

3. If $A = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $C = ABA^T$ and X

$= A^T C^2 A$, then $\det X$ is equal to :

(1) 243

(2) 729

(3) 27

(4) 891

Ans. (2)

Sol.

$$A = \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix} \Rightarrow \det(A) = 3$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \det(B) = 1$$

Now $C = ABA^T \Rightarrow \det(C) = (\det(A))^2 \times \det(B)$

$$|C| = 9$$

Now $|X| = |A^T C^2 A|$

$$= |A^T| |C|^2 |A|$$

$$= |A|^2 |C|^2$$

$$= 9 \times 81$$

$$= 729$$

4. If $\tan A = \frac{1}{\sqrt{x(x^2+x+1)}}$, $\tan B = \frac{\sqrt{x}}{\sqrt{x^2+x+1}}$

and

$$\tan C = (x^{-3} + x^{-2} + x^{-1})^{\frac{1}{2}}, 0 < A, B, C < \frac{\pi}{2}, \text{ then}$$

$A + B$ is equal to :

(1) C

(2) $\pi - C$

(3) $2\pi - C$

(4) $\frac{\pi}{2} - C$

Ans. (1)

Sol.

Finding $\tan(A + B)$ we get

$$\Rightarrow \tan(A + B) =$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{1}{\sqrt{x(x^2+x+1)}} + \frac{\sqrt{x}}{\sqrt{x^2+x+1}}}{1 - \frac{1}{x^2+x+1}}$$

$$\Rightarrow \tan(A + B) = \frac{(1+x)(\sqrt{x^2+x+1})}{(x^2+x)(\sqrt{x})}$$

$$\frac{(1+x)(\sqrt{x^2+x+1})}{(x^2+x)(\sqrt{x})}$$

$$\tan(A + B) = \frac{\sqrt{x^2+x+1}}{x\sqrt{x}} = \tan C$$

$$A + B = C$$

5. If n is the number of ways five different employees can sit into four indistinguishable offices where any office may have any number of persons including zero, then n is equal to:

(1) 47

(2) 53

(3) 51

(4) 43

Ans. (3)

Sol.

Total ways to partition 5 into 4 parts are :

$$5, 0, 0, 0 \Rightarrow 1 \text{ way}$$

$$4, 1, 0, 0 \Rightarrow \frac{5!}{4!} = 5 \text{ ways}$$

$$3, 2, 0, 0 \Rightarrow \frac{5!}{3!2!} = 10 \text{ ways}$$

$$2, 2, 0, 1 \Rightarrow \frac{5!}{2!2!2!} = 15 \text{ ways}$$

$$2, 1, 1, 1 \Rightarrow \frac{5!}{2!(1!)^3 3!} = 10 \text{ ways}$$

$$3, 1, 1, 0 \Rightarrow \frac{5!}{3!2!} = 10 \text{ ways}$$

$$\text{Total} \Rightarrow 1+5+10+15+10+10 = 51 \text{ ways}$$

6. Let $S = \{z \in \mathbb{C} : |z-1| = 1 \text{ and } (\sqrt{2}-1)(z+\bar{z}) - i(z-\bar{z}) = 2\sqrt{2}\}$. Let $z_1, z_2 \in S$ be such that $|z_1| = \max_{z \in S} |z|$ and $|z_2| = \min_{z \in S} |z|$.

Then $|\sqrt{2}z_1 - z_2|^2$ equals :

- (1) 1 (2) 4
(3) 3 (4) 2

Ans. (4)

Sol. Let $Z = x + iy$

Then $(x-1)^2 + y^2 = 1 \rightarrow (1)$

$$\& (\sqrt{2}-1)(2x) - i(2iy) = 2\sqrt{2}$$

$$\Rightarrow (\sqrt{2}-1)x + y = \sqrt{2} \rightarrow (2)$$

Solving (1) & (2) we get

$$\text{Either } x = 1 \text{ or } x = \frac{1}{2-\sqrt{2}} \rightarrow (3)$$

On solving (3) with (2) we get

$$\text{For } x = 1 \Rightarrow y = 1 \Rightarrow Z_2 = 1 + i$$

& for

$$x = \frac{1}{2-\sqrt{2}} \Rightarrow y = \sqrt{2} - \frac{1}{\sqrt{2}} \Rightarrow Z_1 = \left(1 + \frac{1}{\sqrt{2}}\right) + \frac{i}{\sqrt{2}}$$

Now

$$\begin{aligned} & |\sqrt{2}z_1 - z_2|^2 \\ &= \left| \left(\frac{1}{\sqrt{2}} + 1\right)\sqrt{2} + i - (1+i) \right|^2 \\ &= (\sqrt{2})^2 \\ &= 2 \end{aligned}$$

