PART : MATHEMATICS

1. If $f(x) = \begin{cases} -2 & ; -2 \le x < 0 \\ x-2 & ; 0 \le x \le 2 \end{cases}$ and $h(x) = f(|x|) + |f(x)|$ then $\int_0^k h(x)dx$ is equal to (where $0 < k \le 2$)

(1) k (2) $\frac{k}{2}$ (3) $\frac{2k}{3}$ (4) 0

Ans. (4)

Sol. $f(|x|) = \begin{cases} -x-2 & -2 \le x < 0 \\ x-2 & 0 \le x \le 2 \end{cases}$

And $|f(x)| = \begin{cases} 2 & -2 \le x < 0 \\ -(x-2) & 0 \le x \le 2 \end{cases}$

So $h(x) = \begin{cases} (-x-2)+(2) = -x & -2 \le x < 0 \\ x-2-x+2 = 0 & 0 \le x \le 2 \end{cases}$

Hence $\int_0^k h(x)dx = \int_0^k 0dx = 0$

2. Let ABC be a triangle . If P_1, P_2, P_3, P_4, P_5 are five points on side AB, $P_6, P_7, \ldots\ldots P_{11}$, are 6 points on side BC and $P_{12}, P_{13}, \ldots\ldots P_{18}$, are 7 points on side AC then find the number of triangles formed by these 18 points taking as vertices

Ans. 751

Sol. Number of triangles = $^{18}C_3 - (^5C_3 + {^6C_3} + {^7C_3}) = \frac{18 \times 17 \times 16}{6} - (10 + 20 + 35) = 816 - 65 = 751$

3. Let y(x) be a curve given by differential equation $\frac{dy}{dx} - y = 1 + 4\sin x$. If y(0) = 1 then value of $y(\pi/2)$ is equal to

(1) $-3 + 4e^{\pi/2}$ (2) $3 - 4e^{\pi/2}$ (3) $3 + 4e^{-\pi/2}$ (4) $3 + 4e^{\pi/2}$

Ans. (1)

Sol. $\frac{dy}{dx} - y = 1 + 4\sin x$ linear differential equation

I.F. $= e^{\int -dx} = e^{-x}$

So solution of linear differential equation is

$ye^{-x} = \int e^{-x}(1 + 4\sin x)dx$

$ye^{-x} = -e^{-x} + 4\int e^{-x}\sin x\, dx$

$ye^{-x} = -e^{-x} + 4\frac{e^{-x}}{2}[-\sin x - \cos x] + c$

$y = -1 - 2(\sin x + \cos x) + c\,e^x$

Now $y(0) = 1 \Rightarrow 1 = -1-2(0+1)+c \Rightarrow 2 = -2 + c \Rightarrow c = 4 \Rightarrow y = -1 - 2(\sin x + \cos x) + 4e^x$

$\Rightarrow y\left(\frac{\pi}{2}\right) = -1 - 2(1+0) + 4e^{\frac{\pi}{2}} \Rightarrow y\left(\frac{\pi}{2}\right) = -3 + 4e^{\frac{\pi}{2}}$

4. Let there are 3 bags A, B and C. Bag A contains 5 black balls and 7 red balls, bag B contains 5 red and 7 black balls and bag C contains 7 red and 7 black balls. A balls is drawn and found to be black, then the probability that it is drawn from bag A, is

(1) $\frac{7}{18}$ (2) $\frac{5}{42}$ (3) $\frac{5}{18}$ (4) $\frac{1}{3}$

Ans. (3)

Sol. prob. that ball drawn from bag is black

$$= \frac{1}{3}\times\frac{5}{12}+\frac{1}{3}\times\frac{7}{12}+\frac{1}{3}\times\frac{7}{14}$$

