GS2024 Exam, School of Mathematics, TIFR

Notation and conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \dots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$.
- All rings are associative, with a multiplicative identity.
- For any ring R, $\mathrm{M}_n(R)$ denotes the ring of $n \times n$ matrices with entries in R. The identity matrix in $\mathrm{M}_n(R)$ will be denoted by Id or by Id_n.
- $\mathrm{M}_n(\mathbb{R})$ will also be viewed as a real vector space, and $\mathrm{M}_n(\mathbb{C})$ as a complex vector space.
- For a ring R, $R[x_1, \dots, x_n]$ denotes the polynomial ring in n variables $x_1, \dots, x_n$ over R.
- If A is a set, $\#A$ stands for the cardinality of A, and equals ∞ if A is infinite.
- If B is a subset of a set A, we write $A \setminus B$ for the set $\{a \in A \mid a \notin B\}$.
- Let G be a finite group, and let $S \subset G$. We say that S generates G if no proper subgroup of G contains S.

Part A — Multiple Choice Questions

(1) What is the number of even positive integers n such that every group of order n is abelian?
(a) 1
✓ (b) 2
(c) Greater than 2, but finite
(d) Infinite

(2) Let n be a positive integer, and let

$$S = \{g \in \mathbb{R}[x] \mid g \text{ is a polynomial of degree at most } n\}.$$

For $g \in S$, let $A_g = \{x \in \mathbb{R} \mid e^x = g(x)\} \subset \mathbb{R}$. Let

$$m = \min\{\#A_g \mid g \in S\}, \qquad \text{and} \qquad M = \max\{\#A_g \mid g \in S\}.$$

Then
(a) $m = 0, M = n$
✓ (b) $m = 0, M = n + 1$
(c) $m = 1, M = n$
(d) $m = 1, M = n + 1$

(3) Let V, W be nonzero finite dimensional vector spaces over $\mathbb{C}$. Let m be the dimension of the space of $\mathbb{C}$-linear transformations $V \to W$, viewed as a real vector space. Let n be the dimension of the space of $\mathbb{R}$-linear transformations $V \to W$, viewed as a real vector space. Then
(a) $n = m$
(b) $2n = m$
✓ (c) $n = 2m$
(d) $4n = m$

(4) Consider the real vector space of infinite sequences of real numbers

$$S = \{(a_0, a_1, a_2, \ldots) \mid a_k \in \mathbb{R}, k = 0, 1, 2, \ldots\}.$$

Let W be the subspace of S consisting of all sequences $(a_0, a_1, a_2, \ldots)$ which satisfy the relation

$$a_{k+2} = 2a_{k+1} + a_k, \quad k = 0, 1, 2, \ldots.$$

What is the dimension of W?
(a) 1
✓ (b) 2
(c) 3
(d) ∞

(5) Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function. If

$$\lim_{n\to\infty} \int_0^1 f(x + n)\, dx = 2,$$

then which of the following statements about the limit

$$\lim_{n\to\infty} \int_0^1 f(nx)\, dx$$

is correct?
(a) The limit exists and equals 0

(b) The limit exists and equals $\frac{1}{2}$

✓ (c) The limit exists and equals 2

(d) None of the remaining three options is correct

(6) Let $f : \mathbb{R} \to [0, \infty)$ be a function such that for any finite set $E \subset \mathbb{R}$ we have

$$\sum_{x \in E} f(x) \leq 1.$$

Let

$$C_f = \{x \in \mathbb{R} \mid f(x) > 0\} \subset \mathbb{R}.$$

Then

(a) C_f is finite

(b) C_f is a bounded subset of $\mathbb{R}$

(c) C_f has at most one limit point

✓ (d) C_f is a countable set

(7) Let p be a prime. Which of the following statements is true?

(a) There exists a noncommutative ring with exactly p elements

(b) There exists a noncommutative ring with exactly p^2 elements

✓ (c) There exists a noncommutative ring with exactly p^3 elements

(d) None of the remaining three statements is correct

(8) Consider the sequence $\{a_n\}$ for $n \geq 1$ defined by

$$a_n = \lim_{N \to \infty} \sum_{k=n}^{N} \frac{1}{k^2}.$$

Which of the following statements about this sequence is true?

(a) $\lim_{n \to \infty} n a_n$ does not exist

(b) $\lim_{n \to \infty} n a_n$ exists and equals 2

✓ (c) $\lim_{n \to \infty} n a_n$ exists and equals 1

(d) $\lim_{n \to \infty} n^2 a_n$ exists and equals 1

(9) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function that is a solution to the ordinary differential equation

$$f'(t) = \sin^2(f(t)) \, (\forall \; t \in \mathbb{R}), \quad f(0) = 1.$$

Which of the following statements is true?

