MATHEMATICS

SECTION - A

Multiple Choice Questions: This section contains 20 multiple choice questions. Each question has 4 choices (1), (2), (3) and (4), out of which **ONLY ONE** is correct.

Choose the correct answer :

1. If $A_r = \begin{vmatrix} r & 1 & \frac{n^2}{2}+\alpha \\ 2r & 2 & n^2-\beta \\ 3r-1 & 3 & \frac{n}{2}(3n-1) \end{vmatrix}$ then the value of $2A_{10} - A_8$ is equal to

   (1) $4\alpha + 2\beta$ (2) $2n$

   (3) 0 (4) $2\alpha + 4\beta$

Answer (1)

Sol. $A_r = \begin{vmatrix} r & 1 & \frac{n^2}{2}+\alpha \\ 2r & 2 & n^2-\beta \\ 3r-2 & 3 & \frac{3n^2}{2}-\frac{n}{2} \end{vmatrix}$

$$A_r = \begin{vmatrix} 0 & 1 & \frac{n^2}{2}+\alpha \\ 0 & 2 & n^2-\beta \\ -2 & 3 & \frac{3n^2}{2}-\frac{n}{2} \end{vmatrix}$$

$$A_r = -2\left[(n^2-\beta)-2\left(\frac{n^2}{2}+\alpha\right)\right] = -2\left[n^2-\beta-n^2-2\alpha\right]$$

$\Rightarrow \quad A_r = (\beta+2\alpha)\cdot 2$

$\Rightarrow \quad 2A_{10} - A_8 = 4(\beta + 2\alpha) - 2(\beta + 2\alpha)$

$= 2(\beta + 2\alpha) = 4\alpha + 2\beta$

2. The value of $\int_0^{\pi/4} \frac{\cos^2 x \sin^2 x}{(\cos^3 x + \sin^3 x)^2} dx$ is equal to

   (1) $\frac{1}{6}$ (2) $\frac{1}{3}$

   (3) $\frac{1}{2}$ (4) 1

Answer (1)

Sol. $\int_0^{\pi/4} \frac{\tan^2 x \sec^2 x}{(1+\tan^3 x)^2} dx$

Put $1 + \tan^3 x = t$

$\Rightarrow 3\tan^2 x \sec^2 x dx = dt$

$\frac{1}{3}\int_1^2 \frac{dt}{t^2}$

$\frac{1}{3}\left[\frac{t^{-1}}{-1}\right]_1^2$

$= \frac{1}{6}$

3. Let α, β be the distinct roots of the quadratic equation $x^2 - (t^2 - 5t + 6)x + 1 = 0$, $a_n = \alpha^n + \beta^n$, then the minimum value of $\frac{a_{2023} + a_{2025}}{a_{2024}}$ is

   (1) $-\frac{1}{4}$ (2) $\frac{1}{4}$

   (3) $-\frac{1}{2}$ (4) $\frac{1}{2}$

Answer (1)

Sol. Given equation

$x^2 - (t^2 - 5t + 6)x + 1 = 0$

$\therefore a_{2025} - (t^2 - 5t + 6)\, a_{2024} + a_{2023} = 0$

$$\Rightarrow \frac{a_{2025}+a_{2023}}{a_{2024}} = \left(t^2 - 5t + 6\right)$$

$$= t^2 - 5t + \frac{25}{4} + 6 - \frac{25}{4}$$

$$= \underbrace{\left(t + \frac{5}{2}\right)^2}_{\geq 0} + \left(-\frac{1}{4}\right)$$

$\therefore$ Minimum value is $-\frac{1}{4}$

4. The shortest distance between two lines

$\frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}$ and $\frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$ is

(1) $4\sqrt{3}$

(2) $8\sqrt{3}$

(3) $6\sqrt{3}$

(4) $2\sqrt{3}$

Answer (1)

Sol. $L_1 : \frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}$

$L_2 : \frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$

Shortest distance $= \dfrac{\begin{vmatrix} -4 & 16 & 0 \\ 2 & -7 & 5 \\ 2 & 1 & -3 \end{vmatrix}}{\sqrt{256+256+256}}$

