FINAL JEE–MAIN EXAMINATION – JANUARY, 2020

(Held On Tuesday 07th JANUARY, 2020) TIME : 2 : 30 PM to 5 : 30 PM

MATHEMATICS — TEST PAPER WITH ANSWER & SOLUTION

1. Let y = y(x) be a function of x satisfying $y\sqrt{1-x^2} = k - x\sqrt{1-y^2}$ where k is a constant and $y\left(\frac{1}{2}\right) = -\frac{1}{4}$. Then $\frac{dy}{dx}$ at $x = \frac{1}{2}$, is equal to:

(1) $\frac{\sqrt{5}}{2}$ (2) $-\frac{\sqrt{5}}{2}$ (3) $\frac{2}{\sqrt{5}}$ (4) $-\frac{\sqrt{5}}{4}$

NTA Ans. (2)

Sol. Put x = sinθ, y = sinα

$$y\sqrt{1-x^2} = k - x\sqrt{1-y^2}$$

$\Rightarrow$ sinα · cosθ + cosα · sinθ = k

$\Rightarrow$ sin(α + θ) = k

$\Rightarrow$ α + θ = $\sin^{-1}k$

$\Rightarrow$ $\sin^{-1}x + \sin^{-1}y = \sin^{-1}k$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-y^2}} \times \frac{dy}{dx} = 0$$

at $x = \frac{1}{2}$, $y = \frac{-1}{4}$

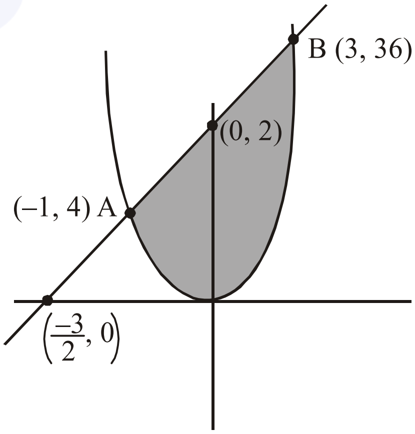
$$\frac{dy}{dx} = \frac{-\sqrt{5}}{2}$$

2. The area (in sq. units) of the region $\{(x, y) \in R^2 | 4x^2 \le y \le 8x + 12)$ is :

(1) $\frac{127}{3}$ (2) $\frac{125}{3}$ (3) $\frac{124}{3}$ (4) $\frac{128}{3}$

NTA Ans. (4)

Sol. $4x^2 - y \le 0$ and $8x - y + 12 \ge 0$



We get A (–1, 4) & B(3, 36)

Required area = area of the shaded region

$$= \int_{-1}^{3} (8x + 12 - 4x^2)dx = \frac{128}{3}$$

3. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three units vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$. If $\lambda = \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$ and $\vec{d} = \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}$, then the ordered pair, $(\lambda, \vec{d})$ is equal to :

(1) $\left(-\frac{3}{2}, 3\vec{a}\times\vec{b}\right)$ (2) $\left(-\frac{3}{2}, 3\vec{c}\times\vec{b}\right)$

(3) $\left(\frac{3}{2}, 3\vec{b}\times\vec{c}\right)$ (4) $\left(\frac{3}{2}, 3\vec{a}\times\vec{c}\right)$

NTA Ans. (1)

Sol. $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}\cdot\vec{b}) + 2(\vec{b}\cdot\vec{c}) + 2(\vec{c}\cdot\vec{a}) = 0$$

$$\lambda = \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a} = \frac{-3}{2}$$

$$\vec{d} = \vec{a}\times\vec{b} + \vec{b}\times\vec{c} + \vec{c}\times\vec{a}$$

$$\vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\Rightarrow \vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$$

$$\Rightarrow \vec{d} = 3(\vec{a}\times\vec{b})$$

4. If the sum of the first 40 terms of the series, 3 + 4 + 8 + 9 + 13 + 14 + 18 +19 +… is (102)m, then m is equal to :

(1) 20 (2) 5 (3) 10 (4) 25

NTA Ans. (1)

Sol. Sum of the 40 terms of

3 + 4 + 8 + 9 + 13 + 14 + 18 + 19…

= (3 + 8 + 13 +…upto 20 term)

+ [4 + 9 + 15 +… upto 20 terms]

= 10 [{6 + 19 × 5} + {8 + 19 × 5}]

5. The value of c in the Lagrange's mean value theorem for the function $f(x) = x^3 - 4x^2 + 8x + 11$, when $x \in [0, 1]$ is :

