FINAL JEE–MAIN EXAMINATION – JANUARY, 2024

(Held On Saturday 27th January, 2024) **TIME : 3 : 00 PM to 6 : 00 PM**

SECTION-A

1. Considering only the principal values of inverse trigonometric functions, the number of positive real values of x satisfying $\tan^{-1}(x)+\tan^{-1}(2x)=\frac{\pi}{4}$ is :

(1) More than 2

(2) 1

(3) 2

(4) 0

Ans. (2)

Sol. $\tan^{-1}x+\tan^{-1}2x=\frac{\pi}{4}$; $x>0$

$\Rightarrow \tan^{-1}2x=\frac{\pi}{4}-\tan^{-1}x$

Taking tan both sides

$\Rightarrow 2x=\frac{1-x}{1+x}$

$\Rightarrow 2x^2+3x-1=0$

$x=\frac{-3\pm\sqrt{9+8}}{8}=\frac{-3\pm\sqrt{17}}{8}$

Only possible $x=\frac{-3+\sqrt{17}}{8}$

2. Consider the function $f:(0,2)\to R$ defined by $f(x)=\frac{x}{2}+\frac{2}{x}$ and the function g(x) defined by

$$g(x)=\begin{cases}\min\{f(t)\}, & 0<t\le x \text{ and } 0<x\le 1\\ \frac{3}{2}+x, & 1<x<2\end{cases}$$. Then

(1) g is continuous but not differentiable at x = 1

(2) g is not continuous for all

(3) g is neither continuous nor differentiable at x = 1

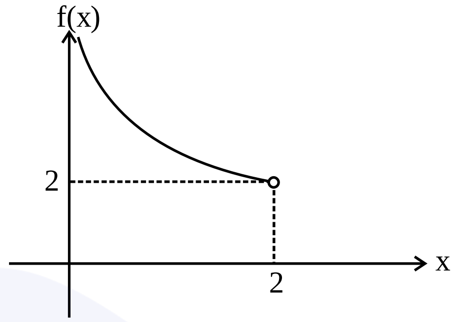
(4) g is continuous and differentiable for all $x\in(0,2)$

Ans. (1)

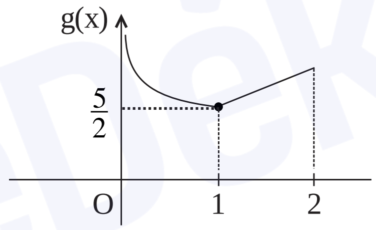
Sol. $f:(0,2)\to R$; $f(x)=\frac{x}{2}+\frac{2}{x}$

$f'(x)=\frac{1}{2}-\frac{2}{x^2}$

$\therefore f(x)$ is decreasing in domain.



$$g(x)=\begin{cases}\frac{x}{2}+\frac{2}{x} & 0<x\le 1\\ \frac{3}{2}+x & 1<x<2\end{cases}$$



3. Let the image of the point (1, 0, 7) in the line $\frac{}{1}=\frac{}{2}=\frac{}{3}$ be the point (α, β, γ). Then which one of the following points lies on the line passing through (α, β, γ) and making angles $\frac{2\pi}{3}$ and $\frac{3\pi}{4}$ with y-axis and z-axis respectively and an acute angle with x-axis ?

(1) $\left(1,-2,1+\sqrt{2}\right)$

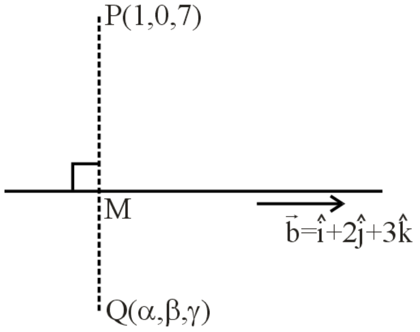
(2) $\left(1,2,1-\sqrt{2}\right)$

(3) $\left(3,4,3-2\sqrt{2}\right)$

(4) $\left(3,-4,3+2\sqrt{2}\right)$

Ans. (3)

Sol. $L_1 = \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda$



$M(\lambda, 1+2\lambda, 2+3\lambda)$

$\overrightarrow{PM} = (\lambda-1)\hat{i} + (1+2\lambda)\hat{j} + (3\lambda-5)\hat{k}$

$\overrightarrow{PM}$ is perpendicular to line L_1

$\overrightarrow{PM}.\vec{b} = 0 \quad (\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k})$

$\Rightarrow \lambda - 1 + 4\lambda + 2 + 9\lambda - 15 = 0$

$14\lambda = 14 \Rightarrow \lambda = 1$

$\therefore M = (1,3,5)$

$\vec{Q} = 2\vec{M} - \vec{P}$ [M is midpoint of $\vec{P}$ & $\vec{Q}$]

$\vec{Q} = 2\hat{i} + 6\hat{j} + 10\hat{k} - \hat{i} - 7\hat{k}$

$\vec{Q} = \hat{i} + 6\hat{j} + 3\hat{k}$

$\therefore (\alpha, \beta, \gamma) = (1,6,3)$

Required line having direction cosine (l, m, n)

