FINAL JEE–MAIN EXAMINATION – JANUARY, 2020

(Held On Tuesday 07th JANUARY, 2020) TIME : 9 : 30 AM to 12 : 30 PM

MATHEMATICS

TEST PAPER WITH ANSWER & SOLUTION

1. If $g(x) = x^2 + x - 1$ and $(gof)(x) = 4x^2 - 10x + 5$, then $f\left(\frac{5}{4}\right)$ is equal to

(1) $\frac{3}{2}$ (2) $-\frac{1}{2}$ (3) $-\frac{3}{2}$ (4) $\frac{1}{2}$

NTA Ans. (2)

Sol. $g(x) = x^2 + x-1$

$g(f(x)) = 4x^2 - 10x + 5$

$= (2x-2)^2 + (2-2x) - 1$

$= (2-2x)^2 + (2-2x) - 1$

$\Rightarrow f(x) = 2 - 2x$

$f\left(\frac{5}{4}\right) = \frac{-1}{2}$

2. If $\text{Re}\left(\frac{z-1}{2z+i}\right) = 1$, where $z = x + iy$, then the point (x,y) lies on a :

(1) circle whose centre is at $\left(-\frac{1}{2},-\frac{3}{2}\right)$

(2) circle whose diameter is $\frac{\sqrt{5}}{2}$

(3) straight line whose slope is $\frac{3}{2}$

(4) straight line whose slope is $-\frac{2}{3}$

NTA Ans. (2)

Sol. $\text{Re}\left(\frac{z-1}{2z+i}\right) = 1$

Put $z = x + iy$

$\text{Re}\left(\frac{(x+iy)-1}{2(x+iy)+i}\right) = 1$

$\text{Re}\left(\left(\frac{(x-1)+iy}{2x+i(2y+1)}\right)\left(\frac{2x-i(2y+1)}{2x-i(2y+1)}\right)\right) = 1$

$\Rightarrow 2x^2 + 2y^2 + 2x + 3y + 1 = 0$

$x^2 + y^2 + x + \frac{3}{2}y + \frac{1}{2} = 0$

$\Rightarrow$ locus is a circle whose

Centre is $\left(-\frac{1}{2},-\frac{3}{4}\right)$ and radius $\frac{\sqrt{5}}{4}$

$\sqrt{}$

3. Five numbers are in A.P., whose sum is 25 and product is 2520. If one of these five numbers is $-\frac{1}{2}$, then the greatest number amongst them is :

(1) $\frac{21}{2}$ (2) 27 (3) 16 (4) 7

NTA Ans. (3)

Sol. Let the A.P is

$a - 2d, a - d, a, a + d, a + 2d$

$\because$ sum $= 25 \Rightarrow a = 5$

Product $= 2520$

$(25 - 4d^2)(25 - d^2) = 504$

$4d^4 - 125d^2 + 121 = 0$

$\Rightarrow d^2 = 1, \frac{121}{4}$

$\Rightarrow d = \pm 1, \pm\frac{11}{2}$

$d = \pm 1$ is rejected because none of the term can be $\frac{-1}{2}$.

$\Rightarrow d = \pm \frac{11}{2}$

$\Rightarrow$ AP will be $-6, -\frac{1}{2}, 5, \frac{21}{2}, 16$

Largest term is 16.

4. If

$y(\alpha) = \sqrt{2\left(\frac{\tan\alpha + \cot\alpha}{1+\tan^2\alpha}\right) + \frac{1}{\sin^2\alpha}}, \alpha \in \left(\frac{3\pi}{4}, \pi\right)$,

then $\frac{dy}{d\alpha}$ at $\alpha = \frac{5\pi}{6}$ is :

(1) 4 (2) $-\frac{1}{4}$ (3) $\frac{4}{3}$ (4) -4

Sol. $y(\alpha)=\sqrt{2\frac{(\tan\alpha+\cot\alpha)}{1+\tan^2\alpha}+\frac{1}{\sin^2\alpha}}$, $\alpha\in\left(\frac{3\pi}{4},\pi\right)$

