FINAL JEE–MAIN EXAMINATION – SEPTEMBER, 2020

(Held On Saturday 05th SEPTEMBER, 2020) TIME : 9 AM to 12 PM

MATHEMATICS | TEST PAPER WITH SOLUTION

1. If $3^{2\sin 2\alpha - 1}$, 14 and $3^{4 - 2\sin 2\alpha}$ are the first three terms of an A.P. for some α, then the sixth term of this A.P. is :

(1) 66 (2) 65

(3) 81 (4) 78

Official Ans. by NTA (1)

Sol. Given that

$3^{4 - \sin 2\alpha} + 3^{2\sin 2\alpha - 1} = 28$

Let $3^{2\sin 2\alpha} = t$

$\frac{81}{t} + \frac{t}{3} = 28$

$t = 81, 3$

$3^{2\sin 2\alpha} = 3^1, 3^4$

$2\sin 2\alpha = 1, 4$

$\sin 2\alpha = \frac{1}{2}$, 2 (rejected)

First term $a = 3^{2\sin 2\alpha - 1}$

$a = 1$

Second term = 14

$\therefore$ common difference d = 13

$T_6 = a + 5d$

$T_6 = 1 + 5 \times 13$

$T_6 = 66$

2. If the function $f(x) = \begin{cases} k_1(x-\pi)^2 - 1, & x \le \pi \\ k_2 \cos x, & x > \pi \end{cases}$

is twice differentiable, then the ordered pair (k_1, k_2) is equal to :

(1) $\left(\frac{1}{2}, 1\right)$ (2) (1, 1)

(3) $\left(\frac{1}{2}, -1\right)$ (4) (1, 0)

Official Ans. by NTA (1)

Sol. f(x) is continuous and differentiable

$f(\pi^-) = f(\pi) = f(\pi^+)$

$-1 = -k_2$

$\boxed{k_2 = 1}$

$f'(x) = \begin{cases} 2k_1(x-\pi) ; & x \le \pi \\ -k_2 \sin x ; & x > \pi \end{cases}$

$f'(\pi^-) = f'(\pi^+)$

$0 = 0$

so, differentiable at x = 0

$f''(x) = \begin{cases} 2k_1 ; & x \le \pi \\ -k_2 \cos x ; & x > \pi \end{cases}$

$f''(\pi^-) = f''(\pi^+)$

$2k_1 = k_2$

$\boxed{k_1 = \frac{1}{2}}$

$(k_1, k_2) = \left(\frac{1}{2}, 1\right)$

3. If the common tangent to the parabolas, $y^2 = 4x$ and $x^2 = 4y$ also touches the circle, $x^2 + y^2 = c^2$, then c is equal to :

(1) $\frac{1}{2}$ (2) $\frac{1}{2\sqrt{2}}$

(3) $\frac{1}{\sqrt{2}}$ (4) $\frac{1}{4}$

Official Ans. by NTA (3)

Sol. $y = mx + \frac{1}{m}$ (tangent at $y^2 = 4x$)

$y = mx - m^2$ (tangent at $x^2 = 4y$)

$\frac{1}{m} = -m^2$ (for common tangent)

$m^3 = -1$

$y = -x - 1$

$x + y + 1 = 0$

This line touches circle

$\therefore$ apply p = r

$c = \left|\frac{0+0+1}{\sqrt{2}}\right| = \frac{1}{\sqrt{2}}$

4. The negation of the Boolean expression $x \leftrightarrow \sim y$ is equivalent to :

(1) $(\sim x \wedge y) \vee (\sim x \wedge \sim y)$

(2) $(x \wedge \sim y) \vee (\sim x \wedge y)$

(3) $(x \wedge y) \vee (\sim x \wedge \sim y)$

(4) $(x \wedge y) \wedge (\sim x \vee \sim y)$

Official Ans. by NTA (3)

Sol. $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$x \leftrightarrow \sim y \equiv (x \rightarrow \sim y) \wedge (\sim y \rightarrow x)$

