FINAL JEE–MAIN EXAMINATION – JUNE, 2022

(Held On Saturday 25th June, 2022) **TIME : 9 : 00 AM to 12 : 00 PM**

MATHEMATICS

TEST PAPER WITH SOLUTION

SECTION-A

1. Let a circle C touch the lines $L_1 : 4x - 3y + K_1 = 0$ and $L_2 : 4x - 3y + K_2 = 0$, $K_1, K_2 \in R$. If a line passing through the centre of the circle C intersects L_1 at $(-1, 2)$ and L_2 at $(3, -6)$, then the equation of the circle C is

(A) $(x - 1)^2 + (y - 2)^2 = 4$

(B) $(x + 1)^2 + (y - 2)^2 = 4$

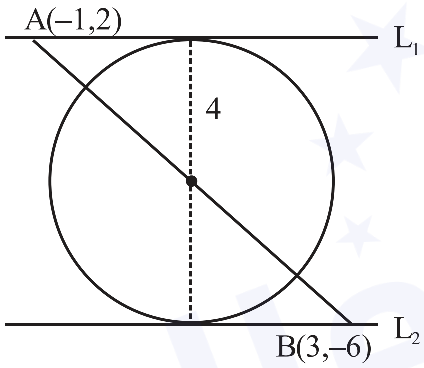
(C) $(x - 1)^2 + (y + 2)^2 = 16$

(D) $(x - 1)^2 + (y - 2)^2 = 16$

al Ans. by NTA (C)

Ans. (C)

Sol.



$L_1 : 4x - 3y + K_1 = 0$

$L_2 : 4x - 3y + K_2 = 0$

now

$-4 - 6 + K_1 = 0 \Rightarrow K_1 = 10$

$12 + 18 + K_2 = 0 \Rightarrow K_2 = -30$

$\Rightarrow$ Tangent to the circle are

$4x - 3y + 10 = 0$

$4x - 3y - 30 = 0$

Length of diameter $2r = \frac{|10+30|}{5} = 8$

$\Rightarrow r = 4$

Now centre is mid point of A & B

$x = 1, y = -2$

Equation of circle

$(x - 1)^2 + (y + 2)^2 = 16$ **Ans.**

2. The value of $\int_0^{\pi} \frac{e^{\cos x}\sin x}{(1+\cos^2 x)(e^{\cos x}+e^{-\cos x})}dx$ is equal to

(A) $\frac{\pi^2}{4}$

(B) $\frac{\pi^2}{2}$

(C) $\frac{\pi}{4}$

(D) $\frac{\pi}{2}$

al Ans. by NTA (C)

Ans. (C)

Sol. $\int_0^{\pi} \frac{e^{\cos x}\sin x}{(1+\cos^2 x)(e^{\cos x}+e^{-\cos x})}dx$ ....(1)

Use King's property

$I = \int_0^{\pi} \frac{e^{-\cos x}\sin x}{(1+\cos^2 x)(e^{-\cos x}+e^{\cos x})}dx$ ....(2)

On adding equation (1) and (2), we get

$2I = \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx = 2\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$

On putting $\cos x = t$, we get

$I = \int_0^1 \frac{dt}{1+t^2} = \left(\tan^{-1} t\right)_0^1 = \frac{\pi}{4}$

3. Let a, b and c be the length of sides of a triangle ABC such that $\frac{a+b}{7} = \frac{b+c}{8} = \frac{c+a}{9}$. If r and R are the radius of incircle and radius of circumcircle of the triangle ABC, respectively, then the value of $\frac{R}{r}$ is equal to

(A) $\frac{5}{2}$

(B) 2

(C) $\frac{3}{2}$

(D) 1

al Ans. by NTA (A)

Ans. (A)

Sol. $\frac{a+b}{7}=\frac{b+c}{8}=\frac{c+a}{9}=\lambda$

$a+b=7\lambda,\ b+c=8\lambda,\ a+c=9\lambda$

$\Rightarrow\ a+b+c=12\lambda$

Now $a=4\lambda,\ b=3\lambda,\ c=5\lambda$

$\because\ c^2=b^2+a^2$

$\angle C=90^\circ$

$$\Delta=\frac{1}{2}ab\sin C=\frac{1}{2}ab$$

$$\frac{R}{r}=\frac{c}{2\sin C}\times\frac{s}{\Delta}=\frac{c}{2}\times\frac{6\lambda}{\frac{1}{2}ab}=\frac{c}{ab}\times 6\lambda=\frac{5}{2}$$

4. Let $f: N\to R$ be a function such that $f(x+y)=2f(x)f(y)$ for natural numbers x and y. If $f(1)=2$, then the value of α for which

$$\sum_{k=1}^{10} f(\alpha+k)=\frac{512}{3}(2^{20}-1)$$

holds, is

(A) 2 (B) 3

(C) 4 (D) 6

al Ans. by NTA (C)

Ans. (C)

$f: N\to R,\ f(x+y)=2f(x)f(y)$ ....(1)

$f(1)=2,$

$$\sum_{k=1}^{10} f(\alpha+k)=2f(\alpha)\sum_{k=1}^{10} f(k)$$

$=2f(\alpha)\left(f(1)+f(2)+.....+f(10)\right)$ ....(2)

From (1)