Prob. that black ball drawn from bag A is

$$= \frac{\frac{1}{3}\times\frac{5}{12}}{\frac{1}{3}\times\frac{5}{12}+\frac{1}{3}\times\frac{7}{12}+\frac{1}{3}\times\frac{7}{14}}$$

$$= \frac{5}{5+7+6}$$

$$= \frac{5}{18}$$

5. The number of rational terms in the expansion of $\left(2^{\frac{1}{5}}+3^{\frac{1}{3}}\right)^{15}$

Ans. **(2)**

Sol. $T_{r+1} = {}^{15}C_r 2^{\frac{15-r}{5}} \times 3^{\frac{r}{3}}$

for rational numbers $\frac{15-r}{5}$ & $\frac{r}{3}$

should be integers

so $r = 0, 5, 10, 15$ for $\frac{15-r}{5}$

& $r = 0, 3, 6, 9, 12, 15$ for $\frac{r}{3}$

common values 0, 15

So only 2 terms are rational

6. 2 and 6 are roots of the equation $ax^2 + bx + 1 = 0$ then the quadratic equation whose roots are $\frac{1}{a+3b}$ and $\frac{1}{a+6b}$ is

(1) $1081x^2 + 840x - 144 = 0$ (2) $1081x^2 + 840x + 144 = 0$

(3) $1081x^2 - 840x + 144 = 0$ (4) $1081x^2 - 840x - 144 = 0$

Ans. **(2)**

Sol. Sum of roots $-\frac{b}{a} = 2 + 6 = 8 \Rightarrow b = -8a$

Product of roots $\frac{1}{a} = 12 \Rightarrow a = \frac{1}{12}$ $b = -\frac{2}{3}$

Hence new roots $\frac{1}{a+3b} = \frac{1}{\frac{1}{12}-2} = \frac{12}{-23}$

and $\frac{1}{a+6b} = \frac{1}{\frac{1}{12}-4} = -\frac{12}{47}$

so other quadratic equation is

$$x^2 + \left(\frac{12}{23}+\frac{12}{47}\right)x + \frac{144}{23\times 47} = 0$$

$\Rightarrow 1081\,x^2 + 840x + 144 = 0$

7. Let $f(x)=\begin{cases}\dfrac{1-\cos 2x}{x^2} & x<0\\ \alpha & x=0\\ \beta\left(\dfrac{\sqrt{1-\cos x}}{x}\right) & x>0\end{cases}$

If f(x) is continuous at x = 0 then value of $4|\alpha^2+\beta^2|$ is

(1) 48 (2) 36 (3) 28 (4) 16

Ans. (1)

Sol. f(x) is continuous at x = 0

$\Rightarrow \quad f(0)=\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+} f(x) \quad \Rightarrow \quad \alpha=\lim_{x\to 0}\frac{1-\cos 2x}{x^2}=\lim_{x\to 0}\frac{\beta\sqrt{1-\cos x}}{x}$

$\Rightarrow \quad \alpha=\lim_{x\to 0^-}\frac{2\sin^2 x}{x^2}=\lim_{x\to 0}\frac{\sqrt{2}\,\beta\sin\frac{x}{2}}{x} \Rightarrow \quad \alpha=2=\frac{\beta}{\sqrt{2}}$

$\Rightarrow \quad \alpha=2\ \&\ \beta=2\sqrt{2}$ So $4|\alpha^2+\beta^2|=4|4+8|=48$

8. One points of intersection of curves $y = 1 + 3x - 2x^2$ and $y=\frac{1}{x}$ is $\left(\frac{1}{2},2\right)$ and area of region bounded by both curves is $\frac{1}{24}\left(\ell\sqrt{5}+m\right)-n\ell n\left(1+\sqrt{5}\right)$ then value of $(\ell+m+n)$ is

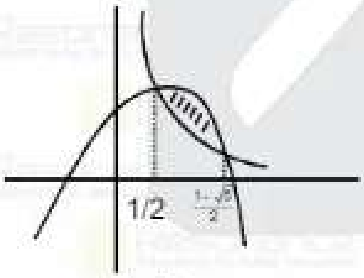
Ans. (30)