$\because (p \rightarrow q \equiv \sim p \vee q)$

$x \leftrightarrow \sim y \equiv (\sim x \vee \sim y) \wedge (y \vee x)$

$\sim (x \leftrightarrow \sim y) \equiv (x \wedge y) \vee (\sim x \wedge \sim y)$

5. If the volume of a parallelopiped, whose coterminus edges are given by the vectors $\vec{a} = \hat{i} + \hat{j} + n\hat{k}$, $\vec{b} = 2\hat{i} + 4\hat{j} - n\hat{k}$ and $\vec{c} = \hat{i} + n\hat{j} + 3\hat{k}$ $(n \geq 0)$, is 158 cu. units, then :

(1) $\vec{a} \cdot \vec{c} = 17$ (2) $\vec{b} \cdot \vec{c} = 10$

(3) n = 7 (4) n = 9

Official Ans. by NTA (2)

Sol. $v = \left[\vec{a}\ \vec{b}\ \vec{c}\right]$

$$158 = \begin{vmatrix} 1 & 1 & n \\ 2 & 4 & -n \\ 1 & n & 3 \end{vmatrix}, n \geq 0$$

$158 = 1\ (12 + n^2) - (6 + n) + n(2n - 4)$

$158 = n^2 + 12 - 6 - n + 2n^2 - 4n$

$3n^2 - 5n - 152 = 0$

$\vec{a} \cdot \vec{c} = 1 + n + 3n = 1 + 4n = 33$

$\vec{b} \cdot \vec{c} = 2 + 4n - 3n = 2 + n = 10$

6. If y = y(x) is the solution of the differential equation $\frac{5+e^x}{2+y} \cdot \frac{dy}{dx} + e^x = 0$ satisfying y(0) = 1, then a value of $y(\log_e 13)$ is :

(1) 1 (2) –1

(3) 2 (4) 0

Official Ans. by NTA (2)

Sol. $\frac{(5+e^x)}{2+y}\ \frac{dy}{dx} = -e^x$

$$\int \frac{dy}{2+y} = \int \frac{-e^x}{e^x+5}\ dx$$

$\ln (y + 2) = -\ln(e^x + 5) + k$

$(y + 2)(e^x + 5) = C$

$\because$ y(0) = 1

$\Rightarrow$ C = 18

$y + 2 = \frac{18}{e^x+5}$

at $x = \ln 13$

$y + 2 = \frac{18}{13+5} = 1$

$\boxed{y = -1}$

7. A survey shows that 73% of the persons working in an office like coffee, whereas 65% like tea. If x denotes the percentage of them, who like both coffee and tea, then x cannot be:

(1) 63 (2) 38

(3) 54 (4) 36

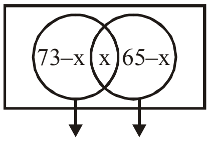
Official Ans. by NTA (4)

Sol. C $\rightarrow$ person like coffee

T $\rightarrow$ person like Tea

n(C) = 73

n(T) = 65



$n(C) + n(T) - n(C \cap T) \le 100$

$73 + 65 - x \le 100$

$x \ge 38$

$73 - x \ge 0 \Rightarrow x \le 73$

$65 - x \ge 0 \Rightarrow x \le 65$

$\boxed{38 \le x \le 65}$

8. The product of the roots of the equation $9x^2 - 18|x| + 5 = 0$, is

(1) $\frac{25}{9}$ (2) $\frac{25}{81}$

(3) $\frac{5}{27}$ (4) $\frac{5}{9}$

Official Ans. by NTA (2)

Sol. $9x^2 - 18|x| + 5 = 0$

$9|x|^2 - 15|x| - 3|x| + 5 = 0$ ($\because x^2 = |x|^2$)

$3|x|(3|x| - 5) - (3|x| - 5) = 0$

$|x| = \frac{1}{3}, \frac{5}{3}$

$x = \pm\frac{1}{3}, \pm\frac{5}{3}$

Product of roots $= \frac{25}{81}$

9. If $\int (e^{2x} + 2e^{x} - e^{-x} - 1)e^{(e^x + e^{-x})}dx$

$= g(x)e^{(e^x + e^{-x})} + c$, where c is a constant of integration, then g(0) is equal to :

