

FINAL JEE–MAIN EXAMINATION – JANUARY, 2023

(Held On Tuesday 31st January, 2023)

TIME : 3 : 00 PM to 6 : 00 PM

SECTION-A

61. If $\phi(x) = \frac{1}{\sqrt{x}} \int_{\frac{\pi}{4}}^x (4\sqrt{2} \sin t - 3\phi'(t)) dt$, $x > 0$,

then $\phi'\left(\frac{\pi}{4}\right)$ is equal to :

(1) $\frac{8}{\sqrt{\pi}}$

(2) $\frac{4}{6 + \sqrt{\pi}}$

(3) $\frac{8}{6 + \sqrt{\pi}}$

(4) $\frac{4}{6 - \sqrt{\pi}}$

Official Ans. by NTA (3)

Ans. (3)

Sol. $\phi'(x) = \frac{1}{\sqrt{x}} \left[(4\sqrt{2} \sin x - 3\phi'(x)) \cdot 1 - 0 \right] - \frac{1}{2} x^{-3/2}$

$$\int_{\frac{\pi}{4}}^x (4\sqrt{2} \sin t - 3\phi'(t)) dt,$$

$$\phi'\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{\pi}} \left[4 - 3\phi'\left(\frac{\pi}{4}\right) \right] + 0$$

$$\left(1 + \sqrt{\pi} \right) \phi'\left(\frac{\pi}{4}\right) = \frac{8}{\sqrt{\pi}}$$

$$\phi'\left(\frac{\pi}{4}\right) = \frac{8}{\sqrt{\pi} + 6}$$

62. If a point $P(\alpha, \beta, \gamma)$ satisfying

$$(\alpha \ \beta \ \gamma) \begin{pmatrix} 2 & 10 & 8 \\ 9 & 3 & 8 \\ 8 & & \end{pmatrix} = (0 \ 0 \ 0) \text{ lies on the plane}$$

$2x + 4y + 3z = 5$, then $6\alpha + 9\beta + 7\gamma$ is equal to:

(1) -1

(2) $\frac{11}{5}$

(3) $\frac{5}{4}$

(4) 11

Official Ans. by NTA (4)

Ans. (4)

Sol. $2\alpha + 4\beta + 3\gamma = 5$ (1)

$2\alpha + 9\beta + 8\gamma = 0$ (2)

$10\alpha + 3\beta + 4\gamma = 0$ (3)

$8\alpha + 8\beta + 8\gamma = 0$ (4)

Subtract (4) from (2)

$-6\alpha + \beta = 0$

$\beta = 6\alpha$ (5)

From equation (4)

$8\alpha + 48\alpha + 8\gamma = 0$

$\gamma = -7\alpha$ (6)

From equation (1)

$2\alpha + 24\alpha - 21\alpha = 5$

$5\alpha = 5$

$\alpha = 1$

$\beta = +6, \ \gamma = -7$

$\therefore 6\alpha + 9\beta + 7\gamma$

$= 6 + 54 - 49$

$= 11$

63. Let a_1, a_2, a_3, \dots be an A.P. If $a_7 = 3$, the product $a_1 a_4$ is minimum and the sum of its first n terms is zero, then $n! - 4a_{n(n+2)}$ is equal to :

(1) 24

(2) $\frac{33}{4}$

(3) $\frac{381}{4}$

(4) 9

Official Ans. by NTA (1)

Ans. (1)

Sol. $a+6d=3, \dots\dots\dots(1)$

$$\begin{aligned} Z &= a(a+3d) \\ &= (3-6d)(3-3d) \\ &= 18d^2 - 27d + 9 \end{aligned}$$

Differentiating with respect to d
 $\Rightarrow 36d - 27 = 0$

$$\Rightarrow d = \frac{3}{4}, \text{ from (1) } a = \frac{-3}{2}, (Z = \text{minimum})$$

$$\text{Now, } S_n = \frac{n}{2} \left(-3 + (n-1) \frac{3}{4} \right) = 0$$

$$\Rightarrow n = 5$$

Now,

$$\begin{aligned} n! - 4a &= 120 - 4(a_{35}) \\ &= 120 - 4 \left(a + (35-1)d \right) \\ &= 120 - 4 \left(\frac{-3}{2} + 34 \cdot \left(\frac{3}{4} \right) \right) \\ &= 120 - 4 \left(\frac{-6 + 102}{4} \right) \\ &= 120 - 96 = 24 \end{aligned}$$

64. Let $(a, b) \subset (0, 2\pi)$ be the largest interval for which $\sin^{-1}(\sin \theta) - \cos^{-1}(\sin \theta) > 0, \theta \in (0, 2\pi)$, holds. If

$$\alpha x^2 + \beta x + \sin^{-1}(x^2 - 6x + 10) + \cos^{-1}(x^2 - 6x + 10) = 0$$

and $\alpha - \beta = b - a$, then α is equal to :

(1) $\frac{\pi}{48}$

(2) $\frac{\pi}{16}$

(3) $\frac{\pi}{8}$

(4) $\frac{\pi}{12}$

Official Ans. by NTA (4)

Ans. (4)

