FINAL JEE–MAIN EXAMINATION – APRIL, 2024

(Held On Thursday 04th April, 2024) TIME : 9 : 00 AM to 12 : 00 NOON

SECTION-A

1. Let $f : R \to R$ be a function given by

$$f(x)=\begin{cases}\dfrac{1-\cos 2x}{x^2}, & x<0\\ \alpha, & x=0\\ \dfrac{\beta\sqrt{1-\cos x}}{x}, & x>0\end{cases}$$

, where $\alpha, \beta \in R$. If f is continuous at $x = 0$, then $\alpha^2 + \beta^2$ is equal to :

(1) 48 (2) 12

(3) 3 (4) 6

Ans. (2)

Sol. $f(0^-) = \lim\limits_{x\to 0^-} \dfrac{2\sin^2 x}{x^2} = 2 = \alpha$

$f(0^+) = \lim\limits_{x\to 0^+} \beta \times \sqrt{2}\dfrac{\sin\frac{x}{2}}{2\frac{x}{2}} = \dfrac{\beta}{\sqrt{2}} = 2$

$\Rightarrow \beta = 2\sqrt{2}$

$\alpha^2 + \beta^2 = 4 + 8 = 12$

2. Three urns A, B and C contain 7 red, 5 black; 5 red, 7 black and 6 red, 6 black balls, respectively. One of the urn is selected at random and a ball is drawn from it. If the ball drawn is black, then the probability that it is drawn from urn A is :

(1) $\dfrac{4}{17}$ (2) $\dfrac{5}{18}$

(3) $\dfrac{7}{18}$ (4) $\dfrac{5}{16}$

Ans. (2)

Sol.

A	B	C
7R, 5B	5R, 7B	6R, 6B

$P(B) = \dfrac{1}{3}.\dfrac{5}{12} + \dfrac{1}{3}.\dfrac{7}{12} + \dfrac{1}{3}.\dfrac{6}{12}$

required probability $= \dfrac{\frac{1}{3}.\frac{5}{12}}{\frac{1}{3}.\left[\frac{5}{12}+\frac{7}{12}+\frac{6}{12}\right]} = \dfrac{5}{18}$

3. The vertices of a triangle are A(–1, 3), B(–2, 2) and C(3, –1). A new triangle is formed by shifting the sides of the triangle by one unit inwards. Then the equation of the side of the new triangle nearest to origin is :

(1) $x - y - \left(2+\sqrt{2}\right) = 0$

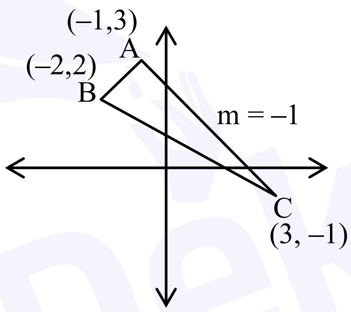
(2) $-x + y - \left(2-\sqrt{2}\right) = 0$

(3) $x + y - \left(2-\sqrt{2}\right) = 0$

(4) $x + y + \left(2-\sqrt{2}\right) = 0$

Ans. (3)

Sol.



equation of AC $\to x + y = 2$

equation of line parallel to AC $x + y = d$

$\left|\dfrac{d-2}{\sqrt{2}}\right| = 1$

$d = 2-\sqrt{2}$

eq^n of new required line

$\boxed{x+y=2-\sqrt{2}}$

4. If the solution $y = y(x)$ of the differential equation $(x^4 + 2x^3 + 3x^2 + 2x + 2)dy - (2x^2 + 2x + 3)dx = 0$ satisfies $y(-1) = -\dfrac{\pi}{4}$, then $y(0)$ is equal to :

(1) $-\dfrac{\pi}{12}$ (2) 0

(3) $\dfrac{\pi}{4}$ (4) $\dfrac{\pi}{2}$

Ans. (3)

