FINAL JEE–MAIN EXAMINATION – JANUARY, 2020

(Held On Wednesday 08th JANUARY, 2020) TIME : 2 : 30 PM to 5 : 30 PM

MATHEMATICS — TEST PAPER WITH ANSWER & SOLUTION

1. Let $\vec{a}=\hat{i}-2\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+\hat{k}$ be two vectors. If $\vec{c}$ is a vector such that $\vec{b}\times\vec{c}=\vec{b}\times\vec{a}$ and $\vec{c}\cdot\vec{a}=0$, then $\vec{c}\cdot\vec{b}$ is equal to

(1) $\frac{1}{2}$ (2) -1 (3) $-\frac{1}{2}$ (4) $-\frac{3}{2}$

NTA Ans. (3)

Sol. $\vec{b}\times\vec{c}-\vec{b}\times\vec{a}=\vec{0}$

$\vec{b}\times(\vec{c}-\vec{a})=\vec{0}$

$\vec{b}=\lambda(\vec{c}-\vec{a})$ ...(i)

$\vec{a}\cdot\vec{b}=\lambda(\vec{a}\cdot\vec{c}-\vec{a}^2)$

$4=\lambda(0-6)\Rightarrow\lambda=\frac{-4}{6}=\frac{-2}{3}$

from (i) $\vec{b}=\frac{-2}{3}(\vec{c}-\vec{a})$

$\vec{c}=\frac{-3}{2}\vec{b}+\vec{a}=\frac{-1}{2}(\hat{i}+\hat{j}+\hat{k})$

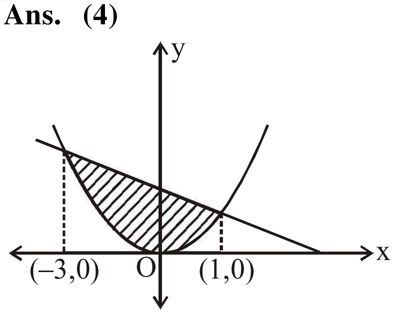
$\boxed{\vec{b}\cdot\vec{c}=\frac{-1}{2}}$ (3) Option

2. The area (in sq. units) of the region $\{(x,y)\in R^2 : x^2 \le y \le 3-2x\}$, is

(1) $\frac{29}{3}$ (2) $\frac{31}{3}$ (3) $\frac{34}{3}$ (4) $\frac{32}{3}$

NTA Ans. (4)

Sol.



Area $=\int_{-3}^{1}(3-2x-x^2)dx=\frac{32}{3}$

3. The length of the perpendicular from the origin, on the normal to the curve, $x^2+2xy-3y^2=0$ at the point (2,2) is

(1) $4\sqrt{2}$ (2) $2\sqrt{2}$

(3) 2 (4) $\sqrt{2}$

NTA Ans. (2)

Sol. $x^2+2xy-3y^2=0$

m_N = slope of normal drawn to curve at (2,2) is –1

L : x + y = 4.

perpendicular distance of L from (0,0)

$=\frac{|0+0-4|}{\sqrt{2}}=2\sqrt{2}$

(2) Option

4. If $I=\int_1^2\frac{dx}{\sqrt{2x^3-9x^2+12x+4}}$, then :

(1) $\frac{1}{9}<I^2<\frac{1}{8}$ (2) $\frac{1}{16}<I^2<\frac{1}{9}$

(3) $\frac{1}{6}<I^2<\frac{1}{2}$ (4) $\frac{1}{8}<I^2<\frac{1}{4}$

NTA Ans. (1)

Sol. $f(x)=\frac{1}{\sqrt{2x^3-9x^2+12x+4}}$

$f'(x)=\frac{-6(x-1)(x-2)}{2(2x^3-9x^2+12x+4)^{3/2}}$

$\therefore f(x)$ is decreasing in (1,2)

$f(1)=\frac{1}{3};\ f(2)=\frac{1}{\sqrt{8}}$

$\frac{1}{3}<I<\frac{1}{\sqrt{8}}\Rightarrow I^2\in\left(\frac{1}{9},\frac{1}{8}\right)$