$f(2)=2f^2(1)=2^3$

$f(3)=2f(2)f(1)=2^5$

$\vdots \qquad \vdots$

$f(10)=2^9 f^{10}(1)=2^{19}$

$f(\alpha)=2^{2\alpha-1};\ \alpha\in N$

from (2)

$$\sum_{k=1}^{10} f(\alpha+k)=2\left(2^{2\alpha-1}\right)\left(2+2^3+2^5+....+2^{19}\right)$$

$$\frac{512}{3}(2^{20}-1)=2^{2\alpha}\left(2\frac{(2^{20}-1)}{3}\right)$$

Hence $\alpha=4$

5. Let A be a 3×3 real matrix such that

$$A\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix};A\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}-1\\0\\1\end{pmatrix}\text{ and }A\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}1\\1\\2\end{pmatrix}.$$

If $X=(x_1,x_2,x_3)^T$ and I is an identity matrix of order 3, then the system $(A-2I)X=\begin{pmatrix}4\\1\\1\end{pmatrix}$ has

(A) no solution

(B) infinitely many solutions

(C) unique solution

(D) exactly two solutions

al Ans. by NTA (B)

Ans. (B)

Sol. $A=\begin{bmatrix}a_1&b_1&c_1\\a_2&b_2&c_2\\a_3&b_3&c_3\end{bmatrix}$

$$A\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}1\\1\\2\end{bmatrix}$$

$\Rightarrow\ c_1=1,\ c_2=1,\ c_3=2$

$$A\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}c_1+a_1\\c_2+a_2\\c_3+a_3\end{bmatrix}=\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$\Rightarrow\ a_1=-2,\ a_2=-1,\ a_3=-1$

$$A\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}a_1+b_1\\a_2+b_2\\a_3+b_3\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$$

$\Rightarrow\ b_1=3,\ b_2=2,\ b_3=1$

$$\Rightarrow\ A=\begin{bmatrix}-2&3&1\\-1&2&1\\-1&1&2\end{bmatrix}$$

$$\Rightarrow\ A-2I=\begin{bmatrix}-4&3&1\\-1&0&1\\-1&1&0\end{bmatrix}$$

$|A-2I|=0$

Now, $\begin{bmatrix}-4&3&1\\-1&0&1\\-1&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}4\\1\\1\end{bmatrix}$

$-4x_1+3x_2+x_3=4$ ....(1)

$-x_1+x_3=1$ .....(2)

$-x_1+x_2=1$ ....(3)

$(1)-[(2)+3(3)]$

$0=0\ \Rightarrow$ infinite solutions

6. Let $f : R \to R$ be defined as $f(x) = x^3 + x - 5$. If $g(x)$ is a function such that $f(g(x)) = x$, $\forall x \in R$, then $g'(63)$ is equal to ________.

(A) $\frac{1}{49}$ (B) $\frac{3}{49}$

(C) $\frac{43}{49}$ (D) $\frac{91}{49}$

al Ans. by NTA (A)

Ans. (A)

$f(x) = x^3 + x - 5$

$\Rightarrow f'(x) = 3x^2 + 1 \Rightarrow$ increasing function

$\Rightarrow$ invertible

$\Rightarrow$ $g(x)$ is inverse of $f(x)$

$\Rightarrow g(f(x)) = x$

$\Rightarrow g'(f(x))f'(x) = 1$

$f(x) = 63$

$\Rightarrow x^3 + x - 5 = 63$

$\Rightarrow x = 4$

put $x = 4$

$g'(f(4))f'(4) = 1$

$g'(63) \times 49 = 1 \quad \{f'(4) = 49\}$

$g'(63) = \frac{1}{49}$

7. Consider the following two propositions:

$P1 : \sim(p \to \sim q)$

$P2 : (p \wedge \sim q) \wedge ((\sim p) \vee q)$

If the proposition $p \to ((\sim p) \vee q)$ is evaluated as FALSE, then:

(A) P1 is TRUE and P2 is FALSE

(B) P1 is FALSE and P2 is TRUE

(C) Both P1 and P2 are FALSE

(D) Both P1 and P2 are TRUE

al Ans. by NTA (C)

Ans. (C)

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$p \to (\sim p \vee q)$	$p \to \sim q$	$\sim(p \to \sim q)$	$p \wedge \sim q$	p_2
T	T	F	F	T	T	F	T	F	F
T	F	F	T	F	F	T	F	T	F
F	T	T	F	T	T	T	F	F	F
F	F	T	T	T	T	T	F	F	F

$p \to (\sim p \vee q)$ is F when p is true q is false

From table

P1 & P2 both are false

8. If $\frac{1}{2 \cdot 3^{10}} + \frac{1}{2^2 \cdot 3^9} + \dots \frac{1}{2^{10} \cdot 3} = \frac{K}{2^{10} \cdot 3^{10}}$, then the remainder when K is divided by 6 is

(A) 1 (B) 2

(C) 3 (D) 5

al Ans. by NTA (D)

Ans. (D)

Sol. $\frac{1}{2 \cdot 3^{10}} + \frac{1}{2^2 \cdot 3^9} + \frac{1}{2^3 \cdot 3^8} + \dots\dots + \frac{1}{2^{10} \cdot 3} = \frac{K}{2^{10} \cdot 3^{10}}$

$K = 2^9 + 2^8 \cdot 3 + 2^7 \cdot 3^2 + \dots\dots + 3^9$

$$= \frac{2^9\left(\left(\frac{3}{2}\right)^{10} - 1\right)}{\frac{3}{2} - 1} = 3^{10} - 2^{10}$$

Now, $3^{10} - 2^{10} = (3^5 - 2^5)(3^5 + 2^5)$

$= (211)(275)$

$= (35 \times 6 + 1)(45 \times 6 + 5)$

$= 6\lambda + 5$

Remainder is 5.