$l^2 + m^2 + n^2 = 1$

$\Rightarrow l^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = 1$

$l^2 = \frac{1}{4}$

$\therefore l = \frac{1}{2}$ [Line make acute angle with x-axis]

Equation of line passing through (1, 6, 3) will be

$\vec{r} = (\hat{i} + 6\hat{j} + 3\hat{k}) + \mu\left(\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} - \frac{1}{\sqrt{2}}\hat{k}\right)$

Option (3) satisfying for $\mu = 4$

4. Let R be the interior region between the lines $3x - y + 1 = 0$ and $x + 2y - 5 = 0$ containing the origin. The set of all values of a, for which the points $(a^2, a + 1)$ lie in R, is :

(1) $(-3,-1) \cup \left(-\frac{1}{3}, 1\right)$

(2) $(-3,0) \cup \left(\frac{1}{3}, 1\right)$

(3) $(-3,0) \cup \left(\frac{2}{3}, 1\right)$

(4) $(-3,-1) \cup \left(\frac{1}{3}, 1\right)$

Ans. **(2)**

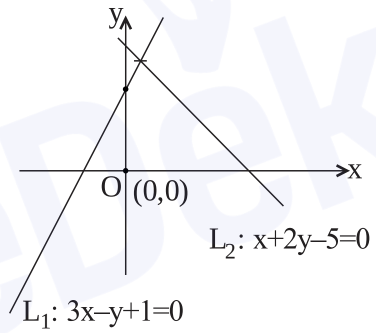
Sol. $P(a^2, a + 1)$

$L_1 = 3x - y + 1 = 0$

Origin and P lies same side w.r.t. L_1

$\Rightarrow L_1(0) \,.\, L_1(P) > 0$

$\therefore 3(a^2) - (a + 1) + 1 > 0$



$\Rightarrow 3a^2 - a > 0$

$a \in (-\infty, 0) \cup \left(\frac{1}{3}, \infty\right)$ ....(1)

Let $L_2 : x + 2y - 5 = 0$

Origin and P lies same side w.r.t. L_2

$\Rightarrow L_2(0).L_2(P) > 0$

$\Rightarrow a^2 + 2(a+1) - 5 < 0$

$\Rightarrow a^2 + 2a - 3 < 0$

$\Rightarrow \quad + \quad - \quad <$

$\therefore a \in (-3,1)$ ....(2)

Intersection of (1) and (2)

$a \in (-3,0) \cup \left(\frac{1}{3}, 1\right)$

5. The 20th term from the end of the progression $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, ..., -129\frac{1}{4}$ is :-

(1) -118

(2) -110

(3) -115

(4) -100

Ans. (3)

Sol. $20, 19\frac{1}{4}, 18\frac{1}{2}, 17\frac{3}{4}, ..., -129\frac{1}{4}$

This is A.P. with common difference

$d_1 = -1 + \frac{1}{4} = -\frac{3}{4}$

$-129\frac{1}{4}, ..., 19\frac{1}{4}, 20$

This is also A.P. $a = -129\frac{1}{4}$ and $d = \frac{3}{4}$

Required term =

$-129\frac{1}{4} + (20-1)\left(\frac{3}{4}\right)$

$= -129 - \frac{1}{4} + 15 - \frac{3}{4} = -115$

6. Let $f: R - \left\{\frac{-1}{2}\right\} \to R$ and $g: R - \left\{\frac{-5}{2}\right\} \to R$ be defined as $f(x) = \frac{2x+3}{2x+1}$ and $g(x) = \frac{|x|+1}{2x+5}$. Then the domain of the function fog is :

(1) $R - \left\{-\frac{5}{2}\right\}$

(2) R

(3) $R - \left\{-\frac{7}{4}\right\}$

(4) $R - \left\{-\frac{5}{2}, -\frac{7}{4}\right\}$

Ans. (1)

Sol. $f(x) = \frac{2x+3}{2x+1}; x \neq -\frac{1}{2}$

$g(x) = \frac{|x|+1}{2x+5}, x \neq -\frac{5}{2}$

Domain of f(g(x))

$f(g(x)) = \frac{2g(x)+3}{2g(x)+1}$

$x \neq -\frac{5}{2}$ and $\frac{|x|+1}{2x+5} \neq -\frac{1}{2}$

$x \in R - \left\{-\frac{5}{2}\right\}$ and $x \in R$

$\therefore$ Domain will be $R - \left\{-\frac{5}{2}\right\}$

7. For $0 < a < 1$, the value of the integral $\int_0^{\pi} \frac{dx}{1 - 2a\cos x + a^2}$ is :

(1) $\frac{\pi^2}{\pi + a^2}$

(2) $\frac{\pi^2}{\pi - a^2}$

(3) $\frac{\pi}{1 - a^2}$

(4) $\frac{\pi}{1 + a^2}$

Ans. (3)