Sol. $\int dy = \int \frac{(2x^2+2x+3)}{x^4+2x^3+3x^2+2x+2}dx$

$y = \int \frac{(2x^2+2x+3)}{(x^2+1)(x^2+2x+2)}dx$

$y = \int \frac{dx}{x^2+2x+2} + \int \frac{dx}{x^2+1}$

$y = \tan^{-1}(x+1) + \tan^{-1}x + C$

$y(-1) = \frac{-\pi}{4}$

$\frac{-\pi}{4} = 0 - \frac{\pi}{4} + C \Rightarrow C = 0$

$\Rightarrow y = \tan^{-1}(x+1) + \tan^{-1}x$

$y(0) = \tan^{-1}1 = \frac{\pi}{4}$

5. Let the sum of the maximum and the minimum values of the function $f(x) = \frac{2x^2-3x+8}{2x^2+3x+8}$ be $\frac{m}{n}$, where gcd(m, n) = 1. Then m + n is equal to :

(1) 182 (2) 217

(3) 195 (4) 201

Ans. (4)

Sol. $y = \frac{2x^2-3x+8}{2x^2+3x+8}$

$x^2(2y-2) + x(3y+3) + 8y - 8 = 0$

use $D \geq 0$

$(3y+3)^2 - 4(2y-2)(8y-8) \geq 0$

$(11y-5)(5y-11) \leq 0$

$\Rightarrow y \in \left[\frac{5}{11}, \frac{11}{5}\right]$

y = 1 is also included

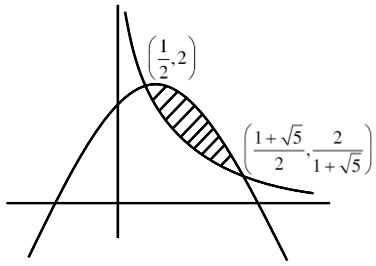
6. One of the points of intersection of the curves $y = 1 + 3x - 2x^2$ and $y = \frac{1}{x}$ is $\left(\frac{1}{2}, 2\right)$. Let the area of the region enclosed by these curves be $\frac{1}{24}\left(\ell\sqrt{5}+m\right) - n\log_e\left(1+\sqrt{5}\right)$, where ℓ, m, n $\in$ N. Then ℓ + m + n is equal to

(1) 32 (2) 30

(3) 29 (4) 31

Ans. (2)

Sol.



$$A = \int_{\frac{1}{2}}^{\frac{1+\sqrt{5}}{2}} \left(1+3x-2x^2-\frac{1}{x}\right)dx$$

$$A = \left[x + \frac{3x^2}{2} - \frac{2x^3}{3} - \ell n\, x\right]_{\frac{1}{2}}^{\frac{1+\sqrt{5}}{2}}$$

$$A = \frac{1+\sqrt{5}}{2} + \frac{3}{2}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \frac{2}{3}\left(\frac{1+\sqrt{5}}{2}\right)^3 - \ell n\left(\frac{1+\sqrt{5}}{2}\right)$$

$$-\frac{1}{2} - \frac{3}{2}\left(\frac{1}{4}\right) + \frac{2}{3}\left(\frac{1}{8}\right) + \ell n\left(\frac{1}{2}\right)$$

$$A = \frac{1}{2} + \frac{\sqrt{5}}{2} + \frac{3}{8} + \frac{3}{4}\sqrt{5} + \frac{15}{8} - \frac{4}{3} - \frac{2}{3}\sqrt{5}$$

$$-\frac{1}{2} - \frac{3}{8} + \frac{1}{12} - \ell n\left(1+\sqrt{5}\right)$$

$$= \sqrt{5}\left(\frac{1}{2} + \frac{3}{4} - \frac{2}{3}\right) + \frac{15}{8} - \frac{4}{3} + \frac{1}{12} - \ell n\left(1+\sqrt{5}\right)$$

$$= \frac{14}{24}\sqrt{5} + \frac{15}{24} - \ell n\left(1+\sqrt{5}\right)$$

7. If the system of equations

$x + \left(\sqrt{2}\sin\alpha\right)y + \left(\sqrt{2}\cos\alpha\right)z = 0$

$x + (\cos\alpha)y + (\sin\alpha)z = 0$

$x + (\sin\alpha)y - (\cos\alpha)z = 0$

has a non-trivial solution, then $\alpha \in \left(0, \frac{\pi}{2}\right)$ is equal to :

(1) $\frac{3\pi}{4}$ (2) $\frac{7\pi}{24}$

(3) $\frac{5\pi}{24}$ (4) $\frac{11\pi}{24}$

Ans. (3)