Sol. $\sin^{-1} \sin \theta - \left(\frac{\pi}{2} - \sin^{-1} \sin \theta \right) > 0$

$$\Rightarrow \frac{\pi}{4}$$

$$\Rightarrow \sin \theta > \frac{1}{\sqrt{2}}$$

$$\text{So, } \theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4} \right)$$

$$\theta \in \left(\frac{\pi}{4}, \frac{3\pi}{4} \right) = (a, b)$$

$$b - a = \frac{\pi}{2} = \alpha - \beta$$

$$\Rightarrow \beta = \alpha - \frac{\pi}{2}$$

$$\Rightarrow \alpha x^2 + \beta x + \sin^{-1}[(x-3)^2 + 1] + \cos^{-1}[(x-3)^2 + 1] = 0$$

$$x = 3, 9\alpha + 3\beta + \frac{\pi}{2} + 0 = 0$$

$$\Rightarrow 9\alpha + 3\left(\alpha - \frac{\pi}{2}\right) + \frac{\pi}{2} = 0$$

$$\Rightarrow 12\alpha - \pi = 0$$

$$\alpha = \frac{\pi}{12}$$

65. Let $y = y(x)$ be the solution of the differential equation $(3y^2 - 5x^2)y dx + 2x(x^2 - y^2) dy = 0$ such that $y(1) = 1$. then $\left| (y(2))^3 - 12y(2) \right|$ is equal to:

(1) $32\sqrt{2}$

(2) 64

(3) $16\sqrt{2}$

(4) 32

Official Ans. by NTA (1)

Ans. (1)

Sol. $(3y^2 - 5x^2)y \cdot dx + 2x(x^2 - y^2)dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{y(5x^2 - 3y^2)}{2x(x^2 - y^2)}$$

Put $y = mx$

$$\Rightarrow m + x \cdot \frac{dm}{dx} = \frac{m(5 - 3m^2)}{2(1 - m^2)}$$

$$x \cdot \frac{dm}{dx} = \frac{(5 - 3m^2)m - 2m(1 - m^2)}{2(1 - m^2)}$$

$$\Rightarrow \frac{dx}{x} = \frac{2(m^2 - 1)}{m(m^2 - 3)} dm$$

$$\Rightarrow \frac{dx}{x} = \left(\frac{2}{m} - \frac{4}{3} + \frac{4m}{m^2 - 3} \right) dm$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{\left(\frac{2}{3}\right)}{m} + \int \frac{2}{3} \left(\frac{2m}{m^2 - 3} \right) dm$$

$$\Rightarrow \ln |x| = \frac{2}{3} \ln |m| + \frac{2}{3} \ln |m^2 - 3| + C$$

$$\text{Or, } \ln |x| = \frac{2}{3} \ln \left| \frac{y}{x} \right| + \frac{2}{3} \ln \left| \left(\frac{y}{x} \right)^2 - 3 \right| + C$$

Put $(x = 1, y = 1)$: we get $c = -\frac{2}{3} \ln(2)$

$$\Rightarrow \ln |x| = \frac{2}{3} \ln \left| \frac{y}{x} \right| + \frac{2}{3} \ln \left| \left(\frac{y}{x} \right)^2 - 3 \right| - \frac{2}{3} \ln(2)$$

$$\Rightarrow \left(\frac{y}{x} \right) \left[\left(\frac{y}{x} \right)^2 - 3 \right] = 2 \cdot (x^{3/2})$$

Put $x = 2$ to get $y(2)$

$$\Rightarrow y(y^2 - 12) = 4 \times 2 \times 2 \times 2\sqrt{2}$$

$$\Rightarrow y^3 - 12y = 32\sqrt{2}$$

$$\Rightarrow |y^3(2) - 12y(2)| = 32\sqrt{2}$$

66. The set of all values of a^2 for which the line $x + y = 0$ bisects two distinct chords drawn from a point $P\left(\frac{1+a}{2}, \frac{1-a}{2}\right)$ on the circle

$$2x^2 + 2y^2 - (1+a)x - (1-a)y = 0$$
 is equal to:

(1) $(8, \infty)$

(2) $(4, \infty)$

(3) $(0, 4]$

(4) $(2, 12]$

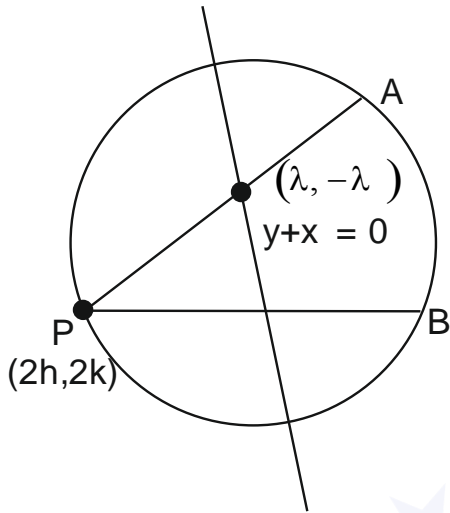
Official Ans. by NTA (1)

Ans. (1)