(1) 2 (2) e^2

(3) e (4) 1

Official Ans. by NTA (1)

Sol. $e^{2x} + 2e^x - e^{-x} - 1$

$= e^x(e^x + 1) - e^{-x}(e^x + 1) + e^x$

$= [(e^x + 1)(e^x - e^{-x}) + e^x]$

so $I = \int (e^x + 1)(e^x - e^{-x})e^{e^x + e^{-x}} + \int e^x \cdot e^{e^x + e^{-x}} dx$

$= (e^x + 1)e^{e^x + e^{-x}} - \int e^x . e^{e^x + e^{-x}} dx + \int e^x . e^{e^x + e^{-x}} dx$

$= (e^x + 1)e^{e^x + e^{-x}} + C$

$\therefore g(x) = e^x + 1 \Rightarrow g(0) = 2$

10. If the minimum and the maximum values of the function $f : \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \to R$, defined by :

$$f(\theta) = \begin{vmatrix} -\sin^2\theta & -1-\sin^2\theta & 1 \\ -\cos^2\theta & -1-\cos^2\theta & 1 \\ 12 & 10 & -2 \end{vmatrix}$$

are m and M respectively, then the ordered pair (m, M) is equal to :

(1) (0, 4) (2) (–4, 4)

(3) $(0, 2\sqrt{2})$ (4) (–4, 0)

Official Ans. by NTA (4)

Sol. $C_3 \to C_3 - (C_1 - C_2)$

$$f(\theta) = \begin{vmatrix} -\sin^2\theta & -1-\sin^2\theta & 0 \\ -\cos^2\theta & -1-\cos^2\theta & 0 \\ 12 & 10 & -4 \end{vmatrix}$$

$= -4[(1 + \cos^2\theta)\sin^2\theta - \cos^2\theta(1 + \sin^2\theta)]$

$= -4[\sin^2\theta + \sin^2\theta\cos^2\theta - \cos^2\theta - \cos^2\theta\sin^2\theta]$

$f(\theta) = 4\cos 2\theta$

$\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

$2\theta \in \left[\frac{\pi}{2}, \pi\right]$

$f(\theta) \in [-4, 0]$

$(m, M) = (-4, 0)$

11. Let $\lambda \in R$. The system of linear equations

$2x_1 - 4x_2 + \lambda x_3 = 1$

$x_1 - 6x_2 + x_3 = 2$

$\lambda x_1 - 10x_2 + 4x_3 = 3$

is inconsistent for :

(1) exactly one negative value of λ.

(2) exactly one positive value of λ.

(3) every value of λ.

(4) exactly two values of λ.

Official Ans. by NTA (1)

Sol. $D = \begin{vmatrix} 2 & -4 & \lambda \\ 1 & -6 & 1 \\ \lambda & -10 & 4 \end{vmatrix}$

$= 2(3\lambda + 2)(\lambda - 3)$

$D_1 = -2(\lambda - 3)$

$D_2 = -2(\lambda + 1)(\lambda - 3)$

$D_3 = -2(\lambda - 3)$

When $\boxed{\lambda = 3}$, then

$D = D_1 = D_2 = D_3 = 0$

$\Rightarrow$ Infinite many solution

when $\boxed{\lambda = -\frac{2}{3}}$ then D_1, D_2, D_3 none of them is zero so equations are inconsistant

$\boxed{\therefore \lambda = -\frac{2}{3}}$

12. If S is the sum of the first 10 terms of the series

$\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \tan^{-1}\left(\frac{1}{21}\right) + \dots,$

then tan(S) is equal to :

(1) $\frac{5}{11}$ (2) $-\frac{6}{5}$

(3) $\frac{10}{11}$ (4) $\frac{5}{6}$

Sol. $S = \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \dots$

$S = \tan^{-1}\left(\frac{2-1}{1+1.2}\right) + \tan^{-1}\left(\frac{3-2}{1+2\times 3}\right) + \tan^{-1}\left(\frac{4-3}{1+3\times 4}\right) + \dots + \tan^{-1}\left(\frac{11-10}{1+10\times 11}\right)$