9. Let $f(x)$ be a polynomial function such that $f(x) + f'(x) + f''(x) = x^5 + 64$. Then, the value of $\lim_{x \to 1} \frac{f(x)}{x - 1}$

(A) – 15 (B) – 60

(C) 60 (D) 15

Official Ans. by NTA (A)

Ans. (A)

Sol. $\underset{x \to 1}{\text{Lt}} \frac{f(x)}{x - 1} = f'(1) \left(\text{and } f(1) = 0\right)$

$f(x) + f'(x) + t''(x) = x^5 + 64$

$f'(x) + f''(x) + f'''(x) = 5x^4$

$f''(x) + f'''(x) + f^{iv}(x) = 20x^3$

$f'''(x) + f^{iv}(x) + f^{v}(x) = 60x^2$

$\therefore f^{v}(x) - f''(x) = 60x^2 - 20x^3$

$\Rightarrow 120 - f''(1) = 40 \Rightarrow f''(1) = 80$

Also $f(1) + f'(1) + f''(1) = 65 \Rightarrow f'(1) = -15$. **Ans.**

10. Let E_1 and E_2 be two events such that the conditional probabilities $P(E_1|E_2)=\frac{1}{2}$, $P(E_2|E_1)=\frac{3}{4}$ and $P(E_1 \cap E_2)=\frac{1}{8}$. Then:

(A) $P(E_1 \cap E_2)=P(E_1)\cdot P(E_2)$

(B) $P(E'_1 \cap E'_2)=P(E'_1)\cdot P(E_2)$

(C) $P(E_1 \cap E'_2)=P(E_1)\cdot P(E_2)$

(D) $P(E'_1 \cap E_2)=P(E_1)\cdot P(E_2)$

al Ans. by NTA (C)

Ans. (C)

(A) $P(E_1)\cdot P(E_2)=\frac{1}{6}\cdot\frac{1}{4}=\frac{1}{24}\neq P(E_1 \cap E_2)$

(B) $P(E'_1 \cap E'_2)=1-P(E_1 \cup E_2)$

$=1-\left(P(E_1)+P(E_2)-P(E_1 \cap E_2)\right)$

$=1-\left(\frac{1}{6}+\frac{1}{4}-\frac{1}{8}\right)=\frac{17}{24}$

$P(E'_1)P(E_2)=\frac{5}{6}\times\frac{1}{4}=\frac{5}{24}$

(C) $P(E_1 \cap E'_2)=P(E_1)-P(E_1 \cap E_2)=\frac{1}{6}-\frac{1}{8}=\frac{1}{24}$

(D) $P(E'_1 \cap E_2)=P(E_2)-P(E_1 \cap E_2)=\frac{1}{4}-\frac{1}{8}=\frac{1}{8}$

11. Let $A=\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}$. If M and N are two matrices given by $M=\sum_{k=1}^{10}A^{2k}$ and $N=\sum_{k=1}^{10}A^{2k-1}$ then MN^2 is

(A) a non-identity symmetric matrix

(B) a skew-symmetric matrix

(C) neither symmetric nor skew-symmetric matrix

(D) an identify matrix

al Ans. by NTA (A)

Ans. (A)

Sol. $A=\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}$

$A^2=\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}=\begin{bmatrix}-4 & 0\\ 0 & -4\end{bmatrix}=-4I$

$A^3=-4A$

$A^4=(-4I)(-4I)=(-4)^2I$

$A^5=(-4)^2A,\ A^6=(-4)^3I$

$M=\sum_{k=1}^{10}A^{2k}=A^2+A^4+....+A^{20}$

$=[-4+(-4)^2+(-4)^3+....+(-4)^{20}]I$

$=-4\lambda I$

$\Rightarrow$ M is symmetric matrix

$N=\sum_{k=1}^{10}A^{2k-1}=A+A^3+.....+A^{19}$

$=A[1+(-4)+(-4)^2+.....+(-4)^9]$

$=\lambda A \Rightarrow$ skew symmetric

$\Rightarrow$ N^2 is symmetric matrix

$\Rightarrow$ MN^2 is non identity symmetric matrix

12. Let $g:(0,\infty)\to R$ be a differentiable function such that

$$\int\left(\frac{x(\cos x-\sin x)}{e^x+1}+\frac{g(x)(e^x+1-xe^x)}{(e^x+1)^2}\right)dx=\frac{xg(x)}{e^x+1}+c,$$

for all $x>0$, where c is an arbitrary constant. Then.

(A) g is decreasing in $\left(0,\frac{\pi}{4}\right)$

(B) g' is increasing in $\left(0,\frac{\pi}{4}\right)$

(C) g + g' is increasing in $\left(0,\frac{\pi}{2}\right)$

(D) g – g' is increasing in $\left(0,\frac{\pi}{2}\right)$

Official Ans. by NTA (D)

Ans. (D)

Sol.

$$\int\left(\frac{x(\cos x-\sin x)}{e^x+1}+\frac{g(x)(e^x+1-xe^x)}{(e^x+1)^2}\right)dx=\frac{xg(x)}{e^x+1}+c$$

On differentiating both sides w.r.t. x, we get

$$\left(\frac{x(\cos x-\sin x)}{e^x+1}+\frac{g(x)(e^x+1-xe^x)}{(e^x+1)^2}\right)$$

$$=\frac{(e^x+1)\big(g(x)+xg'(x)\big)-e^x\cdot x\cdot g(x)}{\left(e^x+1\right)^2}$$