Sol. $x^2 + y^2 - \frac{(1+a)x}{2} - \frac{(1-a)y}{2} = 0$

Centre $\left(\frac{1+a}{4}, \frac{1-a}{4}\right) \Rightarrow (h, k)$

P $\left(\frac{1+a}{2}, \frac{1-a}{2}\right) \Rightarrow (2h, 2k)$



Equation of chord $\Rightarrow T = S_1$

$$\Rightarrow (x-y)\lambda - \frac{2h(x+\lambda)}{2} - \frac{(2k)(y-\lambda)}{2}$$

$$= 2\lambda^2 - 2h(\lambda) + 2k\lambda$$

Now, $\lambda(2h, 2k)$ satisfies the chord

$$\therefore (2h - 2k)\lambda - h(x + \lambda) - k(y - \lambda)$$

$$\Rightarrow 2\lambda^2 + 4k\lambda - 4h\lambda + h\lambda - k\lambda + hx + ky = 0$$

$$\Rightarrow 2\lambda^2 + \lambda(3k - 3h) + ky + hx = 0$$

$$\Rightarrow D > 0$$

$$\Rightarrow 9(k-h)^2 - 8(ky+hx) > 0$$

$$\Rightarrow 9(k-h)^2 - 8(2k^2 + 2h^2) > 0$$

$$\Rightarrow -7k^2 - 7h^2 - 18kh > 0$$

$$\Rightarrow 7k^2 + 7h^2 + 18kh < 0$$

$$\Rightarrow 7\left(\frac{1-a}{4}\right)^2 + 7\left(\frac{1+a}{4}\right)^2 + 18\left(\frac{1-a^2}{16}\right) < 0$$

$$\Rightarrow 7\left[\frac{2(1+a^2)}{16}\right] + \frac{18(1-a^2)}{16} < 0, \quad a^2 = t$$

$$\Rightarrow \frac{7}{8}(1+t) + \frac{18(1-t)}{16} < 0$$

$$\Rightarrow \frac{14+14t+18-18t}{16} < 0$$

$$\Rightarrow 4t > 32$$

$$t > 8 \quad a^2 > 8$$

67. Among the relations

$$S = \left\{ (a, b) : a, b \in \mathbb{R} - \{0\}, 2 + \frac{a}{b} > 0 \right\}$$

$$\text{And } T = \{(a, b) : a, b \in \mathbb{R}, a^2 - b^2 \in \mathbb{Z}\},$$

- (1) S is transitive but T is not
- (2) T is symmetric but S is not
- (3) Neither S nor T is transitive
- (4) Both S and T are symmetric

Official Ans. by NTA (2)

Ans. (2)

Sol. For relation $T = a^2 - b^2 = -I$

Then, (b, a) on relation R

$$\Rightarrow b^2 - a^2 = -I$$

\therefore T is symmetric

$$S = \left\{ (a, b) : a, b \in \mathbb{R} - \{0\}, 2 + \frac{a}{b} > 0 \right\}$$

$$2 + \frac{a}{b} > 0 \Rightarrow \frac{a}{b} > -2, \Rightarrow \frac{b}{a} < \frac{-1}{2}$$

If (b, a) \in S then

$$2 + \frac{b}{a} \text{ not necessarily positive}$$

\therefore S is not symmetric

68. The equation

$$e^{4x} + 8e^{3x} + 13e^{2x} - 8e^x + 1 = 0, x \in \mathbb{R} \text{ has:}$$

- (1) two solutions and both are negative
- (2) no solution
- (3) four solutions two of which are negative
- (4) two solutions and only one of them is negative

Official Ans. by NTA (1)

Ans. (1)

Sol. $e^{4x} + 8e^{3x} + 13e^{2x} - 8e^x + 1 = 0$

Let $e^x = t$

Now, $t^4 + 8t^3 + 13t^2 - 8t + 1 = 0$

Dividing equation by t^2 ,

$$t^2 + 8t + 13 - \frac{8}{t} + \frac{1}{t^2} = 0$$

$$t^2 + \frac{1}{t^2} + 8\left(t - \frac{1}{t}\right) + 13 = 0$$

$$\left(t - \frac{1}{t}\right)^2 + 2 + 8\left(t - \frac{1}{t}\right) + 13 = 0$$

Let $t - \frac{1}{t} = z$

$$z^2 + 8z + 15 = 0$$

$$(z + 3)(z + 5) = 0$$

$$z = -3 \text{ or } z = -5$$

So, $t - \frac{1}{t} = -3$ or $t - \frac{1}{t} = -5$

$$t^2 + 3t - 1 = 0 \text{ or } t^2 + 5t - 1 = 0$$

$$\frac{\sqrt{13} - 3}{2} \text{ or } \frac{\sqrt{29} - 5}{2}$$

as $t = e^x$ so t must be positive,

$$t = \frac{\sqrt{13} - 3}{2} \text{ or } \frac{\sqrt{29} - 5}{2}$$

$$\text{So, } x = \ln\left(\frac{\sqrt{13} - 3}{2}\right) \text{ or } x = \ln\left(\frac{\sqrt{29} - 5}{2}\right)$$

Hence two solution and both are negative.

69. The number of values of $r \in \{p, q, \sim p, \sim q\}$ for which $((p \wedge q) \Rightarrow (r \vee q)) \wedge ((p \wedge r) \Rightarrow q)$ is a tautology, is:

(1) 3

(2) 2

(3) 1

(4) 4

Official Ans. by NTA (2)

Ans. (2)

Sol. $((p \wedge q) \Rightarrow (r \vee q)) \wedge ((p \wedge r) \Rightarrow q)$

We know, $p \Rightarrow q$ is equivalent to

$$\sim p \vee q$$

$$(\sim(p \wedge q) \vee (r \vee q)) \wedge (\sim(p \wedge r) \vee q)$$

$$\Rightarrow (\sim p \vee \sim q \vee r \vee q) \wedge (\sim p \vee \sim r \vee q)$$

$$\Rightarrow (\sim p \vee r \vee t) \wedge (\sim p \vee \sim r \vee q)$$

$$\Rightarrow (t) \wedge (\sim p \vee \sim r \vee q)$$

For this to be tautology, $(\sim p \vee \sim r \vee q)$ must be

always true which follows for $r = \sim p$ or $r = q$.