(a) f is neither bounded nor periodic

(b) f is bounded and periodic

✓ (c) f is bounded, but not periodic

(d) None of the remaining three statements is correct

(10) Let B denote the set of invertible upper triangular 2×2 matrices with entries in $\mathbb{C}$, viewed as a group under matrix multiplication. Which of the following subgroups of B is the normalizer of itself in B?

✓ (a) $\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| a, b \in \mathbb{C} \setminus \{0\} \right\}$

(b) $\left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \middle| a \in \mathbb{C} \setminus \{0\} \right\}$

(c) $\left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \middle| c \in \mathbb{C} \right\}$

(d) $\left\{\begin{pmatrix} a & c \\ 0 & a \end{pmatrix} \middle| a \in \mathbb{C}\setminus\{0\}, c \in \mathbb{C}\right\}$

(11) What is the least positive integer $n > 1$ such that x^n and x are conjugate, for every $x \in S_{11}$? Here, S_{11} denotes the symmetric group on 11 letters.
(a) 10
(b) 11
(c) 12
✓ (d) 13

(12) Consider the following statements:
(A) Let G be a group and let $H \subset G$ be a subgroup of index 2. Then $[G, G] \subseteq H$.
(B) Let G be a group and let $H \subset G$ be a subgroup that contains the commutator subgroup $[G, G]$ of G. Then H is a normal subgroup of G.
Which of the following statements is correct?
✓ (a) (A) and (B) are both true
(b) (A) and (B) are both false
(c) (A) is true and (B) is false
(d) (A) is false and (B) is true

(13) For any symmetric real matrix A, let $\lambda(A)$ denote the largest eigenvalue of A. Let S be the set of positive definite symmetric 3×3 real matrices. Which of the following assertions is correct?
(a) There exist $A, B \in S$ such that $\lambda(A + B) < \max(\lambda(A), \lambda(B))$
✓ (b) For all $A, B \in S$, $\lambda(A + B) > \max(\lambda(A), \lambda(B))$
(c) There exist $A, B \in S$ such that $\lambda(A + B) = \max(\lambda(A), \lambda(B))$
(d) None of the remaining three assertions is correct

(14) Let $\theta \in (0, \pi/2)$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which sends a vector v to its reflection with respect to the line through $(0, 0)$ and $(\cos\theta, \sin\theta)$. Then the matrix of T with respect to the standard basis of $\mathbb{R}^2$ is given by

✓ (a) $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$
(b) $\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$
(c) $\begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix}$
(d) $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$

(15) For a polynomial $f(x, y) \in \mathbb{R}[x, y]$, let $X_f = \{(a, b) \in \mathbb{R}^2 \mid f(a, b) = 1\} \subset \mathbb{R}^2$. Which of the following statements is correct?
(a) If $f(x, y) = x^2 + 4xy + 3y^2$, then X_f is compact
✓ (b) If $f(x, y) = x^2 - 3xy + 3y^2$, then X_f is compact
(c) If $f(x, y) = x^2 - 4xy - y^2$, then X_f is compact
(d) None of the remaining three statements is correct

(16) What is the number of distinct subfields of $\mathbb{C}$ isomorphic to $\mathbb{Q}[\sqrt[3]{2}]$?
(a) 1
(b) 2
✓ (c) 3
(d) Infinite

(17) Let $\mathbb{F}_3$ denote the finite field with 3 elements. What is the number of one dimensional vector subspaces of the vector space $\mathbb{F}_3^5$ over $\mathbb{F}_3$?
(a) 5
✓ (b) 121
(c) 81
(d) None of the remaining three options

(18) For a positive integer n, let a_n, b_n, c_n, d_n be the real numbers such that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}.$$

Which of the following numbers equals $\lim_{n\to\infty} a_n/b_n$?
(a) 1
(b) e
(c) 3/2
✓ (d) None of the remaining three options

(19) Consider the complex vector space

$$V = \{f \in \mathbb{C}[x] \mid f \text{ has degree at most 50, and } f(ix) = -f(x) \text{ for all } x \in \mathbb{C}\}.$$

Then the dimension of V equals
(a) 50
(b) 25
✓ (c) 13
(d) 47

(20) Let S denote the set of sequences $a = (a_1, a_2, \dots)$ of real numbers such that a_k equals 0 or 1 for each k. Then the function $f : S \to \mathbb{R}$ defined by

$$f((a_1, a_2, \dots)) = \frac{a_1}{10} + \frac{a_2}{10^2} + \dots$$

is
✓ (a) injective but not surjective
(b) surjective but not injective
(c) bijective
(d) neither injective nor surjective

Part B — True/False Questions

T (1) If G is a group of order 361, then G has a normal subgroup H such that $H \cong G/H$.

