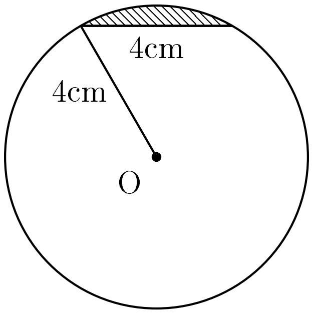
Common Section

1. If O is the center of the circle, what is the area of the shaded portion in square cm?



(a) $4\pi - 4\sqrt{2}$

(b) $\frac{7}{2}\pi - 4\sqrt{2}$

(c) $\frac{7}{2}\pi - 4\sqrt{3}$

(d) $\frac{8}{3}\pi - 4\sqrt{3}$ ✓

(e) $\frac{8}{3}\pi - 4\sqrt{2}$

2. Let σ be a uniform random permutation of $\{1, \ldots, 100\}$. What is the probability that $\sigma(1) < \sigma(2) < \sigma(3)$ (i.e., what is the probability that the first three elements are in increasing order)?

In the options below, $n! = 1 \times 2 \times \ldots \times n$.

(a) $\dfrac{3}{100!}$

(b) $\dfrac{3!}{100!}$

(c) $\dfrac{6}{100}$

(d) $\dfrac{1}{6}$ ✓

(e) $\dfrac{1}{3}$

3. There is a 100cm long ruler that has 11 ants on positions 0cm, 10cm, 20cm, 30cm, ..., 100cm. The ant at the 0cm mark is facing towards the 100cm mark, and the ant at the 100cm is facing towards the 0cm mark; all other ants is either facing towards 0cm or towards the 100cm mark.

All ants start moving in the direction they are facing at a speed of 1cm per second. Whenever an ant collides with another ant, the two instantly reverse their direction and continue moving in the other direction. All ants continue moving until they fall off the ruler when they move past one of the two ends (at 0cm or 100cm marks).

How long will it take for all ants to fall off the ruler?

(a) At most 50 seconds, but cannot be determined without knowing the directions of all ants.

(b) 50 seconds.

(c) At least 50 seconds and at most 100 seconds, but cannot be determined without knowing the directions of all ants.

(d) 100 seconds. ✓

(e) More than 100 seconds, but cannot be determined without knowing the directions of all ants.

4. Let $z_1, z_2, z_3, \ldots, z_{2023}$ be a permutation of the numbers $1, 2, 3, \ldots, 2023$. Which of the following is true about the product $\prod_{i=1}^{2023}(z_i - i)$?

Note: The *parity* of an integer n just denotes whether n is even or odd. Formally, the parity of n is said to be odd if n is odd, and even if n is even.

(a) The above product is always even. ✓

(b) The above product is always odd.

(c) The parity of the above product always changes if we swap the values of any two variables among $z_1, z_2, \ldots, z_{2023}$.

(d) There always exist two variables among $z_1, z_2, \ldots, z_{2023}$ such that the parity of the above product changes if we swap their values, but there may also exist two variables among the $z_1, z_2, \ldots, z_{2023}$ such that swapping their values does not change the parity of the above product.

(e) None of the above statements is true.

5. Let $p(x)$ be a polynomial with real coefficients which satisfies $p(r) = p(-r)$ for every real number r. Let $n \geq 5$ be a positive integer. Suppose that $p(i) = i$ for all $1 \leq i \leq n$. What is the maximum possible value of the absolute value of the coefficient of x^5 in $p(x)$?

(a) 0 ✓

(b) 5

(c) 10

(d) n

(e) $n + 1$

6. For each month in the year (i.e., January, February, March,...), let us assume the probability that a person's birthday falls in that particular month is exactly 1/12, and let us assume that this is independent for different persons.

What is the smallest value of the natural number n such that, among n independently chosen persons the probability that there is a pair of them born in the same month is at least 1/2?

(a) 3

(b) 4

(c) 5 ✓

(d) 6

(e) 7

7. Let $S := \{(a,b) \mid 0 \leq a \leq 1 \ , \ 0 \leq b \leq 1\}$, a unit square, in $\mathbb{R}^2$. Let $B := \{(x,y) \mid x^2 + y^2 \leq 1\}$, a unit disk, in $\mathbb{R}^2$. Define the set $S + B$ as follows:

$$S + B := \{(u,v) \mid \exists (a,b) \in S \ , (x,y) \in B \text{ such that } u = a + x \ , \ v = b + y\} \, .$$

What is the area of $S + B$?