$= \frac{192}{16\sqrt{3}}$

$= \frac{12}{\sqrt{3}}$

$= 4\sqrt{3}$

5. R is defined on set $X = \{1, 2, \ldots, 20\}$

$R_1 = \{(x, y) : 2x - 3y = 2\}$

$R_2 = \{(x, y) : 5x - 4y = 0\}$

If M, N represent the number of elements to be added to make R_1 and R_2 symmetric respectively. Then value of $M + N$ equals to

(1) 10 (2) 8

(3) 12 (4) 11

Answer (1)

Sol. R is defined on $X = \{1, 2, 3, \ldots., 20\}$

$R_1 = \{(x, y) : 2x - 3y = 2\}$

$R_2 = \{(x, y) : 5x + 4y = 0\}$

As $2x - 3y = 2$

So $2x$ and $3y$ both has to be even or odd simultaneously and $2x$ can't be odd so $2x$ and $3y$ both will be even.

So $R_1 = \{(4, 3), (7, 4), (10, 6), (13, 8), (16, 10), (19, 12)\}$

for symmetric we need to add 6 elements here as (3, 4), (4, 7) and (6, 10), (8, 13), (10, 16), (12, 19)

So $M = 6$

For R_2 $5x - 4y = 0$

So $5x$ and $4y$ has to be equal. $4y$ is always even so $5x$ will also be even

$R_2 = \{(4, 5), (8, 10), (12, 15), (16, 20)\}$

So 4 elements (5, 4), (10, 8) (15, 12), (20, 16) need to be added

$N = 4$

$M + N = 10$

6. If $\frac{dy}{dx} + \frac{y}{x\ell nx} = \frac{1}{x^2\ell nx}$ and $y(e^{-1}) = 0$. Then $y(e)$ equals to

(1) $\frac{e^2+1}{e}$ (2) $\frac{e^2-1}{e}$

(3) $\frac{e^2+2}{e}$ (4) $\frac{e^2-2}{e}$

Answer (2)

Sol. I.F = $e^{\int \frac{1}{x\ell nx}dx}$

Put $\ell nx = t$

$$\frac{1}{x}dx = dt.$$

I.F $= e^{\int \frac{1}{t}dt}$

$= e^{\ell nt}$

$= t$

$= \ell nx$.

$$\therefore \quad y.\ell nx = \int \ell nx \cdot \left(\frac{1}{x^2 \ell nx}\right)dx$$

$$y.\ell nx = \int \frac{1}{x^2}dx$$

$$y\ell nx = \frac{-1}{x} + c \qquad \ldots(2)$$

Given y $(e^{-1}) = 0$

$0 = -e + c$

$c = e$

$\therefore$ From (2)

We get,

$$y\,\ell nx = \frac{-1}{x} + e$$

$\therefore$ Put $x = e$

$$y = \frac{-1}{e} + e$$

$$y = \frac{e^2 - 1}{e}$$

Option (b) is correct

7. Interval in which x^x is strictly increasing is

(1) $(0, \infty)$ (2) $\left(0, \frac{1}{e}\right]$

(3) $\left[\frac{1}{C^2}, \infty\right)$ (4) $\left[\frac{1}{e}, \infty\right)$

Answer (4)

Sol. Let $f(x) = x^x$

$\Rightarrow$ $f'(x) = x^x(1 + \ln x)$

For strictly increasing, $f'(x) > 0$

$\Rightarrow$ $1 + \ln x > 0$

$\Rightarrow$ $x > \frac{1}{e}$

8. If $\frac{dy}{dx} + \frac{y}{1+x^2} = e^{-\tan^{-1}x}$, then which of the following is true

(1) $ye^{\tan^{-1}x} = \frac{x^2}{2} + c$ (2) $ye^{\tan^{-1}x} = \frac{1}{x} + c$

(3) $ye^{\tan^{-1}x} = x + c$ (4) $ye^{\tan^{-1}x} = -x + c$

Answer (3)