(1) $\frac{2}{3}$ (2) $\frac{\sqrt{7}-2}{3}$

(3) $\frac{4-\sqrt{5}}{3}$ (4) $\frac{4-\sqrt{7}}{3}$

NTA Ans. (4)

Sol. $f(0) = 11$

$f(1) = 16$

$$\frac{f(1)-f(0)}{1-0} = 3c^2 - 8c + 8$$

$\Rightarrow 3c^2 - 8c + 8 = 5$

$\Rightarrow 3c^2 - 8c + 3 = 0$

$c \in [0, 1] \Rightarrow c = \frac{4-\sqrt{7}}{3}$

6. If θ_1 and θ_2 be respectively the smallest and the largest values of θ in $(0, 2\pi) - \{\pi\}$ which satisfy the equation, $2\cot^2\theta - \frac{5}{\sin\theta} + 4 = 0$, then $\int_{\theta_1}^{\theta_2} \cos^2 3\theta d\theta$ is equal to :

(1) $\frac{2\pi}{3}$ (2) $\frac{\pi}{3} + \frac{1}{6}$ (3) $\frac{\pi}{9}$ (4) $\frac{\pi}{3}$

NTA Ans. (4)

Sol. $2\cos^2\theta - 5\sin\theta + 4\sin^2\theta = 0$

$3\sin^2\theta - 5\sin\theta + 2 = 0$

$\sin\theta = \frac{1}{2}$, 2 (Rejected)

$$\int_{\theta_1}^{\theta_2} \cos^2 3\theta d\theta = \int_{\pi/6}^{5\pi/6} \frac{1+\cos 6\theta}{2} d\theta$$

$$= \frac{1}{2}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right) = \frac{2\pi}{6} = \frac{\pi}{3}$$

7. The number of ordered pairs (r, k) for which $6 \cdot {}^{35}C_r = (k^2 - 3) \cdot {}^{36}C_{r+1}$, where k is an integer, is :

(1) 3 (2) 2 (3) 4 (4) 6

Sol. $6 \times {}^{35}C_r = (k^2 - 3)\, {}^{36}C_{r+1}$

$k^2 - 3 > 0 \Rightarrow k^2 > 3$

$$k^2 - 3 = \frac{6 \times {}^{35}C_r}{{}^{36}C_{r+1}} = \frac{r+1}{6}$$

Possible values of r for integral values of k, are

r = 5, 35

number of ordered pairs are 4

(5, 2), (5, –2), (35, 3), (35, –3)

8. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two 3 × 3 real matrices such that $b_{ij} = (3)^{(i+j-2)} a_{ji}$, where i, j = 1, 2, 3. If the determinant of B is 81, then the determinant of A is :

(1) 3 (2) 1/3 (3) 1/81 (4) 1/9

NTA Ans. (4)

Sol. $b_{ij} = (3)^{(i+j-2)} a_{ij}$

$$B = \begin{bmatrix} a_{11} & 3a_{12} & 3^2 a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 3^2 a_{31} & 3^2 a_{32} & 3^2 a_{33} \end{bmatrix}$$

$$\Rightarrow |B| = 3 \times 3^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 3a_{21} & 3a_{22} & 3a_{23} \\ 3^2 a_{31} & 3^2 a_{32} & 3^2 a_{33} \end{vmatrix}$$

$= 3^6|A|$

$$\Rightarrow |A| = \frac{81}{27 \times 27} = \frac{1}{9}$$

9. Let $a_1, a_2, a_3, \ldots$ be a G.P. such that $a_1 < 0$, $a_1 + a_2 = 4$ and $a_3 + a_4 = 16$. If $\sum_{i=1}^{9} a_i = 4\lambda$, then λ is equal to :

(1) –171 (2) 171 (3) $\frac{511}{3}$ (4) –513

NTA Ans. (1)

Sol. $a_1 + a_2 = 4$

$r^2 a_1 + r^2 a_2 = 16$

$\Rightarrow r^2 = 4 \Rightarrow r = -2$ as $a_1 < 0$

and $a_1 + a_2 = 4$

$a_1 + a_1(-2) = 4 \Rightarrow a_1 = -4$

$$4\lambda = (-4)\left(\frac{(-2)^9 - 1}{}\right) = (-4) \times \frac{513}{}$$

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10. Let A, B, C and D be four non-empty sets. The contrapositive statement of "If $A \subseteq B$ and $B \subseteq D$, then $A \subseteq C$" is :