Sol. $I = \int_0^{\pi} \frac{dx}{1 - 2a\cos x + a^2}; 0 < a < 1$

$I = \int_0^{\pi} \frac{dx}{1 + 2a\cos x + a^2}$

$2I = 2\int_0^{\pi/2} \frac{2(1+a^2)}{(1+a^2)^2 - 4a^2\cos^2 x} dx$

$\Rightarrow I = \int_0^{\pi/2} \frac{2(1+a^2).\sec^2 x}{(1+a^2)^2.\sec^2 x - 4a^2} dx$

$\Rightarrow I = \int_0^{\pi/2} \frac{2.(1+a^2).\sec^2 x}{(1+a^2)^2.\tan^2 x + (1-a^2)^2} dx$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\frac{2.\sec^2 x}{1+a^2}.dx}{\tan^2 x + \left(\frac{1-a^2}{1+a^2}\right)^2}$$

$$\Rightarrow I = \frac{2}{(1-a^2)}\left[\frac{\pi}{2} - 0\right]$$

$$I = \frac{\pi}{1-a^2}$$

8. Let $g(x) = 3f\left(\frac{x}{3}\right) + f(3-x)$ and $f''(x) > 0$ for all $x \in (0,3)$. If g is decreasing in (0, α) and increasing in (α, 3), then 8α is

(1) 24

(2) 0

(3) 18

(4) 20

Ans. **(3)**

Sol. $g(x) = 3f\left(\frac{x}{3}\right) + f(3-x)$ and $f''(x) > 0 \; \forall \; x \in (0, 3)$

$\Rightarrow f'(x)$ is increasing function

$$g'(x) = 3 \times \frac{1}{3}.f'\left(\frac{x}{3}\right) - f'(3-x)$$

x)

If g is decreasing in (0, α)

$g'(x) < 0$

$$f'\left(\frac{x}{3}\right) - f'(3-x) < 0$$

$$f'\left(\frac{x}{3}\right) < f'(3-x)$$

$$\Rightarrow \frac{x}{3} < 3 - x$$

$$\Rightarrow x < \frac{9}{4}$$

Therefore $\alpha = \frac{9}{4}$

Then $8\alpha = 8 \times \frac{9}{4} = 18$

9. If $\lim_{x\to 0} \frac{3 + \alpha \sin x + \beta \cos x + \log_e(1-x)}{3\tan^2 x} = \frac{1}{3}$, then 2α–β is equal to :

(1) 2

(2) 7

(3) 5

(4) 1

Ans. **(3)**

Sol. $$\lim_{x\to 0} \frac{3 + \alpha \sin x + \beta \cos x + \log_e(1-x)}{3\tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \lim_{x\to 0} \frac{3 + \alpha\left[x - \frac{x^3}{3!} + ...\right] + \beta\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!}....\right] + \left(-x - \frac{x^2}{2} - \frac{x^3}{3}...\right)}{3\tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \lim_{x\to 0} \frac{(3+\beta) + (\alpha - 1)x + \left(-\frac{1}{2} - \frac{\beta}{2}\right)x^2 + ...}{3x^2} \times \frac{x^2}{\tan^2 x} = \frac{1}{3}$$

$$\Rightarrow \beta + 3 = 0,\ \alpha - 1 = 0 \text{ and } \frac{-\frac{1}{2} - \frac{\beta}{2}}{3} = \frac{1}{3}$$

$$\Rightarrow \beta = -3, \alpha = 1$$

$$\Rightarrow 2\alpha - \beta = 2 + 3 = 5$$

10. If α, β are the roots of the equation, $x^2 - x - 1 = 0$ and $S_n = 2023\alpha^n + 2024\beta^n$, then

(1) $2S_{12} = S_{11} + S_{10}$

(2) $S_{12} = S_{11} + S_{10}$

(3) $2S_{11} = S_{12} + S_{10}$

(4) $S_{11} = S_{10} + S_{12}$

Ans. **(2)**

Sol. $x^2 - x - 1 = 0$

$S_n = 2023\alpha^n + 2024\beta^n$

$S_{n-1} + S_{n-2} = 2023\alpha^{n-1} + 2024\beta^{n-1} + 2023\alpha^{n-2} + 2024\beta^{n-2}$

$= 2023\alpha^{n-2}[1+\alpha] + 2024\beta^{n-2}[1+\beta]$

$= 2023\alpha^{n-2}[\alpha^2] + 2024\beta^{n-2}[\beta^2]$

$= 2023\alpha^n + 2024\beta^n$

$S_{n-1} + S_{n-2} = S_n$

Put n = 12

$S_{11} + S_{10} = S_{12}$

11. Let A and B be two finite sets with m and n elements respectively. The total number of subsets of the set A is 56 more than the total number of subsets of B. Then the distance of the point P(m, n) from the point Q(–2, –3) is

(1) 10

(2) 6

(3) 4

(4) 8

Ans. (1)