7. Let the median and the mean deviation about the median of 7 observation 170, 125, 230, 190, 210, a, b be 170 and $\frac{205}{7}$ respectively. Then the mean

deviation about the mean of these 7 observations is :

- (1) 31
(2) 28
(3) 30
(4) 32

Ans. (3)

Sol. Median = 170 \Rightarrow 125, a, b, 170, 190, 210, 230

Mean deviation about

Median =

$$\frac{0+45+60+20+40+170-a+170-b}{7} = \frac{205}{7}$$

$$\Rightarrow a + b = 300$$

$$\text{Mean} = \frac{170+125+230+190+210+a+b}{7} = 175$$

Mean deviation

About mean =

$$\frac{50+175-a+175-b+5+15+35+55}{7} = 30$$

8. Let $\vec{a} = -5\hat{i} + \hat{j} - 3\hat{k}, \vec{b} = \hat{i} + 2\hat{j} - 4\hat{k}$ and

$\vec{c} = \left(\left((\vec{a} \times \vec{b}) \times \hat{i} \right) \times \hat{i} \right) \times \hat{i}$. Then $\vec{c} \cdot (-\hat{i} + \hat{j} + \hat{k})$ is equal to

- (1) -12 (2) -10
(3) -13 (4) -15

Ans. (1)

Sol. $\vec{a} = -5\hat{i} + \hat{j} - 3\hat{k}$

$$\vec{b} = \hat{i} + 2\hat{j} - 4\hat{k}$$

$$(\vec{a} \times \vec{b}) \times \hat{i} = (\vec{a} \cdot \hat{i})\vec{b} - (\vec{b} \cdot \hat{i})\vec{a}$$

$$= -5\vec{b} - \vec{a}$$

$$= \left(\left((-5\vec{b} - \vec{a}) \times \hat{i} \right) \times \hat{i} \right)$$

$$= \left((-11\hat{j} + 23\hat{k}) \times \hat{i} \right) \times \hat{i}$$

$$\Rightarrow (11\hat{k} + 23\hat{j}) \times \hat{i}$$

$$\Rightarrow (11\hat{j} - 23\hat{k})$$

$$\vec{c} \cdot (-\hat{i} + \hat{j} + \hat{k}) = 11 - 23 = -12$$

9. Let $S = \{x \in \mathbf{R} : (\sqrt{3} + \sqrt{2})^x + (\sqrt{3} - \sqrt{2})^x = 10\}$.

Then the number of elements in S is :

- (1) 4 (2) 0
(3) 2 (4) 1

Ans. (3)

Sol. $(\sqrt{3} + \sqrt{2})^x + (\sqrt{3} - \sqrt{2})^x = 10$

Let $(\sqrt{3} + \sqrt{2})^x = t$

$$t + \frac{1}{t} = 10$$

$$t^2 - 10t + 1 = 0$$

$$t = \frac{10 \pm \sqrt{100 - 4}}{2} = 5 \pm 2\sqrt{6}$$

$$(\sqrt{3} + \sqrt{2})^x = (\sqrt{3} \pm \sqrt{2})^2$$

$$x = 2 \text{ or } x = -2$$

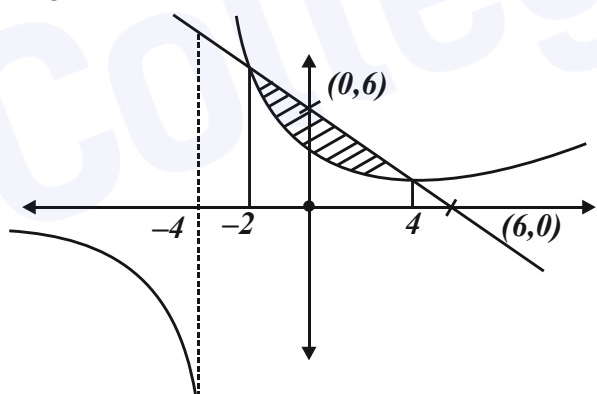
Number of solutions = 2

10. The area enclosed by the curves $xy + 4y = 16$ and $x + y = 6$ is equal to :

- (1) $28 - 30 \log_e 2$ (2) $30 - 28 \log_e 2$
(3) $30 - 32 \log_e 2$ (4) $32 - 30 \log_e 2$

Ans. (3)

Sol. $xy + 4y = 16$, $x + y = 6$
 $y(x + 4) = 16$ (1) , $x + y = 6$ (2)
 on solving, (1) & (2)
 we get $x = 4$, $x = -2$



$$\begin{aligned} \text{Area} &= \int_{-2}^4 \left((6-x) - \left(\frac{16}{x+4} \right) \right) dx \\ &= 30 - 32 \ln 2 \end{aligned}$$

