GS2025 Exam, School of Mathematics, TIFR

Notation and Conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \ldots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- $\mathbb{R}^n$ denotes the Euclidean space of dimension n. Subsets of $\mathbb{R}^n$ are viewed as metric spaces using the standard Euclidean distance on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, $\|x\|$ denotes the standard Euclidean norm of x, i.e., the distance from x to 0.
- All rings are associative, with a multiplicative identity. Ring homomorphisms are assumed to respect multiplicative identities.
- For any ring R, $\mathrm{M}_n(R)$ denotes the ring of $n \times n$ matrices with entries in R. The identity matrix in $\mathrm{M}_n(R)$ will be denoted by Id or by Id_n. For a matrix $A \in \mathrm{M}_n(R)$, A^t will stand for its tranpose, $\mathrm{trace}(A)$ for its trace, and $\det(A)$ for its determinant.
- $\mathrm{M}_n(\mathbb{R})$ will also be viewed as a real vector space, and $\mathrm{M}_n(\mathbb{C})$ as a complex vector space. $\mathrm{M}_n(\mathbb{R})$ is given the topology such that any $\mathbb{R}$-linear isomorphism $\mathrm{M}_n(\mathbb{R}) \to \mathbb{R}^{n^2}$ is a homeomorphism. Subsets of $\mathrm{M}_n(\mathbb{R})$ are given the subspace topology.
- For a ring R, $R[x_1, \ldots, x_n]$ denotes the polynomial ring in n variables $x_1, \ldots, x_n$ over R.
- If A is a set, $\#A$ stands for the cardinality of A, and equals ∞ if A is infinite.
- If B is a subset of a set A, we write $A \backslash B$ for the set $\{a \in A \mid a \notin B\}$.
- Let G be a group, and let $S \subset G$. The subgroup of G generated by S is defined to be the smallest subgroup of G that contains S.

(1) For a positive integer m, let $d(m)$ denote the number of divisors of m, including 1 and m. Define a sequence $\{a_n\}_{n=1}^{\infty}$ by

$$a_n = \#\{1 \leqslant m \leqslant n \mid \ d(m) \text{ is odd}\}.$$

Find

$$\lim_{n\to\infty} \frac{a_n}{n}.$$

✓ (a) 0
(b) 1
(c) 1/2
(d) None of the remaining three options

(2) Let S_3 be the symmetric group on 3 letters. Consider the real vector space

$$V = \{f : S_3 \to \mathbb{R} \mid f(g) = f(g^3), \forall\, g \in S_3\}.$$

What is the dimension of this space?
(a) 1
(b) 2
(c) 3
✓ (d) Greater than 3

(3) Consider the set

$$S = \{(x, y) \in \mathbb{R}^2 \mid (x^3 + 1)^3 = (y^5 + 1)^5\}.$$

How many connected components does S have?
✓ (a) 1
(b) 3
(c) Greater than 3, but finite
(d) Infinite

(4) Define a set S of real polynomials as follows:

$$S = \{p \in \mathbb{R}[x] \mid p \text{ is not constant}, |p(0)| < 1\}.$$

For p in S, define

$$U_p = \{x \in \mathbb{R} \mid |p(x)| < 1\}.$$

Which of the following statements is correct?
(a) There exists $p \in S$ such that U_p is a union of infinitely many disjoint open intervals.
(b) For each $p \in S$, U_p is a union of infinitely many disjoint open intervals.
✓ (c) For each $p \in S$, U_p is a union of finitely many open intervals.
(d) There exists $p \in S$ such that U_p is an unbounded set.

(5) What is the number of functions $f : \mathbb{R}\backslash\{0, 1\} \to \mathbb{R}\backslash\{0, 1\}$ such that $|f(x) - f(y)| = |x - y|$ for all $x, y \in \mathbb{R}\backslash\{0, 1\}$?
(a) 1
✓ (b) 2
(c) Greater than 2, but finite
(d) Infinite

(6) Let $x_n = \left(1 - \frac{1}{\sqrt{n}}\right)^n \exp(n^{\frac{1}{4}})$. Which of the following statements is correct?
✓ (a) $\lim\limits_{n\to\infty} x_n = 0$ and $\sum_{n=1}^{\infty} x_n$ converges.