$(e^x + 1)x(\cos x - \sin x) + g(x)(e^x + 1 - xe^x)$

$= (e^x+1)\big(g(x)+xg'(x)\big)-e^x\cdot x\cdot g(x)$

$\Rightarrow g'(x) = \cos x - \sin x$

$\Rightarrow g(x) = \sin x + \cos x + C$

$g(x)$ is increasing in $(0, \pi/4)$

$g''(x) = -\sin x - \cos x < 0$

$\Rightarrow$ $g'(x)$ is decreasing function

let $h(x) = g(x) + g'(x) = 2\cos x + C$

$\Rightarrow h'(x) = g'(x) + g''(x) = -2\sin x < 0$

$\Rightarrow$ h is decreasing

let $\phi(x) = g(x) - g'(x) = 2\sin x + C$

$\Rightarrow \phi'(x) = g'(x) - g''(x) = 2\cos x > 0$

$\Rightarrow$ ϕ is increasing

Hence option D is correct.

13. Let $f : R \to R$ and $g : R \to R$ be two functions defined by $f(x) = \log_e(x^2 + 1) - e^{-x} + 1$ and $g(x)=\dfrac{1-2e^{2x}}{e^x}$. Then, for which of the following range of α, the inequality

$$f\left(g\left(\frac{(\alpha-1)^2}{3}\right)\right)>f\left(g\left(\alpha-\frac{5}{3}\right)\right)\text{ holds?}$$

(A) (2, 3) (B) (–2, – 1)

(C) (1, 2) (D) (–1, 1)

Official Ans. by NTA (A)

Ans. (A)

Sol. $f(x) = \log_e(x^2 + 1) - e^{-x} + 1$

$\Rightarrow f'(x) = \dfrac{2x}{x^2+1}+e^{-x} > 0 \quad \forall x \in R$

$\Rightarrow$ f is strictly increasing

$g(x) = \dfrac{1-2e^{2x}}{e^x} = e^{-x} - 2e^x$

$\Rightarrow g'(x) = -\left(2e^x+e^{-x}\right) < 0 \quad \forall x \in R$

$\Rightarrow$ g is decreasing

Now $f\left(g\left(\dfrac{(\alpha-1)^2}{3}\right)\right)>f\left(g\left(\alpha-\dfrac{5}{3}\right)\right)$

$\Rightarrow g\left(\dfrac{(\alpha-1)^2}{3}\right)>g\left(\alpha-\dfrac{5}{3}\right)$

$\Rightarrow \dfrac{(\alpha-1)^2}{3}<\alpha-\dfrac{5}{3}$

$\Rightarrow \alpha^2 - 5\alpha + 6 < 0$

$\Rightarrow (\alpha - 2)(\alpha - 3) < 0$

$\Rightarrow \alpha \in (2, 3)$

14. Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ $a_i > 0$, $i = 1, 2, 3$ be a vector which makes equal angles with the coordinates axes OX, OY and OZ. Also, let the projection of $\vec{a}$ on the vector $3\hat{i}+4\hat{j}$ be 7. Let $\vec{b}$ be a vector obtained by rotating $\vec{a}$ with 90°. If $\vec{a}$, $\vec{b}$ and x-axis are coplanar, then projection of a vector $\vec{b}$ on $3\hat{i}+4\hat{j}$ is equal to

(A) $\sqrt{7}$ (B) $\sqrt{2}$

(C) 2 (D) 7

al Ans. by NTA (B)

Ans. (B)

Sol. $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$$\vec{a}=\lambda\left(\frac{1}{\sqrt{3}}\hat{i}+\frac{1}{\sqrt{3}}\hat{j}+\frac{1}{\sqrt{3}}\hat{k}\right)=\frac{\lambda}{\sqrt{3}}\left(\hat{i}+\hat{j}+\hat{k}\right)$$

Now projection of $\vec{a}$ on $\vec{b}=7$

$\Rightarrow \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = 7$

$$\frac{\lambda}{\sqrt{3}}\frac{\left(\hat{i}+\hat{j}+\hat{k}\right)\cdot\left(3\hat{i}+4\hat{j}\right)}{5}=7$$

$\lambda = 5\sqrt{3}$

$\vec{a} = 5(\hat{i} + \hat{j} + \hat{k})$

now $\vec{b} = 5\alpha(\hat{i}+\hat{j}+\hat{k}) + \beta(\hat{i})$

$\vec{a}\cdot\vec{b} = 0$

$\Rightarrow 25\alpha(3) + 5\beta = 0$

$\Rightarrow 15\alpha + \beta = 0 \quad \Rightarrow \quad \beta = -15\alpha$

$\vec{b} = 5\alpha(-2\hat{i}+\hat{j}+\hat{k})$

$\left|\vec{b}\right| = 5\sqrt{3}$

$\Rightarrow \alpha = \pm\dfrac{1}{\sqrt{2}}$

$\vec{b} = \pm\dfrac{5}{\sqrt{2}}(-2\hat{i}+\hat{j}+\hat{k})$

Projection of $\vec{b}$ on $3\hat{i}+4\hat{j}$ is

$$\frac{\vec{b}\cdot(3\hat{i}+4\hat{j})}{5} = \pm\frac{5}{\sqrt{2}}\left(\frac{-6+4}{5}\right) = \pm\sqrt{2}$$

15. Let $y = y(x)$ be the solution of the differential equation $(x + 1)y' - y = e^{3x}(x + 1)^2$, with $y(0) = \frac{1}{3}$. Then, the point $x = -\frac{4}{3}$ for the curve $y = y(x)$ is:

