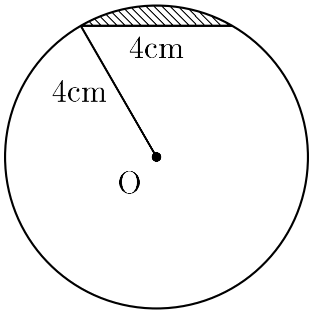
Common Section

1. If O is the center of the circle, what is the area of the shaded portion in square cm?



(a) $4\pi - 4\sqrt{2}$

(b) $\frac{7}{2}\pi - 4\sqrt{2}$

(c) $\frac{7}{2}\pi - 4\sqrt{3}$

(d) $\frac{8}{3}\pi - 4\sqrt{3}$ ✓

(e) $\frac{8}{3}\pi - 4\sqrt{2}$

2. Let σ be a uniform random permutation of $\{1, \ldots, 100\}$. What is the probability that $\sigma(1) < \sigma(2) < \sigma(3)$ (i.e., what is the probability that the first three elements are in increasing order)?

In the options below, $n! = 1 \times 2 \times \ldots \times n$.

(a) $\frac{3}{100!}$

(b) $\frac{3!}{100!}$

(c) $\frac{6}{100}$

(d) $\frac{1}{6}$ ✓

(e) $\frac{1}{3}$

3. There is a 100cm long ruler that has 11 ants on positions 0cm, 10cm, 20cm, 30cm, ..., 100cm. The ant at the 0cm mark is facing towards the 100cm mark, and the ant at the 100cm is facing towards the 0cm mark; all other ants is either facing towards 0cm or towards the 100cm mark.

All ants start moving in the direction they are facing at a speed of 1cm per second. Whenever an ant collides with another ant, the two instantly reverse their direction and continue moving in the other direction. All ants continue moving until they fall off the ruler when they move past one of the two ends (at 0cm or 100cm marks).

How long will it take for all ants to fall off the ruler?

(a) At most 50 seconds, but cannot be determined without knowing the directions of all ants.

(b) 50 seconds.

(c) At least 50 seconds and at most 100 seconds, but cannot be determined without knowing the directions of all ants.

(d) 100 seconds. ✓

(e) More than 100 seconds, but cannot be determined without knowing the directions of all ants.

4. Let $z_1, z_2, z_3, \ldots, z_{2023}$ be a permutation of the numbers $1, 2, 3, \ldots, 2023$. Which of the following is true about the product $\prod_{i=1}^{2023}(z_i - i)$?

Note: The *parity* of an integer n just denotes whether n is even or odd. Formally, the parity of n is said to be odd if n is odd, and even if n is even.

(a) The above product is always even. ✓

(b) The above product is always odd.

(c) The parity of the above product always changes if we swap the values of any two variables among $z_1, z_2, \ldots, z_{2023}$.

(d) There always exist two variables among $z_1, z_2, \ldots, z_{2023}$ such that the parity of the above product changes if we swap their values, but there may also exist two variables among the $z_1, z_2, \ldots, z_{2023}$ such that swapping their values does not change the parity of the above product.

(e) None of the above statements is true.

5. Let $p(x)$ be a polynomial with real coefficients which satisfies $p(r) = p(-r)$ for every real number r. Let $n \geq 5$ be a positive integer. Suppose that $p(i) = i$ for all $1 \leq i \leq n$. What is the maximum possible value of the absolute value of the coefficient of x^5 in $p(x)$?

(a) 0 ✓

(b) 5

(c) 10

(d) n

(e) $n + 1$

6. For each month in the year (i.e., January, February, March,...), let us assume the probability that a person's birthday falls in that particular month is exactly 1/12, and let us assume that this is independent for different persons.

What is the smallest value of the natural number n such that, among n independently chosen persons the probability that there is a pair of them born in the same month is at least 1/2?

(a) 3

(b) 4

(c) 5 ✓

(d) 6

(e) 7

7. Let $S := \{(a,b) \mid 0 \leq a \leq 1 \ , \ 0 \leq b \leq 1\}$, a unit square, in $\mathbb{R}^2$. Let $B := \{(x,y) \mid x^2+y^2 \leq 1\}$, a unit disk, in $\mathbb{R}^2$. Define the set $S+B$ as follows:

$$S+B := \{(u,v) \mid \exists (a,b) \in S \ , (x,y) \in B \text{ such that } u = a+x \ , \ v = b+y\}\,.$$

What is the area of $S+B$?

(a) $\pi+4$

(b) $\pi+5$ ✓

(c) $\pi+3$

(d) $\pi+2$

(e) None of the above.