Sol. $\begin{vmatrix} 1 & \sqrt{2}\sin\alpha & \sqrt{2}\cos\alpha \\ 1 & \sin\alpha & -\cos\alpha \\ 1 & \cos\alpha & \sin\alpha \end{vmatrix} = 0$

$\Rightarrow 1-\sqrt{2}\sin\alpha(\sin\alpha+\cos\alpha)+\sqrt{2}\cos\alpha(\cos\alpha-\sin\alpha)=0$

$\Rightarrow 1+\sqrt{2}\cos 2\alpha-\sqrt{2}\sin 2\alpha=0$

$\cos 2\alpha-\sin 2\alpha=-\frac{1}{\sqrt{2}}$

$\cos\left(2\alpha+\frac{\pi}{4}\right)=-\frac{1}{2}$

$2\alpha+\frac{\pi}{4}=2n\pi\pm\frac{2\pi}{3}$

$\alpha+\frac{\pi}{8}=n\pi\pm\frac{\pi}{3}$

$n=0,$

$x=\frac{\pi}{3}-\frac{\pi}{8}=\frac{5\pi}{24}$

8. There are 5 points P_1, P_2, P_3, P_4, P_5 on the side AB, excluding A and B, of a triangle ABC. Similarly there are 6 points P_6, P_7, …, P_{11} on the side BC and 7 points P_{12}, P_{13}, …, P_{18} on the side CA of the triangle. The number of triangles, that can be formed using the points P_1, P_2, …, P_{18} as vertices, is :

(1) 776 (2) 751

(3) 796 (4) 771

Ans. (2)

Sol. ${}^{18}C_3-{}^{5}C_3-{}^{6}C_3-{}^{7}C_3$

$=751$

9. Let $f(x)=\begin{cases}-2, & -2\le x\le 0\\ x-2, & 0<x\le 2\end{cases}$ and $h(x)=f(|x|)+|f(x)|$.

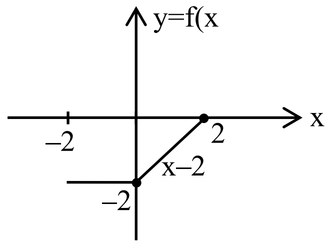
Then $\int_{-2}^{2} h(x)dx$ is equal to :

(1) 2 (2) 4

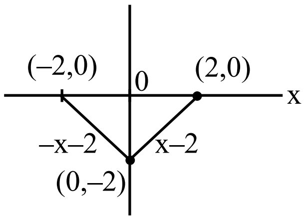
(3) 1 (4) 6

Ans. (1)

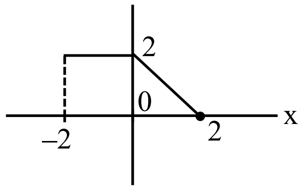
Sol.



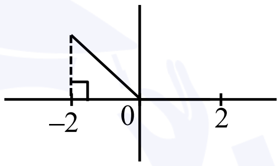
f(|x|) →



|f(x)|



$h(x)=\begin{cases}x-2+2-x=0, & 0\le x\le 2\\ -x-2+2=-x & -2\le x<0\end{cases}$



$\Rightarrow \int_{0}^{2} h(x)dx=0$ and $\int^{0} h(x)dx=2$

10. The sum of all rational terms in the expansion of $\left(2^{\frac{1}{5}}+5^{\frac{1}{3}}\right)^{15}$ is equal to :

(1) 3133 (2) 633

(3) 931 (4) 6131

Ans. (1)

Sol. $T_{r+1}={}^{15}C_r\left(5^{\frac{1}{3}}\right)^r\left(2^{\frac{1}{5}}\right)^{15-r}$

$={}^{15}C_r\,5^{\frac{r}{3}}.2^{\frac{15-r}{5}}$

$R=3\lambda,\ 15\mu$

$\Rightarrow r=0,\ 15$

2 rational terms

$\Rightarrow {}^{15}C_0 2^3+{}^{15}C_{15}(5)^5$

$=8+3125=3133$

11. Let a unit vector which makes an angle of 60° with $2\hat{i}+2\hat{j}-\hat{k}$ and an angle of 45° with $\hat{i}-\hat{k}$ be $\vec{C}$. Then $\vec{C}+\left(-\frac{1}{2}\hat{i}+\frac{1}{3\sqrt{2}}\hat{j}-\frac{\sqrt{2}}{3}\hat{k}\right)$ is :