(A) not a critical point
(B) a point of local minima
(C) a point of local maxima
(D) a point of inflection

al Ans. by NTA (B)

Ans. (B)

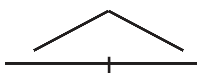
$(x + 1)dy - y\,dx = e^{3x}(x + 1)^2$

$$\frac{(x+1)dy - ydx}{(x+1)^2} = e^{3x}$$

$$d\left(\frac{y}{x+1}\right) = e^{3x} \quad \Rightarrow \quad \frac{y}{x+1} = \frac{e^{3x}}{3} + C$$

$$\left(0, \frac{1}{3}\right) \Rightarrow C = 0 \Rightarrow y = \frac{(x+1)e^{3x}}{3}$$

$$\frac{dy}{dx} = \frac{1}{3}\left((x+1)3e^{3x} + e^{3x}\right) = \frac{3^{3x}}{3}(3x+4)$$



$-4/3$

Clearly, $x = \dfrac{-4}{3}$ is point of local minima

16. If $y = m_1x + c_1$ and $y = m_2x + c_2$, $m_1 \neq m_2$ are two common tangents of circle $x^2 + y^2 = 2$ and parabola $y^2 = x$, then the value of $8|m_1m_2|$ is equal to

(A) $3+4\sqrt{2}$ (B) $-5+6\sqrt{2}$

(C) $-4+3\sqrt{2}$ (D) $7+6\sqrt{2}$

Official Ans. by NTA (C)

Ans. (C)

Sol. $C_1: x^2 + y^2 = 2$

$C_2 : y^2 = x$

Let tangent to parabola be $y = mx + \dfrac{1}{4m}$.

It is also a tangent of circle so distance from centre of circle (0, 0) will be $\sqrt{2}$.

$$\left|\frac{\frac{1}{4m}}{\sqrt{1+m^2}}\right| = \sqrt{2} \quad \Rightarrow \quad 1 = 32m^2 + 32m^4$$

by solving

$$m^2 = \frac{3\sqrt{2}-4}{8}, \quad m^2 = \frac{-3\sqrt{2}-4}{8} \text{ (rejected)}$$

$$m = \pm\sqrt{\frac{3\sqrt{2}-4}{8}}$$

so, $8|m_1m_2| = 3\sqrt{2}-4$

17. Let Q be the mirror image of the point P(1, 0, 1) with respect to the plane $S : x + y + z = 5$. If a line L passing through (1, –1, –1), parallel to the line PQ meets the plane S at R, then QR^2 is equal to:

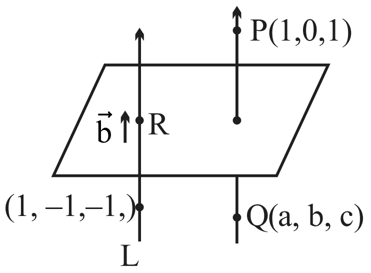
(A) 2 (B) 5

(C) 7 (D) 11

Official Ans. by NTA (B)

Ans. (B)

Sol.



Let parallel vector of L = $\vec{b}$

mirror image of Q on given plane x+y+z=5

$$\frac{a-1}{1}=\frac{b-0}{1}=\frac{c-1}{1}=\frac{-2(2-5)}{3}$$

a = 3, b = 2, c = 3

Q≡(3, 2, 3)

$\because \quad \vec{b}\,||\,\overrightarrow{PQ}$

so, $\vec{b}=(1,1,1)$

Equation of line

$$L : \frac{x-1}{1}=\frac{y+1}{1}=\frac{z+1}{1}$$

Let point R, $(\lambda+1, \lambda-1, \lambda-1)$

lying on plane x + y + z = 5,

so, 3λ – 1 = 5

$\Rightarrow \quad \lambda = 2$

Point R is (3, 1, 1)

$QR^2 = 5$ **Ans.**

18. If the solution curve y = y(x) of the differential equation $y^2dx + (x^2 - xy + y^2)dy = 0$, which passes through the point (1, 1) and intersects the line $y = \sqrt{3}\ x$ at the point $(\alpha, \sqrt{3}\alpha)$, then value of $\log_e(\sqrt{3}\alpha)$ is equal to

(A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$

(C) $\frac{\pi}{12}$ (D) $\frac{\pi}{6}$

Official Ans. by NTA (C)

Ans. (C)

Sol. $y^2dx - xy\,dy = -(x^2 + y^2)dy$

$y(y\,dx - x\,dy) = -(x^2 + y^2)\,dy$

$-y(x\,dx - y\,dx) = -(x^2 + y^2)dy$

$$\frac{xdy-ydx}{x^2}=\left(1+\frac{y^2}{x^2}\right)\frac{dy}{y}$$

$$\Rightarrow \frac{d(y/x)}{1+\frac{y^2}{x^2}}=\frac{dy}{y}$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \ell n\, y + C$$

$$\left(\alpha, \sqrt{3}\alpha\right) \Rightarrow \frac{\pi}{3}=\ln\left(\sqrt{3}\alpha\right)+\frac{\pi}{4}$$

$$\therefore \quad \ln\left(\sqrt{3}\alpha\right)=\frac{\pi}{12}$$

19. Let $x = 2t,\ y=\frac{t^2}{3}$ be a conic. Let S be the focus and B be the point on the axis of the conic such that SA ⊥ BA, where A is any point on the conic. If k is the ordinate of the centroid of ΔSAB, then $\lim\limits_{t\to1} k$ is equal to