5. If a line, $y = mx + c$ is a tangent to the circle, $(x-3)^2 + y^2 = 1$ and it is perpendicular to a line L_1, where L_1 is the tangent to the circle, $x^2 + y^2 = 1$ at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, then

(1) $c^2 - 6c + 7 = 0$
(2) $c^2 + 6c + 7 = 0$
(3) $c^2 + 7c + 6 = 0$
(4) $c^2 - 7c + 6 = 0$

NTA Ans. (2)

Sol. Slope of tangent to $x^2 + y^2 = 1$ at $P\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$$2x + 2yy' = 0 \Rightarrow m_T|_P = -1$$

$y = mx + c$ is tangent to $(x-3)^2 + y^2 = 1$
$y = x + c$ is tangent to $(x-3)^2 + y^2 = 1$

$$\left|\frac{c+3}{\sqrt{2}}\right| = 1 \Rightarrow c^2 + 6c + 7 = 0$$

(2) Option

6. Let S be the set of all functions $f : [0,1] \to R$, which are continuous on [0,1] and differentiable on (0,1). Then for every f in S, there exists a $c \in (0,1)$, depending on f, such that

(1) $|f(c) - f(1)| < (1 - c)|f'(c)|$
(2) $|f(c) - f(1)| < |f'(c)|$
(3) $|f(c) + f(1)| < (1 + c)|f'(c)|$
(4) $\frac{f(1) - f(c)}{1 - c} = f'(c)$

NTA Ans. (2)

Ans. (BONUS)

Sol. Option (1), (2), (3) are incorrect for $f(x)$ = constant and option (4) is incorrect

$$\frac{f(1) - f(c)}{1 - c} = f'(a) \text{ where } c < a < 1 \text{ (use LMVT)}$$

Also for $f(x) = x^2$ option (4) is incorrect.

7. Which of the following statements is a tautology?

(1) $\sim(p \vee \sim q) \to p \vee q$
(2) $\sim(p \wedge \sim q) \to p \vee q$
(3) $\sim(p \vee \sim q) \to p \wedge q$
(4) $p \vee (\sim q) \to p \wedge q$

Sol. $\sim(p \vee \sim q) \to p \vee q$
$(\sim p \wedge q) \to p \vee q$
$\sim\{(\sim p \wedge q) \wedge (\sim p \wedge \sim q)\}$
$\sim(\sim p \wedge f)$
(1) Option

8. If the 10^{th} term of an A.P. is $\frac{1}{20}$ and its 20^{th} term is $\frac{1}{10}$, then the sum of its first 200 terms is

(1) $50\frac{1}{4}$
(2) $100\frac{1}{2}$
(3) 50
(4) 100

NTA Ans. (2)

Sol. $T_{10} = \frac{1}{20} = a + 9d$ ...(i)

$T_{20} = \frac{1}{10} = a + 19d$ ...(ii)

$$a = \frac{1}{200} = d$$

Hence, $S_{200} = \frac{200}{2}\left[\frac{2}{200} + \frac{199}{200}\right] = \frac{201}{2}$

(2) Option

9. Let $f : (1,3) \to R$ be a function defined by $f(x) = \frac{x[x]}{1+x^2}$, where [x] denotes the greatest integer $\leq x$. Then the range of f is

(1) $\left(\frac{3}{5}, \frac{4}{5}\right)$
(2) $\left(\frac{2}{5}, \frac{3}{5}\right] \cup \left(\frac{3}{4}, \frac{4}{5}\right)$
(3) $\left(\frac{2}{5}, \frac{4}{5}\right]$
(4) $\left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{4}{5}\right]$

NTA Ans. (4)

Sol. $$f(x) = \begin{cases} \frac{x}{x^2+1} & ; \quad x \in (1,2) \\ \frac{2x}{x^2+1} & ; \quad x \in [2,3) \end{cases}$$

$f(x)$ is decreasing function

$$\therefore f(x) \in \left(\frac{2}{\ }, \frac{1}{\ }\right) \cup \left(\frac{3}{\ }, \frac{4}{\ }\right]$$