(a) $\pi + 4$

(b) $\pi + 5$ ✓

(c) $\pi + 3$

(d) $\pi + 2$

(e) None of the above.

8. A palindrome is a string that reads the same in reverse (e.g. ABBA or KAYAK or MALAYALAM).

How many strings of length 5 using the letters from $\{A, B, C, D, E\}$ have no palindromic substring of length at least 2?

(a) 243

(b) 405

(c) 540 ✓

(d) 675

(e) 1280

9. Compute $\int_{16}^{\infty} \frac{1}{x} \cdot \frac{1}{\sqrt{\sqrt{x}-1}} \, dx$.

(a) 0

(b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$

(d) $\frac{2\pi}{3}$ ✓

(e) 2π

10. Let M be a 3×3 matrix over the real numbers such that $M^T M = \mathbf{I}$. Consider the following statements.

(i) There exists a non-zero vector $x \in \mathbb{R}^3$ such that $Mx = \mathbf{0}$.

(ii) There exist two non-zero vectors $x, y \in \mathbb{R}^3$ such that the angle between them and that between Mx and My are different.

(iii) There exists a non-zero vector x such that its length is different from that of Mx.

Which of the above statements is/are true?

(a) Only (i).

(b) Only (ii).
(c) Only (iii).
(d) All three statements.
(e) None of the three statements. ✓

11. Consider the following sequence of polynomials with real coefficients.

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 2x \\ P_{n+1}(x) &= 2xP_n(x) - P_{n-1}(x), \text{ for all natural numbers } n \geq 1. \end{aligned}$$

What is the dimension of the linear span of the set

$$\{P_0(x), P_1(x), P_2(x), P_3(x), P_4(x)\}$$

in the vector space of polynomials in variable x with real coefficients?

(a) 1
(b) 2
(c) 3
(d) 4
(e) 5 ✓

12. A subset S of the rational numbers is said to be "nice" if for every infinite sequence of $x_1, x_2, \ldots$ of elements from S, there is always two indices $i < j$ such that $x_i \leq x_j$. Consider the following statements.

(i) The set of natural numbers $\mathbb{N}$ is "nice".
(ii) The set of integers $\mathbb{Z}$ is "nice".
(iii) The set of positive rational numbers is "nice".

Which of the above statements is/are true?

(a) Only (i). ✓
(b) Only (i) and (ii).
(c) Only (i) and (iii).
(d) All three statements are true.
(e) None of the three statements is true.

13. Let $n \geq 100$ be a positive integer. Let $X_1, X_2, \ldots, X_n$ be independent random variables, each taking values in the set $\{0, 1\}$ such that $\Pr[X_i = 1] = \frac{2}{3}$ for each $1 \leq i \leq n$. Define $S := \sum_{i=1}^{n} X_i$, and let $p(x)$ be a polynomial such that for each non-negative integer j, the coefficient of x^j in $p(x)$ is $\Pr[S = j]$. What is $p'(1)$?

Note: $p'(x)$ denotes the polynomial obtained by differentiating p with respect to x.

(a) 0

(b) n

(c) $\frac{2n}{3}$ ✓

(d) $\frac{4n^2+2n}{9}$

(e) $\frac{4n^2-4n}{9}$

14. Let A and B be two $n \times n$ invertible matrices with real entries such that every row in A sums to 1 and every row in B sums to 2. Consider the following three statements:

(i) Every row in the matrix AB sums to 2.

(ii) Every row in the matrix $A^{-1}B$ sums to 2.

(iii) Every row in the matrix $A^{-1}B^{-1}$ sums to $\frac{1}{2}$.

Which of the above statements is/are true?

(a) None of the statements (i), (ii), (iii) is true.

(b) All the three statements (i), (ii), and (iii) are true. ✓

(c) Statement (i) is true but not necessarily statements (ii) or (iii).

(d) Statements (i) and (ii) are true but not necessarily statement (iii).

(e) Statements (i) and (iii) are true but not necessarily statement (ii).