Sol. $2^m - 2^n = 56$

$2^n(2^{m-n} - 1) = 2^3 \times 7$

$2^n = 2^3$ and $2^{m-n} - 1 = 7$

$\Rightarrow n = 3$ and $2^{m-n} = 8$

$\Rightarrow n = 3$ and $m - n = 3$

$\Rightarrow n = 3$ and $m = 6$

P(6,3) and Q(–2, –3)

$PQ = \sqrt{8^2 + 6^2} = \sqrt{100} = 10$

Hence option (1) is correct

12. The values of α, for which $\begin{vmatrix} 1 & \frac{3}{2} & \alpha + \frac{3}{2} \\ 1 & \frac{1}{3} & \alpha + \frac{1}{3} \\ 2\alpha + 3 & 3\alpha + 1 & 0 \end{vmatrix} = 0$, lie in the interval

(1) (–2, 1)

(2) (–3, 0)

(3) $\left(-\frac{3}{2}, \frac{3}{2}\right)$

(4) (0, 3)

Ans. (2)

Sol. $\begin{vmatrix} 1 & \frac{3}{2} & \alpha + \frac{3}{2} \\ 1 & \frac{1}{3} & \alpha + \frac{1}{3} \\ 2\alpha + 3 & 3\alpha + 1 & 0 \end{vmatrix} = 0$

$\Rightarrow (2\alpha + 3)\left\{\frac{7\alpha}{6}\right\} - (3\alpha + 1)\left\{\frac{-7}{6}\right\} = 0$

$\Rightarrow (2\alpha + 3).\frac{7\alpha}{6} + (3\alpha + 1).\frac{7}{6} = 0$

$\Rightarrow 2\alpha^2 + 3\alpha + 3\alpha + 1 = 0$

$\Rightarrow 2\alpha^2 + 6\alpha + 1 = 0$

$\Rightarrow \alpha = \frac{-3 + \sqrt{7}}{2}, \frac{-3 - \sqrt{7}}{2}$

Hence option (2) is correct.

13. An urn contains 6 white and 9 black balls. Two successive draws of 4 balls are made without replacement. The probability, that the first draw gives all white balls and the second draw gives all black balls, is :

(1) $\frac{5}{256}$

(2) $\frac{5}{715}$

(3) $\frac{3}{715}$

(4) $\frac{3}{256}$

Ans. (3)

Sol. $\frac{{}^6C_4}{{}^{15}C_4} \times \frac{{}^9C_4}{{}^{11}C_4} = \frac{3}{715}$

Hence option (3) is correct.

14. The integral $\int \frac{(x^8 - x^2)dx}{(x^{12} + 3x^6 + 1)\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)}$ is equal to :

(1) $\log_e\left(\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right|\right)^{1/3} + C$

(2) $\log_e\left(\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right|\right)^{1/2} + C$

(3) $\log_e\left(\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right|\right) + C$

(4) $\log_e\left(\left|\tan^{-1}\left(x^3 + \frac{1}{x^3}\right)\right|\right)^3 + C$

Ans. (1)

Sol. $I=\int \frac{x^8-x^2}{(x^{12}+3x^6+1)\tan^{-1}\left(x^3+\frac{1}{x^3}\right)}dx$

Let $\tan^{-1}\left(x^3+\frac{1}{x^3}\right)=t$

$\Rightarrow \frac{1}{1+\left(x^3+\frac{1}{x^3}\right)^2}\cdot\left(3x^2-\frac{3}{x^4}\right)dx=dt$

$\Rightarrow \frac{x^6}{x^{12}+3x^6+1}\cdot\frac{3x^6-3}{x^4}dx=dt$

$I=\frac{1}{3}\int\frac{dt}{t}=\frac{1}{3}\ln|t|+C$

$I=\frac{1}{3}\ln\left|\tan^{-1}\left(x^3+\frac{1}{x^3}\right)\right|+C$

$I=\ln\left|\tan^{-1}\left(x^3+\frac{1}{x^3}\right)\right|^{1/3}+C$

Hence option (1) is correct

15. If $2\tan^2\theta-5\sec\theta=1$ has exactly 7 solutions in the interval $\left[0,\frac{n\pi}{2}\right]$, for the least value of $n\in N$ then $\sum_{k=1}^{n}\frac{k}{2^k}$ is equal to :

(1) $\frac{1}{2^{15}}(2^{14}-14)$

(2) $\frac{1}{14}(2^{15}-15)$

(3) $1-\frac{15}{2^{13}}$

(4) $\frac{1}{2^{13}}(2^{14}-15)$

Ans. **(4)**

Sol. $2\tan^2\theta-5\sec\theta-1=0$

$\Rightarrow 2\sec^2\theta-5\sec\theta-3=0$

$\Rightarrow (2\sec\theta+1)(\sec\theta-3)=0$

$\Rightarrow \sec\theta=-\frac{1}{2},3$

$\Rightarrow \cos\theta=-2,\frac{1}{3}$

$\Rightarrow \cos\theta=\frac{1}{3}$

For 7 solutions n = 13

So, $\sum_{k=1}^{13}\frac{k}{2^k}=S$ (say)

