Notation and Conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \dots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.

- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$.

- $\mathrm{M}_n(\mathbb{R})$ denotes the real vector space of $n \times n$ real matrices, and $\mathrm{M}_n(\mathbb{C})$ the complex vector space of $n \times n$ complex matrices. $\mathrm{M}_n(\mathbb{R})$ gets the topology transferred from any $\mathbb{R}$-linear isomorphism $\mathrm{M}_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, and its subsets get the subspace topology.

- Id denotes the identity matrix in $\mathrm{M}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{C})$.

- For a metric space X, $C(X, \mathbb{R})$ denotes the set of continuous functions from X to $\mathbb{R}$, viewed as a ring under pointwise addition and multiplication. Similarly, $C(X, \mathbb{C})$ will denote the ring of continuous functions from X to $\mathbb{C}$, again with pointwise addition and multiplication.

- A matrix $T \in \mathrm{M}_n(\mathbb{C})$, or a linear transformation $T : V \to V$ from a vector space V to itself, is called idempotent if $T^2 = T$, and nilpotent if $T^m = 0$ for some positive integer m.

- All rings are associative, with a multiplicative identity. A subring of a ring R is assumed by definition to contain the multiplicative identity of R.

- If q is a power of a prime number, $\mathbb{F}_q$ will stand for the finite field with q elements.

- For a ring R, $R[x_1, \dots, x_n]$ denotes the polynomial ring in n variables $x_1, \dots, x_n$ over R, and $R^\times$ denotes the multiplicative group of units of R.

- All logarithms are natural logarithms.

- If B is a subset of a set A, we write $A \setminus B$ for the set $\{a \in A \mid a \notin B\}$.

- By the group of isometries of a metric space (X, d), we mean the group whose elements are bijective maps $f : X \to X$ satisfying $d(x, y) = d(f(x), f(y))$ for all $x, y \in X$, and whose multiplication is defined by composition.

- If f is a real or complex valued function defined on an open interval (a, b) of $\mathbb{R}$, then f is said to be continuously differentiable on (a, b) if its derivative exists and is continuous on (a, b).

- If f is a real or complex valued function defined on an open interval of $\mathbb{R}$, then f' will stand for the first derivative of f (wherever it exists), and f'' for the second derivative of f (wherever it exists).

PART A

Answer the following multiple choice questions.

1. Consider the following properties of a metric space (X, d):

   (I) (X, d) is complete as a metric space.

   (II) For any sequence $\{Z_n\}_{n\in\mathbb{N}}$ of closed nonempty subsets of X, such that $Z_1 \supseteq Z_2 \supseteq \dots$ and

   $$\lim_{n\to\infty}\left(\sup_{x,y\in Z_n} d(x,y)\right) = 0,$$

   $\bigcap_{n=1}^{\infty} Z_n$ is a singleton set.

   Which of the following sentences is true?

   (a) (I) implies (II) and (II) implies (I).

   (b) (I) implies (II) but (II) does not imply (I).

   (c) (I) does not imply (II) but (II) implies (I).

   (d) (I) does not imply (II) and (II) does not imply (I).

2. Consider the following assertions:

   (I) $\{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$ is connected.

   (II) $\{(x, y) \in \mathbb{C}^2 \mid xy = 1\}$ is connected.

   Which of the following sentences is true?

   (a) Both (I) and (II) are true.

   (b) (I) is true but (II) is false.

   (c) (I) is false but (II) is true.

   (d) Both (I) and (II) are false.

3. What is the number of solutions of:

   $$x = \frac{x^2}{50} - \cos\frac{x}{2} + 2$$

   in $[0, 10]$?

   (a) 0

   (b) 1

   (c) 2

   (d) ∞

4. Let A be an element of $M_4(\mathbb{R})$ with characteristic polynomial $t^4 - t$. What is the characteristic polynomial of A^2?

   (a) $t^4 - t$

(b) $t^4 - 2t^3 + t^2$

(c) $t^4 - t^2$

(d) None of the other three options

5. Let n be a positive integer, and let $V = \{f \in \mathbb{R}[x] \mid \deg f \leq n\}$ be the real vector space of real polynomials of degree at most n. Let $\mathrm{End}_{\mathbb{R}}(V)$ denote the real vector space of linear transformations from V to itself. For $m \in \mathbb{Z}$, let $T_m \in \mathrm{End}_{\mathbb{R}}(V)$ be such that $(T_m f)(x) = f(x+m)$ for all $f \in V$. Then the dimension of the vector subspace of $\mathrm{End}_{\mathbb{R}}(V)$ given by

$$\mathrm{Span}\left(\{T_m \mid m \in \mathbb{Z}\}\right)$$

is

(a) 1

(b) n

(c) $n+1$

(d) n^2

6. Let T be the linear transformation from the real vector space $\mathbb{R}[x]$ to itself, given by $T(f) = f'$, where f' is the derivative of f. Consider the following statements about T:

(I) T is nilpotent.