Sol. Solving curves $y = 1+3x-2x^2$ and $y=\frac{1}{x}$

$2x^3-3x^2-x+1=0$

$(2x-1)(x^2-x-1)=0$

$x=\frac{1}{2}, \quad x=\frac{1\pm\sqrt{5}}{2}$



$\text{Area}=\int_{\frac{1}{2}}^{\frac{\sqrt{5}+1}{2}}\left(1+3x-2x^2-\frac{1}{x}\right)dx$

$=\left(x+\frac{3x^2}{2}-\frac{2x^3}{3}-\ell nx\right)\Big|_{1/2}^{\frac{\sqrt{5}+1}{2}}$

$=\frac{\sqrt{5}+1}{2}+\frac{3}{8}\left(\sqrt{5}+1\right)^2-\frac{1}{12}\left(\sqrt{5}+1\right)^3-\ell n\left(\frac{\sqrt{5}+1}{2}\right)-\left(\frac{1}{2}+\frac{3}{8}-\frac{1}{12}-\ell n\frac{1}{2}\right)$

$= \frac{1}{24}\left[12(\sqrt{5}+1)+9(\sqrt{5}+1)^2-2(\sqrt{5}+1)^3-12-9+2\right]-\ell n\left(\frac{\sqrt{5}+1}{2}\times 2\right)$

$= \frac{1}{24}\left[12(\sqrt{5}+1)+9(6+2\sqrt{5})-2(5\sqrt{5}+1+3\sqrt{5}(\sqrt{5}+1))-19\right]-\ell n(\sqrt{5}+1)$

$= \frac{1}{24}\left[14\sqrt{5}+15\right]-\ell n(\sqrt{5}+1)$

Now, $\ell = 14$, $m = 15$, $n = 1$

then $\ell + m + n = 30$

9. If α and β are sum and product of non-zero solution of the equation $\bar{z}^2+|z|=0$ where z is a complex number then value of $4(\alpha^2+\beta^2)$ is equal to

(1) 2 (2) 6 (3) 4 (4) 8

Ans. (3)

Sol. $\bar{z}^2+|z|=0$ ....(1)

$z^2+|\bar{z}|=0$ ....(2)

By (1) and (2) as $|z|=|\bar{z}|$

$\Rightarrow \bar{z}^2=z^2$

$\Rightarrow z=\bar{z}$ or $z=-\bar{z}$

$\Rightarrow I(z)=0$ or $R(z)=0$

Case-I Let $I(z)=0 \Rightarrow z=x,\ x\in R-\{0\}$

By (1) $x^2+|x|=0 \Rightarrow x=0$ only rejected

Case-II Let $R(z)=0 \Rightarrow z=iy,\ y\in R-\{0\}$

By (1) $-y^2+|y|=0$

$y^2=|y|$

$y=\pm 1$, as $y\neq 0$

Hence $z=\pm i$

$\Rightarrow$ sum $\alpha = +i-i=0$

And product $\beta=(i)+(-i)=1$

$\Rightarrow 4(\alpha^2+\beta^2)=4$

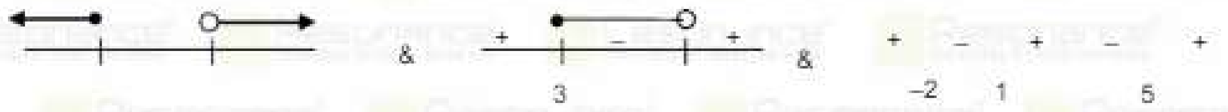
10. If domain of the function $f(x)=\sin^{-1}\left(\frac{3x-22}{2x-19}\right)+\ell n\left(\frac{3x^2+8x+5}{x^2-3x-10}\right)$ is $(\alpha,\beta]$ then value of $(3\alpha+10\beta)$ is

Ans. (97)

Sol. $-1\le\frac{3x-22}{2x-19}\le 1$ & $\frac{3x^2-8x+5}{x^2-3x-10}>0$

$\frac{3x-22}{2x-19}\ge -1$ and $\frac{3x-22}{2x-19}\le 1$ & $\frac{(x-1)(3x-5)}{(x-5)(x+2)}>0$