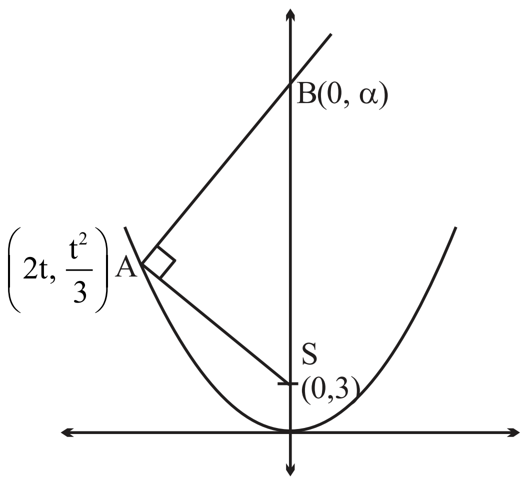
(A) $\frac{17}{18}$ (B) $\frac{19}{18}$

(C) $\frac{11}{18}$ (D) $\frac{13}{18}$

al Ans. by NTA (D)

Ans. (D)

Sol.



parabola $x^2 = 12y$

SA ⊥ SB

so, $m_{AS} \cdot m_{AB} = -1$

$$\frac{\left(3-\frac{t^2}{3}\right)}{(0-2t)} \cdot \frac{\left(\alpha-\frac{t^2}{3}\right)}{(0-2t)} = -1$$

by solving

$$3\alpha = \frac{27t^2+t^4}{t^2-9}$$

ordinate of centriod of $\Delta SAB = K = \dfrac{\alpha + \frac{t^2}{3} + 3}{3}$

$$k = \frac{9+3\alpha+t^2}{9}$$

$$\lim_{t\to 1} k = \lim_{t\to 1} \frac{1}{9}\left(9+t^2+\frac{27t^2+t^4}{(t^2-9)}\right) = \frac{13}{18}$$

20. Let a circle C in complex plane pass tltrough the points $z_1 = 3 + 4i$, $z_2 = 4 + 3i$ and $z_3 = 5i$. If $z(\neq z_1)$ is a point on C such that the line through z and z_1 is perpendicular to the line through z_2 and z_3, then arg(z) is equal to :

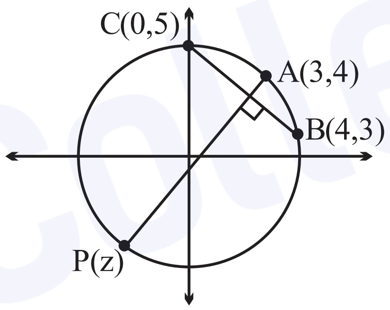
(A) $\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)-\pi$ (B) $\tan^{-1}\left(\frac{24}{7}\right)-\pi$

(C) $\tan^{-1}(3)-\pi$ (D) $\tan^{-1}\left(\frac{3}{4}\right)-\pi$

al Ans. by NTA (B)

Ans. (B)

Sol.



Slope of BC $= \frac{3-5}{4-0} = -\frac{1}{2}$

Slope of AP = 2

equation of AP : $y - 4 = 2(x - 3)$

$\Rightarrow y = 2(x - 1)$

P lies on circle $x^2 + y^2 = 25$

$\Rightarrow x^2 + (2(x - 1))^2 = 25$

$\Rightarrow x = -\frac{7}{5}$ and $y = -\frac{24}{5}$

$\Rightarrow \arg(z) = \tan^{-1}\left(\frac{24}{7}\right)-\pi$

SECTION-B

1. Let C_r denote the binomial coefficient of x^r in the expansion of $(1 + x)^{10}$. If $\alpha, \beta \in R$. $C_1 + 3\cdot2C_2 + 5\cdot3C_3 + ...$ upto 10 terms

$$= \frac{\alpha\times2^{11}}{2^\beta-1}\left(C_0+\frac{C_1}{2}+\frac{C_2}{3}+.....\text{upto 10 terms}\right)$$

then the value of $\alpha + \beta$ is equal to

Official Ans. by NTA (286)

Ans. (BONUS)

Sol. $(1 + x)^{10} = C_0 + C_1x + C_2x^2 + ... + C_{10}x^{10}$

Differentiating

$10(1 + x)^9 = C_1 + 2C_2x + 3C_3x^2 + ... + 10C_{10}x^9$

replace $x \to x^2$

$10(1+x^2)^9 = C_1 + 2C_2x^2 + 3C_3x^4 + ... + 10C_{10}x^{18}$

$10\cdot x(1+x^2)^9 = C_1x + 2C_2x^3 + 3C_3x^5 + ... + 10C_{10}x^{19}$

Differentiating

$$10\left((1+x^2)^9\cdot1 + x\cdot9(1+x^2)^8 2x\right)$$

$= C_1x + 2C_2\cdot3x^3 + 3\cdot5\cdot C_3x^4 + ... + 10\cdot19C_{10}x^{18}$

putting $x = 1$

$10(2^9 + 18\cdot2^8)$

$= C_1 + 3\cdot2\cdot C_2 + 5\cdot3\cdot C_3 + ... + 19\cdot10\cdot C_{10}$

$C_1 + 3\cdot2\cdot C_2 + ... + 19\cdot10\cdot C_{10}$

$= 10\cdot2^9\cdot10 = 100\cdot2^9$

$$C_0+\frac{C_1}{2}+\frac{C_2}{3}+.....+\frac{C_9}{11}+\frac{C_{10}}{11}=\frac{2^{11}-1}{11}$$

$\uparrow$ 10th term, $\uparrow$ 11th term

$$C_0+\frac{C_1}{2}+\frac{C_2}{3}+.....+\frac{C_9}{11}=\frac{2^{11}-2}{11}$$

Now, $100\cdot2^9 = \dfrac{\alpha\cdot2^{11}}{2^\beta-1}\left(\dfrac{2^{11}-2}{11}\right)$

Eqn. of form $y = k(2^x - 1)$.