8. A palindrome is a string that reads the same in reverse (e.g. ABBA or KAYAK or MALAYALAM).

How many strings of length 5 using the letters from $\{A, B, C, D, E\}$ have no palindromic substring of length at least 2?

(a) 243

(b) 405

(c) 540 ✓

(d) 675

(e) 1280

9. Compute $\int_{16}^{\infty} \frac{1}{x} \cdot \frac{1}{\sqrt{\sqrt{x}-1}}\, dx$.

(a) 0

(b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$

(d) $\frac{2\pi}{3}$ ✓

(e) 2π

10. Let M be a 3×3 matrix over the real numbers such that $M^T M = \mathbf{I}$. Consider the following statements.

(i) There exists a non-zero vector $x \in \mathbb{R}^3$ such that $Mx = \mathbf{0}$.

(ii) There exist two non-zero vectors $x, y \in \mathbb{R}^3$ such that the angle between them and that between Mx and My are different.

(iii) There exists a non-zero vector x such that its length is different from that of Mx.

Which of the above statements is/are true?

(a) Only (i).

(b) Only (ii).

(c) Only (iii).

(d) All three statements.

(e) None of the three statements. ✓

11. Consider the following sequence of polynomials with real coefficients.

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 2x \\ P_{n+1}(x) &= 2xP_n(x) - P_{n-1}(x), \text{ for all natural numbers } n \geq 1. \end{aligned}$$

What is the dimension of the linear span of the set

$$\{P_0(x), P_1(x), P_2(x), P_3(x), P_4(x)\}$$

in the vector space of polynomials in variable x with real coefficients?

(a) 1

(b) 2

(c) 3

(d) 4

(e) 5 ✓

12. A subset S of the rational numbers is said to be "nice" if for every infinite sequence of $x_1, x_2, \ldots$ of elements from S, there is always two indices $i < j$ such that $x_i \leq x_j$. Consider the following statements.

(i) The set of natural numbers $\mathbb{N}$ is "nice".

(ii) The set of integers $\mathbb{Z}$ is "nice".

(iii) The set of positive rational numbers is "nice".

Which of the above statements is/are true?

(a) Only (i). ✓

(b) Only (i) and (ii).

(c) Only (i) and (iii).

(d) All three statements are true.

(e) None of the three statements is true.

13. Let $n \geq 100$ be a positive integer. Let $X_1, X_2, \ldots, X_n$ be independent random variables, each taking values in the set $\{0, 1\}$ such that $\Pr[X_i = 1] = \frac{2}{3}$ for each $1 \leq i \leq n$. Define $S := \sum_{i=1}^{n} X_i$, and let $p(x)$ be a polynomial such that for each non-negative integer j, the coefficient of x^j in $p(x)$ is $\Pr[S = j]$. What is $p'(1)$?

Note: $p'(x)$ denotes the polynomial obtained by differentiating p with respect to x.

(a) 0

(b) n

(c) $\frac{2n}{3}$ ✓

(d) $\frac{4n^2+2n}{9}$

(e) $\frac{4n^2-4n}{9}$

14. Let A and B be two $n \times n$ invertible matrices with real entries such that every row in A sums to 1 and every row in B sums to 2. Consider the following three statements:

(i) Every row in the matrix AB sums to 2.

(ii) Every row in the matrix $A^{-1}B$ sums to 2.

(iii) Every row in the matrix $A^{-1}B^{-1}$ sums to $\frac{1}{2}$.

Which of the above statements is/are true?

(a) None of the statements (i), (ii), (iii) is true.

(b) All the three statements (i), (ii), and (iii) are true. ✓

(c) Statement (i) is true but not necessarily statements (ii) or (iii).

(d) Statements (i) and (ii) are true but not necessarily statement (iii).

(e) Statements (i) and (iii) are true but not necessarily statement (ii).

15. Suppose Michelle gives Asna and Badri two different numbers from $\mathbb{N} = \{1, 2, 3, \ldots\}$. It is commonly known to both Asna and Badri that they each know only their own number and that it is different from the other one. The following conversation ensues.

Michelle: I privately gave each of you a different natural number. Which of you has the larger of the two numbers?

Asna: I don't know.

Badri: I don't know either.

Asna: Oh, then I know who has the larger number.

Badri: In that case, I know both numbers. What numbers were Asna and Badri respectively given?

(a) Asna was given 2, Badri was given 3. ✓

(b) Asna was given 3, Badri was given 2.

(c) Asna was given 3, Badri was given 4.

(d) Asna was given 4, Badri was given 3.

(e) None of the above.