$S = (\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + (\tan^{-1}4 - \tan^{-1}3) + \dots + (\tan^{-1}(11) - \tan^{-1}(10))$

$S = \tan^{-1}11 - \tan^{-1}1 = \tan^{-1}\left(\frac{11-1}{1+11}\right)$

$\tan(S) = \frac{11-1}{1+11\times 1} = \frac{10}{12} = \frac{5}{6}$

13. If the four complex numbers z, $\bar{z}$, $\bar{z} - 2\mathrm{Re}(\bar{z})$ and $z - 2\mathrm{Re}(z)$ represent the vertices of a square of side 4 units in the Argand plane, then $|z|$ is equal to :

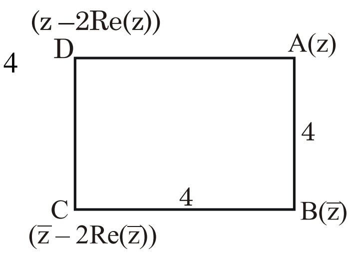
(1) 4 (2) 2

(3) $4\sqrt{2}$ (4) $2\sqrt{2}$

Official Ans. by NTA (4)

Sol. Let $z = x + iy$

Length of side = 4



$AB = 4$

$|z - \bar{z}| = 4$

$|2y| = 4$; $|y| = 2$

$BC = 4$

$|\bar{z} - (\bar{z} - 2\mathrm{Re}(\bar{z}))| = 4$

$|2x| = 4$; $|x| = 2$

$|z| = \sqrt{x^2 + y^2} = \sqrt{4+4} = 2\sqrt{2}$

14. If the point P on the curve, $4x^2 + 5y^2 = 20$ is farthest from the point Q(0, –4), then PQ^2 is equal to :

(1) 21 (2) 36

(3) 48 (4) 29

Official Ans. by NTA (2)

Sol. Given ellipse is $\frac{x^2}{5} + \frac{y^2}{4} = 1$

Let point P is $(\sqrt{5}\cos\theta, 2\sin\theta)$

$(PQ)^2 = 5\cos^2\theta + 4(\sin\theta + 2)^2$

$(PQ)^2 = \cos^2\theta + 16\sin\theta + 20$

$(PQ)^2 = -\sin^2\theta + 16\sin\theta + 21$

$= 85 - (\sin\theta - 8)^2$

will be maximum when $\sin\theta = 1$

$\Rightarrow (PQ)^2_{max} = 85 - 49 = 36$

15. The mean and variance of 7 observations are 8 and 16, respectively. If five observations are 2, 4, 10, 12, 14, then the absolute difference of the remaining two observations is :

(1) 2 (2) 4

(3) 3 (4) 1

Official Ans. by NTA (1)

Sol. $\bar{x} = \frac{2+4+10+12+14+x+y}{7} = 8$

$x + y = 14$ .....(i)

$(\sigma)^2 = \frac{\sum (x_i)^2}{n} - \left(\frac{\sum x_i}{n}\right)^2$

$16 = \frac{4+16+100+144+196+x^2+y^2}{7} - 8^2$

$16 + 64 = \frac{460+x^2+y^2}{7}$

$560 = 460 + x^2 + y^2$

$x^2 + y^2 = 100$ ....(ii)

Clearly by (i) and (ii), $|x - y| = 2$

Ans. 1

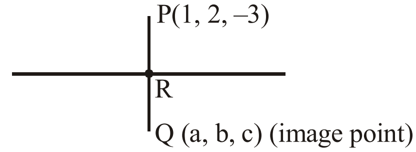
16. If (a, b, c) is the image of the point (1, 2, –3) in the line, $\frac{x+1}{2} = \frac{y-3}{-2} = \frac{z}{-1}$, then a + b + c is equal to

(1) –1 (2) 2

(3) 3 (4) 1

Official Ans. by NTA (2)

Sol.



