Notation and Conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \dots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$.
- $\mathrm{M}_n(\mathbb{R})$ denotes the real vector space of $n \times n$ real matrices, and $\mathrm{M}_n(\mathbb{C})$ the complex vector space of $n \times n$ complex matrices. $\mathrm{M}_n(\mathbb{R})$ gets the topology transferred from any $\mathbb{R}$-linear isomorphism $\mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, and its subsets get the subspace topology.
- Id denotes the identity matrix in $\mathrm{M}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{C})$.
- A matrix $A \in \mathrm{M}_n(\mathbb{R})$ is called idempotent if $A^2 = A$, and nilpotent if $A^m = 0$ for some positive integer m.
- All rings are associative, with a multiplicative identity.
- For a ring R, $R[x]$ denotes the polynomial ring in one variable over R, and $R^\times$ denotes the multiplicative group of units of R.
- All logarithms are natural logarithms.
- If B is a subset of a set A, we write $A \setminus B$ for the set $\{a \in A \mid a \notin B\}$.
- An isometry from a metric space (X, d) to a metric space (X', d') is a (not necessarily surjective) map $f : X \to X'$ such that for all $a, b \in X$, we have $d'(f(a), f(b)) = d(a, b)$.

PART A

Answer the following multiple choice questions.

1. For each positive integer n, let

$$s_n = \frac{1}{\sqrt{4n^2 - 1^2}} + \frac{1}{\sqrt{4n^2 - 2^2}} + \cdots + \frac{1}{\sqrt{4n^2 - n^2}}.$$

Then $\lim_{n\to\infty} s_n$ equals

(a) $\pi/2$
✔ (b) $\pi/6$
(c) $1/2$
(d) ∞

2. The number of bijective maps $g : \mathbb{N} \to \mathbb{N}$ such that

$$\sum_{n=1}^{\infty} \frac{g(n)}{n^2} < \infty$$

is

✔ (a) 0
(b) 1
(c) 2
(d) ∞

3. The value of

$$\lim_{n\to\infty} \prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right)$$

is

✔ (a) $1/2$
(b) 1
(c) $1/4$
(d) 0

4. The set

$$S = \{x \in \mathbb{R} \mid x > 0 \text{ and } (1 + x^2)\tan(2x) = x\}$$

is

(a) empty
(b) nonempty but finite
✔ (c) countably infinite

(d) uncountable

5. The dimension of the real vector space

$$V = \{f : (-1,1) \to \mathbb{R} \mid f \text{ is infinitely differentiable on } (-1,1) \text{ and } f^{(n)}(0) = 0 \text{ for all } n \geq 0\}$$

is

(a) 0
(b) 1
(c) greater than one, but finite
(d) infinite

6. For a positive integer n, let a_n denote the unique positive real root of $x^n + x^{n-1} + \cdots + x - 1 = 0$. Then

(a) the sequence $\{a_n\}_{n=1}^{\infty}$ is unbounded
(b) $\lim_{n\to\infty} a_n = 0$
(c) $\lim_{n\to\infty} a_n = 1/2$
(d) $\lim_{n\to\infty} a_n$ does not exist

7. Let A be the set of all real numbers $\lambda \in [0,1]$ such that

$$\lim_{p\to 0} \frac{\log(\lambda 2^p + (1-\lambda)3^p)}{p} = \lambda \log 2 + (1-\lambda)\log 3.$$

Then

(a) $A = \{0, 1\}$
(b) $A = \left\{0, \frac{1}{2}, 1\right\}$
(c) $A = \left\{0, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\}$
(d) $A = [0,1]$

8. Let $X \subseteq \mathbb{R}$ be a subset. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : X \to \mathbb{R}$, that converges uniformly to a function $f : X \to \mathbb{R}$. For each positive integer n, let $D_n \subseteq X$ denote the set of points at which f_n is not continuous. Let $D \subseteq X$ denote the set of points at which f is not continuous. Which one of the following statements is correct?

(a) If each D_n is finite, then D is finite.
(b) If each D_n has at most 7 elements, then D has at most 7 elements.
(c) If each D_n is uncountable, then D is uncountable.
(d) None of the other three statements is correct.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary function. Consider the following assertions:

(I) f is continuous.

(II) The set

$$\mathrm{Graph}(f) = \big\{(x, f(x)) \in \mathbb{R}^2 \mid x \in \mathbb{R}\big\}$$

is a connected subset of $\mathbb{R}^2$.