11. Let $f : \mathbf{R} \rightarrow \mathbf{R}$ and $g : \mathbf{R} \rightarrow \mathbf{R}$ be defined as

$$f(x) = \begin{cases} \log_e x & , x > 0 \\ e^{-x} & , x \leq 0 \end{cases} \text{ and}$$

$$g(x) = \begin{cases} x & , x \geq 0 \\ e^x & , x < 0 \end{cases} \text{ Then, } \text{gof} : \mathbf{R} \rightarrow \mathbf{R} \text{ is :}$$

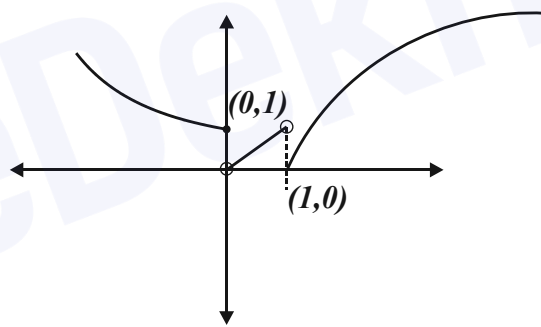
- (1) one-one but not onto
(2) neither one-one nor onto
(3) onto but not one-one
(4) both one-one and onto

Ans. (2)

Sol.

$$g(f(x)) = \begin{cases} f(x), f(x) \geq 0 \\ e^{f(x)}, f(x) < 0 \end{cases}$$

$$g(f(x)) = \begin{cases} e^{-x}, (-\infty, 0] \\ e^{\ln x}, (0, 1) \\ \ln x, [1, \infty) \end{cases}$$



Graph of $g(f(x))$

$g(f(x)) \Rightarrow$ Many one into

12. If the system of equations

$$2x + 3y - z = 5$$

$$x + \alpha y + 3z = -4$$

$$3x - y + \beta z = 7$$

has infinitely many solutions, then $13\alpha\beta$ is equal to

- (1) 1110 (2) 1120
(3) 1210 (4) 1220

Ans. (2)

Sol. At $x = 1$, $f(x)$ is continuous therefore,

$$f(1^-) = f(1) = f(1^+) \quad \dots(1)$$

$$f(1) = 3 + c \quad \dots(1)$$

$$f(1^+) = \lim_{h \rightarrow 0} 2(1+h) + 1$$

$$f(1^+) = \lim_{h \rightarrow 0} 3 + 2h = 3 \quad \dots(2)$$

from (1) & (2)

$$c = 0$$

at $x = 0$, $f(x)$ is continuous therefore,

$$f(0^-) = f(0) = f(0^+) \quad \dots(3)$$

$$f(0) = f(0^+) = 2 \quad \dots(4)$$

$f(0^-)$ has to be equal to 2

$$\lim_{h \rightarrow 0} \frac{a - b \cos(2h)}{h^2}$$

$$\lim_{h \rightarrow 0} \frac{a - b \left\{ 1 - \frac{4h^2}{2!} + \frac{16h^4}{4!} + \dots \right\}}{h^2}$$

$$\lim_{h \rightarrow 0} \frac{a - b + b \left\{ 2h^2 - \frac{2}{3}h^4 \dots \right\}}{h^2}$$

for limit to exist $a - b = 0$ and limit is $2b \quad \dots(5)$

from (3), (4) & (5)

$$a = b = 1$$

checking differentiability at $x = 0$

$$\text{LHD : } \lim_{h \rightarrow 0} \frac{1 - \cos 2h}{h^2} - 2$$

$$\lim_{h \rightarrow 0} \frac{1 - \left(1 - \frac{4h^2}{2!} + \frac{16h^4}{4!} \dots \right) - 2h^2}{-h^3} = 0$$

$$\text{RHD : } \lim_{h \rightarrow 0} \frac{(0+h)^2 + 2 - 2}{h} = 0$$

Function is differentiable at every point in its domain

$$\therefore m = 0$$

$$m + a + b + c = 0 + 1 + 1 + 0 = 2$$

16. Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$ be an ellipse, whose eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is $\sqrt{14}$. Then the square of the eccentricity of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is :

(1) 3 (2) 7/2

(3) 3/2 (4) 5/2

Ans. (3)

Sol.