10. The system of linear equations

$\lambda x + 2y + 2z = 5$

$2\lambda x + 3y + 5z = 8$

$4x + \lambda y + 6z = 10$ has

(1) infinitely many solutions when $\lambda = 2$

(2) a unique solution when $\lambda = -8$

(3) no solution when $\lambda = 8$

(4) no solution when $\lambda = 2$

NTA Ans. (4)

Sol. $D = \begin{vmatrix} \lambda & 3 & 2 \\ 2\lambda & 3 & 5 \\ 4 & \lambda & 6 \end{vmatrix} = (\lambda+8)(2-\lambda)$

for $\lambda = 2$; $D_1 \neq 0$

Hence, no solution for $\lambda = 2$

(4) Option

11. If α and β be the coefficients of x^4 and x^2 respectively in the expansion of

$\left(x+\sqrt{x^2-1}\right)^6 + \left(x-\sqrt{x^2-1}\right)^6$, then

(1) $\alpha + \beta = 60$ (2) $\alpha + \beta = -30$

(3) $\alpha - \beta = -132$ (4) $\alpha - \beta = 60$

NTA Ans. (3)

Sol. $2[{}^6C_0 . x^6 + {}^6C_2 x^4(x^2-1) + {}^6C_4 x^2(x^2-1)^2 + {}^6C_6(x^2-1)^3]$

$\alpha = -96$ & $\beta = 36$

$\therefore \alpha - \beta = -132$

(3) Option

12. $\lim_{x\to 0} \frac{\int_0^x t\sin(10t)dt}{x}$ is equal to

(1) 0 (2) $-\frac{1}{5}$

(3) $-\frac{1}{10}$ (4) $\frac{1}{10}$

NTA Ans. (1)

Sol. Using L.H. Rule

$\lim_{x\to 0} \frac{x\sin(10x)}{1} = 0$

13. If $A = \begin{pmatrix} 2 & 2 \\ 9 & 4 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $10A^{-1}$ is equal to

(1) 4I – A (2) A – 6I

(3) 6I – A (4) A – 4I

NTA Ans. (2)

Sol. $A = \begin{pmatrix} 2 & 2 \\ 9 & 4 \end{pmatrix}$; $|A| = 8 - 18 = -10$

$A^{-1} = \frac{\text{adj}A}{|A|} = \frac{\begin{pmatrix} 4 & -2 \\ -9 & 2 \end{pmatrix}}{-10}$

$10A^{-1} = \begin{pmatrix} -4 & 2 \\ 9 & -2 \end{pmatrix} = A - 6I$

(2) Option

14. The mean and variance of 20 observations are found to be 10 and 4, respectively. On rechecking, it was found that an observation 9 was incorrect and the correct observation was 11. Then the correct variance is

(1) 3.99 (2) 3.98

(3) 4.02 (4) 4.01

NTA Ans. (1)

Sol. $\frac{\sum x_i}{20} = 10 \Rightarrow \Sigma x_i = 200$ ...(i)

$\frac{\sum x_i^2}{20} - 100 = 4 \Rightarrow \Sigma x_i^2 = 2080$ ...(ii)

Actual mean = $\frac{200-9+11}{20} = \frac{202}{20}$

Variance = $\frac{2080-81+121}{20} - \left(\frac{202}{20}\right)^2 = 3.99$

(1) Option

15. If a hyperbola passes through the point P(10,16) and it has vertices at (±6,0), then the equation of the normal to it at P is

(1) $x + 2y = 42$ (2) $3x + 4y = 94$

Sol. $\frac{x^2}{36}-\frac{y^2}{b^2}=1$ ...(i)

P(10,16) lies on (i) get $b^2 = 144$

$\frac{x^2}{36}-\frac{y^2}{144}=1$

Equation of normal is

$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2e^2$

$2x + 5y = 100$

(3) Option

16. Let A and B be two events such that the probability that exactly one of them occurs is $\frac{2}{5}$ and the probability that A or B occurs is $\frac{1}{2}$, then the probability of both of them occur together is

(1) 0.02 (2) 0.01 (3) 0.20 (4) 0.10

NTA Ans. (4)

Sol. $P(A) + P(B) - 2P(A \cap B) = \frac{2}{5}$

$P(A) + P(B) - P(A \cap B) = \frac{1}{2}$

$P(A \cap B) = \frac{1}{10}$

(4) Option

17. The mirror image of the point (1,2,3) in a plane is $\left(-\frac{7}{3},-\frac{4}{3},-\frac{1}{3}\right)$. Which of the following points lies on this plane ?