15. Suppose Michelle gives Asna and Badri two different numbers from $\mathbb{N} = \{1, 2, 3, \ldots\}$. It is commonly known to both Asna and Badri that they each know only their own number and that it is different from the other one. The following conversation ensues.

Michelle: I privately gave each of you a different natural number. Which of you has the larger of the two numbers?

Asna: I don't know.

Badri: I don't know either.

Asna: Oh, then I know who has the larger number.

Badri: In that case, I know both numbers. What numbers were Asna and Badri respectively given?

(a) Asna was given 2, Badri was given 3. ✓

(b) Asna was given 3, Badri was given 2.

(c) Asna was given 3, Badri was given 4.

(d) Asna was given 4, Badri was given 3.

(e) None of the above.

CS Section

1. Consider the following three functions defined for all positive integers $n \geq 0$.

$$f(n) = |\sin(n) + n|,$$
$$g(n) = n,$$
$$h(n) = |\sin(n)|.$$

Which of the following statements about these functions is / are true?

(i) $f(n) = O(g(n))$

(ii) $g(n) = O(f(n))$

(iii) $h(n) = O(g(n))$

(a) Only (i) is true.

(b) Only (ii) is true.

(c) Only (i) and (ii) are true.

(d) Only (ii) and (iii) are true.

(e) All of (i), (ii), and (iii) are true. ✓

2. Let S be the set of all 4-digit numbers created using just the digits $1, 2, 3, 4, 5$ such that no two successive digits are the same. If the numbers in S are arranged in ascending order, what is the 100th number in this sequence?

(a) 2135

(b) 2324

(c) 2315

(d) 2352

(e) 2415 ✓

3. For any positive integer N, let $p(N)$ be the probability that a uniformly random number $a \in \{1, \ldots, N\}$ has an odd number of factors (including 1 and the number itself).

Which of the following is true about the function $p : \mathbb{N} \to \mathbb{R}$?

(a) $\lim_{N\to\infty} p(N) = \frac{1}{2}$.

(b) $p(N) = \Theta\left(\frac{\log N}{N}\right)$.

(c) $p(N) = \Theta\left(\frac{1}{N}\right)$.

(d) $p(N) = \Theta\left(\frac{1}{\sqrt{N}}\right)$**.** ✓

(e) $p(N) = \Theta\left(\frac{1}{\log N}\right)$.

4. Consider functions f and g from the set of positive real numbers to itself, recursively defined as follows.

$$\begin{array}{lrcl} \forall n \leq 1 & f(n),\, g(n) & = & 1, \\ \forall n > 1 & f(n) & = & 3f(n/3) + n^2, \\ \forall n > 1 & g(n) & = & n^{2/3} g(n^{1/3}) + n. \end{array}$$

Which of the following options is correct about their asymptotic behaviour?

(a) $f(n) = \Theta(n^2)$ **and** $g(n) = \Theta(n \log\log n)$ ✓

(b) $f(n) = \Theta(n^2)$ and $g(n) = \Theta(n \log n)$

(c) $f(n) = \Theta(n^2 \log n)$ and $g(n) = \Theta(n \log\log\log n)$

(d) $f(n) = \Theta(n^2 \log n)$ and $g(n) = \Theta(n \log\log n)$

(e) $f(n) = \Theta(n^2 \log n)$ and $g(n) = \Theta(n \log n)$

5. For two languages A, B over the alphabet Σ, let the *perfect shuffle* of A and B be the language

$$\left\{ w \ : \ \begin{array}{c} w = a_1 b_1 a_2 b_2 \cdots a_k b_k \text{ where } a_1 a_2 \cdots a_k \in A \text{ and } b_1 b_2 \cdots b_k \in B \\ \text{and } k \in \mathbb{N} \end{array} \right\}.$$

Consider the following statements.

(i) If A and B are regular, then perfect shuffle of A and B is regular.

(ii) If A and B are regular, then perfect shuffle of A and B is context-free.

(iii) If A and B are decidable, then perfect shuffle of A and B is decidable.

Which of above statements is/are true?

(a) All of (i), (ii) and (iii). ✓

(b) Only (i) and (ii).

(c) Only (ii).

(d) Only (ii) and (iii).