$$e = \frac{1}{\sqrt{2}} = \sqrt{1 - \frac{b^2}{a^2}} \Rightarrow \frac{1}{2} = 1 - \frac{b^2}{a^2}$$

$$\frac{2b^2}{a} = 14$$

$$e_H = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{1}{2}} = \sqrt{\frac{3}{2}}$$

$$(e_H)^2 = \frac{3}{2}$$

17. Let 3, a, b, c be in A.P. and 3, a - 1, b + 1, c + 9 be in G.P. Then, the arithmetic mean of a, b and c is :

(1) -4 (2) -1

(3) 13 (4) 11

Ans. (4)

Sol.

$$3, a, b, c \rightarrow \text{A.P} \Rightarrow 3, 3+d, 3+2d, 3+3d$$

$$3, a-1, b+1, c+9 \rightarrow \text{G.P} \Rightarrow 3, 2+d, 4+2d, 12+3d$$

$$a = 3 + d \quad (2+d)^2 = 3(4+2d)$$

$$b = 3 + 2d \quad d = 4, -2$$

$$c = 3 + 3d$$

$$\text{If } d = 4 \quad \text{G.P} \Rightarrow 3, 6, 12, 24$$

$$a = 7$$

$$b = 11$$

$$c = 15$$

$$\frac{a+b+c}{3} = 11$$

18. Let $C : x^2 + y^2 = 4$ and $C' : x^2 + y^2 - 4\lambda x + 9 = 0$ be two circles. If the set of all values of λ so that the circles C and C' intersect at two distinct points, is $\mathbf{R} - [a, b]$, then the point $(8a + 12, 16b - 20)$ lies on the curve :

- (1) $x^2 + 2y^2 - 5x + 6y = 3$
- (2) $5x^2 - y = -11$
- (3) $x^2 - 4y^2 = 7$
- (4) $6x^2 + y^2 = 42$

Ans. (4)

Sol. $x^2 + y^2 = 4$

$C(0, 0) \quad r_1 = 2$

$C'(2\lambda, 0) \quad r_2 = \sqrt{4\lambda^2 - 9}$

$|r_1 - r_2| < CC' < |r_1 + r_2|$

$|2 - \sqrt{4\lambda^2 - 9}| < |2\lambda| < 2 + \sqrt{4\lambda^2 - 9}$

$4 + 4\lambda^2 - 9 - 4\sqrt{4\lambda^2 - 9} < 4\lambda^2$

True $\lambda \in \mathbf{R} \dots (1)$

$4\lambda^2 < 4 + 4\lambda^2 - 9 + 4\sqrt{4\lambda^2 - 9}$

$5 < 4\sqrt{4\lambda^2 - 9}$ and $\lambda^2 \geq \frac{9}{4}$

$\frac{16}{16} < 4\lambda^2 - 9 \quad \lambda \in \left(-\infty, -\frac{3}{2}\right] \cup \left[\frac{3}{2}, \infty\right)$

$\frac{169}{64} < \lambda^2$

$\lambda \in \left(-\infty, -\frac{13}{8}\right) \cup \left(\frac{13}{8}, \infty\right) \dots (2)$

from (1) and (2) $\lambda \in$

$\lambda \in \left(-\infty, -\frac{13}{8}\right) \cup \left(\frac{13}{8}, \infty\right) \Rightarrow \mathbf{R} - \left[-\frac{13}{8}, \frac{13}{8}\right]$

as per question $a = -\frac{13}{8}$ and $b = \frac{13}{8}$

\therefore required point is $(-1, 6)$ with satisfies option (4)

19. If $5f(x) + 4f\left(\frac{1}{x}\right) = x^2 - 2, \forall x \neq 0$ and $y = 9x^2f(x)$, then y is strictly increasing in :

- (1) $\left(0, \frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$
- (2) $\left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$
- (3) $\left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(0, \frac{1}{\sqrt{5}}\right)$
- (4) $\left(-\infty, \frac{1}{\sqrt{5}}\right) \cup \left(0, \frac{1}{\sqrt{5}}\right)$

Ans. (2)

Sol. $5f(x) + 4f\left(\frac{1}{x}\right) = x^2 - 2, \forall x \neq 0 \dots (1)$

Substitute $x \rightarrow \frac{1}{x}$

$5f\left(\frac{1}{x}\right) + 4f(x) = \frac{1}{x^2} - 2 \dots (2)$

On solving (1) and (2)

$f(x) = \frac{5x^4 - 2x^2 - 4}{9x^2}$

$y = 9x^2f(x)$

$y = 5x^4 - 2x^2 - 4 \dots (3)$

dy

for strictly increasing

$\frac{dy}{dx} > 0$

$4x(5x^2 - 1) > 0$

$x \in \left(-\frac{1}{\sqrt{5}}, 0\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$