(1) $-\frac{\sqrt{2}}{3}\hat{i}+\frac{\sqrt{2}}{3}\hat{j}+\left(\frac{1}{2}+\frac{2\sqrt{2}}{3}\right)\hat{k}$

(2) $\frac{\sqrt{2}}{3}\hat{i}+\frac{1}{3\sqrt{2}}\hat{j}-\frac{1}{2}\hat{k}$

(3) $\left(\frac{1}{\sqrt{3}}+\frac{1}{2}\right)\hat{i}+\left(\frac{1}{\sqrt{3}}-\frac{1}{3\sqrt{2}}\right)\hat{j}+\left(\frac{1}{\sqrt{3}}+\frac{\sqrt{2}}{3}\right)\hat{k}$

(4) $\frac{\sqrt{2}}{3}\hat{i}-\frac{1}{2}\hat{k}$

Ans. (4)

Sol. $\vec{C}=C_1\hat{i}+C_2\hat{j}+C_3\hat{k}$

$C_1^2+C_2^2+C_3^2=1$

$\vec{C}.\left(2\hat{i}+2\hat{j}-\hat{k}\right)=|C|\sqrt{9}\cos 60°$

$2C_1+2C_2-C_3=\frac{3}{2}$

$C_1-C_3=1$

$C_1+2C_2=\frac{1}{2}$

$C_1=\frac{\sqrt{2}}{3}+\frac{1}{2}$

$C_2=\frac{-1}{3\sqrt{2}}$

$C_3=\frac{\sqrt{2}}{3}-\frac{1}{2}$

12. Let the first three terms 2, p and q, with $q \neq 2$, of a G.P. be respectively the 7^{th}, 8^{th} and 13^{th} terms of an A.P. If the 5^{th} term of the G.P. is the n^{th} term of the A.P., then n is equal to

(1) 151 (2) 169

(3) 177 (4) 163

Ans. (4)

Sol. $p^2=2q$

$2=a+6d$ ...(i)

$p=a+7d$ ...(ii)

$q=a+12d$ ...(iii)

$p-2=d$ ((ii) – (i))

$q-p=5d$ ((iii) – (ii))

$q-p=5(p-2)$

$q=6p-10$

$p^2=2(6p-10)$

$p^2-12p+20=0$

$p=10, 2$

$p=10$; $q=50$

$d=8$

$a=-46$

2, 10, 50, 250, 1250

$ar^4=a+(n-1)d$

$1250=-46+(n-1)8$

$n=163$

13. Let $a, b \in R$. Let the mean and the variance of 6 observations –3, 4, 7, –6, a, b be 2 and 23, respectively. The mean deviation about the mean of these 6 observations is :

(1) $\frac{13}{3}$ (2) $\frac{16}{3}$

(3) $\frac{11}{3}$ (4) $\frac{14}{3}$

Ans. (1)

Sol. $\frac{\Sigma x_i}{6}=2$ and $\frac{\Sigma x_i^2}{N}-\mu^2=23$

$\alpha+\beta=10$

$\alpha^2+\beta^2=52$

solving we get $\alpha=4, \beta=6$

$\frac{\Sigma|x_i-\bar{x}|}{6}=\frac{5+2+5+8+2+4}{6}=\frac{13}{3}$

14. If 2 and 6 are the roots of the equation $ax^2 + bx + 1 = 0$, then the quadratic equation, whose roots are $\frac{1}{2a+b}$ and $\frac{1}{6a+b}$, is :

(1) $2x^2 + 11x + 12 = 0$ (2) $4x^2 + 14x + 12 = 0$

(3) $x^2 + 10x + 16 = 0$ (4) $x^2 + 8x + 12 = 0$

Ans. (4)