Which one of the following statements is correct?

✓ (a) (I) implies (II) but (II) does not imply (I).

(b) (II) implies (I) but (I) does not imply (II).

(c) (I) implies (II) and (II) implies (I).

(d) (I) does not imply (II), and (II) does not imply (I).

10. Let $\mathcal{C}$ denote the set of colorings of an 8×8 chessboard, where each square is colored either black or white. Let $\sim$ denote the equivalence relation on $\mathcal{C}$ defined as follows: two colorings are equivalent if and only if one of them can be obtained from the other by a rotation of the chessboard. The cardinality of the set $\mathcal{C}/\sim$ of equivalence classes of elements of $\mathcal{C}$ under $\sim$ is

(a) 2^{62}

✓ (b) $2^{62} + 2^{30} + 2^{15}$

(c) $2^{64} - 2^{32} + 2^{16}$

(d) $2^{63} - 2^{31} + 2^{15}$

11. What is the number of surjective maps from the set $\{1, \ldots, 10\}$ to the set $\{1, 2\}$?

(a) 90

✓ (b) 1022

(c) 98

(d) 1024

12. Let V be a vector space over a field F. Consider the following assertions:

(I) V is finite dimensional.

(II) For every linear transformation $T : V \to V$, there exists a nonzero polynomial $p(x) \in F[x]$ such that $p(T) : V \to V$ is the zero map.

Which one of the following statements is correct?

(a) (I) implies (II) but (II) does not imply (I).

(b) (II) implies (I) but (I) does not imply (II).

✓ (c) (I) implies (II) and (II) implies (I).

(d) (I) does not imply (II), and (II) does not imply (I).

13. $T : \mathbb{C}[x] \to \mathbb{C}[x]$ be the $\mathbb{C}$-linear transformation defined on the complex vector space $\mathbb{C}[x]$ of one variable complex polynomials by $Tf(x) = f(x+1)$. How many eigenvalues does T have?

(a) 1
(b) finite but more than 1
(c) countably infinite
(d) uncountable

14. Let $\mathbb{R}^{\mathbb{N}}$ denote the real vector space of sequences $(x_0, x_1, x_2, \dots)$ of real numbers. Define a linear transformation $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by

$$(x_0, x_1, \dots,) \mapsto (x_0 + x_1, x_1 + x_2, \dots).$$

Which one of the following statements is correct?

(a) The kernel of T is infinite dimensional.
(b) The image of T is infinite dimensional.
(c) The quotient vector space $\mathbb{R}^{\mathbb{N}}/T(\mathbb{R}^{\mathbb{N}})$ is infinite dimensional.
(d) None of the other three statements is correct.

15. Which one of the following statements is correct?

(a) There exists a $\mathbb{C}$-linear isomorphism $\mathbb{C}^2 \to \mathbb{C}$.
(b) There exists no $\mathbb{C}$-linear isomorphism $\mathbb{C}^2 \to \mathbb{C}$, but there exists an $\mathbb{R}$-linear isomorphism $\mathbb{C}^2 \to \mathbb{C}$.
(c) There exists no $\mathbb{R}$-linear isomorphism $\mathbb{C}^2 \to \mathbb{C}$, but there exists a $\mathbb{Q}$-linear isomorphism $\mathbb{C}^2 \to \mathbb{C}$.
(d) None of the other three statements is correct.

16. The matrix

$$\begin{pmatrix} 4 & -3 & -3 \\ 3 & -2 & -3 \\ -1 & 1 & 2 \end{pmatrix}$$

is

(a) diagonalizable
(b) nilpotent
(c) idempotent
(d) none of the other three options

17. Which of the following is a necessary and sufficient condition for two real 3×3 matrices A and B to be similar (i.e., $PAP^{-1} = B$ for an invertible real 3×3 matrix P)?

(a) They have the same characteristic polynomial.
(b) They have the same minimal polynomial.
(c) They have the same minimal and characteristic polynomials.
(d) None of the other three conditions.

18. Consider the following two subgroups A, B of the group $\mathbb{Q}[x]$ of one variable rational polynomials under addition:

$$A = \{p(x) \in \mathbb{Z}[x] \mid p \text{ has degree at most } 2\}, \text{ and}$$

$$B = \{p(x) \in \mathbb{Q}[x] \mid p \text{ has degree at most } 2, \text{ and } p(\mathbb{Z}) \subseteq \mathbb{Z}\}.$$

Then the index $[B : A]$ of A in B equals

(a) 1
✓ (b) 2
(c) 4
(d) none of the other three options

19. Let G be any finite group of order 2021. For which of the following positive integers m is the map $G \to G$, given by $g \mapsto g^m$, a bijection?