Sol. $\frac{dy}{dx} + \frac{y}{1+x^2} = e^{-\tan^{-1}x}$

I.F $= e^{\int \frac{1}{1+x^2}dx}$

$= e^{\tan^{-1}x}$

Now,

$$y\,e^{\tan^{-1}x} = \int e^{\tan^{-1}x} \,.\, e^{-\tan^{-1}x}dx$$

$$\Rightarrow ye^{\tan^{-1}x} = \int dx$$

$$\Rightarrow ye^{\tan^{-1}x} = x + c$$

9. A company produces automobiles. It has two factories factory 'A' produces 60% of the automobiles and rest is produced by factory B. 80% of the automobiles produced by 'A' is upto the standards and 90% of the automobiles by 'B' is upto the standards. If an automobile is selected we found it as standard, the probability it came from B is P. Then 126 P equals to

(1) 54 (2) 52

(3) 48 (4) 27

Answer (1)

Sol. P (standard automobile from A) $= \frac{6}{10} \times \frac{8}{10} = \frac{12}{25}$

P (standard automobile from B) $= \frac{4}{10} \times \frac{9}{10} = \frac{9}{25}$

Required probability $= \dfrac{\frac{9}{25}}{\frac{12}{25} + \frac{9}{25}}$

$P = \frac{9}{21} = \frac{3}{7}$

$126\,P = 54$

10. If σ = 4(standard deviation) and $\bar{x}$ = 10 (mean) of 20 observations. One term was taken wrong *i.e*, instead of 12 they have taken 8. Then the correct standard deviation is

(1) 1.8 (2) $\sqrt{3.96}$

(3) $\sqrt{3.84}$ (4) 1.93

Answer (2)

Sol. Mean = $\bar{x} = 10$

$\sigma = 4$, $n = 20$

Take observations as $x_1, x_2, \ldots x_{20}$

$$\frac{x_1 + x_2 + \ldots + x_{20}}{20} = 200$$

$x_1 + x_2 + \ldots + x_{20} = 200$

One term, say x_{20} is wrongly written as 8

So, $x_1 + x_2 + \ldots + x_{19} = 200 - 8$

$x_1 + x_2 + \ldots + x_{19} = 192$

Now $(x_{20})_{new} = 12$

So, $\bar{x}_{new} = \frac{192+12}{20} = \frac{204}{20} = \frac{102}{10} = 10.2$

$$\sigma^2 = \frac{\sum x_1^2}{n} - (\bar{x})^2 = 4$$

$$\frac{x_1^2 + x_2^2 + \ldots + x_{20}^2}{20} = 4 + 100$$

$$\Rightarrow \frac{x_1^2 + x_2^2 + \ldots + x_{19}^2 + 64}{20} = 104$$

$\Rightarrow x_1^2 + x_2^2 + \ldots + x_{19}^2 + 64 = 2080$

$\Rightarrow x_1^2 + x_2^2 + \ldots + x_{19}^2 = 2016$

Now $\sigma_{new}^2 = \frac{2016+144}{20} - (10.2)^2$

$\sigma_{new}^2 = 108 - 104.04$

$\sigma = \sqrt{3.96}$

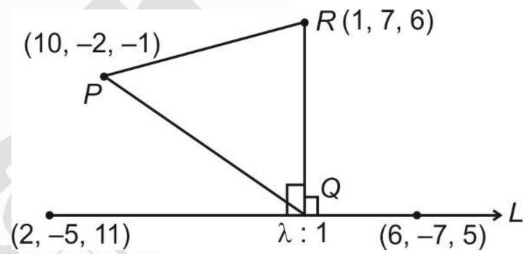
11. A point $P(10, -2, -1)$ and $R(1, 7, 6)$, if Q is a foot of perpendicular from R to the line joining points $(2, -5, 11)$ and $(6, -7, 5)$. Then $(PQ)^2$ is

(1) $\frac{3509}{14}$ (2) $\frac{3600}{7}$

(3) $\frac{3509}{7}$ (4) $\frac{3409}{7}$

Answer (1)

Sol.



$$\left(\frac{6\lambda+2}{\lambda+1}, \frac{-7\lambda-5}{\lambda+1}, \frac{5\lambda+11}{\lambda+1}\right)$$