20. If the shortest distance between the lines $\frac{x-\lambda}{-2} = \frac{y-2}{1} = \frac{z-1}{1}$ and $\frac{x-\sqrt{3}}{1} = \frac{y-1}{-2} = \frac{z-2}{1}$ is 1, then the sum of all possible values of λ is :

- (1) 0
- (2) $2\sqrt{3}$
- (3) $3\sqrt{3}$
- (4) $-2\sqrt{3}$

Ans. (2)

Sol. Passing points of lines L_1 & L_2 are

$$(\lambda, 2, 1) \& (\sqrt{3}, 1, 2)$$

$$\text{S.D} = \frac{\begin{vmatrix} \sqrt{3}-\lambda & -1 & 1 \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}}{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix}}$$

$$1 = \left| \frac{\sqrt{3}-\lambda}{\sqrt{3}} \right|$$

$$\lambda = 0, \lambda = 2\sqrt{3}$$

SECTION-B

21. If $x = x(t)$ is the solution of the differential equation $(t+1)dx = (2x + (t+1)^4) dt$, $x(0) = 2$, then, $x(1)$ equals _____.

Ans. (14)

Sol. $(t+1)dx = (2x + (t+1)^4)dt$

$$\frac{dx}{dt} = \frac{2x + (t+1)^4}{t+1}$$

$$\frac{dx}{dt} - \frac{2x}{t+1} = (t+1)^3$$

$$I \cdot F = e^{-\int \frac{2}{t+1} dt} = e^{-2\ln(t+1)} = \frac{1}{(t+1)^2}$$

$$\frac{x}{(t+1)^2} = \int \frac{1}{(t+1)^2} (t+1)^3 dt + c$$

$$\frac{x}{(t+1)^2} = \frac{(t+1)^2}{2} + c$$

$$\Rightarrow c = \frac{3}{2}$$

$$x = \frac{(t+1)^4}{2} + \frac{3}{2}(t+1)^2$$

$$\text{put, } t = 1$$

$$x = 2^3 + 6 = 14$$

22. The number of elements in the set

$$S = \{(x, y, z) : x, y, z \in \mathbf{Z}, x + 2y + 3z = 42, x, y, z \geq 0\} \text{ equals } \underline{\hspace{2cm}}.$$

Ans. (169)

Sol. $x + 2y + 3z = 42, \quad x, y, z \geq 0$

$$z = 0 \quad x + 2y = 42 \Rightarrow 22$$

$$z = 1 \quad x + 2y = 39 \Rightarrow 20$$

$$z = 2 \quad x + 2y = 36 \Rightarrow 19$$

$$z = 3 \quad x + 2y = 33 \Rightarrow 17$$

$$z = 4 \quad x + 2y = 30 \Rightarrow 16$$

$$z = 5 \quad x + 2y = 27 \Rightarrow 14$$

$$z = 6 \quad x + 2y = 24 \Rightarrow 13$$

$$z = 7 \quad x + 2y = 21 \Rightarrow 11$$

$$z = 8 \quad x + 2y = 18 \Rightarrow 10$$

$$z = 9 \quad x + 2y = 15 \Rightarrow 8$$

$$z = 10 \quad x + 2y = 12 \Rightarrow 7$$

$$z = 11 \quad x + 2y = 9 \Rightarrow 5$$

$$z = 12 \quad x + 2y = 6 \Rightarrow 4$$

$$z = 13 \quad x + 2y = 3 \Rightarrow 2$$

$$z = 14 \quad x + 2y = 0 \Rightarrow 1$$

Total : 169

23. If the Coefficient of x^{30} in the expansion of

$$\left(1 + \frac{1}{x}\right)^6 (1 + x^2)^7 (1 - x^3)^8; \quad x \neq 0 \text{ is } \alpha, \text{ then } |\alpha| \text{ equals } \underline{\hspace{2cm}}.$$

Ans. (678)

Sol. coeff of x^{30} in $\frac{(x+1)^6(1+x^2)^7(1-x^3)^8}{x^6}$

coeff. of x^{36} in $(1+x)^6(1+x^2)^7(1-x^3)^8$

General term

$${}^6C_{r_1} {}^7C_{r_2} {}^8C_{r_3} (-1)^{r_3} x^{r_1+2r_2+3r_3}$$

$$r_1 + 2r_2 + 3r_3 = 36$$

	r_1	r_2	r_3	
	0	6	8	
Case-I :	2	5	8	$r_1 + 2r_2 = 12$ (Taking $r_3 = 8$)
	4	4	8	
	6	3	8	

