FINAL JEE–MAIN EXAMINATION – JANUARY, 2020

(Held On Wednesday 08th JANUARY, 2020) TIME : 9 : 30 AM to 12 : 30 PM

MATHEMATICS — TEST PAPER WITH ANSWER & SOLUTION

1. Let the line y = mx and the ellipse $2x^2 + y^2 = 1$ intersect at a ponit P in the first quadrant. If the normal to this ellipse at P meets the co-ordinate axes at $\left(-\frac{1}{3\sqrt{2}}, 0\right)$ and $(0,\beta)$, then β is equal to

(1) $\frac{2}{\sqrt{3}}$ (2) $\frac{2\sqrt{2}}{3}$ (3) $\frac{2}{3}$ (4) $\frac{\sqrt{2}}{3}$

NTA Ans. (4)

Sol. Any normal to the ellipse is

$$\frac{x\sec\theta}{\sqrt{2}} - y\,\text{cosec}\,\theta = -\frac{1}{2}$$

$$\Rightarrow \frac{x}{\left(\frac{-\cos\theta}{\sqrt{2}}\right)} + \frac{y}{\left(\frac{\sin\theta}{2}\right)} = 1$$

$$\Rightarrow \frac{\cos\theta}{\sqrt{2}} = \frac{1}{3\sqrt{2}} \text{ and } \frac{\sin\theta}{2} = \beta$$

$$\Rightarrow \beta = \frac{\sqrt{2}}{3}$$

2. Let $f : R \to R$ be such that for all $x \in R$ $(2^{1+x} + 2^{1-x})$, $f(x)$ and $(3^x + 3^{-x})$ are in A.P., then the minimum value of $f(x)$ is

(1) 0 (2) 3 (3) 2 (4) 4

NTA Ans. (2)

Sol. $f(x) = \frac{2\left(2^x + 2^{-x}\right) + \left(3^x + 3^{-x}\right)}{2} \geq 3$

(A.M ≥ G.M)

3. Let the volume of a parallelopiped whose coterminous edges are given by $\vec{u} = \hat{i} + \hat{j} + \lambda\hat{k}, \vec{v} = \hat{i} + \hat{j} + 3\hat{k}$ and $\vec{w} = 2\hat{i} + \hat{j} + \hat{k}$ be 1 cu. unit. If θ be the angle between the edges $\vec{u}$ and $\vec{w}$, then cosθ can be

(1) $\frac{7}{\sqrt{\ }}$ (2) $\frac{5}{\ }$ (3) $\frac{7}{\sqrt{\ }}$ (4) $\frac{5}{\sqrt{\ }}$

Sol. $\left\|\begin{matrix} 1 & 1 & \lambda \\ 1 & 1 & 3 \\ 2 & 1 & 1 \end{matrix}\right\| = 1 \Rightarrow \lambda = 2, 4$

Now, $\cos\theta = \frac{\vec{u}.\vec{w}}{|\vec{u}||\vec{w}|}$

$= \frac{5}{\sqrt{6}\sqrt{6}} \text{ or } \frac{7}{\sqrt{6}\sqrt{18}} = \frac{5}{6} \text{ or } \frac{7}{6\sqrt{3}}$

4. If a,b and c are the greatest value of ${}^{19}C_p, {}^{20}C_q$ and ${}^{21}C_r$ respectively, then

(1) $\frac{a}{11} = \frac{b}{22} = \frac{c}{21}$ (2) $\frac{a}{10} = \frac{b}{11} = \frac{c}{21}$

(3) $\frac{a}{10} = \frac{b}{11} = \frac{c}{42}$ (4) $\frac{a}{11} = \frac{b}{22} = \frac{c}{42}$

NTA Ans. (4)

Sol. $a = {}^{19}C_{10}$, $b = {}^{20}C_{10}$ and $c = {}^{21}C_{10}$

$\Rightarrow a = {}^{19}C_9$, $b = 2({}^{19}C_9)$ and $c = \frac{21}{11}\left({}^{20}C_{10}\right)$

$\Rightarrow b = 2a$ and $c = \frac{21}{11}b = \frac{42a}{11}$

$\Rightarrow a : b : c = a : 2a : \frac{42a}{11} = 11 : 22 : 42$

5. Let $f(x) = (\sin(\tan^{-1}x) + \sin(\cot^{-1}x))^2 - 1$, $|x| > 1$. If $\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\left(\sin^{-1}\left(f(x)\right)\right)$ and $y\left(\sqrt{3}\right) = \frac{\pi}{6}$, then $y\left(-\sqrt{3}\right)$ is equal to

(1) $\frac{5\pi}{6}$ (2) $-\frac{\pi}{6}$

(3) $\frac{\pi}{3}$ (4) $\frac{2\pi}{3}$

NTA Ans. (1)

Sol. Let $\tan^{-1}x = \theta,\ \theta \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$$f(x) = (\sin\theta + \cos\theta)^2 - 1 = \sin 2\theta = \frac{2x}{1+x^2}$$