$S=\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+....+\frac{13}{2^{13}}$

$\frac{1}{2}S=\frac{1}{2^2}+\frac{1}{2^3}+.....+\frac{12}{2^{13}}+\frac{13}{2^{14}}$

$\frac{S}{2}=\frac{1}{2}\cdot\frac{1-\frac{1}{2^{13}}}{1-\frac{1}{2}}-\frac{13}{2^{14}}\Rightarrow\;=\;\frac{(2^{13}-1)}{}-\frac{13}{}$

16. The position vectors of the vertices A, B and C of a triangle are $2\hat{i}-3\hat{j}+3\hat{k}$, $2\hat{i}+2\hat{j}+3\hat{k}$ and $-\hat{i}+\hat{j}+3\hat{k}$ respectively. Let l denotes the length of the angle bisector AD of $\angle BAC$ where D is on the line segment BC, then $2l^2$ equals :

(1) 49

(2) 42

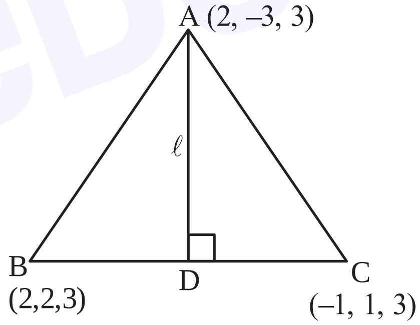
(3) 50

(4) 45

Ans. **(4)**

Sol. AB = 5

AC = 5



$\therefore$ D is midpoint of BC

$D\left(\frac{1}{2},\frac{3}{2},3\right)$

$\therefore l=\sqrt{\left(2-\frac{1}{2}\right)^2+\left(-3-\frac{3}{2}\right)^2+(3-3)^2}$

$l=\sqrt{\frac{45}{2}}$

$\therefore 2l^2=45$

17. If y = y(x) is the solution curve of the differential equation $(x^2-4)dy-(y^2-3y)dx=0$, $x>2, y(4)=\frac{3}{2}$ and the slope of the curve is never zero, then the value of y(10) equals :

(1) $\frac{3}{1+(8)^{1/4}}$

(2) $\frac{3}{1+2\sqrt{2}}$

(3) $\frac{3}{1-2\sqrt{2}}$

(4) $\frac{3}{1-(8)^{1/4}}$

Ans. (1)

Sol. $(x^2-4)dy-(y^2-3y)dx=0$

$$\Rightarrow \int \frac{dy}{y^2-3y}=\int \frac{dx}{x^2-4}$$

$$\Rightarrow \frac{1}{3}\int \frac{y-(y-3)}{y(y-3)}dy=\int \frac{dx}{x^2-4}$$

$$\Rightarrow \frac{1}{3}(\ln|y-3|-\ln|y|)=\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C$$

$$\Rightarrow \frac{1}{3}\ln\left|\frac{y-3}{y}\right|=\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+C$$

At $x=4,\ y=\frac{3}{2}$

$$\therefore \frac{1}{3}\ln\left|\frac{y-3}{y}\right|=\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|+\frac{1}{4}\ln(3)$$

At x = 10

$$\frac{1}{3}\ln\left|\frac{y-3}{y}\right|=\frac{1}{4}\ln\left|\frac{2}{3}\right|+\frac{1}{4}\ln(3)$$

$$\ln\left|\frac{y-3}{y}\right|=\ln 2^{3/4},\ \forall x>2, \frac{dy}{dx}<0$$

as $y(4)=\frac{3}{2}\Rightarrow y\in(0,3)$

$-y+3=8^{1/4}.y$

$$y=\frac{3}{1+8^{1/4}}$$

18. Let e_1 be the eccentricity of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ and e_2 be the eccentricity of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a > b, which passes through the foci of the hyperbola. If $e_1e_2 = 1$, then the length of the chord of the ellipse parallel to the x-axis and passing through (0, 2) is :

(1) $4\sqrt{5}$

(2) $\frac{8\sqrt{5}}{3}$

(3) $\frac{10\sqrt{5}}{3}$

(4) $3\sqrt{5}$

Ans. (3)

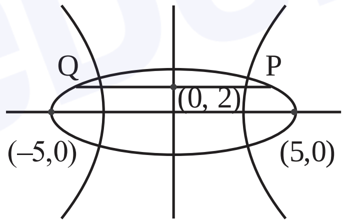
Sol. $H:\frac{x^2}{16}-\frac{y^2}{9}=1 \qquad e_1=\frac{5}{4}$