	r_1	r_2	r_3	
	1	7	7	
Case-II :	3	6	7	$r_1 + 2r_2 = 15$ (Taking $r_3 = 7$)
	5	5	7	

	r_1	r_2	r_3	
	4	7	6	
Case-III :	6	6	6	$r_1 + 2r_2 = 18$ (Taking $r_3 = 6$)

$$\text{Coeff.} = 7 + (15 \times 21) + (15 \times 35) + (35)$$

$$- (6 \times 8) - (20 \times 7 \times 8) - (6 \times 21 \times 8) + (15 \times 28)$$

$$+ (7 \times 28) = -678 = \alpha$$

$$|\alpha| = 678$$

24. Let 3, 7, 11, 15, ..., 403 and 2, 5, 8, 11, ..., 404 be two arithmetic progressions. Then the sum, of the common terms in them, is equal to _____.

Ans. (6699)

Sol. 3, 7, 11, 15,, 403

2, 5, 8, 11,, 404

LCM (4, 3) = 12

11, 23, 35,, let (403)

$$403 = 11 + (n - 1) \times 12$$

$$\frac{392}{12} = n - 1$$

$$33 \cdot 66 = n$$

$$n = 33$$

$$\text{Sum} = \frac{33}{2} (22 + 32 \times 12)$$

$$= 6699$$

25. Let $\{x\}$ denote the fractional part of x and

$$f(x) = \frac{\cos^{-1}(1 - \{x\}^2) \sin^{-1}(1 - \{x\})}{\{x\} - \{x\}^3}, \quad x \neq 0. \text{ If } L$$

and R respectively denotes the left hand limit and the

right hand limit of $f(x)$ at $x = 0$, then $\frac{32}{\pi^2} (L^2 + R^2)$ is

equal to _____.

Ans. (18)

- Sol. Finding right hand limit

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} f(0 + h)$$

$$= \lim_{h \rightarrow 0} f(h)$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1 - h^2) \sin^{-1}(1 - h)}{h(1 - h^2)}$$

$$= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1 - h^2)}{h} \left(\frac{\sin^{-1} 1}{1} \right)$$

$$\text{Let } \cos^{-1}(1 - h^2) = \theta \Rightarrow \cos \theta = 1 - h^2$$

$$= \frac{\pi}{2} \lim_{\theta \rightarrow 0} \frac{\theta}{\sqrt{1 - \cos \theta}}$$

$$= \frac{\pi}{2} \lim_{\theta \rightarrow 0} \frac{1}{\sqrt{\frac{1 - \cos \theta}{\theta^2}}}$$

$$= \frac{\pi}{2} \frac{1}{\sqrt{1/2}}$$

$$R = \frac{\pi}{\sqrt{2}}$$

Now finding left hand limit

$$\begin{aligned}
 L &= \lim_{x \rightarrow 0^-} f(x) \\
 &= \lim_{h \rightarrow 0} f(-h) \\
 &= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1 - \{-h\}^2) \sin^{-1}(1 - \{-h\})}{\{-h\} - \{-h\}^3} \\
 &= \lim_{h \rightarrow 0} \frac{\cos^{-1}(1 - (-h+1)^2) \sin^{-1}(1 - (-h+1))}{(-h+1) - (-h+1)^3} \\
 &= \lim_{h \rightarrow 0} \frac{\cos^{-1}(-h^2 + 2h) \sin^{-1} h}{(1-h)(1-(1-h)^2)} \\
 &= \lim_{h \rightarrow 0} \left(\frac{\pi}{2} \right) \frac{\sin^{-1} h}{-h^2} \\
 &= \frac{\pi}{2} \lim_{h \rightarrow 0} \left(\frac{\sin^{-1} h}{-h^2 + 2h} \right) \\
 &= \frac{\pi}{2} \lim_{h \rightarrow 0} \left(\frac{\sin^{-1} h}{h} \right) \left(\frac{1}{-h+2} \right) \\
 L &= \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 \frac{32}{\pi^2} (L^2 + R^2) &= \frac{32}{\pi^2} \left(\frac{\pi^2}{2} + \frac{\pi^2}{16} \right) \\
 &= 16 + 2 \\
 &= 18
 \end{aligned}$$

26. Let the line $L : \sqrt{2}x + y = \alpha$ pass through the point of the intersection P (in the first quadrant) of the circle $x^2 + y^2 = 3$ and the parabola $x^2 = 2y$. Let the line L touch two circles C_1 and C_2 of equal radius $2\sqrt{3}$. If the centres Q_1 and Q_2 of the circles C_1 and C_2 lie on the y -axis, then the square of the area of the triangle PQ_1Q_2 is equal to _____.