$=\frac{|\sin\alpha+\cos\alpha|}{|\sin\alpha|}=\frac{-(\sin\alpha+\cos\alpha)}{\sin\alpha}$

$= -1-\cot\alpha$

$y'(\alpha)=\text{cosec}^2\alpha$

$y'\left(\frac{5\pi}{6}\right)=4$

5. Let α be a root of the equation $x^2+x+1=0$ and the matrix $A=\frac{1}{\sqrt{3}}\begin{bmatrix}1&1&1\\1&\alpha&\alpha^2\\1&\alpha^2&\alpha^4\end{bmatrix}$, then the matrix A^{31} is equal to:

(1) A^3 (2) A (3) A^2 (4) I_3

NTA Ans. (1)

Sol. $x^2+x+1=0$

$\alpha=\omega$

$\alpha^2=\omega^2$

$A=\frac{1}{\sqrt{3}}\begin{bmatrix}1&1&1\\1&\omega&\omega^2\\1&\omega^2&\omega\end{bmatrix}$

$A^2=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$

$\Rightarrow A^4=A^2.A^2=I_3$

$A^{31}=A^{28}.A^3=A^3.$

6. If $y=mx+4$ is a tangent to both the parabolas, $y^2=4x$ and $x^2=2by$, then b is equal to :

(1) 128 (2) –64 (3) –128 (4) –32

NTA Ans. (3)

Sol. $y=mx+4$ is tangent to $y^2=4x$

$\Rightarrow m=\frac{1}{4}$

$y=\frac{1}{4}x+4$ is tangent to $x^2=2by$

$\Rightarrow x^2-\frac{b}{2}x-8b=0$

$\Rightarrow D=0$

$b^2+128b=0$

7. If the distance between the foci of an ellipse is 6 and the distance between its directrices is 12, then the length of its latus rectum is :

(1) $\sqrt{3}$ (2) $2\sqrt{3}$

(3) $3\sqrt{2}$ (4) $\frac{3}{\sqrt{2}}$

NTA Ans. (3)

Sol. Given $2ae=6\Rightarrow \boxed{ae=3}$ ......(1)

and $\frac{2a}{e}=12\Rightarrow\boxed{a=6e}$ ....(2)

from (1) and (2)

$6e^2=3\Rightarrow \boxed{e=\frac{1}{\sqrt{2}}}$

$\Rightarrow \boxed{a=3\sqrt{2}}$

Now, $b^2=a^2(1-e^2)$

$\Rightarrow b^2=18\left(1-\frac{1}{2}\right)=9$

Length of L.R $=\frac{2(9)}{3\sqrt{2}}=3\sqrt{2}$

8. An unbiased coin is tossed 5 times. Suppose that a variable X is assigned the value k when k consecutive heads are obtained for k = 3, 4, 5 otherwise X takes the value –1. Then the expected value of X, is :

(1) $\frac{3}{16}$ (2) $-\frac{3}{16}$ (3) $\frac{1}{8}$ (4) $-\frac{1}{8}$

NTA Ans. (3)

Sol.

k	0	1	2	3	4	5
P(k)	$\frac{1}{32}$	$\frac{12}{32}$	$\frac{11}{32}$	$\frac{5}{32}$	$\frac{2}{32}$	$\frac{1}{32}$

Expected value $=\sum XP(k)$

$-\frac{1}{32}-\frac{12}{32}-\frac{11}{32}+\frac{15}{32}+\frac{8}{32}+\frac{5}{32}$

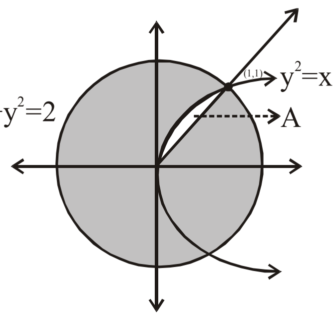
9. The area of the region, enclosed by the circle $x^2 + y^2 = 2$ which is not common to the region bounded by the parabola $y^2 = x$ and the straight line $y = x$, is :

(1) $\frac{1}{3}(12\pi - 1)$ (2) $\frac{1}{6}(12\pi - 1)$

(3) $\frac{1}{6}(24\pi - 1)$ (4) $\frac{1}{3}(6\pi - 1)$

NTA Ans. (2)

Sol.