Sol. Sum $= 8 = -\frac{b}{a}$

Product $= 12 = \frac{1}{a}$ $\Rightarrow a = \frac{1}{12}$

$b = -\frac{2}{3}$

$2a + b = \frac{2}{12} - \frac{2}{3} = -\frac{1}{2}$

$6a + b = \frac{6}{12} - \frac{2}{3} = -\frac{1}{6}$

sum $= -8$

$P = 12$

$x^2 + 8x + 12 = 0$

15. Let α and β be the sum and the product of all the non-zero solutions of the equation $(\bar{z})^2 + |z| = 0$, $z \in C$. Then $4(\alpha^2 + \beta^2)$ is equal to :

(1) 6 (2) 4

(3) 8 (4) 2

Ans. (2)

Sol. $z = x + iy$

$\bar{z} = x - iy$

$\bar{z}^2 = x^2 - y^2 - 2ixy$

$\Rightarrow x^2 - y^2 - 2ixy + \sqrt{x^2 + y^2} = 0$

$\Rightarrow x = 0$ or $y = 0$

$-y^2 + |y| = 0$ $x^2 + |x| = 0$

$|y| = |y|^2$ $\Rightarrow x = 0$

$y = 0, \pm 1$

$\Rightarrow i, -i$ $\Rightarrow \alpha = i - i = 0$

are roots $\beta = i(-i) = 1$

$4(0 + 1) = 4$

16. Let the point, on the line passing through the points P(1, –2, 3) and Q(5, –4, 7), farther from the origin and at a distance of 9 units from the point P, be (α, β, γ). Then $\alpha^2 + \beta^2 + \gamma^2$ is equal to :

(1) 155 (2) 150

(3) 160 (4) 165

Ans. (1)

Sol. PQ line

$\frac{x-1}{4} = \frac{y+2}{-2} = \frac{z-3}{4}$

pt $(4t + 1, -2t - 2, 4t + 3)$

distance$^2 = 16t^2 + 4t^2 + 16t^2 = 81$

$t = \pm\frac{3}{2}$

pt (7, –5, 9)

$\alpha^2 + \beta^2 + \gamma^2 = 155$

option (1)

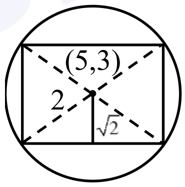
17. A square is inscribed in the circle $x^2 + y^2 - 10x - 6y + 30 = 0$. One side of this square is parallel to $y = x + 3$. If (x_i, y_i) are the vertices of the square, then $\sum(x_i^2 + y_i^2)$ is equal to :

(1) 148 (2) 156

(3) 160 (4) 152

Ans. (4)

Sol.



$y = x + c$ & $x + y + d = 0$

$\left|\frac{5-3+c}{\sqrt{2}}\right| = \sqrt{2}$ $\left|\frac{8+d}{\sqrt{2}}\right| = \sqrt{2}$

$|c + 2| = 2$ $8 + d = \pm 2$

$c = 0, -4$ $d = -10, -6$

pts (5, 5), (3, 3), (7, 3), (5, 1)

$\sum(x_i^2 + y_i^2) = 25 + 25 + 9 + 9 + 49 + 9 + 25 + 1$

$= 152$

Option (4)

18. If the domain of the function $\sin^{-1}\left(\frac{3x-22}{2x-19}\right)+\log_e\left(\frac{3x^2-8x+5}{x^2-3x-10}\right)$ is $(\alpha, \beta]$, then $3\alpha + 10\beta$ is equal to :

(1) 97 (2) 100

(3) 95 (4) 98

Ans. (1)

Sol. $-1 \le \frac{3x-22}{2x-19} \le 1$ $\frac{3x^2-8x+5}{x^2-3x-10} > 0$

$x \in \left(5, \frac{41}{5}\right]$

$3\alpha + 10\beta = 97$

Option (1)

19. Let $f(x) = x^5 + 2e^{x/4}$ for all $x \in R$. Consider a function $g(x)$ such that $(gof)(x) = x$ for all $x \in R$. Then the value of $8g'(2)$ is :

(1) 16 (2) 4

(3) 8 (4) 2

Ans. (1)

Sol. $f(x) = 2$

when $x = 0$

$\because g'(f(x))\, f'(x) = 1$

$g'(2) = \frac{1}{'(0)}$

$\because f'(x) = 5x^4 + \frac{2}{4}e^{x/4}$

$g'(2) = 2$

Ans = 16

Option (1)

20. Let $\alpha \in (0, \infty)$ and $A = \begin{bmatrix} 1 & 2 & \alpha \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.