Ans. (72)

Sol. $x^2 + y^2 = 3$ and $x^2 = 2y$

$$y^2 + 2y - 3 = 0 \Rightarrow (y+3)(y-1) = 0$$

$$y = -3 \text{ or } y = 1$$

$$y = 1 \text{ } x = \sqrt{2} \Rightarrow P(\sqrt{2}, 1)$$

p lies on the line

$$\sqrt{2}x + y = \alpha$$

$$\sqrt{2}(\sqrt{2}) + 1 = \alpha$$

$$\alpha = 3$$

For circle C_1

Q_1 lies on y axis

Let $Q_1 (0, \alpha)$ coordinates

$$R_1 = 2\sqrt{3} \text{ (Given)}$$

Line L act as tangent

Apply $P = r$ (condition of tangency)

$$\Rightarrow \left| \frac{\alpha - 3}{\sqrt{3}} \right| = 2\sqrt{3}$$

$$\Rightarrow |\alpha - 3| = 6$$

$$\alpha - 3 = 6 \quad \text{or} \quad \alpha - 3 = -6$$

$$\Rightarrow \alpha = 9$$

$$\alpha = -3$$

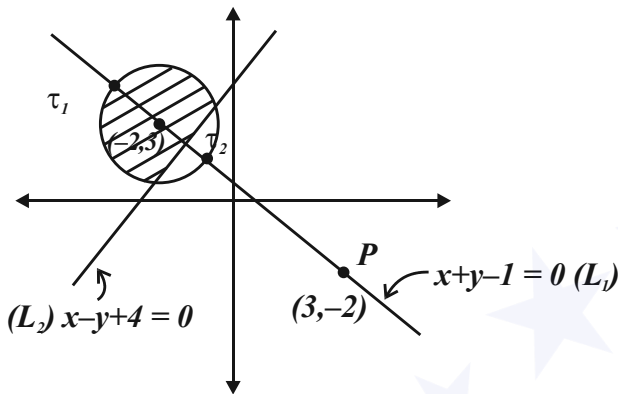
$$\Delta PQ_1Q_2 = \frac{1}{2} \begin{vmatrix} \sqrt{2} & 1 & 1 \\ 0 & 9 & 1 \\ 0 & -3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} (\sqrt{2}(12)) = 6\sqrt{2}$$

27. Let $P = \{z \in \mathbb{C} : |z + 2 - 3i| \leq 1\}$ and $Q = \{z \in \mathbb{C} : z(1+i) + \bar{z}(1-i) \leq -8\}$. Let in $P \cap Q$, $|z - 3 + 2i|$ be maximum and minimum at z_1 and z_2 respectively. If $|z_1|^2 + 2|z_2|^2 = \alpha + \beta\sqrt{2}$, where α, β are integers, then $\alpha + \beta$ equals _____.

Ans. (36)

Sol.



Clearly for the shaded region z_1 is the intersection of the circle and the line passing through P (L_1) and z_2 is intersection of line L_1 & L_2

Circle : $(x + 2)^2 + (y - 3)^2 = 1$

$L_1 : x + y - 1 = 0$

$L_2 : x - y + 4 = 0$

On solving circle & L_1 we get

$z_1 : \left(-2 - \frac{1}{\sqrt{2}}, 3 + \frac{1}{\sqrt{2}}\right)$

On solving L_1 and z_2 is intersection of line L_1 & L_2

we get $z_2 : \left(\frac{-3}{2}, \frac{5}{2}\right)$

$$|z_1|^2 + 2|z_2|^2 = 14 + 5\sqrt{2} + 17 = 31 + 5\sqrt{2}$$

So $\alpha = 31$

$\beta = 5$

28. If $\int_{-\pi/2}^{\pi/2} \frac{8\sqrt{2} \cos x dx}{(1 + e^{\sin x})(1 + \sin^4 x)} = \alpha\pi + \beta \log_e (3 + 2\sqrt{2})$, where α, β are integers, then $\alpha^2 + \beta^2$ equals _____.

Ans. (8)

Sol. $I = \int_{-\pi/2}^{\pi/2} \frac{8\sqrt{2} \cos x}{(1 + e^{\sin x})(1 + \sin^4 x)} dx$