$\therefore e_1e_2=1\Rightarrow e_2=\frac{4}{5}$

Also, ellipse is passing through $(\pm5,0)$

$\therefore$ a = 5 and b = 3

$E:\frac{x^2}{25}+\frac{y^2}{9}=1$



End point of chord are $\left(\pm\frac{5\sqrt{5}}{3},2\right)$

$$\therefore L_{PQ}=\frac{10\sqrt{5}}{3}$$

19. Let $\alpha=\frac{(4!)!}{(4!)^{3!}}$ and $\beta=\frac{(5!)!}{(5!)^{4!}}$. Then :

(1) $\alpha\in N$ and $\beta\notin N$

(2) $\alpha\notin N$ and $\beta\in N$

(3) $\alpha\in N$ and $\beta\in N$

(4) $\alpha\notin N$ and $\beta\notin N$

Ans. (3)

Sol. $\alpha = \frac{(4!)!}{(4!)^{3!}}, \beta = \frac{(5!)!}{(5!)^{4!}}$

$\alpha = \frac{(24)!}{(4!)^{6}}, \beta = \frac{(120)!}{(5!)^{24}}$

Let 24 distinct objects are divided into 6 groups of 4 objects in each group.

No. of ways of formation of group = $\frac{24!}{(4!)^6 . 6!} \in N$

Similarly,
Let 120 distinct objects are divided into 24 groups of 5 objects in each group.
No. of ways of formation of groups

$= \frac{(120)!}{(5!)^{24} . 24!} \in N$

20. Let the position vectors of the vertices A, B and C of a triangle be $2\hat{i} + 2\hat{j} + \hat{k}$, $\hat{i} + 2\hat{j} + 2\hat{k}$ and $2\hat{i} + \hat{j} + 2\hat{k}$ respectively. Let l_1, l_2 and l_3 be the lengths of perpendiculars drawn from the ortho center of the triangle on the sides AB, BC and CA respectively, then $l_1^2 + l_2^2 + l_3^2$ equals :

(1) $\frac{1}{5}$ (2) $\frac{1}{2}$

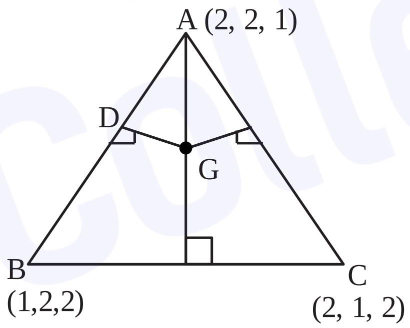
(3) $\frac{1}{4}$ (4) $\frac{1}{3}$

Ans. **(2)**

Sol. ΔABC is equilateral

Orthocentre and centroid will be same

$\left.\frac{5}{3}\right)$



Mid-point of AB is $D\left(\frac{3}{2}, 2, \frac{3}{2}\right)$

$\therefore \ell_1 = \sqrt{\frac{1}{36} + \frac{1}{9} + \frac{1}{36}}$

$\ell_1 = \sqrt{\frac{1}{6}} = \ell_2 = \ell_3$

$\therefore \ell_1^2 + \ell_2^2 + \ell_3^2 = \frac{1}{2}$

SECTION-B

21. The mean and standard deviation of 15 observations were found to be 12 and 3 respectively. On rechecking it was found that an observation was read as 10 in place of 12. If μ and σ^2 denote the mean and variance of the correct observations respectively, then $15(\mu + \mu^2 + \sigma^2)$ is equal to ……………………..

Ans. **(2521)**

Sol. **Let the incorrect mean be** μ' and standard deviation be σ'

We have

$\mu' = \frac{\Sigma x_i}{15} = 12 \Rightarrow \Sigma x_i = 180$

As per given information correct Σx_i = 180–10+12

$\Rightarrow \mu$ (correct mean) = $\frac{182}{15}$

Also

$\sigma' = \sqrt{\frac{\Sigma x_i^2}{15} - 144} = 3 \Rightarrow \Sigma x_i^2 = 2295$

Correct Σx_i^2 = 2295 – 100 + 144 = 2339

σ^2 (correct variance) = $\frac{2339}{15} - \frac{182 \times 182}{15 \times 15}$

Required value

$= 15(\mu + \mu^2 + \sigma^2)$

$= 15\left(\frac{182}{15} + \frac{182 \times 182}{15 \times 15} + \frac{2339}{15} - \frac{182 \times 182}{15 \times 15}\right)$

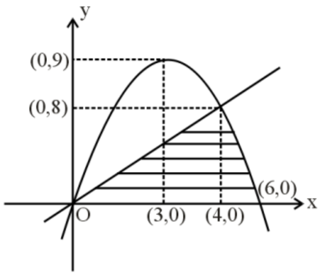
$= 15\left(\frac{182}{15} + \frac{2339}{15}\right)$

$= 2521$

22. If the area of the region $\{(x, y): 0 \le y \le \min\{2x, 6x - x^2\}\}$ is A, then 12 A is equal to..................

Ans. (304)

Sol. We have



$$A = \frac{1}{2} \times 4 \times 8 + \int_4^6 (6x - x^2)dx$$

$$A = \frac{76}{3}$$

$12A = 304$

23. Let A be a 2 × 2 real matrix and I be the identity matrix of order 2. If the roots of the equation $|A - xI| = 0$ be –1 and 3, then the sum of the diagonal elements of the matrix A^2 is..............