(1) If $A \subseteq C$, then $B \subset A$ or $D \subset B$

(2) If $A \not\subseteq C$, then $A \not\subseteq B$ or $B \not\subseteq D$

(3) If $A \not\subseteq C$, then $A \subseteq B$ and $B \subseteq D$

(4) If $A \not\subseteq C$, then $A \not\subseteq B$ and $B \subseteq D$

NTA Ans. (2)

Sol. Contrapositive of $p \to q$ is $\sim q \to \sim p$

$(A \subseteq B) \wedge (B \subseteq D) \longrightarrow (A \subseteq C)$

Contrapositive is

$\sim(A \subseteq C) \longrightarrow \sim(A \subseteq B) \vee \sim(B \subseteq D)$

$A \not\subseteq C \to (A \not\subseteq B) \vee (B \not\subseteq D)$

11. If $3x + 4y = 12\sqrt{2}$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{9} = 1$ for some $a \in R$, then the distance between the foci of the ellipse is :

(1) 4 (2) $2\sqrt{7}$ (3) $2\sqrt{5}$ (4) $2\sqrt{2}$

NTA Ans. (2)

Sol. $3x + 4y = 12\sqrt{12}$ is tangent to $\frac{x^2}{a^2} + \frac{y^2}{9} = 1$

$c^2 = m^2a^2 + b^2$

$\Rightarrow a^2 = 16$

$e = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$

Distance between focii = 2ae = $2\sqrt{7}$

12. The value of α for which $4\alpha \int_{-1}^{2} e^{-\alpha|x|} dx = 5$, is :

(1) $\log_e\left(\frac{3}{2}\right)$ (2) $\log_e\left(\frac{4}{3}\right)$

(3) $\log_e 2$ (4) $\log_e \sqrt{2}$

NTA Ans. (3)

Sol. $4\alpha\left[\int_{-1}^{0} e^{\alpha x} dx + \int_{0}^{2} e^{-\alpha x} dx\right] = 5$

$\Rightarrow 4\alpha\left(\left[\frac{e^{\alpha x}}{\alpha}\right]_{-1}^{0} + \left[\frac{e^{-\alpha x}}{-\alpha}\right]_{0}^{2}\right) = 5$

$\Rightarrow 4e^{-2\alpha} + 4e^{-\alpha} - 3 = 0$

Let $e^{-\alpha} = t$, $4t^2 + 4t - 3 = 0$, $t = \frac{1}{2}, \frac{-3}{2}$ (Rejected)

13. The coefficient of x^7 in the expression $(1 + x)^{10} + x(1 + x)^9 + x^2(1 + x)^8 + \ldots + x^{10}$ is :

(1) 120 (2) 330 (3) 210 (4) 420

NTA Ans. (2)

Sol. Coefficient of x^7 is

$^{10}C_7 + {}^9C_6 + {}^8C_5 + \ldots + {}^4C_1 + {}^3C_0$

$\underbrace{^4C_0 + {}^4C_1}_{^5C_1} + {}^5C_2 + \ldots + {}^{10}C_7 = {}^{11}C_7 = 330$

14. Let α and β be the roots of the equation $x^2 - x - 1 = 0$. If $p_k = (\alpha)^k + (\beta)^k$, $k \geq 1$, then which one of the following statements is not true ?

(1) $(p_1 + p_2 + p_3 + p_4 + p_5) = 26$

(2) $p_5 = 11$

(3) $p_3 = p_5 - p_4$

(4) $p_5 = p_2 \cdot p_3$

NTA Ans. (4)

