GS 2025 Selection Process for PhD in Mathematics at TIFR CAM

Syllabus

Real Analysis: Riemann integration, Riemann-Stieltjes integration in $\mathbb{R}$. Sequences and series of functions, Uniform convergence, Arzelà-Ascoli theorem. Function of several variables, Continuity, Directional derivative, Inverse and implicit function theorems. Fixed point theorems.

Ordinary Differential Equations: Existence and uniqueness of solutions of initial value problems for first order ODEs. General theory of homogenous and non-homogeneous linear ODEs. Stability of linear ODEs. Sturm-Liouville boundary value problem, Green's function.

Linear Algebra: Vector space, Linear transformation, Algebra of matrices, rank and determinant of matrices, linear equations. Eigenvalues and eigenvectors, Cayley-Hamilton theorem, Minimal polynomial, Algebraic and geometric multiplicities. Spectral theorem, Jordan forms.

Topology: Topological spaces, Induced topology, Product topology, Separation axioms. Continuous maps and homeomorphisms, Connectedness, Compactness. Baire category theorem.

Functional Analysis: Normed Linear Spaces, Banach spaces, Continuous linear functional, Dual spaces. Hahn-Banach theorem, Open mapping theorem, Closed graph theorem, Uniform boundedness principle. Weak convergence, Hilbert spaces, Reisz representation theorem for Hilbert space.

Measure Theory: Lebesgue measure, Lebesgue integral. Fatou, Monotone and dominated convergence theorems. Product measures, Fubini theorem. L^p spaces.

Complex Analysis: Cauchy-Riemann equations, Countour integration, Goursat's theorem, Cauchy's theorem, Cauchy's integral formula, Power series representation, Liouville's theorem, Morera's theorem, Schwarz reflection principle. Meromorphic functions, Laurent series, Calculus of residues, Argument principle, Hurwitz's theorem, Maximum modulus principle, Open mapping theorem, Rouché's theorem. Conformal mappings, Schwarz's lemma, Automorphisms of the unit disc and upper half-plane.

Probability: Combinatorial and discrete probability, basics of simple random walk. Random variables and their probability mass function/density function. Distribution function of a random variable. Expectation, variance, and higher moments of random variables. Conditional probability and conditional expectation (for random variables with a continuous density). Borel-Cantelli and its applications. Limit theorems: statements of the law of large numbers and central limit theorem, with applications to estimating probabilities.

Sample Questions

1. Suppose $f : (0, \infty) \to \mathbb{R}$ is strictly convex, that is, if $x_1, x_2 \in (0, \infty)$ and $t \in (0, 1)$ then

$$f(tx_1 + (1-t)x_2) < tf(x_1) + (1-t)f(x_2).$$

Choose all the correct options.

(a) If f is differentiable then $f''(x) > 0$.
(b) f is differentiable.
(c) $f(x) = x^p$ is strictly convex if and only if $p > 1$.
(d) $f(x) - f(x+1)$ is an increasing function.

2. Suppose $\mathscr{P} \subset C[0,1]$ be given by

$$\mathscr{P} = \Big\{ P(x) = \sum_{j=0}^{n} a_j x^j : n \leq 10, |a_j| \leq 1 \text{ for each } j \Big\}.$$

Choose all the correct options.

(a) The closure of $\mathscr{P}$ is compact in $C[0,1]$ under the uniform topology.
(b) $\mathscr{P}$ is dense in $C[0,1]$ under the uniform topology.
(c) $C[0,1] \setminus \mathscr{P}$ is dense in $C[0,1]$ under the uniform topology.
(d) $\mathscr{P}$ is a Banach space under the uniform norm.

3. Let

$$\mathcal{S} = \left\{ f \colon \mathbb{C} \to \mathbb{C} \colon f \text{ is holomorphic and } \int_0^{2\pi} |f(re^{i\theta})| d\theta \leq (1+r)^{3/2}, \text{ for all } r > 0 \right\}$$

Choose all the correct options.

(a) $\{P \text{ polynomial} : \deg P \geq 1\} \subseteq \mathcal{S}$.
(b) $\mathcal{S} \subseteq \{P \text{ polynomial} : \deg P \geq 1\}$.
(c) $\{P \text{ polynomial} : \deg P \geq 2\} \subseteq \mathcal{S}$.
(d) $\mathcal{S}$ contains a non constant function with infinitely many zeroes.

4. Consider the set of all 2×2 matrices A such that three of entries are 1 and one of them is 0. Choose the correct options.

(a) All such matrices are invertible.
(b) All such matrices are diagonalisable.
(c) Some such matrices are positive-definite.
(d) If A is one such matrix which is invertible then A^{-1} is also such a matrix.

5. Let $N(0,1)$ denote the normal distribution with mean 0 and standard deviation 1 and $(X_i)_{i \in \mathbb{N}}$ be any sequence of iid random variables with mean 0 and variance 1. Choose all the correct options.

(a) $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$ has to converge in distribution to $N(0,1)$.

(b) $\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i$ has to converge almost surely to $N(0, 1)$.

(c) $\frac{1}{n} \sum_{i=1}^{n} X_i$ has to converge to 0 almost surely.

(d) $\frac{1}{n} \sum_{i=1}^{n} X_i$ has to converge to 0 in distribution.

Examination Format

The exam will be 120 minutes long consisting of 20 multiple choice questions that may have more than one correct answer.
For a question with only one correct answer, you will get

- 10 points for choosing the correct answer and
- -5 points for choosing any incorrect answer.

For a question with multiple correct answers, you will get

- 10 points for choosing *all* the correct answers,
- 5 points for choosing at least one correct answer (but not all the correct answers) and did not choose any incorrect answer, and
- -5 points if you choose any incorrect answer.

Consider the following illustration.

1. Let $N = 4$. Choose the correct options.

   (a) $N \geq 5$.

   (b) $N \leq 5$.

   (c) $N \geq 1$.

   (d) $N \leq 10$.

   The correct answers are (b), (c) and (d). If someone answers (b), (c) and (d) then they get 10 points, if someone answers only (b) then they get 5 points and if someone answers (a) and (d) then they get -5 points.

The maximum possible score on the exam is 200 while the minimum possible score is -100.