$\Rightarrow \frac{5x-41}{2x-19}\ge 0$ and $\frac{x-3}{2x-19}\le 0$ and $\frac{(x-1)(3x-5)}{(x-5)(x+2)}>0$



$\Rightarrow x\in\left(5,\frac{41}{5}\right]$

Hence $3\alpha+10\beta=15+82=97$

11. The value of $\lim\limits_{x\to4}\dfrac{(5+x)^{\frac{1}{3}}-(1+2x)^{\frac{1}{3}}}{(5+x)^{\frac{1}{2}}-(1+2x)^{\frac{1}{2}}}$ is

(1) $2\left(9^{\frac{1}{3}}\right)$ (2) $\dfrac{2}{9^{\frac{1}{3}}}$ (3) $\dfrac{2\left(9^{\frac{1}{3}}\right)}{9}$ (4) $\dfrac{2}{3^{\frac{1}{3}}}$

Ans. (3)

Sol. $\lim\limits_{x\to4}\dfrac{(5+x)^{\frac{1}{3}}-(1+2x)^{\frac{1}{3}}}{(5+x)^{\frac{1}{2}}-(1+2x)^{\frac{1}{2}}}=\dfrac{9^{\frac{1}{3}}-9^{\frac{1}{3}}}{9^{\frac{1}{2}}-9^{\frac{1}{2}}}=\dfrac{0}{0}$ form

$$=\lim_{x\to4}\frac{[(5+x)-(1+2x)]}{\left\{(5+x)^{\frac{2}{3}}+(5+x)^{\frac{1}{3}}(1+2x)^{\frac{1}{3}}+(1+2x)^{\frac{2}{3}}\right\}}\cdot\frac{(5+x)^{\frac{1}{2}}+(1+2x)^{\frac{1}{2}}}{(5+x)-(1+2x)}$$

$$=\lim_{x\to4}\frac{(4-x)\left\{(5+x)^{\frac{1}{2}}+(1+2x)^{\frac{1}{2}}\right\}}{\left\{(5+x)^{\frac{2}{3}}+(5+x)^{\frac{1}{3}}(1+2x)^{\frac{1}{3}}+(1+2x)^{\frac{2}{3}}\right\}(4-x)}$$

$$=\frac{3+3}{9^{\frac{2}{3}}+9^{\frac{1}{3}}.9^{\frac{1}{3}}+9^{\frac{2}{3}}}=\frac{6}{3.9^{\frac{2}{3}}}=\frac{2}{9^{\frac{2}{3}}}=\frac{2\left(9^{\frac{1}{3}}\right)}{9}$$

12. If the function $f(x)=\begin{cases}\dfrac{1}{|x|} & |x|\geq2\\ ax^2+2b & |x|<2\end{cases}$ differentiable on R then 48(a+b) is equal to

(1) 19 (2) 16 (3) 15 (4) 20

Ans. (3)

Sol. Clearly f(x) must be continuous on R

$\Rightarrow f(2)=\lim\limits_{x\to2}f(x)$ & $f(-2)=\lim\limits_{x\to-2}f(x)$

$\Rightarrow \dfrac{1}{2}=4a+2b$ & $\dfrac{1}{2}=4a+2b$

$\Rightarrow 8a+4b=1$ ....(1)

Also differentiable so Lf′(2) = Rf′(2) & Lf′(–2) = Rf′(–2)

$\Rightarrow 4a=-\dfrac{1}{4}$ & $\dfrac{1}{4}=-4a$

So $a=\dfrac{-1}{16}$ & $b=3/8$

Hence $48(a+b)=48\left(-\dfrac{1}{16}+\dfrac{3}{8}\right)=-3+18=15$

13. Let $\alpha, \beta \in R$. If the mean and the variance of 6 observation, –3, 4, 7, –6, α, β be 2 and 23 respectively, then mean deviation about the mean of the 6 observation is

(1) $\frac{11}{3}$ (2) $\frac{16}{3}$ (3) $\frac{13}{3}$ (4) $\frac{14}{3}$

Ans. (3)