$A = \int_0^1 \left(\sqrt{x} - x\right) dx$

$= \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2}\right]_0^1 = \frac{1}{6}$

Required Area : $\pi r^2 - \frac{1}{6} = \frac{1}{6}(12\pi - 1)$

10. Let $x^k + y^k = a^k$, $(a, K > 0)$ and $\frac{dy}{dx} + \left(\frac{y}{x}\right)^{\frac{1}{3}} = 0$, then k is :

(1) $\frac{3}{2}$ (2) $\frac{1}{3}$ (3) $\frac{2}{3}$ (4) $\frac{4}{3}$

NTA Ans. (3)

Sol. $x^k + y^k = a^k$ $(a, k > 0)$

$kx^{k-1} + ky^{k-1}\frac{dy}{dx} = 0$

$\frac{dy}{dx} + \left(\frac{x}{y}\right)^{k-1} = 0 \Rightarrow k - 1 = -\frac{1}{3} \Rightarrow k = 2/3$

11. If $y = y(x)$ is the solution of the differential equation, $e^y\left(\frac{dy}{dx} - 1\right) = e^x$ such that $y(0) = 0$, then $y(1)$ is equal to :

(1) $2 + \log_e 2$ (2) $2e$

(3) $\log_e 2$ (4) $1 + \log_e 2$

Sol. $e^y\frac{dy}{dx} - e^y = e^x$, Let $e^y = t$

$\Rightarrow e^y\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} - t = e^x$

I.F. $= e^{\int -dx} = e^{-x}$

$t\,e^{-x} = x + c \Rightarrow e^{y-x} = x + c$

$y(0) = 0 \Rightarrow c = 1$

$e^{y-x} = x + 1 \Rightarrow y(1) = 1 + \log_e 2$

12. Total number of 6-digit numbers in which only and all the five digits 1, 3, 5, 7 and 9 appear, is:

(1) $\frac{5}{2}(6!)$ (2) 5^6 (3) $\frac{1}{2}(6!)$ (4) $6!$

NTA Ans. (1)

Sol. Total number of 6-digit numbers in which only and all the five digits 1, 3, 5, 7 and 9 is

${}^5C_1 \times \frac{6!}{2!}$

13. Let P be a plane passing through the points (2, 1, 0), (4, 1, 1) and (5, 0, 1) and R be any point (2, 1, 6). Then the image of R in the plane P is :

(1) (6, 5, –2) (2) (4, 3, 2)

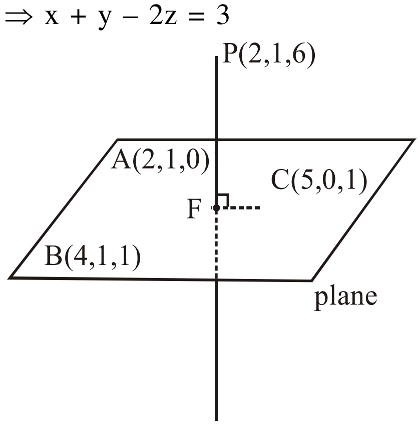
(3) (3, 4, –2) (4) (6, 5, 2)

NTA Ans. (1)

Sol. Plane passing through : (2, 1, 0), (4, 1, 1) and (5, 0, 1)

$$\begin{vmatrix} x-2 & y-1 & z \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{vmatrix} = 0$$

$\Rightarrow x + y - 2z = 3$



Let I and F are respectively image and foot of perpendicular of point P in the plane.