Sol. $\alpha + \beta = 1$, $\alpha\beta = -1$

$P_k = \alpha^k + \beta^k$

$\alpha^2 - \alpha - 1 = 0$

$\Rightarrow \alpha^k - \alpha^{k-1} - \alpha^{k-2} = 0$

& $\beta^k - \beta^{k-1} - \beta^{k-2} = 0$

$\Rightarrow P_k = P_{k-1} + P_{k-2}$

$P_1 = \alpha + \beta = 1$

$P_2 = (\alpha + \beta)^2 - 2\alpha\beta = 1 + 2 = 3$

$P_3 = 4$

$P_4 = 7$

$P_5 = 11$

15. The locus of the mid-points of the perpendiculars drawn from points on the line, $x = 2y$ to the line $x = y$ is :

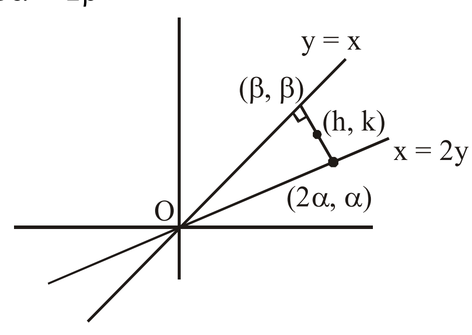
(1) $2x - 3y = 0$ (2) $7x - 5y = 0$

(3) $5x - 7y = 0$ (4) $3x - 2y = 0$

NTA Ans. (3)

Sol. $\frac{\alpha - \beta}{2\alpha - \beta} = -1$

$3\alpha = 2\beta$



$$h = \frac{2\alpha+\beta}{2}$$

$$2h = \frac{7\alpha}{2}$$

$$k = \frac{\alpha+\beta}{2}$$

$$2k = \frac{5\alpha}{2}$$

$$\frac{h}{k} = \frac{7}{5}$$

$5x = 7y$

16. If $\frac{3+i\sin\theta}{4-i\cos\theta}, \theta \in [0, 2\pi]$, is a real number, then an argument of $\sin\theta + i\cos\theta$ is :

(1) $-\tan^{-1}\left(\frac{3}{4}\right)$ (2) $\tan^{-1}\left(\frac{4}{3}\right)$

(3) $\pi - \tan^{-1}\left(\frac{4}{3}\right)$ (4) $\pi - \tan^{-1}\left(\frac{3}{4}\right)$

NTA Ans. (3)

Sol. $\frac{3+i\sin\theta}{4-i\cos\theta}$ is a real number

$\Rightarrow 3\cos\theta + 4\sin\theta = 0$

$\Rightarrow \tan\theta = \frac{-3}{4}$

argument of $\sin\theta + i\cos\theta = \pi - \tan^{-1}\frac{4}{3}$

17. Let $y = y(x)$ be the solution curve of the differential equation, $(y^2 - x)\frac{dy}{dx} = 1$, satisfying $y(0) = 1$. This curve intersects the x-axis at a point whose abscissa is :

(1) $2 + e$ (2) 2 (3) $2 - e$ (4) $-e$

NTA Ans. (3)

Sol. $(y^2 - x)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dx}{dy} + x = y^2$

I.F. $= e^{\int dy} = e^y$

Solution is given by

$x e^y = \int y^2 e^y dy + C$

$\Rightarrow x e^y = (y^2 - 2y + 2)e^y + C$

$x = 0, y = 1,$ gives $C = -e$

18. Let f(x) be a polynomial of degree 5 such that $x = \pm 1$ are its critical points. If $\lim_{x\to 0}\left(2 + \frac{f(x)}{x^3}\right) = 4$, then which one of the following is not true?

(1) f is an odd function

(2) $x = 1$ is a point of minima and $x = -1$ is a point of maxima of f.

(3) $x = 1$ is a point of maxima and $x = -1$ is a point of minimum of f.

(4) $f(1) - 4f(-1) = 4$

NTA Ans. (2)

Sol. $\lim_{x\to 0}\left(2 + \frac{f(x)}{x^3}\right) = 4$

$\Rightarrow f(x) = 2x^3 + ax^4 + bx^5$

$f'(x) = 6x^2 + 4ax^3 + 5bx^4$

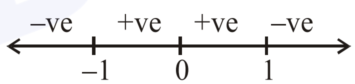
$f'(1) = 0, f'(-1) = 0$

$a = 0, b = \frac{-6}{5} \Rightarrow f(x) = 2x^3 - \frac{6}{5}x^5$

$f'(x) = 6x^2 - 6x^4$

$= 6x^2(1 - x)(1 + x)$

Sign scheme for f'(x)



Minima at $x = -1$

Maxima at $x = 1$

19. In a workshop, there are five machines and the probability of any one of them to be out of service on a day is $\frac{1}{4}$. If the probability that at most two machines will be out of service on the same day is $\left(\frac{3}{4}\right)^3 k$, then k is equal to :

(1) $\frac{17}{2}$ (2) 4

(3) $\frac{17}{8}$ (4) $\frac{17}{4}$

Sol. Probability that at most 2 machines are out of service

$$= {}^5C_0\left(\frac{3}{4}\right)^5 + {}^5C_1\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + {}^5C_2\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2$$

$$= \left(\frac{3}{4}\right)^4 \times \frac{17}{8} \qquad \Rightarrow k = \frac{17}{8}$$