Ans. (10)

Sol. $|A - xI| = 0$

Roots are –1 and 3

Sum of roots = tr(A) = 2

Product of roots = |A| = –3

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

We have a + d = 2

ad – bc = –3

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

We need $a^2 + bc + bc + d^2$

$= a^2 + 2bc + d^2$

$= (a + d)^2 - 2ad + 2bc$

$= 4 - 2(ad - bc)$

$= 4 - 2(-3)$

$= 4 + 6$

$= 10$

24. If the sum of squares of all real values of α, for which the lines 2x – y + 3 = 0, 6x + 3y + 1 = 0 and αx + 2y –2 = 0 do not form a triangle is p, then the greatest integer less than or equal to p is ....

Ans. (32)

Sol. $2x - y + 3 = 0$

$6x + 3y + 1 = 0$

$\alpha x + 2y - 2 = 0$

Will not form a Δ if $\alpha x + 2y - 2 = 0$ is concurrent with $2x - y + 3 = 0$ and $6x + 3y + 1 = 0$ or parallel to either of them so

Case-1: Concurrent lines

$$\begin{vmatrix} 2 & -1 & 3 \\ 6 & 3 & 1 \\ \alpha & 2 & -2 \end{vmatrix} = 0 \Rightarrow \alpha = \frac{4}{5}$$

Case-2 : Parallel lines

$$-\frac{\alpha}{2} = \frac{-6}{3} \text{ or } -\frac{\alpha}{2} = 2$$

$\Rightarrow \alpha = 4$ or $\alpha = -4$

$$P = 16 + 16 + \frac{16}{25}$$

$$[P] = \left[32 + \frac{16}{25}\right] = 32$$

25. The coefficient of x^{2012} in the expansion of $(1-x)^{2008}(1+x+x^2)^{2007}$ is equal to

Ans. (0)

Sol. $(1-x)(1-x)^{2007}(1+x+x^2)^{2007}$

$(1-x)(1-x^3)^{2007}$

$(1-x)({}^{2007}C_0 - {}^{2007}C_1(x^3) + ...)$

General term

$(1-x)((-1)^r\, {}^{2007}C_r x^{3r})$

$(-1)^{r\,2007}C_r x^{3r} - (-1)^{r\,2007}C_r x^{3r+1}$

3r = 2012

$r \neq \frac{2012}{3}$

3r + 1 = 2012

3r = 2011

$r \neq \frac{2011}{3}$

Hence there is no term containing x^{2012}.

So coefficient of $x^{2012} = 0$

26. If the solution curve, of the differential equation $\frac{dy}{dx} = \frac{x+y-2}{x-y}$ passing through the point (2, 1) is $\tan^{-1}\left(\frac{y-1}{x-1}\right) - \frac{1}{\beta}\log_e\left(\alpha + \left(\frac{y-1}{x-1}\right)^2\right) = \log_e|x-1|$, then $5\beta+\alpha$ is equal to

Ans. **(11)**

Sol. $\frac{dy}{dx} = \frac{x+y-2}{x-y}$

$x = X+h, y = Y+k$

$\frac{dY}{dX} = \frac{X+Y}{X-Y}$

$\left.\begin{matrix} h+k-2=0 \\ h-k=0 \end{matrix}\right\} h=k=1$

$Y = vX$

$v + \frac{dv}{dX} = \frac{1+v}{1-v} \Rightarrow X - \frac{dv}{dX} = \frac{1+v^2}{1-v}$

$\frac{1-v}{1+v^2}dv = \frac{dX}{X}$

$\tan^{-1}v - \frac{1}{2}\ln(1+v^2) = \ln|X| + C$

As curve is passing through (2, 1)

$\tan^{-1}\left(\frac{y-1}{x-1}\right) - \frac{1}{2}\ln\left(1+\left(\frac{y-1}{x-1}\right)^2\right) = \ln|x-1|$

$\therefore \alpha = 1$ and $\beta = 2$

$\Rightarrow 5\beta + \alpha = 11$

27. Let $f(x) = \int_0^x g(t)\log_e\left(\frac{1-t}{1+t}\right)dt$, where g is a continuous odd function.

If $\int_{-\pi/2}^{\pi/2}\left(f(x) + \frac{x^2\cos x}{1+e^x}\right)dx = \left(\frac{\pi}{\alpha}\right)^2 - \alpha$, then α is equal to............