If $\det(\text{adj}(2A - A^T).\text{adj}(A - 2A^T)) = 2^8$, then $(\det(A))^2$ is equal to :

(1) 1 (2) 49

(3) 16 (4) 36

Ans. (3)

Sol. $|\text{adj}(A - 2A^T)\,(2A - A^T)| = 28$

$|(A - 2A^T)\,(2A - A^T)| = 24$

$|A - 2A^T|\,|2A - A^T| = \pm 16$

$(A - 2A^T)^T = A^T - 2A$

$|A - 2A^T| = |A^T - 2A|$

$\Rightarrow |A - 2A^T|^2 = 16$

$|A - 2A^T| = \pm 4$

$$\begin{bmatrix} 1 & 2 & \alpha \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 0 \\ 4 & 0 & 2 \\ 2\alpha & 2 & 4 \end{bmatrix}$$

$$\begin{vmatrix} -1 & 0 & \alpha \\ -3 & 0 & -1 \\ -2\alpha & -1 & -2 \end{vmatrix}$$

$1 + 3\alpha = 4$

$3\alpha = 3$

$\alpha = 1$

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{vmatrix} = -1 - 3 = -4$$

$|A|^2 = 16$

SECTION-B

21. If $\lim_{x\to 1}\frac{(5x+1)^{1/3}-(x+5)^{1/3}}{(2x+3)^{1/2}-(x+4)^{1/2}} = \frac{m\sqrt{5}}{n(2n)^{2/3}}$, where $\gcd(m, n) = 1$, then $8m + 12n$ is equal to ______

Ans. (100)

Sol. $$\lim_{x\to 1}\frac{\frac{1}{3}(5x+1)^{-2/3}5 - \frac{1}{3}(x+5)^{-2/3}}{\frac{1}{2}(2x+3)^{-1/2}.2 - \frac{1}{2}(x+4)^{-1/2}}$$

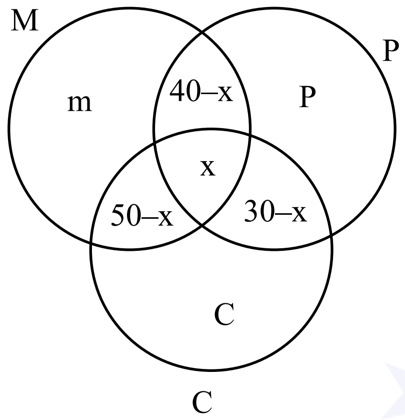
$= \frac{8}{3}\frac{\sqrt{5}}{6^{2/3}}$ $m = 8$, $n = 3$

$8m + 12n = 100$

22. In a survey of 220 students of a higher secondary school, it was found that at least 125 and at most 130 students studied Mathematics; at least 85 and at most 95 studied Physics; at least 75 and at most 90 studied Chemistry; 30 studied both Physics and Chemistry; 50 studied both Chemistry and Mathematics; 40 studied both Mathematics and Physics and 10 studied none of these subjects. Let m and n respectively be the least and the most number of students who studied all the three subjects. Then m + n is equal to _____

Ans. (45)

Sol.



$125 \le m + 90 - x \le 130$

$85 \le P + 70 - x \le 95$

$75 \le C + 80 - x \le 90$

$m + P + C + 120 - 2x = 210$

$\Rightarrow 15 \le x \le 45 \ \& \ 30 - x \ge 0$

$\Rightarrow 15 \le x \le 30$

$30 + 15 = 45$

23. Let the solution $y = y(x)$ of the differential equation $\frac{dy}{dx} - y = 1 + 4\sin x$ satisfy $y(\pi) = 1$. Then $y\left(\frac{\pi}{2}\right) + 10$ is equal to _____

Ans. (7)

Sol. $ye^{-x} = \int \left(e^{-x} + 4e^{-x}\sin x\right)dx$

$ye^{-x} = -e^{-x} - 2(e^{-x}\sin x \ \ e^{-x}\cos x) + C$

$y = -1 - 2(\sin x + \cos x) + ce^{x}$

$\because y(\pi) = 1 \Rightarrow c = 0$

$y(\pi/2) = -1 - 2 = -3$

Ans = 10 – 3 = 7

24. If the shortest distance between the lines $\frac{x+2}{2} = \frac{y+3}{3} = \frac{z-5}{4}$ and $\frac{x-3}{1} = \frac{y-2}{-3} = \frac{z+4}{2}$ is $\frac{38}{3\sqrt{5}}k$ and $\int_0^k [x^2]dx = \alpha - \sqrt{\alpha}$, where [x] denotes the greatest integer function, then $6\alpha^3$ is equal to _____

