GS2025 - Mathematics PhD at CAM

Notations and Conventions

- $\mathbb{N}$ denotes the set of natural numbers $\{0, 1, \ldots\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers, and $\mathbb{C}$ the set of complex numbers. These sets are assumed to carry the usual algebraic and metric structures.
- For sets A and B, $A \setminus B = \{x \in A \colon x \notin B\}$.
- $i = \sqrt{-1}$.
- $\mathbb{R}^n$, $n \geqslant 1$, is equipped with the Euclidean topology.
- For a subset A of a topological space, $\overline{A}$ denotes the closure of A.
- For a real normed linear space X,
  - the dual of X is $X^* = \{\varphi \colon X \to \mathbb{R} : \ \varphi \text{ is linear and continuous}\}$, and
  - $\langle Tx, y\rangle := (Tx)(y)$ for $T : X \to X^*$ and $x, y \in X$.
- $C([0,1])$ denotes the set of continuous real-valued functions on the interval $[0,1]$.
- For $k \in \mathbb{N}$, $C^k([0,1])$ denotes the set of real-valued functions having k continuous derivatives on $(0,1)$, and $f^{(k)} \in C([0,1])$.
- For a set $S \subseteq \mathbb{R}$ of positive Lebesgue measure and $1 \leqslant p < \infty$,

$$L^p(S) = \{f \colon S \to \mathbb{R} : \ f \text{ is measurable and } \int_S |f(x)|^p \,\mathrm{d}x < \infty\}.$$

- For an $n \times n$ matrix A, $\det(A)$ denotes the determinant of A.

1. For $0 < a < 1$, let $\lambda = \frac{1}{a} + a\mathrm{i}$ and

$$z_n = \int_{\mathbb{R}} e^{\mathrm{i}nx} e^{-\lambda x^2} \mathrm{d}x.$$

What is $\lim_{n \to \infty} z_n$?

(a) a
(b) ∞
(c) 0
(d) 1

2. Let $\mathbb{D} = \{(x, y) \in \mathbb{R}^2 \colon x^2 + y^2 < 1\}$. For $p \in \mathbb{R}$, define

$$C_p = \int_{\mathbb{D}} \frac{\sin^2(x^2 + y^2)}{(x^2 + y^2)^p} \mathrm{d}x\mathrm{d}y.$$

Choose the correct option.

(a) $C_p < \infty$ for all $p > 0$.
(b) $C_p < \infty$ for all $p > 1$.
(c) $C_p < \infty$ for all $p < 3$.
(d) $C_p < \infty$ for all $p < 4$.

3. Let $p(x) = x^3 - x^2 - x + 1$. Consider the following statements.

(I) *If $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and $p(T) = 0$, then T is invertible.*
(II) *If $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation and $p(T) = 0$, then T is diagonalizable.*

Choose the correct option.

(a) (I) and (II) are both true.
(b) (I) and (II) are both false.
(c) (I) is true and (II) is false.
(d) (I) is false and (II) is true.

4. What is the cardinality of $\{A \subseteq \mathbb{R} \colon \mathbb{R} \setminus \overline{A} = \overline{\mathbb{R} \setminus A}\}$?

(a) 0
(b) 1
(c) 2
(d) infinite

5. Let $y \in C^2([0,1])$ be a non identically zero solution to

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}(t) + \big(e^{t^2} \sin t\big) y(t) = 0, \quad t \in (0,1).$$

Choose the correct option.

(a) y can have infinitely many zeroes.

(b) Either y has no zeroes or finitely many zeroes.

(c) The third derivate of y does not exist at $1/2$.

(d) The equation has a unique non identically zero solution.

6. Let f be an entire function which satisfies $|f(z)| \geqslant |z|^2$ for all $z \in \mathbb{C}$, and assume moreover that $f(i) = -i$. Then,

(a) $f(0) \neq 0$

(b) $f(1) = -3$

(c) $f(2)$ cannot be uniquely determined from the given information.

(d) $f(3) = 9i$

7. Let $\mathbb{H} = \{x + iy \colon x \in \mathbb{R},\ y > 0\}$ and $f : \mathbb{H} \to \mathbb{H}$ be a bijective holomorphic function such that $\lim_{|z|\to\infty} |f(z)| = \infty$. Choose the correct option.