70. Let $f: \mathbb{R} - \{2, 6\} \rightarrow \mathbb{R}$ be real valued function

defined as $f(x) = \frac{x^2 + 2x + 1}{x^2 - 8x + 12}$. Then range of f is

(1) $\left(-\infty, -\frac{21}{4}\right] \cup [0, \infty)$

(2) $\left(-\infty, -\frac{21}{4}\right) \cup (0, \infty)$

(3) $\left(-\infty, -\frac{21}{4}\right] \cup \left[\frac{21}{4}, \infty\right)$

(4) $\left(\frac{21}{4}\right]$

Official Ans. by NTA (1)

Ans. (1)

Sol. Let $y = \frac{x^2 + 2x + 1}{x^2 - 8x + 12}$

By cross multiplying

$$yx^2 - 8xy + 12y - x^2 - 2x - 1 = 0$$

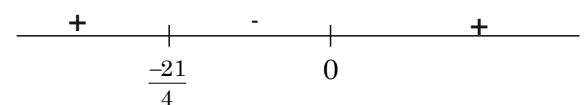
$$x^2(y - 1) - x(8y + 2) + (12y - 1) = 0$$

Case 1, $y \neq 1$

$$D \geq 0$$

$$\Rightarrow (8y + 2)^2 - 4(y - 1)(12y - 1) \geq 0$$

$$\Rightarrow \geq 0$$



$$y \in \left(-\infty, -\frac{21}{4}\right] \cup [0, \infty) - \{1\}$$

Case 2, $y = 1$

$$x^2 + 2x + 1 = x^2 - 8x + 12$$

$$10x = 11$$

$$x = \frac{11}{10} \quad \text{So, } y \text{ can be } 1$$

$$\text{Hence } y \in \left(-\infty, \frac{-21}{4}\right] \cup [0, \infty)$$

71.
$$\lim_{x \rightarrow \infty} \frac{(\sqrt{3x+1} + \sqrt{3x-1})^6 + (\sqrt{3x+1} - \sqrt{3x-1})^6}{(x + \sqrt{x^2-1})^6 + (x - \sqrt{x^2-1})^6} x^3$$

(1) is equal to 9

(2) is equal to 27

(3) does not exist

(4) is equal to $\frac{27}{2}$

Official Ans. by NTA (2)

Ans. (2)

Sol.
$$\lim_{x \rightarrow \infty} \frac{(\sqrt{3x+1} + \sqrt{3x-1})^6 + (\sqrt{3x+1} - \sqrt{3x-1})^6}{(x + \sqrt{x^2-1})^6 + (x - \sqrt{x^2-1})^6} x^3$$

$$\lim_{x \rightarrow \infty} x^3 \times \frac{\left\{ \left(\sqrt{3 + \frac{1}{x}} + \sqrt{3 - \frac{1}{x}} \right)^6 + \left(\sqrt{3 + \frac{1}{x}} - \sqrt{3 - \frac{1}{x}} \right)^6 \right\}}{\left\{ \left(1 + \sqrt{1 - \frac{1}{x^2}} \right)^6 + \left(1 - \sqrt{1 - \frac{1}{x^2}} \right)^6 \right\}}$$

$$= \frac{(2\sqrt{3})^6 + 0}{2^6 + 0} = 3^3 = (27)$$

72. Let P be the plane, passing through the point $(1, -1, -5)$ and perpendicular to the line joining the points $(4, 1, -3)$ and $(2, 4, 3)$. Then the distance of P from the point $(3, -2, 2)$ is

(1) 6

(2) 4

(3) 5

(4) 7

Official Ans. by NTA (3)

Ans. (3)

Sol. Equation of Plane :

$$2(x-1) - 3(y+1) - 6(z+5) = 0$$

Or $2x - 3y - 6z = 35$

\Rightarrow Required distance =

$$\frac{|2(3) - 3(-2) - 6(2) - 35|}{\sqrt{4 + 9 + 36}}$$

$$= 5$$

73. The absolute minimum value, of the function $f(x) = |x^2 - x + 1| + [x^2 - x + 1]$, where $[t]$ denotes the greatest integer function, in the interval $[-1, 2]$, is :

(1) $\frac{3}{4}$

(2) $\frac{3}{2}$

(3) $\frac{1}{4}$

(4) $\frac{5}{4}$

Official Ans. by NTA (1)

Ans. (1)

Sol. $f(x) = |x^2 - x + 1| + [x^2 - x + 1]; x \in [-1, 2]$

Let $g(x) = x^2 - x + 1$

$$= \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\therefore |x^2 - x + 1| \text{ and } [x^2 - x + 2]$$

Both have minimum value at $x = 1/2$

$$\Rightarrow \text{Minimum } f(x) = \frac{3}{4} + 0$$

$$= \frac{3}{4}$$

74. Let the plane $P : 8x + \alpha_1 y + \alpha_2 z + 12 = 0$ be parallel to the line $L : \frac{x+2}{2} = \frac{y-3}{3} = \frac{z+4}{5}$. If

the intercept of P on the y -axis is 1, then the distance between P and L is :