(e) None of (i), (ii), (iii) is true.

6. The four nucleotides in DNA are called A, C, G, and T. Consider the following languages over the alphabet {A, C, G, and T}.

$$\begin{aligned} L_1 &= \{(\mathsf{AC})^n\,(\mathsf{GT})^n \mid n \geq 0\} \\ L_2 &= \{\mathsf{A}^n \mathsf{C}^n \mathsf{G}^n \mathsf{T}^n \mid n \geq 0\} \\ L_3 &= \{(\mathsf{AT})^n\,(\mathsf{CG})^m \mid m \neq n \text{ and } m, n \geq 0\} \end{aligned}$$

Which of the above languages is/are context-free?

(a) Only L_1.

(b) Only L_3.

(c) Only L_1 and L_3. ✓

(d) Only L_1 and L_2.

(e) All three of L_1, L_2, L_3.

7. Consider the following algorithm that takes as input a positive integer n.

```
if (n == 1) {
  return "Neither prime nor composite."
}
m = 2
while (m < n) {
  if (m divides n) {
    return "Composite."
  }
  m = m+1
}
return "Prime."
```

If n is a number of the form $n = p^2q^3r^4$ where p, q, r are natural numbers greater than 1, how many times does the while loop in the algorithm run?

In the options below, for any real number m, $\lceil m \rceil$ denotes the least integer greater than or equal to m.

(a) The while loop runs at most $\lceil n^{1/9} \rceil$ times for all natural numbers p, q, r greater than one. ✓

(b) The while loop runs at most $\lceil n^{1/9} \rceil$ times only if p, q, r are all distinct.

(c) The while loop runs at most $\lceil n^{1/9} \rceil$ times only if at least two of p, q, r are distinct.

(d) The while loop runs at most $\lceil n^{1/9} \rceil$ times only if p, q, r are distinct primes.

(e) The while loop runs at most $\lceil n^{1/9} \rceil$ times only if p, q, r are distinct primes or distinct prime powers.

8. In the following pseudocode, assume that for any pair of integers $x \leq y$, the function `random (x , y)` produces an integer uniformly chosen from the set $\{x, x+1, \ldots, y\}$.

```
n = 9
for (i = 1 to n) {
  A[i] = i
}
for (i = 1 to n) {
  r = random (i , n)
  temp = A[i]
  A[i] = A[r]
  A[r] = temp
  print A[i]
}
```

Which of the following statements is TRUE of the output of the code?

(a) It outputs all permutations of 123456789 with equal probability. ✓

(b) It never outputs 123456789.

(c) It outputs all cyclic permutations of 123456789 with equal probability, and does not print any other output.

(d) The output is always 987654321.

(e) The output may not be a permutation of 123456789.

9. Given m vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m$ in $\mathbb{R}^d$, we construct an undirected graph $G = (V, E)$ as follows. Each vector $\vec{x}_i$ is represented by a vertex v_i. We add an edge between vertices v_i and v_j if the corresponding pair of vectors $\vec{x}_i$ and $\vec{x}_j$ are linearly independent.

Which of the following statements regarding the graph G must be TRUE, for any such vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m$? Recall that a clique in a graph is a set of vertices $S \subseteq V$ that have an edge between every pair of vertices in S.

(a) If $m > d$, the graph must be disconnected.

(b) Any clique has size at most d

(c) Any clique has size at most $m/2$

(d) The maximum degree of any vertex in G is at most d

(e) None of the above ✓

10. Arun has a non-empty subset S of the numbers $\{1, 2, 3, \ldots, 1000\}$. Bela wants to find any number x in Arun's set S.

To do this, Arun and Bela decide to play a game which proceeds in rounds. In each round, Bela can give Arun a subset $T \subseteq \{1, 2, 3, \ldots, 1000\}$, and Arun answers with a "YES" if $S \cap T$ is non-empty, and with "NO" otherwise.

In the worst case, how many rounds will Bela need to find out some x in Arun's set S?

(a) 9

(b) 10 ✓

(c) 11

(d) 1023

(e) 1024

11. Let $\mathbb{C}$ denote the set of complex numbers and let k be a positive integer. Given a non-zero univariate polynomial $f(x)$ with coefficients in $\mathbb{C}$ and an $a \in \mathbb{C}$, we say that a is a zero of f of multiplicity k if $f(a) = 0$, $\frac{d^k f}{dx^k}(a) \neq 0$, and for all $i \in \{1, \ldots, k-1\}$, $\frac{d^i f}{dx^i}(a) = 0$.