(b) $\lim_{n\to\infty} x_n = 0$ but $\sum_{n=1}^{\infty} x_n$ diverges.
(c) $\lim_{n\to\infty} x_n$ exists but is non-zero.
(d) $\lim_{n\to\infty} x_n$ does not exist.

(7) Let $f : [0,1] \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} \frac{1}{2^n}, & \text{if } x = \frac{p}{2^n}, \text{ where } p \text{ is an odd integer, and} \\ 0, & \text{otherwise.} \end{cases}$$

Which of the following statements is correct?
(a) f is continuous at $x \in [0,1]$ if and only if x is rational.
(b) f is continuous at $x \in [0,1]$ if and only if x is irrational.
(c) f is not continuous at x, for any $x \in [0,1]$.
✓ (d) None of the remaining three statements is true.

(8) Let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of the rationals. In other words, $\mathbb{Q} = \{q_n \mid n \in \mathbb{N}\backslash\{0\}\}$, and $q_m \neq q_n$ if $m \neq n$. Set

$$X = \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n^2}, q_n + \frac{1}{n^2} \right).$$

Which of the following statements is correct?
✓ (a) X is dense in $\mathbb{R}$, but not equal to $\mathbb{R}$.
(b) $X = \mathbb{R}$.
(c) X contains an open interval of length 10.
(d) None of the remaining three statements is correct.

(9) Consider the linear map $T : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $T(p(x)) = \frac{d}{dx}(xp(x))$. Which of the following statements is correct?
(a) T is injective but not surjective.
(b) T is surjective but not injective.
✓ (c) T is bijective.
(d) T is neither injective nor surjective.

(10) Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

Consider the following statements:
(i) A has a positive eigenvalue.
(ii) A has a negative eigenvalue.

Which of the following statements is correct?

✓ (a) (i) and (ii) are true.
(b) (i) and (ii) are false.
(c) (i) is true and (ii) is false.
(d) (ii) is true and (i) is false.

(11) Let A be a nonzero $n \times n$ matrix with real entries, where $n > 1$. Consider the following statements.
(i) If $A^2 = 0$ then $\text{rank}(A) \leqslant \lfloor \frac{n}{2} \rfloor$.
(ii) If $\text{rank}(A) \leqslant \lfloor \frac{n}{2} \rfloor$ then $A^2 = 0$.

Which of the following statements is correct?

(a) (i) and (ii) are true.
(b) (i) and (ii) are false.

✓ (c) (i) is true and (ii) is false.
(d) (ii) is true and (i) is false.

(12) Consider the following subset of $\mathbb{R}^2$:

$$S = \left\{ \left(x, \sin\frac{1}{x} \right) \middle| x > 0 \right\} \cup \left\{ \left(0, \pm\frac{1}{n} \right) \mid n \geqslant 1 \right\}.$$

Which of the following statements is correct?
(a) S is path connected.
✓ (b) S is connected, but not path connected.
(c) S is not connected, but has finitely many connected components.
(d) S has infinitely many connected components.

(13) Consider the following statements:

(i) Any linear map $\mathbb{R}^3 \to \mathbb{R}^3$ has a one-dimensional invariant subspace.
(ii) Any linear map $\mathbb{R}^3 \to \mathbb{R}^3$ has a two-dimensional invariant subspace.

Which of the following statements is correct?
✓ (a) (i) and (ii) are both true.
(b) (i) and (ii) are both false.
(c) (i) is true and (ii) is false.
(d) (i) is false and (ii) is true.

(14) Consider the following statements:
(i) For any linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ with a two-dimensional kernel, the trace of T is an eigenvalue of T.
(ii) For any linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ with a two-dimensional kernel, X^2 divides the characteristic polynomial of T.

Which of the following statements is correct?
✓ (a) (i) and (ii) are both true.
(b) (i) and (ii) are both false.
(c) (i) is true and (ii) is false.
(d) (i) is false and (ii) is true.