(a) f is a linear function.

(b) f is a rational function that is not a polynomial.

(c) $f(\{it \colon t > 0\})$ is a subset of a circle of finite radius.

(d) $f(\{it \colon t > 0\})$ is a subset of a horizontal line.

8. Let $f \colon \mathbb{R} \to \mathbb{R}$ be defined by $f(y) = y \sin(1/y)$ for $y \neq 0$, and $f(0) = 0$. Then, the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t}(t) = f(y(t)),\ t \in \mathbb{R}, \\ y(0) = 0, \end{cases}$$

has

(a) at least one strictly monotonic solution.

(b) a unique solution; which is $y \equiv 0$.

(c) infinitely many solutions.

(d) at least one oscillatory solution (i.e., the solution is not identically zero and has infinitely many zeroes).

9. Choose the correct option.

For any real Banach space X and any linear operator $T : X \to X^*$, if $\langle Tx, y\rangle = \langle Ty, x\rangle$ for all $x, y \in X$, then

(a) T is invertible.

(b) T is surjective.

(c) T is bounded.

(d) $T = 0$.

10. Choose the correct option.

(a) If $f : \mathbb{R} \to \mathbb{R}$ is a bijective differentiable map then its inverse is also a bijective differentiable map.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is a bijective differentiable map then for all $x \in \mathbb{R}$ there exists an $\epsilon > 0$ and a differentiable map $g : (x - \epsilon, x + \epsilon) \to \mathbb{R}$ such that $f \circ g(t) = t$ for all $t \in (x - \epsilon, x + \epsilon)$.

(c) If $f : \mathbb{R} \to \mathbb{R}$ is a bijective differentiable map then f is either monotonically increasing or monotonically decreasing.

(d) If $f : \mathbb{R} \to \mathbb{R}$ is a bijective differentiable map then $f'(x) > 0$ for all $x \in \mathbb{R}$ or $f'(x) < 0$ for all $x \in \mathbb{R}$.

11. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following condition: for each x_0 and for each $r > 0$,

$$f(x_0) = \frac{1}{r} \int_{x_0 - r}^{x_0 + r} f(y) \mathrm{d}y.$$

Choose the correct statement.

(a) There exists a non constant function $f \in L^2(\mathbb{R})$ satisfying the above property.

(b) There exists at most one function in $L^2(\mathbb{R})$ that satisfies the above property.

(c) The space of functions $f \in L^2(\mathbb{R})$ and satisfying the above property is one dimensional.

(d) The space of functions $f \in L^2(\mathbb{R})$ and satisfying the above property is infinite dimensional.

12. Consider the 11×11 matrix:

$$A = \begin{pmatrix} -2 & 1 & -\frac{1}{2^2} & \frac{1}{3^2} & \cdots & -\frac{1}{10^2} \\ \frac{1}{11^2} & 2 & -\frac{1}{12^2} & \frac{1}{13^2} & \cdots & -\frac{1}{20^2} \\ \frac{1}{21^2} & -\frac{1}{22^2} & -2 & \frac{1}{23^2} & \cdots & -\frac{1}{30^2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{101^2} & \cdots & \cdots & \cdots & -\frac{1}{110^2} & -2 \end{pmatrix}.$$

Consider the following statements.

(I) *The eigenvalues of A are all non-negative.*

(II) *A is invertible.*

Choose the correct option.

(a) (I) and (II) are both true.

(b) (I) and (II) are both false.

(c) (I) is true and (II) is false.

(d) (I) is false and (II) is true.

13. For sets $A, B \subseteq \mathbb{R}^n$, define

$$A + B = \{a + b : \ a \in A \text{ and } b \in B\}.$$

Choose the correct option.

(a) If A is compact and B is closed, then $A + B$ is compact.

(b) If A is closed and B is closed, then $A + B$ is closed.

(c) If A is open and B is compact, then $A + B$ is open.

(d) If A is open and B is compact, then $A + B$ is compact.

14. Choose the correct option.

If $g\colon \mathbb{R} \to \mathbb{R}$ is a positive continuous function such that $\int_0^\infty e^{\alpha x} g(x) \mathrm{d}x < \infty$ for some $\alpha > 0$, then

(a) $\int_0^\infty \frac{g(x)}{x^n} \mathrm{d}x < \infty$ for all $n \in \mathbb{N}$.