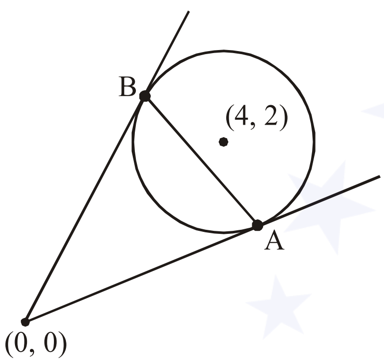
20. Let the tangents drawn from the origin to the circle, $x^2 + y^2 - 8x - 4y + 16 = 0$ touch it at the points A and B. The $(AB)^2$ is equal to :

(1) $\frac{52}{5}$ (2) $\frac{32}{5}$ (3) $\frac{56}{5}$ (4) $\frac{64}{5}$

NTA Ans. (4)

Sol. $R = \sqrt{16+4-16} = 2$

$L = \sqrt{S_1} = 4$



$$AB(\text{Chord of contact}) = \frac{2LR}{\sqrt{L^2+R^2}} = \frac{8}{\sqrt{5}}$$

$$(AB)^2 = \frac{64}{5}$$

21. If the system of linear equations,

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$3x + 2y + \lambda z = \mu$$

has more two solutions, then $\mu - \lambda^2$ is equal to ________

NTA Ans. (13.00)

Sol. System has intfinitely many solution

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 1$$

$$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & 1 \end{vmatrix} = 0$$

$\mu = 14$

$\mu - \lambda^2 = 13$

22. If the function f defined on $\left(-\frac{1}{3}, \frac{1}{3}\right)$ by

$$f(x) = \begin{cases} \frac{1}{x}\log_e\left(\frac{1+3x}{1-2x}\right) & ,\text{when } x \neq 0 \\ k & ,\text{when } x = 0 \end{cases}$$

is continuous, then k is equal to_______

NTA Ans. (5.00)

Sol. $k = \lim_{x\to 0}\left(\frac{\ell n(1+3x)}{x} - \frac{\ell n(1-2x)}{x}\right)$

$k = 3 + 2 = 5$

23. If the mean and variance of eight numbers 3, 7, 9, 12, 13, 20, x and y be 10 and 25 respectively, then x·y is equal to_____

NTA Ans. (54.00)

Sol. $\frac{3+7+9+12+13+20+x+y}{8} = 10$

$x + y = 16$

$$\frac{\Sigma x^2}{n} - \left(\frac{\Sigma x}{n}\right)^2 = 25$$

$3^2 + 7^2 + 9^2 + 12^2 + 13^2 + 20^2 + x^2 + y^2 = 1000$

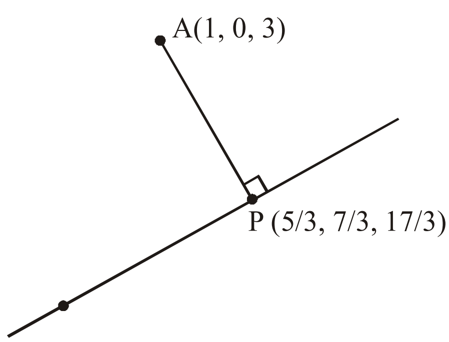
$x^2 + y^2 = 148$

$xy = 54$

24. If the foot of the perpendicular drawn from the point (1, 0, 3) on a line passing through (α, 7, 1) is $\left(\frac{5}{3}, \frac{7}{3}, \frac{17}{3}\right)$, then α is equal to_____

NTA Ans. (4.00)

Sol.



D.R. of BP = $<\frac{5}{3}-\alpha, \frac{7}{3}-7, \frac{17}{3}-1>$

D.R. of AP = $<\frac{5}{3}-1, \frac{7}{3}-0, \frac{17}{3}-3>$

BP $\perp^r$ AP

$\Rightarrow \alpha = 4$

25. Let X = {n ∈ N : 1 ≤ n ≤ 50}. If
A = {n ∈ X : n is a multiple of 2} and
B = {n ∈ X : n is a multiple of 7}, then the number of elements in the smallest subset of X containing both A and B is_____

NTA Ans. (29.00)

Sol. n(A) = 25

n(B) = 7

n(A ∩ B) = 3

n(A ∪ B) = 25 + 7 – 3 = 29