F (2) There exists a metric space X such that the number of open subsets of X is exactly 2024.

F (3) The function $d\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $d(x, y) = |e^x - e^y|$ defines a metric on $\mathbb{R}$, and $(\mathbb{R}, d)$ is a complete metric space.

F (4) Let n be a positive integer, and A an $n \times n$ matrix over $\mathbb{R}$ such that $A^3 = \mathrm{Id}$. Then A is diagonalizable in $\mathrm{M}_n(\mathbb{R})$, i.e., there exists $P \in \mathrm{M}_n(\mathbb{R})$ such that P is invertible and PAP^{-1} is a diagonal matrix.

T (5) If $A \in \mathrm{M}_n(\mathbb{Q})$ is such that the characteristic polynomial of A is irreducible over $\mathbb{Q}$, then A is diagonalizable in $\mathrm{M}_n(\mathbb{C})$, i.e., there exists $P \in \mathrm{M}_n(\mathbb{C})$ such that P is invertible and PAP^{-1} is a diagonal matrix.

T (6) The complement of any countable union of lines in $\mathbb{R}^3$ is path connected.

T (7) The subsets $\{(x,y) \in \mathbb{R}^2 \mid (y^2 - x)(y^2 - x - 1) = 0\}$ and $\{(x,y) \in \mathbb{R}^2 \mid y^2 - x^2 = 1\}$ of $\mathbb{R}^2$ (with the induced metric) are homeomorphic.

F (8) $\mathbb{Q} \cap [0,1]$ is a compact subset of $\mathbb{Q}$.

T (9) Suppose $f : X \to Y$ is a function between metric spaces, such that whenever a sequence $\{x_n\}$ converges to x in X, the sequence $\{f(x_n)\}$ converges in Y (but it is not given that the limit of $\{f(x_n)\}$ is $f(x)$). Then f is continuous.

T (10) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and assume that $|f'(x)| \geq 1$ for all $x \in \mathbb{R}$. Then for each compact set $C \subset \mathbb{R}$, the set $f^{-1}(C)$ is compact.

F (11) There exists a function $f : [0,1] \to \mathbb{R}$, which is not Riemann integrable and satisfies

$$\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^2 < 1,$$

for every choice of a positive integer n and of $0 \leq t_0 < t_1 < t_2 < \cdots < t_n \leq 1$.

F (12) Let $E \subset [0,1]$ be the subset consisting of numbers that have a decimal expansion which does not contain the digit 8. Then E is dense in $[0,1]$.

T (13) Let G be a proper subgroup of $(\mathbb{R}, +)$ which is closed as a subset of $\mathbb{R}$. Then G is generated by a single element.

T (14) There exists a unique function $f : \mathbb{R} \to \mathbb{R}$ such that f is continuous at $x = 0$, and such that for all $x \in \mathbb{R}$

$$f(x) + f\left(\frac{x}{2}\right) = x.$$

F (15) A map $f : V \to W$ between finite dimensional vector spaces over $\mathbb{Q}$ is a linear transformation if and only if $f(x) = f(x-a) + f(x-b) - f(x-a-b)$, for all $x, a, b \in V$.

F (16) Let R be the ring $\mathbb{C}[x]/(x^2)$ obtained as the quotient of the polynomial ring $\mathbb{C}[x]$ by its ideal generated by x^2. Let $R^\times$ be the multiplicative group of units of this ring. Then there is an injective group homomorphism from $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ into $R^\times$.

T (17) Let $A \in \mathrm{M}_2(\mathbb{Z})$ be such that $|A_{ij}(n)| \leq 50$ for all $1 \leq n \leq 10^{50}$ and all $1 \leq i,j \leq 2$, where $A_{ij}(n)$ denotes the (i,j)-th entry of the 2×2 matrix A^n. Then $|A_{ij}(n)| \leq 50$ for all positive integers n.

F (18) Let A, B be subsets of $\{0, \ldots, 9\}$. It is given that, on choosing elements $a \in A$ and $b \in B$ at random, $a + b$ takes each of the values $0, \ldots, 9$ with equal probability. Then one of A or B is singleton.

T (19) If $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous, then there exists $M > 0$ such that for all $x \in \mathbb{R} \setminus [-M, M]$, we have $f(x) < x^{100}$.

T (20) If a sequence $\{f_n\}$ of continuous functions from $[0,1]$ to $\mathbb{R}$ converges uniformly on $(0,1)$ to a continuous function $f : [0,1] \to \mathbb{R}$, then $\{f_n\}$ converges uniformly on $[0,1]$ to f.