(b) $\lim_{x \to \infty} g(x)$ exists.

(c) $\int_0^\infty \frac{1}{g^\beta(x)} \mathrm{d}x < \infty$ for some $\beta > 0$.

(d) there exist $C > 0$ and $\beta > 0$ such that

$$\int_t^\infty g(x) \mathrm{d}x \leqslant Ce^{-\beta t},$$

for all $t > 0$.

15. Let $f_n : \mathbb{R} \to \mathbb{R}$ be sequence of differentiable functions such that

$$\int_0^1 f_n(x)\, dx = 0 \quad \text{and} \quad |f_n'(x)| \leqslant x^{-1/2} \text{ for all } x \in (0, 1].$$

Consider the following statements.

(I) *$\{f_n\}$ is uniformly bounded on $[0,1]$.*

(II) *$\{f_n\}$ has a subsequence that converges uniformly on $[0,1]$.*

Choose the correct option.

(a) (I) and (II) are both true.

(b) (I) and (II) are both false.

(c) (I) is true and (II) is false.

(d) (I) is false and (II) is true.

16. Let $n \geqslant 2$, and A be an $n \times n$ real symmetric and positive definite matrix satisfying $\operatorname{trace}(A) = \det(A)$.

Choose the correct statement that holds for every such A.

(a) $\operatorname{trace}(A) < n^{\frac{n}{n-1}}$.

(b) $\operatorname{trace}(A) \leqslant n$.

(c) $\det(A) \geqslant n^{\frac{n}{n-1}}$.

(d) all of the eigenvalues of A are $\leqslant \frac{1}{2}$.

17. A box contains 2024 balls in which 200 are blue and the rest are red. Balls are chosen from the box one by one at random, and discarded.

Consider the following statements.

(I) *The probability of picking a red ball at the k-th pick is the same for all $k = 1, \ldots, 2024$.*

(II) *The probability of picking a blue ball at the 200-th pick is the same as the probability of picking a red ball at 1824-th pick.*

(III) *The probability of picking all the blue balls in the first 200 draws is the same as the probability of picking all the red balls in the first 1824 draws.*

Choose the correct option.

(a) (I) and (II) are true, (III) is false.

(b) (I) and (III) are true, (II) is false.

(c) (II) and (III) are true, (I) is false.

(d) (I), (II), and (III) are all true.

18. Let X denote the inner product space $\big(C([0,1]), \langle \cdot, \cdot \rangle\big)$ where $\langle f, g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$.

Define $T\colon X \to X$ by $(Tf)(x) = \int_x^1 f(t)\,\mathrm{d}t$ for $f \in X$ and $x \in [0,1]$. Choose the correct statement.

(a) T is surjective.

(b) There exists a bounded linear operator $U: X \to X$ such that $U \circ T$ is the identity map.

(c) $T(X)$ is dense in X.

(d) $T(X)$ is a closed subset of X.

19. Let $T : L^2(0,1) \mapsto L^2(0,1)$ defined by $(Tf)(x) = f(\sqrt{x})$. We denote the operator norm of T by $\|T\|$.

Choose the correct option.

(a) $\|T\| = \sqrt{2}$.

(b) $\|T\| = 2$.

(c) $\|T\| = \frac{1}{\sqrt{2}}$.

(d) $\|T\| = \frac{1}{2}$.

20. Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^{-\frac{1}{2}}, & \text{if } 0 < x < 1, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Let r_n be an enumeration of rational numbers and correspondingly define

$$h(x) = \sum_{n=1}^{\infty} 2^{-n} f(x - r_n), \quad \text{for } x \in \mathbb{R}.$$

Choose the correct statement.

(a) $h \in L^1(\mathbb{R})$ and $h \in L^2(\mathbb{R})$.

(b) $h \in L^1(\mathbb{R})$ and $h \notin L^2(\mathbb{R})$.

(c) $h \notin L^1(\mathbb{R})$ and $h \in L^2(\mathbb{R})$.

(d) $h \notin L^1(\mathbb{R})$ and $h \notin L^2(\mathbb{R})$.