eq^n of line PI $\frac{x-2}{1}=\frac{y-1}{1}=\frac{z-6}{-2}=\lambda(\text{say})$

Let I $(\lambda + 2 , \lambda + 1, -2\lambda + 6)$

$\Rightarrow F\left(2+\frac{\lambda}{2},1+\frac{\lambda}{2},-\lambda+6\right)$

F lies in the plane

$\Rightarrow 2+\frac{\lambda}{2}+1+\frac{\lambda}{2}+2\lambda-12-3=0$

$\Rightarrow \lambda = 4$

$\Rightarrow$ I (6, 5, –2)

14. A vector $\vec{a}=\alpha\hat{i}+2\hat{j}+\beta\hat{k}(\alpha,\beta\in R)$ lies in the plane of the vectros $\vec{b}=\hat{i}+\hat{j}$ and $\vec{c}=\hat{i}-\hat{j}+4\hat{k}$. If $\vec{a}$ bisects the angle between $\vec{b}$ and $\vec{c}$, then:

(1) $\vec{a}.\hat{i}+1=0$ (2) $\vec{a}.\hat{i}+3=0$

(3) $\vec{a}.\hat{k}+4=0$ (4) $\vec{a}.\hat{k}+2=0$

NTA Ans. (4)

Ans. (BONUS)

Sol. $\vec{a}=\lambda\left(\hat{b}+\hat{c}\right)=\lambda\left(\frac{\hat{i}+\hat{j}}{\sqrt{2}}+\frac{\hat{i}-\hat{j}+4\hat{k}}{3\sqrt{2}}\right)$

$\vec{a}=\frac{\lambda}{3\sqrt{2}}\left(4\hat{i}+2\hat{j}+4\hat{k}\right)\Rightarrow\frac{\lambda}{3\sqrt{2}}(4\hat{i}+2\hat{j}+4\hat{k})$

$=\alpha\hat{i}+2\hat{j}+\beta\hat{k}$

$\Rightarrow \alpha = 4$ and $\beta = 4$

So, $\vec{a}=4\hat{i}+2\hat{j}+4\hat{k}$

None of the given options is correct

15. If $f(a + b + 1 - x) = f(x)$, for all x, where a and b are fixed positive real numbers, then $\frac{1}{a+b}\int_a^b x(f(x)+f(x+1))dx$ is equal to :

(1) $\int_{a+1}^{b+1} f(x)dx$ (2) $\int_{a+1}^{b+1} f(x+1)dx$

(3) $\int_{a-1}^{b-1} f(x+1)dx$ (4) $\int_{a-1}^{b-1} f(x)dx$

Ans. (1 OR 3)

Sol. $f(x + 1) = f(a + b - x)$

$$I=\frac{1}{(a+b)}\int_a^b x\left(f(x)+f(x+1)\right)dx \quad .....(1)$$

$$I=\frac{1}{(a+b)}\int_a^b (a+b-x)\left(f(x+1)+f(x)\right)dx \quad ...(2)$$

from (1) and (2)

$$2I=\int_a^b\left(f(x)+f(x+1)\right)dx$$

$$2I=\int_a^b f\left(a+b-x\right)dx+\int_a^b f(x+1)dx$$

$$2I=2\int_a^b f(x+1)dx\Rightarrow I=\int_a^b f(x+1)dx$$

$$=\int_{a+1}^{b+1} f(x)dx$$

OR

$$I=\frac{1}{(a+b)}\int_a^b x\left(f(x)+f(x+1)\right)dx \quad ...(1)$$

$$=\frac{1}{(a+b)}\int_a^b\left(a+b-x\right)\left(f(a+b-x)+f(a+b+1-x)\right)dx$$

$$I=\frac{1}{(a+b)}\int_a^b (a+b-x)\left(f(x+1)+f(x)\right)dx \quad ...(2)$$

equation (1) + (2)