- (1) $\sqrt{14}$
- (2) $\frac{6}{\sqrt{14}}$
- (3) $\sqrt{\frac{2}{7}}$
- (4) $\sqrt{\frac{7}{2}}$

Official Ans. by NTA (1)

Ans. (1)

Sol. $P: 8x + \alpha_1 y + \alpha_2 z + 12 = 0$

$$L: \frac{x+2}{2} = \frac{y-3}{3} = \frac{z+4}{5}$$

$\therefore P$ is parallel to L

$$\Rightarrow 3\alpha_1 + 5(\alpha_2) = -16$$

Also y -intercept of plane P is 1

$$\Rightarrow \alpha_1 = -12$$

And $\alpha_2 = 4$

$$\Rightarrow \text{Equation of plane } P \text{ is } 2x - 3y + z + 3 = 0$$

\Rightarrow Distance of line L from Plane P is

$$= \frac{|0 - 3(6) + 1 + 3|}{\sqrt{4 + 9 + 1}} = \sqrt{14}$$

75. The foot of perpendicular from the origin O to a plane P which meets the co-ordinate axes at the points A, B, C is $(2, a, 4)$, $a \in \mathbb{N}$. If the volume of the tetrahedron $OABC$ is 144 unit^3 , then which of the following points is NOT on P ?

- (1) $(2, 2, 4)$
- (2) $(0, 4, 4)$
- (3) $(3, 0, 4)$
- (4) $(0, 6, 3)$

Official Ans. by NTA (3)

Ans. (3)

Sol. Equation of Plane:

$$(2\hat{i} + a\hat{j} + 4\hat{k}) \cdot [(x-2)\hat{i} + (y-a)\hat{j} + (z-4)\hat{k}] = 0$$

$$\Rightarrow 2x + ay + 4z = 20 + a^2$$

$$\Rightarrow A \equiv \left(\frac{20+a^2}{2}, 0, 0 \right)$$

$$B \equiv \left(0, \frac{20+a^2}{a}, 0 \right)$$

$$C \equiv \left(0, 0, \frac{20+a^2}{4} \right)$$

\Rightarrow Volume of tetrahedron

$$= \frac{1}{6} [\vec{a} \cdot \vec{b} \times \vec{c}]$$

$$= \frac{1}{6} \vec{a} \cdot (\vec{b} \times \vec{c})$$

$$\Rightarrow \frac{1}{6} \left(\frac{20+a^2}{2} \right) \cdot \left(\frac{20+a^2}{a} \right) \cdot \left(\frac{20+a^2}{4} \right) = 144$$

$$\Rightarrow (20+a^2)^3 = 144 \times 48 \times a$$

$$\Rightarrow a = 2$$

$$\Rightarrow \text{Equation of plane is } 2x + 2y + 4z = 24$$

$$\text{Or } x + y + 2z = 12$$

$$\Rightarrow (3, 0, 4) \text{ Not lies on the Plane}$$

$$x + y + 2z = 12$$

76. Let the mean and standard deviation of marks of class A of 100 students be respectively 40 and $\alpha (> 0)$, and the mean and standard deviation of marks of class B of n students be respectively 55 and $30 - \alpha$. If the mean and variance of the marks of the combined class of $100 + n$ students are respectively 50 and 350, then the sum of variances of classes A and B is:

- (1) 500
- (2) 650
- (3) 450
- (4) 900

Official Ans. by NTA (1)

Ans. (1)

79. Let $\alpha > 0$. If $\int_0^\alpha \frac{x}{\sqrt{x+\alpha} - \sqrt{x}} dx = \frac{16+20\sqrt{2}}{15}$,

then α is equal to :

- (1) 2
- (2) 4
- (3) $\sqrt{2}$
- (4) $2\sqrt{2}$

Official Ans. by NTA (1)

Ans. (1)

Sol. After rationalising

$$\int_0^\alpha \frac{x}{\alpha} (\sqrt{x+\alpha} + \sqrt{x})$$

$$\int_0^\alpha \frac{1}{\alpha} [(x+\alpha)^{3/2} - \alpha(x+\alpha)^{1/2} + x^{3/2}]$$

$$\frac{1}{\alpha} \left[\frac{2}{5} (x+\alpha)^{5/2} - \alpha \frac{2}{3} (x+\alpha)^{3/2} + \frac{2}{5} x^{5/2} \right] \Big|_0^\alpha$$

$$= \frac{1}{\alpha} \left(\frac{5}{2} (2\alpha)^{5/2} - \frac{2\alpha}{3} (2\alpha)^{3/2} + \frac{2}{5} \alpha^{5/2} - \frac{2}{5} \alpha^{5/2} + \frac{2}{3} \alpha^{5/2} \right)$$

$$= \frac{1}{\alpha} \left(\frac{2^2 2^{5/2} \alpha^{5/2}}{5} + \frac{2}{3} \alpha^{5/2} \right)$$

$$= \alpha^{3/2} \left(\frac{2^{7/2}}{5} - \frac{2^{5/2}}{3} + \frac{2}{3} \right)$$

$$= \frac{\alpha^{3/2}}{15} (24\sqrt{2} - 20\sqrt{2} + 10) = \frac{\alpha^{3/2}}{15} (4\sqrt{2} + 10)$$

Now,

$$\frac{\alpha^{3/2}}{15} (4\sqrt{2} + 10) = \frac{16+20\sqrt{2}}{15}$$

$$\Rightarrow \alpha = 2$$

80. The complex number $z = \frac{i-1}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}}$ is equal

to:

