^2} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - 1)^2$ and it is biased.

(D) $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n}(X_i - 1)^2$ and it is unbiased.

26. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed exponential random variables with mean λ. Then maximum likelihood estimator of the median of the distribution is

(A) Sample median. (B) $(\log 2)\bar{X}$. (C) $\log 2/\bar{X}$. (D) $\bar{X}/\log 2$.

27. Suppose X is a discrete random variable taking values $-1, 0, 1$, each with probability $\frac{1}{3}$. Let $Y = |X|$. Which one of the following statements is correct?

(A) Correlation between X and Y is ± 1.

(B) X and Y are positively correlated.

(C) X and Y are negatively correlated.

(D) X and Y are uncorrelated.

ROUGH WORK

28. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with probability density function

$$f(x) = \frac{1}{2\lambda} e^{-|\frac{x-1}{\lambda}|}, \qquad -\infty < x < \infty,$$

where $\lambda > 0$ is an unknown parameter. Which one of the following is a form of the most powerful test of its size for testing $H_0 : \lambda = 2$ vs $H_1 : \lambda = 1$?

(A) Reject H_0 if $\sum_{i=1}^{n}(X_i - 1)^2 > C$.

(B) Reject H_0 if $\sum_{i=1}^{n}|X_i - 1| > C$.

(C) Reject H_0 if $\sum_{i=1}^{n}(X_i - 1)^2 < C$.

(D) Reject H_0 if $\sum_{i=1}^{n}|X_i - 1| < C$.

29. Suppose X and Y are independent and identically distributed continuous random variables and are independent of Z. Assume $P(Z = \pm 1) = 1/2$. Then which of the following statements is always true?

(A) $X + Y$ has the same distribution as $Z(X + Y)$.

(B) $X - Y$ has the same distribution as $Z(X - Y)$.

(C) X/Y has the same distribution as $Z(X/Y)$.

(D) XY has the same distribution as $Z(XY)$.

30. Suppose T_1 and T_2 are two different real valued statistics each of which is sufficient for an unknown parameter θ. Then, which of the following is always true?

(A) $T_1 + T_2$ is sufficient for θ.

(B) $\max\{T_1, T_2\}$ is sufficient for θ.

(C) (T_1, T_2) is sufficient for θ.

(D) $\min\{T_1, T_2\}$ is sufficient for θ.

ROUGH WORK

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