Sol. Mean = $2 = \frac{-3+4+7+(-6)+\alpha+\beta}{6}$

$12 = 2 + \alpha + \beta$

$\alpha + \beta = 10$ _____(1)

again variance

$23 = \frac{9+16+49+36+\alpha^2+\beta^2}{6} - (2)^2$

$162 = \alpha^2 + \beta^2 + 110$

$\alpha^2 + \beta^2 = 52$

$\alpha = 6, \quad \beta = 4$ or $\alpha = 4, \quad \beta = 6$

Now mean deviation about mean.

$= \frac{|-3-2|+|4-2|+|7-2|+|-6-2|+|6-2|+|4-2|}{6}$

$= \frac{26}{6} = \frac{13}{3}$

14. A square is inscribed in the circle $x^2 + y^2 - 10x - 6y + 30 = 0$ such that one side of the square is parallel to $y = x + 3$. If (x_i, y_i) are the vertices of the square then $\sum (x_i^2 + y_i^2)$ is equal to

(1) 148 (2) 156 (3) 152 (4) 160

Ans. (3)

Sol. diagonal of square makes 45° angle with side

Let slope of diagonal = m

Then $\tan 45° = \left|\frac{m-1}{1+m}\right|$

So m = not defined and m = 0

So diagonals x = 5 and y = 3 are passing through centre

Now solving diagonals with circle we get vertices of square

x = 5 and $x^2 + y^2 - 10x - 6y + 30 = 0$

$y^2 - 6y + 5 = 0, \ y = 1, 5$

two vertices (5, 1) & (5, 5)

and with y = 3 and

$x^2 + y^2 - 10x - 6y + 30 = 0$

$x^2 - 10x + 21 = 0$

x = 3, 7

= 25 + 1+25+25+9+9+49+9 = 152

15. Let $f(x) = x^5 + 2e^{x/4} \ \forall x \in R$. Consider a function (gof) (x) = x, $\forall x \in R$ then the value of 8 g' (2) is

(1) 2 (2) 4 (3) 8 (4) 16

Ans. (1)

Sol. $g(f(x)) = x$

$\Rightarrow g'(f(x)) \cdot f'(x) = 1$

$\Rightarrow g'(f(x)) = \frac{1}{f'(x)}$

Now to find g'(2) let at x = a, f(x) = 2

$\Rightarrow a^5 + 2e^{a/4} = 2$

$\Rightarrow a = 0$

$\Rightarrow \quad f(0) = 2$

Hence $g'(f(0)) = \frac{1}{f'(0)}$

$\Rightarrow \quad g'(2) = \frac{1}{\left(5x^4 + \frac{1}{2}e^{x/4}\right)_{(x=0)}} = \frac{1}{0+\frac{1}{2}} = 2$

16. Let $f(x) = \frac{2x^2 - 3x + 9}{2x^2 + 3x + 4}$, $x \in R$, if maximum and minimum value of f(x) is m and n respectively then m+n is equal to

(1) $\frac{60}{23}$ (2) $\frac{122}{23}$ (3) $\frac{120}{23}$ (4) $\frac{5}{23}$

Ans. **(2)**

Sol. $y = \frac{2x^2 - 3x + 9}{2x^2 + 3x + 4}$

$x^2(2y-2) + x(3y+3) + 4y - 9 = 0$

as $x \in R$ so $D \geq 0$

$(3y+3)^2 - 4(2y-2)(4y-9) \geq 0$

$9y^2 + 18y + 9 - 8(y-1)(4y-9) \geq 0$

$9y^2 + 18y + 9 - 8(4y^2 - 13y + 9) \geq 0$

$+ 23y^2 - 122y + 63 \leq 0$

as $m \leq y \leq n$

by comparing $y^2 - (m+n)y + mn \leq 0$

So $m + n = \frac{122}{23}$

17. $\int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{1 + \sin x.\cos x}.dx =$