(1) (–1, –1, –1) (2) (–1, –1, 1)

(3) (1, 1, 1) (4) (1, –1, 1)

NTA Ans. (4)

Sol. Point on plane $R\left(\frac{-2}{3},\frac{1}{3},\frac{4}{3}\right)$

Normal vector of plane is $\frac{10}{3}\hat{i}+\frac{10}{3}\hat{j}+\frac{10}{3}\hat{k}$

Equation of require plane is $x + y + z = 1$

Hence (1, –1, 1) lies on plane

(4) Option

18. Let S be the set of all real roots of the equation, $3^x(3^x - 1) + 2 = |3^x - 1| + |3^x - 2|$. Then S :

(1) is an empty set.

(2) contains at least four elements.

(3) contains exactly two elements.

(4) is a singleton.

NTA Ans. (4)

Sol. Let $3^x = t$; $t > 0$

$t(t - 1) + 2 = |t - 1| + |t - 2|$

$t^2 - t + 2 = |t - 1| + |t - 2|$

Case-I : $t < 1$

$t^2 - t + 2 = 1 - t + 2 - t$

$t^2 + 2 = 3 - t$

$t^2 + t - 1 = 0$

$t=\frac{-1\pm\sqrt{5}}{2}$

$t=\frac{\sqrt{5}-1}{2}$ is only acceptable

Case-II : $1 \le t < 2$

$t^2 - t + 2 = t - 1 + 2 - t$

$t^2 - t + 1 = 0$

$D < 0$ no real solution

Case-III : $t \ge 2$

$t^2 - t + 2 = t - 1 + t - 2$

$t^2 - 3t \;\; 5 = 0 \Rightarrow D < 0$ no real solution

(4) Option

19. Let $\alpha = \frac{-1+i\sqrt{3}}{2}$. If $a = (1+\alpha)\sum_{k=0}^{100}\alpha^{2k}$ and $b = \sum_{k=0}^{100}\alpha^{3k}$, then a and b are the roots of the quadratic equation :

(1) $x^2 - 102x + 101 = 0$
(2) $x^2 + 101x + 100 = 0$
(3) $x^2 - 101x + 100 = 0$
(4) $x^2 + 102x + 101 = 0$

NTA Ans. (1)

Sol. $\alpha = \omega$

$a = (1 + \omega)(1 + \omega^2 + \omega^4 + ... + \omega^{200})$

$$a = (1+\omega)\frac{\left(1-\left(\omega^2\right)^{101}\right)}{1-\omega^2} = 1$$

$b = 1 + \omega^3 + \omega^6 + ... + \omega^{300} = 101$

$x^2 - 102x + 101 = 0$

(1) Option

20. The differential equation of the family of curves, $x^2 = 4b(y + b)$, $b \in R$, is

(1) $x(y')^2 = x + 2yy'$
(2) $x(y')^2 = 2yy' - x$
(3) $xy'' = y'$
(4) $x(y')^2 = x - 2yy'$

NTA Ans. (1)

Sol. $2x = 4by' \Rightarrow y' = \frac{2x}{4b}$

Required D.E. is $x^2 = \frac{2x}{y'}y + \left(\frac{x}{y'}\right)^2$

$x(y')^2 = 2yy' + x$

(1) Option

21. If $\frac{\sqrt{2}\sin\alpha}{\sqrt{1+\cos 2\alpha}} = \frac{1}{7}$ and $\sqrt{\frac{1-\cos 2\beta}{2}} = \frac{1}{\sqrt{10}}$, $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$, then $\tan(\alpha + 2\beta)$ is equal to _____.