Ans. (48)

Sol. $\frac{38}{3\sqrt{5}}\hat{k} = \frac{(5\hat{i} + 5\hat{j} - 9\hat{k})}{\sqrt{5}} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 1 & -3 & 2 \end{vmatrix}$

$\frac{38}{3\sqrt{5}}\hat{k} = \frac{19}{\sqrt{5}}$

$k = \frac{19}{\sqrt{5}}$

$k = \frac{3}{2}$

$\int_0^{3/2} [x^2] = \int_0^1 0 + \int_1^{\sqrt{2}} 1 + \int_{\sqrt{2}}^{3/2} 2$

$= \sqrt{2} - 1 + 2\left(\frac{3}{2} - \sqrt{2}\right)$

$= 2 - \sqrt{2}$

$\alpha = 2$

$\Rightarrow 6\alpha^3 = 48$

25. Let A be a square matrix of order 2 such that |A| = 2 and the sum of its diagonal elements is –3. If the points (x, y) satisfying $A^2 + xA + yI = 0$ lie on a hyperbola, whose transverse axis is parallel to the x-axis, eccentricity is e and the length of the latus rectum is ℓ, then $e^4 + \ell^4$ is equal to ______

Ans. (Bouns)

NTA Ans. (25)

Sol. Given |A| = 2

trace A = –3

and $A^2 + xA + yI = 0$

$\Rightarrow x = 3, y = 2$

so, information is incomplete to determine eccentricity of hyperbola (e) and length of latus rectum of hyperbola (ℓ)

26. Let $a = 1 + \frac{^2C_2}{3!} + \frac{^3C_2}{4!} + \frac{^4C_2}{5!} + ...,$

$$b = 1 + \frac{^1C_0 + {}^1C_1}{1!} + \frac{^2C_0 + {}^2C_1 + {}^2C_2}{2!} + \frac{^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3}{3!} + ...$$

Then $\frac{2b}{a^2}$ is equal to ______

Ans. (8)

Sol. $f(x) = 1 + \frac{(1+x)}{1!} + \frac{(1+x)^2}{2!} + \frac{(1+x)^3}{3!} + ...$

$$\frac{e^{(1+x)}}{1+x} = \frac{1}{1+x} + 1 + \frac{(1+x)}{2!} + \frac{(1+x)^2}{3!} + \frac{(1+x)^2}{4!}$$

coef x^2 in RHS : $1 + \frac{^2C_2}{3} + \frac{^3C_2}{4} + ... = a$

coeff. x^2 in L.H.S.

$$e\left(1 + x + \frac{x^2}{2!}\right)....\left(1 - x + \frac{x^2}{2!}......\right)$$

is $e - e + \frac{e}{2!} = a$

$$b = 1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + ... = e^2$$

$$\frac{2b}{a^2} = 8$$

27. Let A be a 3 × 3 matrix of non-negative real elements such that $A\begin{bmatrix}1\\1\\1\end{bmatrix} = 3\begin{bmatrix}1\\1\\1\end{bmatrix}$. Then the maximum value of det(A) is _____

Ans. (27)

Sol. Let $A = \begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}$

$$A\begin{bmatrix}1\\1\\1\end{bmatrix} = 3\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$\Rightarrow a_1 + a_2 + a_3 = 3$ .....(1)

$\Rightarrow b_1 + b_2 + b_3 = 3$ .....(2)

$\Rightarrow c_1 + ca_2 + c_3 = 3$ .....(3)

Now,

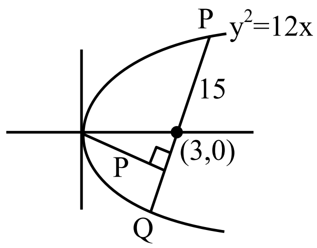
$|A| = (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2)$
$- (a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)$