(15) For a positive integer n, define the function $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \sin\left(\frac{x}{n}\right)$. Consider the sequences $\{f_n\}_{n=1}^{\infty}$ and $\{f'_n\}_{n=1}^{\infty}$ of functions from $\mathbb{R}$ to $\mathbb{R}$. Which of the following statements is correct?
(a) The sequences $\{f_n\}_{n=1}^{\infty}$ and $\{f'_n\}_{n=1}^{\infty}$ both converge uniformly on $\mathbb{R}$.
(b) Neither of the sequences $\{f_n\}_{n=1}^{\infty}$ and $\{f'_n\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$.
(c) The sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$, but the sequence $\{f'_n\}_{n=1}^{\infty}$ does not.
✓ (d) The sequence $\{f'_n\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$, but the sequence $\{f_n\}_{n=1}^{\infty}$ does not.

(16) Let $n > 1$ be a positive integer. Let $A = [a_{ij}]_{1 \leqslant i,j \leqslant n}$ be an $n \times n$ matrix, such that $a_{i,i+1} = c_i$ for $i = 1, \ldots, (n-1)$, and $a_{n,1} = c_n$, where $c_1, \ldots, c_n$ are real numbers and all the other entries are 0. Which of the following sentences is correct regardless of n?
✓ (a) $\det(\mathrm{Id} - A^n) = 0$ if $\prod_{i=1}^{n} c_i = 1$.

(b) $\det(\mathrm{Id} - A^n) > 0$ if $\prod_{i=1}^{n} c_i > 0$.
(c) $\det(\mathrm{Id} - A^n) < 0$ if $\prod_{i=1}^{n} c_i > 1$.
(d) $\det(\mathrm{Id} - A^n) > 0$ if $\prod_{i=1}^{n} c_i > 1$.

(17) Let $n \geqslant 1$. Consider non-zero $n \times n$ real matrices A such that $\mathrm{trace}(AX) = 0$, for every real matrix X with $\mathrm{trace}(X) = 0$. Consider the following assertions.
(i) For every such matrix A, $\mathrm{trace}(A)^n = n^n \cdot \det(A)$.
(ii) For every such matrix A, the rank of A is n.

Which of the following sentences is correct?

(a) (i) is correct and (ii) is incorrect.
(b) (ii) is correct and (i) is incorrect.
✓ (c) (i) and (ii) are correct.
(d) (i) and (ii) are incorrect.

(18) Let S be the set of functions $f : (0, 1) \to \mathbb{R}$ with the property that there is a sequence $\{f_n\}_{n=1}^{\infty}$ of functions that converges uniformly to f, where each $f_n : (0, 1) \to \mathbb{R}$ is a twice continuously differentiable function (i.e., f_n'' exists and is continuous). Which of the following statements is correct?
✓ (a) Any $f \in S$ is continuous, but there exists $f \in S$ such that f is not differentiable.
(b) Any $f \in S$ is differentiable, but there exists $f \in S$ such that f' is not continuous.
(c) Any $f \in S$ is continuously differentiable, but there exists $f \in S$ such that f'' does not exist.
(d) Any $f \in S$ is twice continuously differentiable.

(19) For a real valued function $f : [0, 1] \mapsto \mathbb{R}$, let $\omega_f : [0, 1] \mapsto \mathbb{R}$ be the *oscillation* of f defined by

$$\omega_f(t) := \sup_{|x-y| \leqslant t} |f(x) - f(y)|.$$

Which one of the following statements is correct for every $f : [0, 1] \to \mathbb{R}$?
✓ (a) f is continuous if and only if $\lim_{t \to 0+} \omega_f(t) = 0$.
(b) If f is continuous then $\lim_{t \to 0+} \omega_f(t) = 0$, but not conversely.
(c) If $\lim_{t \to 0+} \omega_f(t) = 0$ then f is continuous, but not conversely.
(d) None of the remaining three statements is correct.

(20) The number of real cubic polynomials of the form $x^3 + ax + b$, each of whose complex roots lies on the unit circle $S^1 = \{z \in \mathbb{C} \mid z\bar{z} = 1\}$, equals
(a) 0
✓ (b) 2
(c) ∞
(d) None of the remaining three options

F (1) There exists a linear polynomial $p(x) \in \mathbb{R}[x]$, for which there exist exactly two subsets $S \subset \mathbb{R}$ such that $p(S) = S$ and such that $\#S = 2$.