NTA Ans. (1)

Sol. $\frac{\sqrt{2}\sin\alpha}{\sqrt{2}\cos\alpha} = \frac{1}{7} \Rightarrow \tan\alpha = \frac{1}{7}$

$\sin\beta = \frac{1}{\sqrt{10}} \Rightarrow \tan\beta = \frac{1}{3} \Rightarrow \tan 2\beta = \frac{3}{4}$

$$\tan(\alpha + 2\beta) = \frac{\tan\alpha + \tan 2\beta}{1 - \tan\alpha\tan 2\beta} = 1$$

Ans. 1.00

22. Let $f(x)$ be a polynomial of degree 3 such that $f(-1) = 10$, $f(1) = -6$, $f(x)$ has a critical point at $x = -1$ and $f'(x)$ has a critical point at $x = 1$. Then $f(x)$ has a local minima at x = _______.

NTA Ans. (3)

Sol. $f''(x) = \lambda(x - 1)$

$$f'(x) = \frac{\lambda x^2}{2} - \lambda x + C \Rightarrow f'(-1) = 0 \Rightarrow c = \frac{-3\lambda}{2}$$

$$f(x) = \frac{\lambda x^3}{6} - \frac{\lambda x^2}{2} - \frac{3\lambda}{2}x + d$$

$f(1) = -6 \Rightarrow -11\lambda + 6d = -36$ ...(i)

$f(-1) = 10 \Rightarrow 5\lambda + 6d = 60$ ...(ii)

from (i) & (ii) $\lambda = 6$ & $d = 5$

$f(x) = x^3 - 3x^2 - 9x + 5$

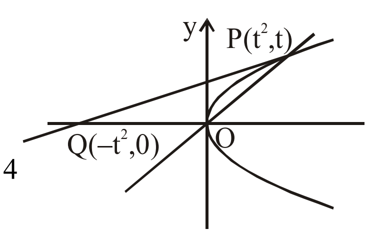
Which has minima at $x = 3$

Ans. 3.00

23. Let a line $y = mx$ $(m > 0)$ intersect the parabola, $y^2 = x$ at a point P, other than the origin. Let the tangent to it at P meet the x-axis at the point Q. If area (ΔOPQ) = 4 sq. units, then m is equal to ___________.

NTA Ans. (0.50)

Sol. $\Delta OPQ = 4$



$$\frac{1}{2}\left\|\begin{matrix} 0 & 0 & 1 \\ t^2 & t & 1 \\ -t^2 & 0 & 1 \end{matrix}\right\| = 4$$

$t = 2$ ($\because$ $t > 0$)

$\therefore$ $m = \frac{1}{2}$

Ans. 0.50

24. The sum, $\sum_{n=1}^{7}\frac{n(n+1)(2n+1)}{4}$ is equal to ________.

NTA Ans. (504)

Sol. $\frac{1}{4}\left(\sum_{n=1}^{7}2n^3+\sum_{n=1}^{7}3n^2+\sum_{n=1}^{7}n\right)$

$=\frac{1}{4}\left(2\left(\frac{7\times8}{2}\right)^2+3\left(\frac{7\times8\times15}{6}\right)+\frac{7\times8}{2}\right)$

= 504

Ans. 504.00

25. The number of 4 letter words (with or without meaning) that can be formed from the eleven letters of the word 'EXAMINATION' is _______.

NTA Ans. (2454)

Sol. N $\rightarrow$ 2, A $\rightarrow$ 2, I $\rightarrow$ 2, E, X, M, T, O $\rightarrow$ 1

Category	Selection	Arrangement
2 alike of one kind and 2 alike of other kind	$^3C_2=3$	$3\times\frac{4!}{2!\,2!}=18$
2 alike and 2 different	$^3C_1\times{}^7C_2$	$^3C_1\times{}^7C_2\times\frac{4!}{2!}=756$
All 4 different	$^8C_4$	$^8C_4\times4!=1680$

Total = 2454

Ans. 2454.00