$\therefore$ From above in formation, clearly $|A|_{max} = 27$, when $a_1 = 3, b_2 = 3, c_3 = 3$

28. Let the length of the focal chord PQ of the parabola $y^2 = 12x$ be 15 units. If the distance of PQ from the origin is p, then $10p^2$ is equal to ____

Ans. (72)

Sol.



length of focal chord = $4a\ \text{cosec}^2\theta = 15$

$12\text{cosec}^2\theta = 15$

$\sin^2\theta = \frac{4}{5}$

$\tan^2\theta = 4$

$\tan\theta = 2$

equation $\frac{y-0}{x-3} = 2$

$y = 2x - 6$

$2x - y - 6 = 0$

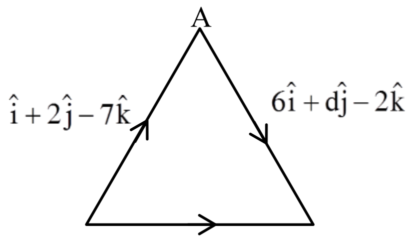
$P = \frac{6}{\sqrt{5}}$

$10p^2 = 10.\frac{36}{5} = 72$

29. Let ABC be a triangle of area $15\sqrt{2}$ and the vectors $\overrightarrow{AB} = \hat{i} + 2\hat{j} - 7\hat{k}$, $\overrightarrow{BC} = a\hat{i} + b\hat{j} + c\hat{k}$ and $\overrightarrow{AC} = 6\hat{i} + d\hat{j} - 2\hat{k}$, d > 0. Then the square of the length of the largest side of the triangle ABC is

Ans. (54)

Sol.



$$\text{Area} = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -7 \\ 6 & d & -2 \end{vmatrix} = 15\sqrt{2}$$

$(-4 + 7d)\,\hat{i} - \hat{j}(-2 + 42) + \hat{k}\,(d - 12)$

$(7d - 4)^2 + (40)^2 + (d - 12)^2 = 1800$

$50d^2 - 80d - 40 = 0$

$5d^2 - 8d - 4 = 0$

$5d^2 - 10d - 2d - 4$

$5d(d - 2) + 2(d - 2) = 0$

$d = 2 \text{ or } d = -\frac{2}{5}$

$\because d > 0, d = 2$

$(a + 1)\,\hat{i} + (b + 2)\,\hat{j} + (c - 7)\,\hat{k} = 6\,\hat{i} + 2\,\hat{j} - 2\,\hat{k}$

$a + 1 = 6 \;\&\; b + 2 = 2,\; c - 7 = -2$

$a = 5 \quad b = 0 \quad c = 5$

$|AB| = \sqrt{1+4+49} = \sqrt{54}$

$|BC| = \sqrt{25+25} = \sqrt{50}$

$|AC| = \sqrt{86+4+4} = \sqrt{44}$

Ans. 54

30. If $\int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{1+\sin x\cos x}dx = \frac{1}{a}\log_e\left(\frac{a}{3}\right) + \frac{\pi}{b\sqrt{3}}$, where a, b $\in$ N, then a + b is equal to ______

Ans. (8)

Sol. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{1+\frac{1}{2}\sin 2x}dx = \int_0^{\frac{\pi}{4}} \frac{1-\cos 2x}{2+\sin 2x}dx$

$\int \frac{1}{2+\sin 2x} - \int \frac{\cos 2x}{2+\sin 2x}$

$(I_1) \quad - \quad (I_2)$

$(I_1) = \int \frac{dx}{2+\frac{2\tan x}{1+\tan^2 x}}$

$\int_0^{\frac{\pi}{4}} \frac{\sec^2 x\, dx}{2\tan^2 x + 2\tan x + 2}$

$\tan x = t$

$\frac{1}{2}\int_0^1 \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{\pi}{6\sqrt{3}}$

$I_2 = \int_0^{\pi/4} \frac{\cos 2x}{2+\sin 2x}dx = \frac{1}{2}\left(\ell n\frac{3}{2}\right)$

$I_1 - I_2 = \frac{1}{\sqrt{3}}\frac{\pi}{6} + \frac{1}{2}\ell n\frac{2}{3}$

$\Rightarrow a = 2, b = 6$

Ans. 8