$\vec{L} = 4\hat{i} - 2\hat{j} - 6\hat{k}$

$$\overrightarrow{RQ} \perp \vec{L} = 4\left(\frac{6\lambda+2}{\lambda+1} - 1\right) - 2\left(\frac{-7\lambda-5}{\lambda+1} - 7\right) - 6\left(\frac{5\lambda+11}{\lambda+1} - 6\right)$$

$$\overrightarrow{RQ} \cdot \vec{L} = \frac{4(5\lambda+1) - 2(-14\lambda-12) - 6(-\lambda+5)}{(\lambda+1)} = 0$$

$$= \frac{\lambda(20+28+6) + (4+24-30)}{\lambda+1} = 0$$

$= \lambda(54) - 2 = 0 \Rightarrow \lambda = \frac{1}{27}$

$$Q = \left(\frac{\frac{6}{27}+2}{\frac{1}{27}+1}, \frac{\frac{-7}{27}-5}{\frac{1}{27}+1}, \frac{\frac{5}{27}+11}{\frac{1}{27}+1}\right)$$

$$\equiv \left(\frac{60}{28}, \frac{-142}{28}, \frac{302}{28}\right)$$

$$= R' = \left(\frac{15}{7}, \frac{71}{14}, \frac{151}{14}\right)$$

$$\Rightarrow \quad (QP) = \sqrt{\left(10-\frac{15}{7}\right)^2 + \left(-2-\frac{71}{14}\right)^2 + \left(-1-\frac{151}{14}\right)^2}$$

$$= \sqrt{\frac{3509}{14}}$$

12.

13.

14.

15.

16.

17.

18.

19.

20.

SECTION - B

Numerical Value Type Questions: This section contains 10 Numerical based questions. The answer to each question should be rounded-off to the nearest integer.

21. If $\cot^{-1}3 + \cot^{-1}4 + \cot^{-1}5 + \cot^{-1} n = \frac{\pi}{4}$, value of n is

Answer (47)

Sol. $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{7}{12}}{1-\frac{1}{12}}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{7}{11}\right) + \tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{23}{24}\right) + \tan^{-1}\left(\frac{1}{n}\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{\frac{23}{24}+\frac{1}{n}}{1-\frac{23}{24\,.\,n}} = 1$$

$$\Rightarrow \frac{23n+24}{24n-23} = 1$$

$\Rightarrow n = 47$

22. If $A = [100, 700]$, how many numbers are in A which are neither multiple of 3 nor 4?

Answer (300)

Sol. Let A_1 denotes multiple of 3,

B_1 denotes multiple of 4

$\therefore$ We need to find $\bar{A}_1 \cap \bar{B}_1$

$\because \quad |\bar{A}_1 \cap \bar{B}_1| = |\overline{A_1 \cup B_1}|$

$|A_1 \cup B_1| = n(A_1) + n(B_1) - n(A_1 \cap B_1)$

$= 200 + 151 - 50$

$= 301$

$\therefore \quad |\overline{A_1 \cup B_1}| = 601 - 301 = 300$

23. If the second, third, fourth terms of the expression $(x + y)^n$ is 135, 30, $\frac{10}{3}$ respectively, then the value of $9(n^3 + x^2 + y)$ is

Answer (1153)

Sol. $(x + y)^n = {}^nC_0\, x^0 y^n + {}^nC_1 x^1 y^{n-1} + {}^nC_2 x^2 y^{n-2}$

$$+\,{}^nC_3x^3y^{1-3}+\ldots$$

First term $\Rightarrow {}^nC_1x^1y^{n-1} = 135 \ldots(1)$

Second term $\Rightarrow {}^nC_2x^2y^{n-2} = 30 \ldots(2)$

Third term $\Rightarrow {}^nC_3x^3y^{n-3} = \dfrac{10}{3} \ldots(3)$

Dividing equation (1) by (2)

$$\frac{2n}{n(n-1)} \times \frac{1}{xy} = \frac{135}{30}$$

$$\frac{2}{(n-1)xy} = \frac{9}{2}$$

$$(n-1)xy = \frac{4}{9} \qquad \ldots(4)$$

Dividing equation (2) by (3)

$$\frac{3}{(n-2)xy} = 9$$

$$(n-2)xy = \frac{1}{3} \qquad \ldots(5)$$

Dividing equation (4) by (5)

$$\frac{n-1}{n-2} = \frac{4}{3}$$

$\Rightarrow\ 3n - 3 = 4n - 8$

$\Rightarrow\ n = 5$

Now equation (1) becomes

$$5 \times y^4 = 135 \Rightarrow x = \frac{27}{y^4} \qquad \ldots(6)$$

And equation (2) becomes

$10x^2y^3 = 30$

$\Rightarrow\ x^2y^3 = 3 \qquad \ldots(7)$

From equation (6) and (7)