Line is $\frac{x+1}{2} = \frac{y-3}{-2} = \frac{z}{-1} = \lambda$: Let point R is $(2\lambda - 1, -2\lambda + 3, -\lambda)$

Direction ratio of PQ$\equiv(2\lambda - 2, -2\lambda + 1, 3 - \lambda)$

PQ is $\perp^r$ to line

$\Rightarrow 2(2\lambda - 2) - 2(-2\lambda + 1) - 1(3 - \lambda) = 0$

$4\lambda - 4 + 4\lambda - 2 - 3 + \lambda = 0$

$9\lambda = 9 \Rightarrow \lambda = 1$

$\Rightarrow$ Point R is (1, 1, –1)

$\frac{a+1}{2} = 1$, $a = 1$ | $\frac{b+2}{2} = 1$, $b = 0$ | $\frac{c-3}{2} = -1$, $c = 1$

$\Rightarrow a + b + c = 2$

17. The value of $\int_{-\pi/2}^{\pi/2} \frac{1}{1+e^{\sin x}}\,dx$ is

(1) π (2) $\frac{3\pi}{2}$

(3) $\frac{\pi}{4}$ (4) $\frac{\pi}{2}$

Official Ans. by NTA (4)

Sol. $I = \int_{-\pi/2}^{\pi/2} \frac{1}{1+e^{\sin x}}dx$ ....(1)

Apply King property

$\int^{\pi/2} \frac{1}{\phantom{1+e^{\sin x}}}$ $\int^{\pi/2} \frac{e^{\sin x}}{\phantom{1+e^{\sin x}}}$

Add (1) & (2)

$$2I = \int_{-\pi/2}^{\pi/2} dx = \pi$$

$$I = \frac{\pi}{2}$$

18. If $2^{10} + 2^9 \cdot 3^1 + 2^8 \cdot 3^2 + \dots + 2 \cdot 3^9 + 3^{10} = S - 2^{11}$, then S is equal to :

(1) $\frac{3^{11}}{2} + 2^{10}$ (2) $3^{11} - 2^{12}$

(3) 3^{11} (4) $2 \cdot 3^{11}$

Official Ans. by NTA (3)

Sol. $a = 2^{10}$; $r = \frac{3}{2}$; $n = 11$ (G.P.)

$$S' = (2^{10}) \frac{\left(\left(\frac{3}{2}\right)^{11} - 1\right)}{\frac{3}{2} - 1} = 2^{11}\left(\frac{3^{11}}{2^{11}} - 1\right)$$

$S' = 3^{11} - 2^{11} = S - 2^{11}$ (Given)

$\therefore$ $S = 3^{11}$

19. If the co-ordinates of two points A and B are $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$ respectively and P is any point on the conic, $9x^2 + 16y^2 = 144$, then PA + PB is equal to :

(1) 8 (2) 6

(3) 16 (4) 9

Official Ans. by NTA (1)

Sol. $\frac{x^2}{16} + \frac{y^2}{9} = 1$

$a = 4$; $b = 3$; $e = \sqrt{\frac{16-9}{16}} = \frac{\sqrt{7}}{4}$

A and B are foci

$\Rightarrow PA + PB = 2a = 2 \times 4 = 8$

20. If α is the positive root of the equation, $p(x) = x^2 - x - 2 = 0$, then $\lim_{x \to \alpha^+} \frac{\sqrt{1 - \cos(p(x))}}{x + \alpha - 4}$ is equal to

(1) $\frac{3}{\sqrt{2}}$ (2) $\frac{3}{2}$

(3) $\frac{1}{\sqrt{2}}$ (4) $\frac{1}{2}$

Official Ans. by NTA (1)

Sol. $x^2 - x - 2 = 0$

roots are 2 & –1

$$\Rightarrow \lim_{x \to 2^+} \frac{\sqrt{1 - \cos(x^2 - x - 2)}}{(x-2)}$$

$$= \lim_{x \to 2^+} \frac{\sqrt{2\sin^2 \frac{(x^2 - x - 2)}{2}}}{(x-2)}$$

$$= \lim_{x \to 2^+} \frac{\sqrt{2} \sin\left(\frac{(x-2)(x+1)}{2}\right)}{(x-2)}$$

$$= \frac{3}{\sqrt{2}}$$

21. Four fair dice are thrown independently 27 times. Then the expected number of times, at least two dice show up a three or a five, is ____.