(1) $\frac{\pi}{6\sqrt{3}} - \ell n\sqrt{\frac{2}{3}}$ (2) $\frac{\pi}{6\sqrt{3}} + \ell n\sqrt{\frac{2}{3}}$ (3) $\frac{\pi}{6\sqrt{3}} + \ell n\sqrt{\frac{3}{2}}$ (4) $\frac{\pi}{6\sqrt{2}} + \ell n\sqrt{\frac{3}{2}}$

Ans. **(2)**

Sol. divide by $\cos^4 x$ in numerator & denominator

$\int_0^{\frac{\pi}{4}} \frac{\tan^2 x \sec^2 x\, dx}{(\tan^2 x + 1 + \tan x)(1 + \tan^2 x)}$

put $t = \tan x$

$dt = \sec^2 x dx$

$$= \frac{1}{2}\int_0^1 \frac{2t^2 dt}{(t^2+t+1)(t^2+1)} = \frac{1}{2}\int_0^1 \frac{t^2+1}{(t^2+1)(t^2+t+1)} + \frac{t^2-1}{(t^2+1)(t^2+t+1)}dt$$

$$= \frac{1}{2}\int_0^1 \frac{dt}{t^2+t+1} + \frac{1}{2}\int \frac{t^2-1}{(t^2+1)(t^2+t+1)}dt = \frac{1}{2}\int_0^1 \frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2}\int_0^1 \frac{1-\frac{1}{t^2}}{\left(t+\frac{1}{t}\right)\left(t+1+\frac{1}{t}\right)}dt$$

$$= \left(\frac{1}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left|\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right|\right)_0^1 + I_1$$

$$\frac{1}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right) + I_1 = \frac{1}{\sqrt{3}}.\frac{\pi}{6} + I_1$$

Now $I_1 = \frac{1}{2}\int \frac{\left(1-\frac{1}{t^2}\right)dt}{\left(t+\frac{1}{t}\right)\left(t+1+\frac{1}{t}\right)}$

put $t+\frac{1}{t} = u$

$$\left(1-\frac{1}{t^2}\right) dt = du$$

$$= \frac{1}{2}\int_{\infty}^{2} \frac{du}{u(u+1)}$$

$$= \frac{1}{2}\int_{\infty}^{2} \left(\frac{1}{u} - \frac{1}{u+1}\right) du$$

$$= \frac{1}{2}(\ell n|u| - \ell n|u+1|)_{\infty}^{2}$$

$$= \frac{1}{2}\left(\ell n\left|\frac{u}{u+1}\right|\right)_{\infty}^{2}$$

$$= \frac{1}{2}\left(\ell n\left(\frac{2}{3}\right) - 0\right)$$

$$= \ell n\sqrt{\frac{2}{3}}$$

So final $= \frac{1}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) + \ell n\sqrt{\frac{2}{3}}$

$$= \frac{\pi}{6\sqrt{3}} + \ell n\sqrt{\frac{2}{3}}$$

18. 2, p, q are in G.P. (where $p \neq q$) and in an A.P., 2 is third term, p is 7^{th} term and q is 8^{th} term, then $64(p^2 + q^2)$

(1) 17 (2) 18 (3) 19 (4) 20

Ans. **(1)**

Sol. Let $p = 2r$ and $q = 2r^2$

Now Let AP be a, a + d, a + 2d, ...

So $a + 2d = 2$ ____(1)

$a + 6d = 2r$ _____(2)

$a + 7d = 2r^2$ _____(3)

Now from (1) & (2)

$4d = 2(r - 1)$

$2d = r - 1$ _______(4)

and from (2) & (3)

$d = 2r(r - 1)$ _____(5)

from (5) & (4)

$2(2r)(r - 1) = r - 1$

$(r - 1)(4r - 1) = 0$

$r = \frac{1}{4}, r = 1$

so $r = \frac{1}{4}$ as $r \neq 1$

so $p = \frac{1}{2}$ & $q = \frac{1}{8} \Rightarrow 64\left(\frac{1}{4} + \frac{1}{64}\right) = 17$