- (1) $\sqrt{2} \left(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right)$
- (2) $\cos \frac{\pi}{12} - i \sin \frac{\pi}{12}$
- (3) $\sqrt{2} \left(\cos \frac{\pi}{12} + i \sin \frac{\pi}{12} \right)$
- (4) $\sqrt{2} i \left(\cos \frac{5\pi}{12} - i \sin \frac{5\pi}{12} \right)$

Official Ans. by NTA (1)

Ans. (1)

Sol. $Z = \frac{i-1}{\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}} = \frac{i-1}{\frac{1}{2} + \frac{\sqrt{3}}{2} i}$

$$= \frac{i-1}{\frac{1}{2} + \frac{\sqrt{3}}{2} i} \times \frac{\frac{1}{2} - \frac{\sqrt{3}}{2} i}{\frac{1}{2} - \frac{\sqrt{3}}{2} i} = \frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}+1}{2} i$$

Apply polar form,

$$r \cos \theta = \frac{\sqrt{3}-1}{2}$$

$$r \sin \theta = \frac{\sqrt{3}+1}{2}$$

$$\text{Now, } \tan \theta = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

So, $\theta = \frac{5\pi}{12}$

81. The Coefficient of x^{-6} , in the expansion of $\left(\frac{4x}{5} + \frac{5}{2x^2} \right)^9$, is _____

Official Ans. by NTA (5040)

Ans. (5040)

Sol: $\left(\frac{4x}{5} + \frac{5}{2x^2}\right)^9$,

Now, $T_{r+1} = {}^9C_r \cdot \left(\frac{4x}{5}\right)^{9-r} \left(\frac{5}{2x^2}\right)^r$

$= {}^9C_r \cdot \left(\frac{4}{5}\right)^{9-r} \left(\frac{5}{2}\right)^r \cdot x^{9-3r}$

Coefficient of x^{-6} i.e. $9-3r = -6 \Rightarrow r = 5$

So, Coefficient of $x^{-6} = {}^9C_5 \left(\frac{4}{5}\right)^4 \cdot \left(\frac{5}{2}\right)^5 = 5040$

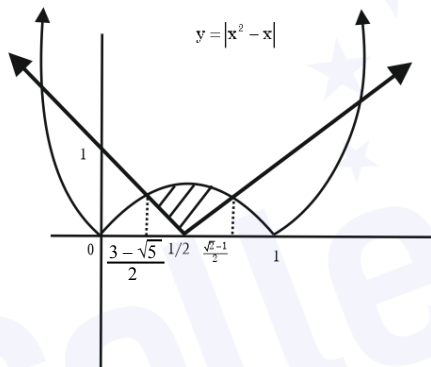
82. Let the area of the region $\{(x, y) : |2x - 1| \leq y \leq |x^2 - x|, 0 \leq x \leq 1\}$ be A.

Then $(6A+11)^2$ is equal to _____.

Official Ans. by NTA (125)

Ans. (125)

Sol: $y \geq |2x - 1|, y \leq |x^2 - x|$



Both curve are symmetric about $x = \frac{1}{2}$ Hence

$$A = 2 \int_{\frac{3-\sqrt{5}}{2}}^{\frac{1}{2}} ((x - x^2) - (1 - 2x)) dx$$

$$A = 2 \int_{\frac{3-\sqrt{5}}{2}}^{\frac{1}{2}} (-x^2 + 3x - 1) dx = 2 \left(\frac{-x^3}{3} + \frac{3}{2}x^2 - x \right) \Big|_{\frac{3-\sqrt{5}}{2}}^{\frac{1}{2}}$$

On solving $6A + 11 = 5\sqrt{5}$

$(6A + 11)^2 = 125$

83. If ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 11 : 21$, then $n^2 + n + 15$ is equal to:

Official Ans. by NTA (45)

Ans. (45)

Sol: $\frac{(2n+1)!(n-1)!}{(n+2)!(2n-1)!} = \frac{11}{21}$

$\Rightarrow \frac{(2n+1)(2n)}{(n+2)(n+1)n} = \frac{11}{21}$

$\Rightarrow \frac{2n+1}{(n+1)(n+2)} = \frac{11}{42}$

$\Rightarrow n = 5$

$\Rightarrow n^2 + n + 15 = 25 + 5 + 15 = 45$

84. If the constant term in the binomial expansion of $\left(\frac{x^{\frac{5}{2}}}{2} - \frac{4}{x^\ell}\right)^9$ is -84 and the Coefficient of $x^{-3\ell}$ is

$2^\alpha \beta$, where $\beta < 0$ is an odd number, Then $|\alpha\ell - \beta|$ is equal to _____

Official Ans. by NTA (98)

Ans. (98)

Sol. In, $\left(\frac{x^{\frac{5}{2}}}{2} - \frac{4}{x^\ell}\right)^9$

$T_{r+1} = {}^9C_r \frac{(x^{5/2})^{9-r}}{2^{9-r}} \left(\frac{-4}{x^\ell}\right)^r$

$= (-1)^r \frac{{}^9C_r}{2^{9-r}} 4^r x^{\frac{45-5r}{2}-lr}$

$= 45 - 5r - 2lr = 0$

$r = \frac{45}{5+2l} \dots\dots (1)$

Now, according to the question, $(-1)^r \frac{{}^9C_r}{2^{9-r}} 4^r = -84$

$= (-1)^r \cdot {}^9C_r \cdot 2^{3r-9} = 21 \times 4$

Only natural value of r possible if $3r - 9 = 0$

$r = 3$ and ${}^9C_3 = 84$

$\therefore 1 = 5$ from equation (1)