Ans. **(2)**

Sol. $f(x) = \int_0^x g(t)\ln\left(\frac{1-t}{1+t}\right)dt$

$f(-x) = \int_0^{-x} g(t)\ln\left(\frac{1-t}{1+t}\right)dt$

$f(-x) = -\int_0^x g(-y)\ln\left(\frac{1+y}{1-y}\right)dy$

$= -\int_0^x g(y)\ln\left(\frac{1-y}{1+y}\right)dy$ (g is odd)

$f(-x) = -f(x) \Rightarrow f$ is also odd

Now,

$I = \int_{-\pi/2}^{\pi/2}\left(f(x) + \frac{x^2\cos x}{1+e^x}\right)dx$ .....(1)

$I = \int_{-\pi/2}^{\pi/2}\left(f(-x) + \frac{x^2e^x\cos x}{1+e^x}\right)dx$ ......(2)

$2I = \int_{-\pi/2}^{\pi/2} x^2\cos x\, dx = 2\int_0^{\pi/2} x^2\cos x\, dx$

$$I=\left(x^2\sin x\right)_0^{\pi/2}-\int_0^{\pi/2}2x\sin x\,dx$$

$$=\frac{\pi^2}{4}-2\left(-x\cos x+\int\cos x\,dx\right)_0^{\pi/2}$$

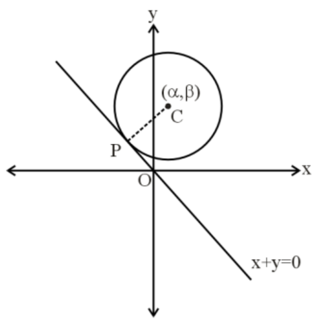
$$=\frac{\pi^2}{4}-2(0+1)=\frac{\pi^2}{4}-2\Rightarrow\left(\frac{\pi}{2}\right)^2-2$$

$\therefore \alpha=2$

28. Consider a circle $(x-\alpha)^2+(y-\beta)^2=50$, where $\alpha,\beta>0$. If the circle touches the line y + x = 0 at the point P, whose distance from the origin is $4\sqrt{2}$, then $(\alpha+\beta)^2$ is equal to................

Ans. **(100)**

Sol.



$S:(x-\alpha)^2+(y-\beta)^2=50$

CP = r

$$\left|\frac{\alpha+\beta}{\sqrt{2}}\right|=5\sqrt{2}$$

$\Rightarrow(\alpha+\beta)^2=100$

29. The lines $\frac{x-2}{2}=\frac{y}{-2}=\frac{z-7}{16}$ and $\frac{x+3}{4}=\frac{y+2}{3}=\frac{z+2}{1}$ intersect at the point P. If the distance of P from the line $\frac{x+1}{2}=\frac{y-1}{3}=\frac{z-1}{1}$ is l, then $14l^2$ is equal to................

Ans. **(108)**

Sol. $\frac{x-2}{1}=\frac{y}{-1}=\frac{z-7}{8}=\lambda$

$\frac{x+3}{4}=\frac{y+2}{3}=\frac{z+2}{1}=k$

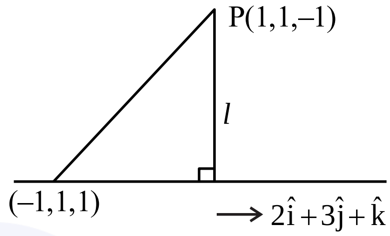
$\Rightarrow\lambda+2=4k-3$

$-\lambda=3k-2$

$\Rightarrow k=1,\lambda=-1$

$8\lambda+7=k-2$

$\therefore P=(1,1,-1)$



Projection of $2\hat{i}-2\hat{k}$ on $2\hat{i}+3\hat{j}+\hat{k}$ is

$$=\frac{4-2}{\sqrt{4+9+1}}=\frac{2}{\sqrt{14}}$$

$$\therefore l^2=8-\frac{4}{14}=\frac{108}{14}$$

$\Rightarrow 14l^2=108$

30. Let the complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on the circles $|z-z_0|^2=4$ and $|z-z_0|^2=16$ respectively, where $z_0=1+i$. Then, the value of $100|\alpha|^2$ is..............

Ans. **(20)**

Sol. $|z-z_0|^2=4$

$\Rightarrow(\alpha-z_0)(\bar{\alpha}-\bar{z}_0)=4$

$\Rightarrow\alpha\bar{\alpha}-\alpha\bar{z}_0-z_0\bar{\alpha}+|z_0|^2=4$

$\Rightarrow|\alpha|^2-\alpha\bar{z}_0-z_0\bar{\alpha}=2$ ....(1)

$|z-z_0|^2=16$

$$\Rightarrow \left(\frac{1}{\bar{\alpha}} - z_0\right)\left(\frac{1}{\alpha} - \bar{z}_0\right) = 16$$

$$\Rightarrow (1-\bar{\alpha}z_0)(1-\alpha\bar{z}_0) = 16|\alpha|^2$$

$$\Rightarrow 1-\bar{\alpha}z_0 - \alpha\bar{z}_0 + |\alpha|^2|z_0|^2 = 16|\alpha|^2$$

$$\Rightarrow 1-\bar{\alpha}z_0 - \alpha\bar{z}_0 = 14|\alpha|^2 \text{ ....(2)}$$

From (1) and (2)

$$\Rightarrow 5|\alpha|^2 = 1$$

$$\Rightarrow 100|\alpha|^2 = 20$$