Which of the following is true for every polynomial f of degree d and every positive integer k?

(a) The number of distinct zeroes in $\mathbb{C}$ of f of multiplicity k is at most d/k, and can be smaller than d/k as well. ✓

(b) The number of distinct zeroes in $\mathbb{C}$ of f of multiplicity k is at least d/k, and can be larger than d/k as well.

(c) The number of distinct zeroes in $\mathbb{C}$ of f of multiplicity k is equal to d/k.

(d) The number of distinct zeroes in $\mathbb{C}$ of f of multiplicity k is at least d, and can be larger than d as well.

(e) The number of distinct zeroes in $\mathbb{C}$ of f of multiplicity k is equal to d.

12. In the n-queens completion problem, the input is an $n \times n$ chess board with queens on some squares, and the goal is to determine if there is a way to place more queens so that the total number of queens is n and no two queens attack each other (two queens are said to attack each other if they are on the same row, or they are on the same column, or they are on the same diagonal).

Consider the following statements:

(i) The n-queens completion problem is decidable.

(ii) The n-queens completion problem is decidable in time $O\left(n^{n^n}\right)$.

(iii) The problem of "checking whether a given program solves the n-queens completion problem" is decidable.

(iv) The problem of "checking whether a given program solves the n-queens completion problem in time $O\left(n^{n^n}\right)$" is decidable.

(v) The problem of "checking whether a given program solves the n-queens completion problem" is decidable in time $O\left(n^{n^n}\right)$.

Which of the above is true?

(a) Only (i) and (ii).

(b) Only (i) and (iii).

(c) Only (i), (ii) and (iv). ✓

(d) Only (iii),(iv) and (v).

(e) Only (i), (iii) and (iv).

13. Suppose we are given a graph $G = (V, E)$ with non-negative edge weights $\{w_e\}_{e \in E}$. Consider the following problems:

P1: Finding a minimum spanning tree of G.

P2: Finding a maximum spanning tree of G.

P3: Finding a cycle of smallest weight in G.

P4: Finding a cycle of largest weight in G.

Note: a cycle consists of distinct vertices.

Assuming $\mathsf{P} \neq \mathsf{NP}$, which of the above problems can be solved in polynomial time?

(a) P1, P2 but not P3, P4.

(b) P1, P3 but not P2, P4.

(c) P1 but not P2,P3, P4.

(d) P1,P2,P3 but not P4. ✓

(e) P1, P4 but not P2, P3.

14. For an undirected graph G, let $\overline{G}$ refer to the *complement* (a graph on the same vertex set as G, with (i, j) as an edge in $\overline{G}$ if and only if it is *not* an edge in G). Consider the following statements.

(i) G has a vertex-cover of size at most k.

(ii) $\overline{G}$ has an independent set of size at least k.

(iii) G has an independent set of size at least $n - k$.

(iv) G has a clique of size at least k.

(v) $\overline{G}$ has a clique of size at least $n - k$.

Which of the following is true for any graph G and its complement graph $\overline{G}$?

(a) (i) is equivalent to (iii) and (iv).

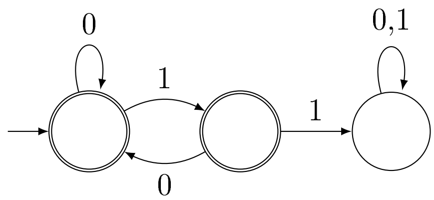
(b) (i) is equivalent to (iii) and (v). ✓

(c) (i) is equivalent to (ii) and (iv).

(d) (i) is equivalent to (ii) and (v)

(e) None of the five statements are equivalent to each other.

15. Consider the following automata:



Let N be the number of 0/1-strings of length exactly 6 accepted by this automata. Which of the following is true about N?

(a) $N \leq 4$.

(b) $4 < N \leq 8$.

(c) $8 < N \leq 16$.

(d) $16 < N \leq 32$**.** ✓

(e) $32 < N \leq 64$.