Now, coefficient of $x^{-31} = x^{\frac{45-5r}{2}-lr}$ at $l = 5$, gives $r = 5$

$$\therefore {}^9C_5 (-1)^{\frac{4^5}{2^4}} = 2^\alpha \times \beta$$

$$= -63 \times 2^7$$

$$\Rightarrow \alpha = 7, \beta = -63$$

\therefore value of $|\alpha l - \beta| = 98$

85. Let $\vec{a}, \vec{b}, \vec{c}$ be three vectors such that $|\vec{a}| = \sqrt{31}$, $4|\vec{b}| = |\vec{c}| = 2$ and $2(\vec{a} \times \vec{b}) = 3(\vec{c} \times \vec{a})$.

If the angle between \vec{b} and \vec{c} is $\frac{2\pi}{3}$, then

$$\left(\frac{\vec{a} \times \vec{c}}{\vec{a} \cdot \vec{b}} \right)^2 \text{ is equal to } \underline{\hspace{2cm}}.$$

Official Ans. by NTA (3)

Ans. (3)

Sol. $2(\vec{a} \times \vec{b}) = 3(\vec{c} \times \vec{a})$

$$\vec{a} \times (2\vec{b} + 3\vec{c}) = 0$$

$$\vec{a} = \lambda(2\vec{b} + 3\vec{c})$$

$$|\vec{a}|^2 = \lambda^2 |2\vec{b} + 3\vec{c}|^2$$

$$|\vec{a}|^2 = \lambda^2 (4|\vec{b}|^2 + 9|\vec{c}|^2 + 12\vec{b} \cdot \vec{c})$$

$$31 = 31\lambda^2 \Rightarrow \lambda = \pm 1$$

$$\vec{a} = \pm(2\vec{b} + 3\vec{c})$$

$$\frac{|\vec{a} \times \vec{c}|}{|\vec{a} \cdot \vec{b}|} = \frac{2|\vec{b} \times \vec{c}|}{2\vec{b} \cdot \vec{b} + 3\vec{c} \cdot \vec{b}}$$

$$|\vec{b} \times \vec{c}|^2 = |\vec{b}|^2 |\vec{c}|^2 - (\vec{b} \cdot \vec{c})^2 = \frac{3}{4}$$

$$\frac{|\vec{a} \times \vec{c}|}{|\vec{a} \cdot \vec{b}|} = \frac{2 \times \frac{\sqrt{3}}{2}}{2 \cdot \frac{1}{4} - \frac{3}{2}} = -\sqrt{3}$$

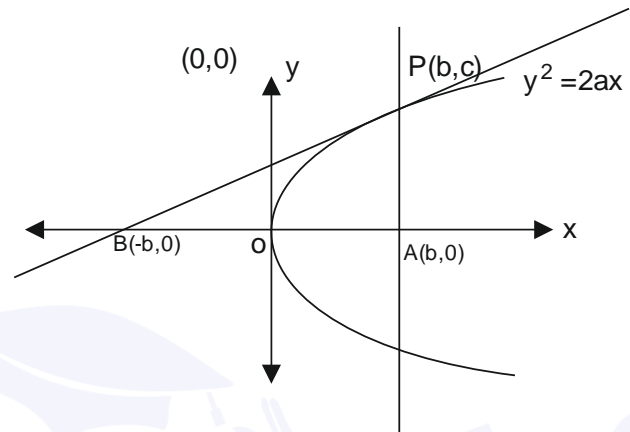
$$\left(\frac{\vec{a} \times \vec{c}}{\vec{a} \cdot \vec{b}} \right)^2 = 3$$

86. Let S be the set of all $a \in \mathbb{N}$ such that the area of the triangle formed by the tangent at the point P (b, c), $b, c \in \mathbb{N}$, on the parabola $y^2 = 2ax$ and the lines $x = b, y = 0$ is 16 unit², then $\sum_{a \in S} a$ is equal to _____.

Official Ans. by NTA (146)

Ans. (146)

Sol.



As P (b, c) lies on parabola so $c^2 = 2ab$ ---- (1)

Now equation of tangent to parabola $y^2 = 2ax$ in point

$$\text{form is } yy_1 = 2a \frac{(x + x_1)}{2}, (x_1, y_1) = (b, c)$$

$$\Rightarrow yc = a(x + b)$$

For point B, put $y = 0$, now $x = -b$

$$\text{So, area of } \Delta PBA, \frac{1}{2} \times AB \times AP = 16$$

$$\Rightarrow \frac{1}{2} \times 2b \times c = 16$$

$$\Rightarrow bc = 16$$

As b and c are natural number so possible values of (b, c) are (1, 16), (2, 8), (4, 4), (8, 2) and (16, 1)

Now from equation (1) $a = \frac{c^2}{2b}$ and $a \in \mathbb{N}$, so

values of (b, c) are (1, 16), (2, 8) and (4, 4) now values of a are 128, 16 and 2.

Hence sum of values of a is 146.

87. The sum

$$1^2 - 2 \cdot 3^2 + 3 \cdot 5^2 - 4 \cdot 7^2 + 5 \cdot 9^2 - \dots + 15 \cdot 29^2$$

is _____.