(II) The only eigenvalue of T is 0.

Which of the following sentences is true?

(a) Both (I) and (II) are true.

(b) (I) is true but (II) is false.

(c) (I) is false but (II) is true.

(d) Both (I) and (II) are false.

7. What is the cardinality of the set of $\theta \in [0, 2\pi)$ such that the linear map $\mathbb{R}^2 \to \mathbb{R}^2$ given by the matrix:

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

has an eigenvector in $\mathbb{R}^2$?

(a) 1

(b) 2

(c) 4

(d) ∞

8. Let p be a prime number, and let A equal $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$, viewed as a 2×2 matrix with integer entries. What is the smallest positive integer n such that the matrix A^n is congruent to the 2×2 identity matrix modulo p?

(a) $p^2 - 1$

(b) $p - 1$

(c) p

(d) $p + 1$

9. What is the largest value of n for which there exists a set $\{A_1, \dots, A_n\}$ of (distinct) nonzero matrices in $\mathrm{M}_2(\mathbb{C})$ such that $A_i^* A_j$ has trace zero for all $1 \le i < j \le n$?

(a) 1

(b) Greater than 1 but at most 4

(c) Greater than 4 but finite

(d) ∞

10. Let p be a prime number. What is the number of elements in the group $\mathbb{Q}/\mathbb{Z}$ that have order exactly p?

(a) 0

(b) $p - 1$

(c) p

(d) ∞

11. Consider the real polynomial

$$f(x) = x^{11} - x^7 + x^2 - 1.$$

Which of the following sentences is correct?

(a) $f(x)$ has exactly one positive root.

(b) $f(x)$ has exactly two positive roots.

(c) $f(x)$ has at least three positive roots.

(d) None of the other three options.

12. Consider polynomials

$$f_1(x,y) = \sum_{i,j=0}^{\infty} a_{ij} x^i y^j \qquad \text{and} \qquad f_2(x,y) = \sum_{i,j=0}^{\infty} b_{ij} x^i y^j \in \mathbb{R}[x,y]$$

(where $a_{ij} = b_{ij} = 0$ for all but finitely many $(i,j) \in \mathbb{N}^2$), such that $f_1(p,q) = f_2(p,q)$ for all $(p,q) \in \mathbb{R}^2$ satisfying $p^2 = q^2$. Which of the following sentences is true for all such f_1 and f_2?

(a) $a_{00} = b_{00}$, but we may not have $a_{ij} = b_{ij}$ for all (i,j) with $i + j = 1$.

(b) $a_{ij} = b_{ij}$ if $i + j \le 1$, but we may not have $a_{ij} = b_{ij}$ for all (i,j) with $i + j = 2$.

(c) $a_{ij} = b_{ij}$ if $i + j \le 2$, but we may not have $a_{ij} = b_{ij}$ for all (i,j) with $i + j = 3$.

(d) $a_{ij} = b_{ij}$ if $i + j \le 3$, but we may not have $f_1 = f_2$.

13. What is the number of bijections $f : \{1, 2, \ldots, 9\} \to \{1, 2, \ldots, 9\}$ such that, for all distinct $i, j \in \{1, \ldots, 9\}$, whenever the squares labelled i and j in the diagram below share an edge, the squares labelled $f(i)$ and $f(j)$ share an edge too?

1	2	3
4	5	6
7	8	9

(a) 8

(b) 9

(c) 4

(d) 24

14. Let m be the number of positive integers n such that $1 \leq n \leq 2022$ and such that n has an odd number of (positive integer) divisors. Then m is

(a) 22

(b) 33

(c) 44

(d) 55

15. Let S be the set of nonnegative continuous functions f on $[0, 1]$ satisfying

$$\int_0^1 \sin^2(x) f(x)\, dx = \int_0^1 \sin(x)\cos(x) f(x)\, dx = \int_0^1 \cos^2(x) f(x)\, dx = 1.$$

Then S is:

(a) an uncountable set

(b) a countably infinite set

(c) a finite and nonempty set

(d) the empty set

16. Let $f(x) = 1 - \sin x$ for $x \in \mathbb{R}$. Define

$$a_n = \sqrt[n]{f\left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) \ldots f\left(\frac{n}{n}\right)}.$$

Then

(a) $\{a_n\}_n$ converges to 0

(b) $\{a_n\}_n$ diverges to ∞

(c) $\{a_n\}_n$ converges and $\lim_{n\to\infty} a_n > 0$

(d) none of the other three options is correct

17. Which of the following is true for every function $u : \mathbb{R} \to \mathbb{R}$ which is continuously differentiable on $\mathbb{R}$ (i.e., u is diffferentiable on $\mathbb{R}$ and its derivative u' is continuous on $\mathbb{R}$), and satisfies

$$u(y) \geq u(x) + u'(x)(y - x)$$

for all $x, y \in \mathbb{R}$?

(a) u' is nonnegative.

(b) u attains a minimum at some $x \in \mathbb{R}$.

(c) u' is nondecreasing.