(a) 43
✓ (b) 45
(c) 47
(d) none of the other three options

20. How many subgroups does $(\mathbb{Z}/13\mathbb{Z}) \times (\mathbb{Z}/13\mathbb{Z})$ have?

(a) 13
✓ (b) 16
(c) 4
(d) 25

PART B

Answer whether the following statements are True or False.

F 1. Let $f_n : [0, 1] \to \mathbb{R}$ be a continuous function for each positive integer n. If

$$\lim_{n\to\infty} \int_0^1 f_n(x)^2 \, dx = 0,$$

then

$$\lim_{n\to\infty} f_n\left(\frac{1}{2}\right) = 0.$$

T 2. Let (X, d) be an infinite compact metric space. Then there exists no function $f : X \to X$, continuous or otherwise, with the property that $d(f(x), f(y)) > d(x, y)$ for all $x \neq y$.

T 3. Every infinite closed subset of $\mathbb{R}^n$ is the closure of a countable set.

T 4. If X is a compact metric space, there exists a surjective (not necessarily continuous) function $\mathbb{R} \to X$.

T 5. If X is a compact metric space, then every isometry $f : X \to X$ is surjective.

F 6. Define a metric on the set of finite subsets of $\mathbb{Z}$ as follows:

$$d(A, B) = \text{ the cardinality of } (A \cup B \setminus (A \cap B)).$$

The resulting metric space admits an isometry into $\mathbb{R}^n$, for some positive integer n.

F 7. There exists a continuous function

$$f : [0, 1] \to \{A \in \mathrm{M}_2(\mathbb{R}) \mid A^2 = A\}$$

such that $f(0) = 0$ and $f(1) = \mathrm{Id}$.

T 8. Let $f : [0, 1] \to \mathbb{R}$ be a monotone increasing (not necessarily continuous) function such that $f(0) > 0$ and $f(1) < 1$. Then there exists $x \in [0, 1]$ such that $f(x) = x$.

F 9. The set

$$\{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x^y \text{ divides } y^x, x \neq y, xy \neq 0, x \neq 1\}$$

is finite.

T 10. Suppose a line segment of a fixed length L is given. It is possible to construct a triangle of perimeter L, whose angles are $105°, 45°$ and $30°$, using only a straight edge and a compass.

T 11. The real vector space $\mathrm{M}_n(\mathbb{R})$ cannot be spanned by nilpotent matrices, for any positive integer n.

T 12. Let $S \subseteq \mathrm{M}_n(\mathbb{R})$ be a nonempty finite set closed under matrix multiplication. Then there exists $A \in S$ such that the trace of A is an integer.

F 13. Given a linear transformation $T : \mathbb{Q}^4 \to \mathbb{Q}^4$, there exists a nonzero proper subspace V of $\mathbb{Q}^4$ such that $T(V) \subseteq V$.

T 14. If G is a finite group such that the group $\mathrm{Aut}(G)$ of automorphisms of G is cyclic, then G is abelian.

T 15. There exists a countable group having uncountably many subgroups.

F 16. There exists a nonzero ideal $I \subseteq \mathbb{Z}[i]$ such that the quotient ring $\mathbb{Z}[i]/I$ is infinite (here i is a square root of -1 in $\mathbb{C}$).

F 17. There exists an injective ring homomorphism from the ring $\mathbb{Q}[x, y]/(x^2 - y^2)$ into the ring $\mathbb{Q}[x, y]/(x - y^2)$.

F 18. The set

$$\{n \in \mathbb{N} \mid n \text{ divides } a^3 - 1, \text{ for all integers } a \text{ such that } \gcd(a, n) = 1\}$$

is infinite.

T 19. The set of polynomials in the ring $\mathbb{Z}[x]$, the sum of whose coefficients is zero, forms an ideal of the ring $\mathbb{Z}[x]$.

T 20. Let $c_1, c_2 > 0$, and let $f, g : \mathbb{R} \to \mathbb{R}$ be functions (not assumed to be continuous) such that for all $x \in \mathbb{R}$

$$f(x + c_1) = f(x) \qquad \text{and} \qquad g(x + c_2) = g(x).$$

Further, assume that

$$\lim_{x\to\infty} (f(x) - g(x)) = 0.$$

Then $f = g$.