Official Ans. by NTA (6952)

Ans. (6952)

Separating odd placed and even placed terms we get

$$S = (1.1^2 + 3.5^2 + \dots + 15.(29)^2) - (2.3^2 + 4.7^2 + \dots + 14.(27)^2)$$

$$S = \sum_{n=1}^8 (2n-1)(4n-3)^2 - \sum_{n=1}^7 (2n)(4n-1)^2$$

Applying summation formula we get
 $= 29856 - 22904 = 6952$

88. Let A be the event that the absolute difference between two randomly chosen real numbers in the sample space $[0, 60]$ is less than or equal to a

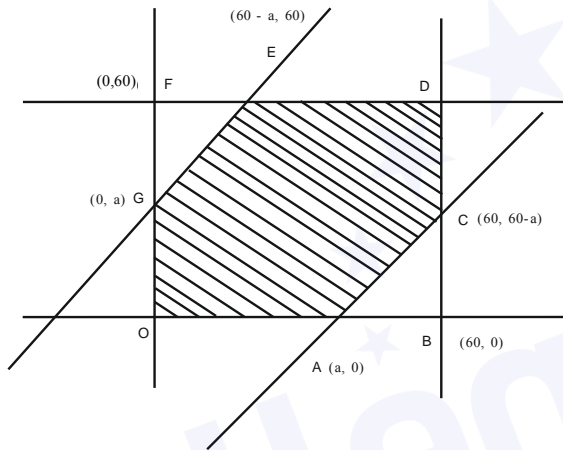
. If $P(A) = \frac{11}{36}$, then a is equal to _____.

Official Ans. by NTA (10)

Ans. (10)

Sol: $|x - y| < a \Rightarrow -a < x - y < a$

$\Rightarrow x - y < a$ and $x - y > -a$



$$P(A) = \frac{\text{ar}(\text{OACDEG})}{(\text{OBDF})}$$

$$= \frac{\text{ar}(\text{OBDF}) - \text{ar}(\text{EFG})}{\text{ar}(\text{OBDF})}$$

$$\Rightarrow \frac{11}{36} = \frac{(60)^2 - \frac{1}{2}(60-a)^2 - \frac{1}{2}(60-a)^2}{3600}$$

$$\Rightarrow 1100 = 3600 - (60 - a)^2$$

$$\Rightarrow (60 - a)^2 = 2500 \Rightarrow 60 - a = 50$$

$$\Rightarrow a = 10$$

89. Let $A = [a_{ij}]$, $a_{ij} \in Z \cap [0, 4]$, $1 \leq i, j \leq 2$. The number of matrices A such that the sum of all entries is a prime number $p \in (2, 13)$ is _____.

Official Ans. by NTA (196)

Ans. (204)

As given $a + b + c + d = 3$ or 5 or 7 or 11
 if sum = 3

$$(1 + x + x^2 + \dots + x^4)^4 \rightarrow x^3$$

$$(1 - x^5)^4 (1 - x)^{-4} \rightarrow x^3$$

$$\therefore {}^{4+3-1}C_3 = {}^6C_3 = 20$$

If sum = 5

$$(1 - 4x^5)(1 - x)^{-4} \rightarrow x^5$$

$$\Rightarrow {}^{4+5-1}C_5 - 4x^{4.4+0-1}C_0 = {}^8C_5 - 4 = 52$$

If sum = 7

$$(1 - 4x^5)(1 - x)^{-4} \rightarrow x^7$$

$$\Rightarrow {}^{4+5-1}C_4 - 4.4^{4+0-1}C_0 = {}^8C_4 - 4 = 52$$

If sum = 11

$$(1 - 4x^5 + 6x^{10})(1 - x)^{-4} \rightarrow x^{11}$$

$$\Rightarrow {}^{4+11-1}C_{11} - 4.{}^{4+6-4}C_6 + 6.{}^{4+1-1}C_1$$

$$= {}^{14}C_{11} - 4.{}^9C_6 + 6.4 = 364 - 336 + 24 = 52$$

$$\therefore \text{Total matrices} = 20 + 52 + 80 + 52 = 204$$

90. Let A be a $n \times n$ matrix such that $|A|=2$. If the determinant of the matrix $\text{Adj}(2 \cdot \text{Adj}(2A^{-1}))$ is 2^{84} , then n is equal to _____.

Official Ans. by NTA (5)

Ans. (5)

Sol. $|\text{Adj}(2 \cdot \text{Adj}(2A^{-1}))|$

$$= |2 \text{Adj}(\text{Adj}(2A^{-1}))|^{n-1}$$

$$= 2^{n(n-1)} |\text{Adj}(2A^{-1})|^{n-1}$$

$$= 2^{n(n-1)} |(2A^{-1})|^{(n-1)(n-1)}$$

$$= 2^{n(n-1)} 2^{n(n-1)(n-1)} |A^{-1}|^{(n-1)(n-1)}$$

$$= 2^{n(n-1)+n(n-1)(n-1)} \frac{1}{|A|^{(n-1)^2}}$$

$$= \frac{2^{n(n-1)+n(n-1)(n-1)}}{2^{(n-1)^2}}$$

$$= 2^{n(n-1)+n(n-1)^2-(n-1)^2}$$

$$= 2^{(n-1)(n^2-n+1)}$$

Now, $2^{(n-1)(n^2-n+1)}$

$$2^{(n-1)(n^2-n+1)} = 2^{84}$$

So, $n = 5$