(d) u' is nonincreasing.

18. Let $\{x_n\}$ be a sequence of positive numbers such that $\lim_{n\to\infty} x_n = x$. Define

$$z_n = \frac{1}{n}\left[x_1\left(1 + \frac{x}{n}\right)^n + x_2\left(1 + \frac{x}{n-1}\right)^{n-1} + \cdots + x_n(1 + x)\right].$$

Then

(a) $\{z_n\}$ converges to xe

(b) $\{z_n\}$ converges to e^x

(c) $\{z_n\}$ does not have a limit

(d) $\{z_n\}$ converges to xe^x

19. The value of

$$\lim_{n\to\infty} \int_0^1 \frac{ne^x}{1 + n^2x^2}\, dx$$

is

(a) πe

(b) $\frac{\pi}{2}$

(c) $\frac{\pi}{2}e$

(d) π

20. What is the number of real solutions of the equation

$$e^{\sin x} = \pi?$$

(a) 0

(b) 1

(c) Countably infinite

(d) Uncountable

PART B

Answer whether the following statements are True or False.

1. $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is connected but not path-connected. False
2. If X is a connected metric space, and F is a subring of $C(X, \mathbb{R})$ that is a field, then every element of $C(X, \mathbb{R})$ that belongs to F is a constant function. True
3. Let $K \subseteq [0, 1]$ be the Cantor set. Then there exists no injective ring homomorphism $C([0, 1], \mathbb{R}) \to C(K, \mathbb{R})$. False
4. There exists a metric space (X, d) such that the group of isometries of X is isomorphic to $\mathbb{Z}$. True
5. Let $A \subset \mathbb{R}^2$ be a nonempty subset such that any continuous function $f : A \to \mathbb{R}$ is constant. Then A is a singleton set. True
6. For a nilpotent matrix $A \in \mathrm{M}_n(\mathbb{R})$, let

$$\exp(A) := \sum_{n=0}^{\infty} \frac{A^n}{n!} = \mathrm{Id} + \frac{A}{1!} + \frac{A^2}{2!} + \cdots \in \mathrm{M}_n(\mathbb{R}).$$

   If A is a nilpotent matrix such that $\exp(A) = \mathrm{Id}$, then A is the zero matrix. True

7. There exists $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{R})$, with $A^2 = A \neq 0$, such that

$$|a| + |b| < 1 \qquad \text{and} \qquad |c| + |d| < 1.$$ False

8. If $A \in \mathrm{M}_3(\mathbb{C})$ is such that A^i has trace zero for all positive integers i, then A is nilpotent. True
9. For any finite cyclic group G, there exists a prime power q such that G is a subgroup of $\mathbb{F}_q^\times$. True
10. There are only finitely many isomorphism classes of finite nonabelian groups, all of whose proper subgroups are abelian. False
11. Every subring of a unique factorization domain is a unique factorization domain. False
12. Let $f_1, f_2, f_3, f_4 \in \mathbb{R}[x]$ be monic polynomials each of degree exactly two. Then there exist a real polynomial $p \in \mathbb{R}[x]$ and a subset $\{i, j\} \subset \{1, 2, 3, 4\}$ with $i \neq j$, such that $f_i \circ p = c f_j$ for some $c \in \mathbb{R}$. True
13. There exists a finite abelian group G such that the group $\mathrm{Aut}(G)$ of automorphisms of G is isomorphic to $\mathbb{Z}/7\mathbb{Z}$. False
14. There exists an integral domain R and a surjective homomorphism $R \to R$ of rings that is not injective. True
15. There exists $f \in C([0, 1], \mathbb{R})$ satisfying the following two conditions:

(i) $\int_0^1 f(x)\,dx = 1$; and

(ii) $\lim\limits_{n\to\infty} \int_0^1 f(x)^n\,dx = 0$. False

16. Let $a_n \geq 0$ for each positive integer n. If the series $\sum_{n=1}^{\infty} \sqrt{a_n}$ converges, then so does the series $\sum_{n=1}^{\infty} \frac{a_n}{n^{1/4}}$. True

17. There exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{x\to\infty} f(x) = 2 \qquad \text{and} \qquad \lim_{x\to\infty} f'(x) = 1.$$

False

18. Let $f : [0,1] \to [0,\infty)$ be continuous on $[0,1]$ and twice differentiable in $(0,1)$. If $f''(x) = 7f(x)$ for all $x \in (0,1)$, then $f(x) \leq \max\{f(0), f(1)\}$ for all $x \in [0,1]$. True

19. There are N balls in a box, out of which n are blue $(1 < n < N)$ and the rest are red. Balls are drawn from the box one by one at random, and discarded. Then the probability of picking all the blue balls in the first n draws is the same as the probability of picking all the red balls in the first $(N-n)$ draws. True

20. The set $\{f(x) \in \mathbb{R}[x] \mid f(n) \in \mathbb{Z} \text{ for all } n \in \mathbb{Z}\}$ is uncountable. False