It has infinite solutions even if we take $x, y \in N$.

2. The number of 3-digit odd numbers, whose sum of digits is a multiple of 7, is _______.

al Ans. by NTA (63)

Ans. (63)

x y z ← odd number

z = 1,3,5,7,9

x+y+ z = 7, 14, 21 [sum of digit multiple of 7]

$\underset{1\text{to}9}{x} + \underset{0\text{to}9}{y} = 6, 4, 2, 13, 11, 9, 7, 5, 20, 18, 16, 14, 12$

x + y = 6 ⇒ (1,5), (2, 4), (3, 3), (4, 2), (5, 1), (6, 0)

→ T.N. = 6

x + y = 4 ⇒ (1,3), (2, 2), (3, 1), (4,0)

→ T.N = 4

x + y = 2 ⇒ (1,1), (2,0)

→ T.N. = 2

x + y = 13 ⇒ (4,9), (5,8), (6,7), (7,6), (8,5),(9,4)

→ T.N. = 6

x + y = 11 ⇒ (2,9), (3,8), (4,7), (5,6), (6,5), (6,5), (7,4), (8,3), (9,2)

→ T.N. = 8

x + y = 9 ⇒ (1,8),(2,7),(3,8),(4,5),(5,4),....(8,1) ,(9,0)

→ T.N. = 9

x + y = 7 ⇒ (1,8),(2,5),(3,4),.... (8, 1), (7,0)

→ T.N. = 7

x + y = 5 ⇒ (1,4),(2,3) , (3, 2), (4,1),(5,0)

→ T.N. = 5

x + y = 20 ⇒ Not possible

x + y = 18 ⇒ (9,9) → T.N. = 1

x + y = 16 ⇒ (7,9),(8,8),(9,7)

→ T.N. = 3

x + y = 14 ⇒ (5,9),(6,8),(7,7),(8,6),(9,5)

→ T.N. = 5

x + y = 12 ⇒ (3,9),(4,8),(5,7),(6,6) ....(9,3)

→ T.N. = 7

3. Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$, where $|\vec{a}| = 4$, $|\vec{b}| = 3$ $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$. Then

$\left|\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)\right|^2 + 4\left(\vec{a}\cdot\vec{b}\right)^2$ is equal to ____

al Ans. by NTA (576)

Ans. (576)

Sol. $|\vec{a}| = 4, |\vec{b}| = 3 \quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$

$$\left|\left(\vec{a}-\vec{b}\right)\times\left(\vec{a}+\vec{b}\right)\right|^2 + 4\left(\vec{a}\cdot\vec{b}\right)^2$$

$$\left|\vec{a}\times\vec{b} - \vec{b}\times\vec{a}\right|^2 + 4a^2b^2\cos^2\theta$$

$$2\left|\vec{a}\times\vec{b}\right|^2 + 4a^2b^2\cos^2\theta$$

$$4a^2b^2\sin^2\theta + 4a^2b^2\cos^2\theta$$

$$4a^2b^2 = 4\times16\times9 = 576$$

4. Let the abscissae of the two points P and Q be the roots of $2x^2 - rx + p = 0$ and the ordinates of P and Q be the roots of $x^2 - sx - q = 0$. If the equation of the circle described on PQ as diameter is $2(x^2 + y^2) - 11x - 14y - 22 = 0$, then $2r + s - 2q + p$ is equal to

al Ans. by NTA (7)

Ans. (7)

Sol. $2x^2 - rx + p = 0 \begin{cases} x_1 \\ x_2 \end{cases}$

$y^2 - sy - q = 0 \begin{cases} y_1 \\ y_2 \end{cases}$

Equation of the circle with PQ as diameter is

$2(x^2 + y^2) - rx - 2sy + p - 2q = 0$

on comparing with the given equation

r = 11, s = 7

p – 2q = – 22

∴ 2r + s – 2q + p = 22 + 7 – 22 = 7

5. The number of values of x in the interval $\left(\frac{\pi}{4}, \frac{7\pi}{4}\right)$ for which $14\text{cosec}^2x - 2\sin^2x = 21 - 4\cos^2x$ holds, is _________

al Ans. by NTA (4)

Ans. (4)