$$2I=\frac{1}{(a+b)}\int_a^b (a+b)\left(f(x+1)+f(x)\right)dx$$

$$I=\frac{1}{2}\left[\int_a^b f(x+1)dx+\int_a^b f(x)dx\right]$$

$$=\frac{1}{2}\left[\int_a^b f(a+b+1-x)dx+\int_a^b f(x)dx\right]$$

$$=\frac{1}{2}\left[\int_a^b f(x)dx+\int_a^b f(x)dx\right]$$

$$I=\int_a^b f(x)dx$$

Let x = T + 1

$$=\int_{a-1}^{b-1} f(T+1)dT$$

16. Let the function, $f : [-7, 0] \to R$ be continuous on $[-7, 0]$ and differentiable on $(-7, 0)$. If $f(-7) = -3$ and $f'(x) \leq 2$, for all $x \in (-7, 0)$, then for all such functions f, $f(-1) + f(0)$ lies in the interval :

(1) $[-6, 20]$ (2) $(-\infty, 20]$

(3) $(-\infty, 11]$ (4) $[-3, 11]$

NTA Ans. (2)

Sol. Using LMVT in $[-7, -1]$

$$\frac{f(-1)-f(-7)}{-1-(-7)} \leq 2$$

$f(-1) - f(-7) \leq 12$

$\Rightarrow f(-1) \leq 9$ ....(1)

Using LMVT in $[-7, 0]$

$$\frac{f(0)-f(-7)}{0-(-7)} \leq 2$$

$f(0) - f(-7) \leq 14$

$f(0) \leq 11$ ....(2)

from (1) and (2)

$f(0) + f(-1) \leq 20$

17. If the system of linear equations

$2x + 2ay + az = 0$

$2x + 3by + bz = 0$

$2x + 4cy + cz = 0,$

where $a, b, c \in R$ are non-zero and distinct; has a non-zero solution, then :

(1) a, b, c are in A.P.

(2) $a + b + c = 0$

(3) a, b, c are in G.P.

(4) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

NTA Ans. (4)

Sol. For non-zero solution

$$\begin{vmatrix} 2 & 2a & a \\ 2 & 3b & b \\ 2 & 4c & c \end{vmatrix} = 0, \Rightarrow \begin{vmatrix} 1 & 2a & a \\ 0 & 3b-2a & b-a \\ 0 & 4c-2a & c-a \end{vmatrix} = 0$$

$\Rightarrow (3b - 2a)(c - a) - (b - a)(4c - 2a) = 0$

$\Rightarrow 2ac = bc + ab$

$\Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$ Hence $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

18. Let α and β be two real roots of the equation $(k + 1)\tan^2 x - \sqrt{2} \cdot \lambda \tan x = (1 - k)$, where $k (\neq -1)$ and λ are real numbers. If $\tan^2(\alpha + \beta) = 50$, then a value of λ is ;

(1) 5 (2) 10 (3) $5\sqrt{2}$ (4) $10\sqrt{2}$

NTA Ans. (2)

Sol. $\tan\alpha + \tan\beta = \frac{\lambda\sqrt{2}}{k+1}$

$\tan\alpha . \tan\beta = \frac{k-1}{k+1}$

$$\tan(\alpha + \beta) = \frac{\frac{\lambda\sqrt{2}}{k+1}}{1-\frac{k-1}{k+1}} = \frac{\lambda\sqrt{2}}{2} = \frac{\lambda}{\sqrt{2}}$$

$$\Rightarrow \frac{\lambda^2}{2} = 50 \Rightarrow \lambda = 10 \,\&\, -10$$

19. The logical statement $(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$ is equivalent to :

(1) p (2) q (3) ~p (4) ~q

NTA Ans. (3)

Sol. $(p \to q) \wedge (q \to \sim p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee \sim p)$

$\equiv \sim p \vee (q \wedge \sim q)$

$\equiv \sim p \vee C \equiv \sim p$

20. The greatest positive integer k, fr which $49^k + 1$ is a factor of the sum $49^{125} + 49^{124} + \ldots 49^2 + 49 + 1$, is :

(1) 32 (2) 60 (3) 63 (4) 65

NTA Ans. (3)

Sol. $1 + 49 + 49^2 + \ldots + 49^{12}$

$$= (49)^{126} - 1 = (49^{63} + 1)\frac{(49^{63}-1)}{(48)}$$

So greatest value of k = 63

21. $\lim_{x\to 2}\frac{3^x+3^{3-x}-12}{3^{-x/2}-3^{1-x}}$ is equal to ____________.