$$\Rightarrow\ \frac{3^6}{y^8}xy^3 = 3$$

$\Rightarrow\ y^5 = 3^5$

$\Rightarrow\ y = 3$

$\Rightarrow\ x = \dfrac{1}{3}$

$\Rightarrow\ 9(n^3 + x^2 + y)$

$$= 9\left(5^3 + \frac{1}{3^2} + 3\right)$$

$$= 9\left(\frac{1153}{9}\right)$$

$= 1153$

24. Find the number of triangles formed whose vertices are also a vertices of regular octagon but the side of triangle is not common with sides of octagon is

Answer (56)

Sol.

$\Rightarrow$ Total triangles = ${}^8C_3$

Triangles with all 3 sides common with octagon = 0

Triangle with 2 sides common with octagon $\Rightarrow$ Choose vertex

$= \left({}^8C_1\right) \Rightarrow$ 8 triangles

Triangle with exactly 1 side common

$= \left({}^8C_1\right)$ ways to choose a side, remaining vertex can be selected in 4 vertices $= {}^4C_1$ ways $= {}^8C_1 \cdot {}^4C_1$

$= 32$

$\Rightarrow 56 - (8 + 0 + 32) = 16$ triangles

25. The number of real solutions of $x\,|x + 5| + 2|x + 7| - 2 = 0$ is equal to

Answer (3)

Sol. (I) $x \geq -5$

$x^2 + 5x + 2x + 14 - 2 = 0$

$x^2 + 7x + 12 = 0$

$x = 3, 4$

(II) $-7 < x < -5$

$-x^2 - 5x + 2x + 14 - 2 = 0$

$-x^2 - 3x + 12 = 0$

$x^2 + 3x - 12 = 0$

$x = 2.275, -5.275$, here $x \neq 2.275$

So, $x = -5.275$

(III) $x \leq -7$

$-x^2 - 5x - 2x - 14 - 2 = 0$

$-x^2 - 7x - 16 = 0$

$x^2 + 7x + 16 = 0$

$D < 0 \rightarrow$ no real roots

Only 3 solutions possible

26. The number of points of discontinuities of $f(x) = 2x^2 + [x^2] - [x]$ where $[\cdot]$ is greatest integer function and $x \in [-1, 2]$ is equals to

Answer (4)

Sol. $f(x) = 2x^2 + [x^2] - [x], x \in [-1, 2]$

This function may be discontinuous at $x = -1, 0, 1, \sqrt{2}, \sqrt{3}$ and 2.

For continuity at $x = -1$

$f(-1) = 4$

$$\lim_{h\to 0} f(-1+h) = \lim_{h\to 0} 2(-1+h)^2 + [(-1+h)^2] - [-1+h]$$

$= 3$

$\therefore f(x)$ is discontinuous at $x = -1$

For continuity at $x = 0$, $f(0) = 0$

$f(0^-) = 1$

$\therefore f(x)$ is discontinuous at $x = 0$

Continuity at $x = 1$

$$\text{L.H.L} = \lim_{h\to 0} f(1-h) = 2(1-h)^2 + [(1-h)^2] - [1-h]$$

$= 2$

$f(1) = 2.1^2 + 1 - 1 = 2$

$$\text{R.H.L} = \lim_{h\to 0} f(1+h) = 2(1+h)^2 + [(1+h)^2] - [1+h]$$

$= 2$

$\therefore f(x)$ is continuous at $x = 1$

For continuity, at $x = \sqrt{2}$ and $\sqrt{3}$ similarly it is discontinuous

For continuity at $x = 2$

$f(2) = 2.2^2 + [2^2] - [2] = 10$

$$\text{L.H.L} = \lim_{h\to 0} 2(2-h)^2 + [(2-h)^2] - [2-h]$$

$= 8 + 3 - 1$

$= 10$

$\therefore f(x)$ is continuous at $x = 2$

$f(x)$ is discontinuous at $x = -1, 0, \sqrt{2}$ and $\sqrt{3}$.

No. of points of discontinuity = 4

27.

28.

29.

30.

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