Sol. $x \in \left(\frac{\pi}{4}, \frac{7\pi}{4}\right)$

$$14\text{cosec}^2x - 2\sin^2x = 21 - 4\cos^2x$$

$$= 21 - 4\left(1 - \sin^2x\right)$$

$$= 17 + 4\sin^2x$$

$$14\text{cosec}^2x - 6\sin^2x = 17$$

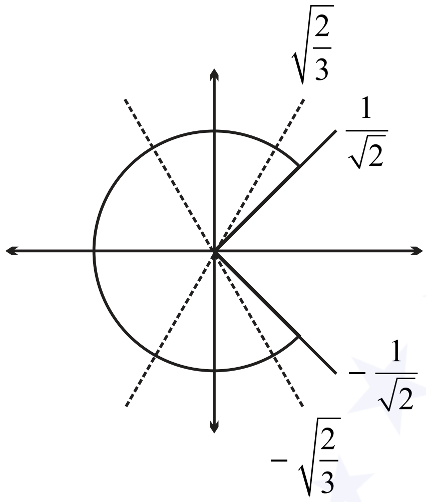
$$\text{let } \sin^2x = p$$

$$\frac{14}{p} - 6p = 17 \Rightarrow 14 - 6p^2 = 17p$$

$$6p^2 + 17p - 14 = 0$$

$$p = -3.5, \frac{2}{3} \Rightarrow \sin^2 x = \frac{2}{3}$$

$$\Rightarrow \sin x = \pm\sqrt{\frac{2}{3}}$$



$\therefore$ Total 4 solutions

6. For a natural number n, let $a_n = 19^n - 12^n$. Then, the value of $\frac{31\alpha_9 - \alpha_{10}}{57\alpha_8}$ is

al Ans. by NTA (4)

Ans. (4)

Sol. $a_n = 19^n - 12^n$

$$\frac{31\alpha_9 - \alpha_{10}}{57\alpha_8} = \frac{31(19^9 - 12^9) - (19^{10} - 12^{10})}{57\alpha_8}$$

$$= \frac{19^9(31-19) - 12^9(31-12)}{57\alpha_8}$$

$$= \frac{19^9 \cdot 12 - 12^{19} \cdot 19}{57\alpha_8}$$

$$= \frac{12 \cdot 19(19^8 - 12^8)}{57\alpha_8} = 4$$

7. Let $f : R \to R$ be a function defined by $f(x) = \left(2\left(1 - \frac{x^{25}}{2}\right)(2 + x^{25})\right)^{\frac{1}{50}}$. If the function $g(x) = f(f(f(x))) + f(f(x))$, the the greatest integer less than or equal to g (1) is ______

al Ans. by NTA (2)

Ans. (2)

Sol. $f(x) = \left[2\left(1 - \frac{x^{25}}{2}\right)(2 + x^{25})\right]^{\frac{1}{50}}$

$$f(x) = \left[(2 - x^{25})(2 + x^{25})\right]^{\frac{1}{50}}$$

$$= (4 - x^{50})^{1/50}$$

$$f(f(x)) = \left(4 - \left((4 - x^{50})^{1/50}\right)^{50}\right)^{1/50} = x$$

$$g(x) = f(f(f(x))) + f(f(x))$$

$$= f(x) + x$$

$$g(1) = f(1) + 1 = 3^{1/50} + 1$$

$$[g(1)] = [3^{1/50} + 1] = 2$$

8. Let the lines

$L_1 : \vec{r} = \lambda(\hat{i} + 2\hat{j} + 3\hat{k}), \lambda \in R$

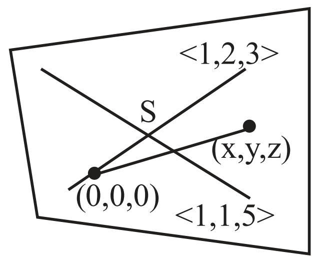
$L_2 : \vec{r} = (\hat{i} + 3\hat{j} + \hat{k}) + \mu(\hat{i} + \hat{j} + 5\hat{k}); \mu \in R$

intersect at the point S. If a plane $ax + by - z + d = 0$ passes through S and is parallel to both the lines L_1 and L_2, then the value of $a + b + d$ is equal to _______

al Ans. by NTA (5)

Ans. (5)

Both the lines lie in the same plane



$\therefore$ equation of the plane

$$\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 1 & 1 & 5 \end{vmatrix} = 0$$

$\Rightarrow 7x - 2y - z = 0$

$\therefore$ $a + b + d = 5$

9. Let A be a 3 × 3 matrix having entries from. the set {–1, 0, 1}. The number of all such matrices A having sum of all the entries equal to 5, is ________

Official Ans. by NTA (414)

Ans. (414)

Sol. **Case-I:** $1 \to 7$ times

and $-1 \to 2$ times

number of possible matrix $= \dfrac{9!}{7!\,2!} = 36$

Case-II: $1 \to 6$ times,

$-1 \to 1$ times

and $0 \to 2$ times

number of possible matrix $= \dfrac{9!}{6!\,2!} = 252$

Case-III: $1 \to 5$ times,

and $0 \to 4$ times

number of possible matrix $= \dfrac{9!}{5!\,4!} = 126$

Hence total number of all such matrix A $= 414$

10. The greatest integer less than or equal to the sum of first 100 terms of the sequence $\dfrac{1}{3}, \dfrac{5}{9}, \dfrac{19}{27}, \dfrac{65}{81}, \ldots\ldots$ is equal to

al Ans. by NTA (98)

Ans. (98)

Sol. $\dfrac{1}{3} + \dfrac{5}{9} + \dfrac{19}{27} + \dfrac{65}{81} + \ldots.$

$$\left(1-\frac{2}{3}\right)+\left(1-\frac{4}{9}\right)+\left(1-\frac{8}{27}\right)+\left(1-\frac{16}{81}\right)\ldots.100 \text{ terms}$$

$$100-\left[\frac{2}{3}+\left(\frac{2}{3}\right)^2+\ldots.\right]$$

$$100-\frac{\frac{2}{3}\left(1-\left(\frac{2}{3}\right)^{100}\right)}{1-\frac{2}{3}}$$

$$100-2\left(1-\left(\frac{2}{3}\right)^{100}\right)$$

$$S = 98 + 2\left(\frac{2}{3}\right)^{100}$$

$$\Rightarrow [S] = 98$$