- T (2) There are exactly 10 abelian groups of order 2025, up to isomorphism.
- T (3) If $A \subset \mathbb{R}$ is an uncountable subset, then there exist uncountably many elements $x \in A$ such that x is a limit point of $A \backslash \{x\}$.
- F (4) There exists an open subset of $\mathbb{R}$ with uncountably many connected components.
- T (5) Let (X, d) be a nonempty complete metric space, and let $T : X \to X$ be a continuous map such that T^2 is a contraction, i.e., there exists $a < 1$ such that for all distinct $x, y \in X$, we have $d(T^2(x), T^2(y)) < ad(x, y)$. Then there exists a unique element $x \in X$ such that $T(x) = x$.
- T (6) Let

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\},$$

and let $d : S^2 \times S^2 \to \mathbb{R}$ be the restriction of the Euclidean metric on $\mathbb{R}^3$. If $f : S^2 \to S^2$ is a map such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in S^2$, then f is surjective.
- F (7) Let $C([0, 1], \mathbb{R})$ be the set of continuous functions from $[0, 1]$ to $\mathbb{R}$, equipped with the metric d given by

$$d(f_1, f_2) = \sup_{x \in [0,1]} |f_1(x) - f_2(x)|.$$

Let $X \subset C([0, 1], \mathbb{R})$ be defined by

$$X := \left\{ f \in C([0, 1], \mathbb{R}) \mid \int_0^1 f(t)\, dt \neq 0 \right\}.$$

Then, with the induced metric, X is connected.
- F (8) Let $SO_4 = \{A \in \mathrm{M}_4(\mathbb{R}) \mid AA^t = \mathrm{Id}, \text{ and } \det(A) > 0\}$. Then every matrix in SO_4 has 1 or -1 as an eigenvalue.
- T (9) Let T be an endomorphism of a finite dimensional real vector space V, and let W denote the image of T. Let T' denote the restriction of T to W. Then T and T' have the same trace.
- F (10) Every subgroup of $S^1 = \{z \in \mathbb{C} \mid z\bar{z} = 1\}$, viewed as a group under multiplication, is finite.
- F (11) If $f : F \to F$ is a homomorphism of fields, then f is surjective.
- T (12) Given any linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, there exist linear transformations $T_1, T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $(1, 0)$ belongs to the kernel of T_1, $(0, 1)$ belongs to the image of T_2, and $T = T_1 + T_2$.
- T (13) For any positive integer d, the set

$$\{10^a - 10^b \mid a, b \text{ are distinct positive integers}\}$$

contains a multiple of d.
- F (14) There exists $\theta \in (0, \pi/2)$ such that

$$\sin\theta + \sec\theta + \cot\theta = 3.$$

- F (15) Let m be a positive integer, and S_m the symmetric group on m letters. Let n be a positive integer such that for every group G of order n, there exists an injective group homomorphism $G \hookrightarrow S_m$. Then $m \geqslant n$.

T (16) Let ℓ_1 be the line in $\mathbb{R}^2$ joining $(0,0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, and ℓ_2 the line in $\mathbb{R}^2$ joining $(0,0)$ and $(\frac{\sqrt{3}}{2}, \frac{1}{2})$. Consider the group of bijections $\mathbb{R}^2 \to \mathbb{R}^2$ under composition, and its subgroup G generated by the reflections about ℓ_1 and ℓ_2. Then G has exactly 12 elements.

F (17) There are only finitely many subsets $I \subset \mathbb{N}$ with the property that

$$\left\{ f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{R}[x] \middle| n \in \mathbb{N}, a_i \in \mathbb{Q} \text{ for all } i \in I \right\}$$

is a subring of $\mathbb{R}[x]$.

T (18) There are uncountably many continuous functions $\mathbb{Q} \to \{0,1\}$.

F (19) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Suppose there exists $C \in \mathbb{R}$ such that f is injective on (C, ∞). Then there exists $D \in \mathbb{R}$ such that f' is injective on (D, ∞).

T (20) If $f, g : [0,1] \to \mathbb{R}$ are continuous and $f \cdot g = 0$, then one of f or g is zero on an open subset of $[0,1]$.