Apply king

$$I = \int_{-\pi/2}^{\pi/2} \frac{8\sqrt{2} \cos x (e^{\sin x})}{(1 + e^{\sin x})(1 + \sin^4 x)} dx \quad \dots(2)$$

adding (1) & (2)

$$2I = \int_{-\pi/2}^{\pi/2} \frac{8\sqrt{2} \cos x}{1 + \sin^4 x} dx$$

$$I = \int_0^{\pi/2} \frac{8\sqrt{2} \cos x}{1 + \sin^4 x} dx,$$

$\sin x = t$

$$I = \int_0^1 \frac{8\sqrt{2}}{1 + t^4} dx$$

$$I = 4\sqrt{2} \int_0^1 \left(\frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} - \frac{1 - \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} \right) dt$$

$$I = 4\sqrt{2} \left[\frac{\left(1 + \frac{1}{t^2}\right)}{t^2 + \frac{1}{t^2}} - \frac{\left(1 - \frac{1}{t^2}\right)}{t^2 + \frac{1}{t^2}} \right] dt$$

$$\begin{aligned}
 &= 4\sqrt{2} \left[\int_{-\infty}^0 \frac{dz}{z^2+2} - \int_{\infty}^2 \frac{dk}{k^2-2} \right] \\
 &= 4\sqrt{2} \left[\frac{1}{\sqrt{2}} \tan^{-1} \frac{z}{\sqrt{2}} \right]_{-\infty}^0 - \left[\frac{1}{2\sqrt{2}} \ln \left(\frac{k-\sqrt{2}}{k+\sqrt{2}} \right) \right]_{\infty}^2 \\
 &= 4\sqrt{2} \left[\frac{\pi}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \left[\ln \frac{2-\sqrt{2}}{2+\sqrt{2}} \right] \right] \\
 &= 2\pi + 2\ln(3+2\sqrt{2}) \\
 \alpha &= 2 \\
 \beta &= 2
 \end{aligned}$$

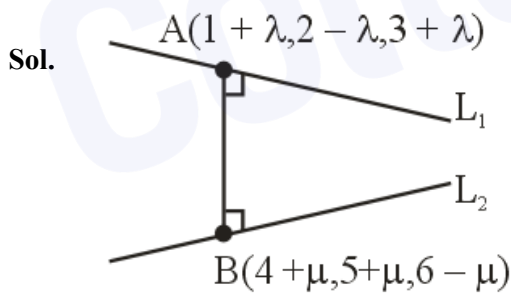
29. Let the line of the shortest distance between the lines

$$L_1 : \vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \text{ and}$$

$$L_2 : \vec{r} = (4\hat{i} + 5\hat{j} + 6\hat{k}) + \mu(\hat{i} + \hat{j} - \hat{k})$$

intersect L_1 and L_2 at P and Q respectively. If (α, β, γ) is the midpoint of the line segment PQ, then $2(\alpha + \beta + \gamma)$ is equal to _____.

Ans. (21)



$$\vec{b} = \hat{i} - \hat{j} + \hat{k} \text{ (DR's of } L_1)$$

$$\vec{d} = \hat{i} + \hat{j} - \hat{k} \text{ (DR's of } L_2)$$

$$\vec{b} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$= 0\hat{i} + 2\hat{j} + 2\hat{k}$ (DR's of Line perpendicular to L_1 and L_2)

DR of AB line

$$= (0, 2, 2) = (3 + \mu - \lambda, 3 + \mu + \lambda, 3 - \mu - \lambda)$$

$$\frac{3 + \mu - \lambda}{0} = \frac{3 + \mu + \lambda}{2} = \frac{3 - \mu - \lambda}{2}$$

Solving above equation we get $\mu = -\frac{3}{2}$ and $\lambda = \frac{3}{2}$

$$\text{point A} = \left(\frac{5}{2}, \frac{1}{2}, \frac{9}{2} \right)$$

$$B = \left(\frac{5}{2}, \frac{7}{2}, \frac{15}{2} \right)$$

$$\text{Point of AB} = \left(\frac{5}{2}, 2, 6 \right) = (\alpha, \beta, \gamma)$$

$$2(\alpha + \beta + \gamma) = 5 + 4 + 12 = 21$$

30. Let $A = \{1, 2, 3, \dots, 20\}$. Let R_1 and R_2 two relation on A such that

$$R_1 = \{(a, b) : b \text{ is divisible by } a\}$$

$$R_2 = \{(a, b) : a \text{ is an integral multiple of } b\}.$$

Then, number of elements in $R_1 - R_2$ is equal to _____.

Ans. (46)

Sol. $n(R_1) = 20 + 10 + 6 + 5 + 4 + 3 + 2 + 2 + 2$

$$+ 2 + \underbrace{1 + \dots + 1}_{10 \text{ times}}$$

$$n(R_1) = 66$$

$$R_1 \cap R_2 = \{(1,1), (2,2), \dots, (20,20)\}$$

$$n(R_1 \cap R_2) = 20$$

$$n(R_1 - R_2) = n(R_1) - n(R_1 \cap R_2)$$

$$= n(R_1) - 20$$

$$= 66 - 20$$

$$R_1 - R_2 = 46 \text{ Pair}$$