LIDS Section

1. Let $f(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$ denote the probability density of a standard Gaussian. Then $\int_{-\infty}^{\infty} f(x)^2 dx$ equals

(a) $\sqrt{\frac{2}{\pi}}$.

(b) $\sqrt{\frac{1}{4\pi}}$. ✓

(c) $2\sqrt{\pi}$.

(d) $4\sqrt{\pi}$.

(e) $8\sqrt{\pi}$.

2. An urn contains 15 white and 10 black balls. Balls are drawn one by one from the urn without replacement. What is the probability that the 5^{th} draw results in a white ball and the 15^{th} draw results in a black ball?

(a) $\frac{2}{15}$

(b) $\frac{1}{4}$ ✓

(c) $\frac{1}{5}$

(d) $\frac{2}{5}$

(e) None of the above

3. Let (x_1, x_2) be a random sample from the uniform probability distribution on a unit square of side 1 centered at 0. Let $M = \begin{bmatrix} \mathbb{E}[x_1^2] & \mathbb{E}[x_1 x_2] \\ \mathbb{E}[x_2 x_1] & \mathbb{E}[x_2^2] \end{bmatrix}$. Let $\lambda_1 \geq \lambda_2$ be the two (possibly equal) eigenvalues of M. Then $\lambda_1 - \lambda_2$ equals the following.

(a) $\frac{1}{4}$.

(b) $\frac{1}{3}$.

(c) $\frac{1}{2}$.

(d) $\frac{1}{12}$.

(e) 0. ✓

4. An $n \times n$ matrix A is called positive semi-definite (p.s.d.) if $x^T A x \geq 0$ for all $x \in \mathbb{R}^n$. Consider the following statements.

1. A p.s.d. matrix is always full-rank.
2. All eigenvalues of a p.s.d. matrix are non-negative.
3. All eigenvalues of a p.s.d. matrix are distinct.

Which of the following is TRUE?

(a) Statement 1 only.

(b) Statement 2 only ✓

(c) Statement 3 only.

(d) Both Statements 1 and 3.

(e) Both Statements 2 and 3.

5. Consider a signal $f(t)$ which peaks in the time domain at a particular time point t_0 in a Gaussian manner. Formally,

$$f(t) = \exp(-(t - t_0)^2).$$

Which of the following is the best description of the frequency domain power spectrum (i.e., the graph of $|F(\omega)|^2$ against the angular frequency ω, where F denotes the continuous time Fourier transform of f) of f?

Note: The following information may be useful. For any complex number b and any complex number a with positive real part,

$$\int_{-\infty}^{\infty} \exp(-a(x-b)^2)\,\mathrm{d}x = \sqrt{\frac{\pi}{a}},$$

where the square root of a is chosen to be the one that has positive real part.

(a) The power spectrum of f does not exist.

(b) The power spectrum of f is shaped like a square pulse.

(c) The power spectrum of f is shaped like the probability density function of some probability distribution whose tails follow an inverse power law.

(d) The power spectrum of f is shaped like a delta function.

(e) The power spectrum of f is shaped like the probability density function of some Gaussian distribution. ✓

6. Let $X_i, i = 1, 2, \ldots,$ be independent and **uniformly** distributed random variables over $[0, 1]$ with mean $\frac{1}{2}$. Let

$$T = \min\left\{i : X_i \geq \frac{1}{2}\right\}.$$

What is the $\mathbb{E}[X_{T-1}]$?

[Hint: Note that T is a random variable and X_{T-1} is not a uniformly distributed random variable over $[0, 1]$.]

(a) $\frac{1}{3}$

(b) $\frac{1}{4}$

(c) $\frac{3}{4}$

(d) $\frac{5}{6}$

(e) None of the above ✓

7. Consider the function

$$f(x) = x^2 - \log_e x$$

for values of x which lie in the interval $[0.5, 1]$. In this domain, suppose the function attains the *maximum* value at x^*. Which of the following is true?

(a) $0.5 < x^* < 1/\sqrt{2}$

(b) $x^* = 1/\sqrt{2}$

(c) $1/\sqrt{2} < x^* < 1$

(d) $x^* \in \{0.5, 1\}$ ✓

(e) The maximum value is attained at more than one value of x

8. Let

$$\mathcal{P} = \left\{(x,y) : x \leq 1,\ y \leq 1, \left(\frac{x}{2}\right)^2 + y^2 \leq 1\right\}.$$

Compute

$$\max_{(x,y)\in\mathcal{P}} x + y.$$

(a) $1 + \frac{\sqrt{5}}{3}$

(b) $1 + \frac{\sqrt{3}}{2}$ ✓

(c) $1 + \frac{2}{\sqrt{5}}$

(d) $\sqrt{5}$

(e) None of the above

9. Recall that the Fourier transform of a signal $f(t)$, $t \in \mathbb{R}$ is given by

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt.$$

Suppose that for a signal $f(t)$, it is known that $F(\omega) = 0$ for $|\omega| > 10$. Consider the following signals:

(i) $\begin{cases} 1 & t = 0 \\ \frac{\sin(t)}{t} & 0 < |t| \leq 10\pi \\ 0 & \text{otherwise} \end{cases}$

(ii) $\begin{cases} \frac{\sin(t)}{t} & \pi \leq t \leq 10\pi \\ 0 & \text{otherwise} \end{cases}$

(iii) $\begin{cases} 1 & t = 0 \\ \frac{\sin(t)}{t} & t \neq 0 \end{cases}$

Which of the above could possibly be $f(t)$?

(a) Only (i)

(b) Only (ii)

(c) Only (iii) ✓

(d) Only (i) and (iii)

(e) Only (i) and (ii)

10. Suppose $A \in \mathbb{C}^{5\times 4}$ such that $\text{Rank}(A) = 3$ (i.e., A is a 5×4 matrix with entries which are complex numbers and of rank 3), and $B \in \mathbb{C}^{4\times 5}$ such that $\text{Rank}(B) = 3$. Of the options below, choose the tightest bound on the rank of AB which holds for all such A and B. Note that to receive credit, you need to choose the tightest bound and not just a correct bound.

(a) $0 \le \text{Rank}(AB) \le 3$

(b) $1 \le \text{Rank}(AB) \le 4$

(c) $1 \le \text{Rank}(AB) \le 3$

(d) $2 \le \textbf{Rank}(AB) \le 3$ ✓

(e) None of the above

11. Let $\alpha = (1, 2, 3, 4)^T$ and $w = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$ be two vectors in $\mathbb{R}^4$. Find the vector z which, among the set of all orthogonal vectors to w, minimizes the Euclidean distance to the vector α.

(a) $(-1.5, -0.5, 0.5, 1.5)^T$ ✓

(b) $(-2, -0.5, 0.5, 2)^T$

(c) $(-1.5, 0.5, 2.5, -1.5)^T$

(d) $(1.5, 2.5, -2.5, -1.5)^T$

(e) $(0.5, 0, 1.5, -2)^T$

12. Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be a continuous and differentiable function such that $\int_{x=0}^{1} \int_{y=0}^{1} f(x, y)dxdy = 1$. Let $\int_{x=0}^{1} \int_{y=0}^{1-x} f(x, y)dxdy = A$. Consider the following statements

1. $A \le 1/2$.
2. $A \ge 1/2$.
3. $A \ge 1/4$.
4. $A \le 1/4$.

Which of the following is TRUE.

(a) Only statement 1 is correct.

(b) Only statement 2 is correct.

(c) Only statement 3 is correct.

(d) Only statement 4 is correct.

(e) None of the above. ✓

13. Let $X = X_0, X_1, X_2, \ldots$ be independent identically distributed random variables, such that $\mathbb{P}[X = -1] = \mathbb{P}[X = 1] = \frac{1}{2}$.. Then $\mathbb{E}[(\sum_{i=1}^{n} X_i)^4]$ is equal to which of the following?

(a) $\binom{n}{1} + \binom{n}{2}$.

(b) $1 + n + \binom{n}{2}$.

(c) $1 + n + \binom{n}{2} + \binom{n}{4}$.
(d) $\binom{n}{2} + \binom{n}{4}$.
(e) None of the above. ✓

14. Let $W_i = (X_i, Y_i)$ be a random variable which takes values in $\mathbb{Z}^2$ for $i = 0, 1, \ldots, n$. Suppose that all $W_i - W_{i-1}$ are independent and take values $(0, 1)$ and $(1, 0)$ with probability $1/2$ each. Suppose W_0 is deterministic and equal to $(0, 0)$.
What is the expected squared distance of W_n from the point $(n/2, n/2)$?

(a) 0.
(b) $n/2$. ✓
(c) n.
(d) $2n$.
(e) None of the above.

15. Consider the function

$$f(x, y) = (x^2 - 4)^2 + y^2.$$

Consider the following statements.

(1) $(-2, 0)$ is a global minimum.
(2) $(+2, 0)$ is a global minimum.
(3) $\nabla f(0, 0) = 0$ but $(0, 0)$ is neither a local minimum nor a local maximum.

Which of the following statements is correct.

(a) Only (1) is true.
(b) Only (2) is true.
(c) Only (1) and (2) are true.
(d) Only (2) and (3) are true.
(e) All of (1), (2), (3), are true. ✓