NTA Ans. (36)

Sol. $\lim_{x\to 2}\frac{3^x+3^{3-x}-12}{3^{-x/2}-3^{1-x}} \Rightarrow \lim_{x\to 2}\frac{3^{2x}-12.3^x+27}{3^{x/2}-3}$

$= \lim_{x\to 2}\frac{(3^x-9)(3^x-3)}{(3^{x/2}-3)}$

$= \lim_{x\to 2}\frac{(3^{x/2}+3)(3^{x/2}-3)(3^x-3)}{(3^{x/2}-3)}$

= 36

22. If the variance of the first n natural numbers is 10 and the variance of the first m even natural numbers is 16, then m + n is equal to ________.

NTA Ans. (18)

Sol. Variance of first 'n' natural numbers = $\frac{n^2-1}{12}=10$

$\Rightarrow$ n = 11

and variance of first 'm' even natural numbers

$= 4\left(\frac{m^2-1}{12}\right) \Rightarrow \frac{m^2-1}{3}=16 \Rightarrow m=7$

m + n = 18

23. If the sum of the coefficients of all even powers of x in the product

$(1+x+x^2+\ldots+x^{2n})(1-x+x^2-x^3+\ldots+x^{2n})$

is 61, then n is equal to __________.

NTA Ans. (30)

Sol. Let $(1+x+x^2+\ldots+x^{2n})(1-x+x^2-x^3+\ldots+x^{2n})$

$= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \ldots + a_{4n}x^{4n}$

So,

$a_0 + a_1 + a_2 + \ldots + a_{4n} = 2n + 1$ …(1)

$a_0 - a_1 + a_2 - a_3 \ldots + a_{4n} = 2n + 1$ …(2)

$\Rightarrow a_0 + a_2 + a_4 + \ldots + a_{4n} = 2n + 1$

$\Rightarrow 2n + 1 = 61$ $\Rightarrow n = 30$

24. Let A(1, 0), B(6, 2) and $C\left(\frac{3}{2},6\right)$ be the vertices of a triangle ABC. If P is a point inside the triangle ABC such that the triangles APC, APB and BPC have equal areas, then the length of the line segment PQ, where Q is the point $\left(-\frac{7}{6},-\frac{1}{3}\right)$, is __________.

NTA Ans. (5)

Sol. P is centroid of the triangle ABC

$\Rightarrow P \equiv \left(\frac{17}{6},\frac{8}{3}\right)$

$\Rightarrow$ PQ = 5

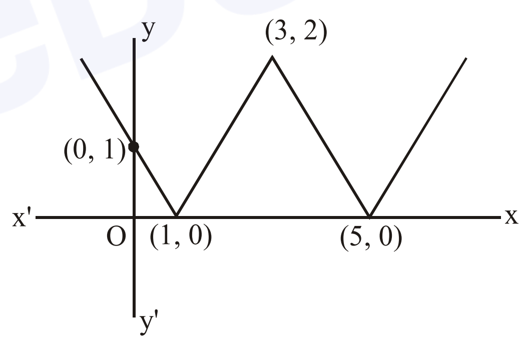
25. Let S be the set of points where the function, $f(x) = \left|2-|x-3|\right|$, $x\in R$, is not differentiable. Then $\sum_{x\in S} f(f(x))$ is equal to __________.

NTA Ans. (3)

Sol. f(x) = |2 – |x – 3||

f is not differentiable at

x = 1, 3, 5



$\Rightarrow \sum_{x\in s} f(f(x)) = f(f(1)) + f(f(3)) + f(f(5))$

= f(0) + f(2) + f(0)

= 1 + 1 + 1 = 3