Official Ans. by NTA (11)

Sol. 4 dice are independently thrown. Each die has probability to show 3 or 5 is

$$p = \frac{2}{6} = \frac{1}{3}$$

$\therefore$ $q = 1 - \frac{1}{3} = \frac{2}{3}$ (not showing 3 or 5)

Experiment is performed with 4 dices independently.

$\therefore$ Their binomial distribution is

4 4 4 3 4 2 2 4 3

$\therefore$ In one throw of each dice probability of showing 3 or 5 at least twice is

$= p^4 + {}^4C_3\, qp^3 + {}^4C_2 q^2 p^2$

$= \dfrac{33}{81}$

$\therefore$ Such experiment performed 27 times

$\therefore$ so expected out comes = np

$= \dfrac{33}{81} \times 27$

$= 11$

22. If the line, $2x - y + 3 = 0$ is at a distance $\dfrac{1}{\sqrt{5}}$ and $\dfrac{2}{\sqrt{5}}$ from the lines $4x - 2y + \alpha = 0$ and $6x - 3y + \beta = 0$, respectively, then the sum of all possible values of α and β is _____

Official Ans. by NTA (30)

Sol. Apply distance between parallel line formula

$4x - 2y + \alpha = 0$

$4x - 2y + 6 = 0$

$\left|\dfrac{\alpha - 6}{255}\right| = \dfrac{1}{55}$

$|\alpha - 6| = 2 \Rightarrow \alpha = 8, 4$

sum = 12

again

$6x - 3y + \beta = 0$

$6x - 3y + 9 = 0$

$\left|\dfrac{\beta - 9}{3\sqrt{5}}\right| = \dfrac{2}{\sqrt{5}}$

$|\beta - 9| = 6 \Rightarrow \beta = 15, 3$

sum = 18

sum of all values of α and β is = 30

23. The natural number m, for which the coefficient of x in the binomial expansion of $\left(x^m + \dfrac{1}{x^2}\right)^{22}$ is 1540, is ________.

Official Ans. by NTA (13)

Sol. $T_{r+1} = {}^{22}C_r (x^m)^{22-r} \left(\dfrac{1}{x^2}\right)^r = {}^{22}C_r x^{22m - mr - 2r}$

$\because\ {}^{22}C_3 = {}^{22}C_{19} = 1540$

$\therefore\ r = 3$ or 19

$22m - mr - 2r = 1$

$m = \dfrac{2r+1}{22-5}$

$r = 3,\quad m = \dfrac{7}{19} \notin N$

$r = 19,\ m = \dfrac{38+1}{22-19} = \dfrac{39}{3} = 13$

$m = 13$

24. The number of words, with or without meaning, that can be formed by taking 4 letters at a time from the letters of the word 'SYLLABUS' such that two letters are distinct and two letters are alike, is ________.

Official Ans. by NTA (240)

Sol. S_2YL_2ABU

ABCC type words

$= \underbrace{{}^2C_1}_{\text{selection of two alike letters}} \times \underbrace{{}^5C_2}_{\text{selection of of two distinct letters}} \times \underbrace{\dfrac{\lfloor 4}{\lfloor 2}}_{\text{arrangement of selected letters}}$

$= 240$

25. Let $f(x) = x \cdot \left[\dfrac{x}{2}\right]$, for $-10 < x < 10$, where [t] denotes the greatest integer function. Then the number of points of discontinuity of f is equal to ________.

Official Ans. by NTA (8)

Sol. $x \in (-10, 10)$

$\dfrac{x}{2} \in (-5, 5) \rightarrow 9$ integers

check continuity at $x = 0$

$\left.\begin{array}{l} f(0) = 0 \\ f(0^+) = 0 \\ f(0^-) = 0 \end{array}\right\}$ continuous at $x = 0$

function will be distcontinuous when

$\dfrac{x}{2} = \pm 4,\ \pm 3,\ \pm 2,\ \pm 1$

8 points of discontinuity