Now, $\frac{dy}{dx} = \frac{1}{2}\frac{d}{dx}\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$= -\frac{1}{1+x^2}, |x| > 1$

Since, we can integrate only in the continuous interval. So we have to take integral in two cases separtely namely for $x < -1$ and for $x > 1$.

$$\Rightarrow y = \begin{cases} -\tan^{-1}x + c_1 & ; \quad x > 1 \\ -\tan^{-1}x + c_2 & ; \quad x < -1 \end{cases}$$

so, $c_1 = \frac{\pi}{2}$ as $y(\sqrt{3}) = \frac{\pi}{6}$

But we cannot find c_2 as we do not have any other additional information for $x < -1$. So, all of the given options may be correct as c_2 is unknown so, it should be bonus.

6. $\lim_{x \to 0}\left(\frac{3x^2+2}{7x^2+2}\right)^{\frac{1}{x^2}}$ is equal to

(1) $\frac{1}{e}$ (2) e^2

(3) e (4) $\frac{1}{e^2}$

NTA Ans. (4)

Sol. Required limit $= e^{\lim_{x\to 0}\left(\frac{3x^2+2}{7x^2+2}-1\right)\frac{1}{x^2}}$

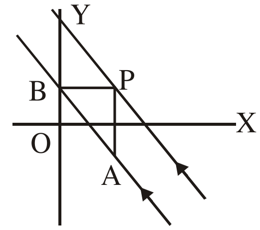
$$= e^{\lim_{x\to 0}\left(\frac{-4}{7x^2+2}\right)} = \frac{1}{e^2}$$

7. Let two points be A(1,–1) and B(0,2). If a point P(x',y') be such that the area of $\Delta PAB = 5$ sq. units and it lies on the line, $3x + y - 4\lambda = 0$, then a value of λ is

(1) 1 (2) 4

(3) 3 (4) –3

Sol. $\overrightarrow{AB} : 3x + y - 2 = 0$



Also, $\frac{1}{2} \times \sqrt{10} \times h = 5$

$\Rightarrow h = \sqrt{10}$

$\Rightarrow \frac{|4\lambda - 2|}{\sqrt{10}} = \sqrt{10} \Rightarrow \lambda = 3, -2$

8. The mean and the standard deviation (s.d.) of 10 observations are 20 and 2 resepectively. Each of these 10 observations is multiplied by p and then reduced by q, where $p \neq 0$ and $q \neq 0$. If the new mean and new s.d. become half of their original values, then q is equal to

(1) –20 (2) 10 (3) –10 (4) –5

NTA Ans. (1)

Sol. $20p - q = 10$ ...(i)

and $2|p| = 1 \Rightarrow p = \pm\frac{1}{2}$ ...(ii)

so, $p = -\frac{1}{2}$ and $q = -20$

9. Let $y = y(x)$ be a solution of the differential equation, $\sqrt{1-x^2}\frac{dy}{dx} + \sqrt{1-y^2} = 0, |x| < 1$.

If $y\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2}$, then $y\left(\frac{-1}{\sqrt{2}}\right)$ is equal to

(1) $-\frac{\sqrt{3}}{2}$ (2) $\frac{1}{\sqrt{2}}$ (3) $\frac{\sqrt{3}}{2}$ (4) $-\frac{1}{\sqrt{2}}$

NTA Ans. (2)

Ans. (BONUS)

Sol. $\frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ so, $\frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0$

Integrating, $\sin^{-1}x + \sin^{-1}y = c$

so, $\frac{\pi}{6} + \frac{\pi}{3} = c$

Hence, $\sin^{-1}x + \sin^{-1}y = \frac{\pi}{2}$

$\frac{1}{\ }$ $\ ^{-1}$ $\frac{3\pi}{\ }$

10. If the equation, $x^2 + bx + 45 = 0$ $(b \in R)$ has conjugate complex roots and they satisfy $|z+1| = 2\sqrt{10}$, then

(1) $b^2 - b = 42$ (2) $b^2 + b = 12$

(3) $b^2 + b = 72$ (4) $b^2 - b = 30$

NTA Ans. (4)

Sol. Assuming z is a root of the given equation,

$$z = \frac{-b \pm i\sqrt{180 - b^2}}{2}$$

$$\text{so, } \left(1 - \frac{b}{2}\right)^2 + \frac{180 - b^2}{4} = 40$$

$\Rightarrow -4b + 184 = 160 \Rightarrow b = 6$

11. For $a > 0$, let the curves $C_1 : y^2 = ax$ and $C_2 : x^2 = ay$ intersect at origin O and a point P. Let the line $x = b (0 < b < a)$ intersect the chord OP and the x-axis at points Q and R, respectively. If the line $x = b$ bisects the area bounded by the curves, C_1 and C_2, and the area of $\Delta OQR = \frac{1}{2}$, then 'a' satisfies the equation

(1) $x^6 - 12x^3 + 4 = 0$

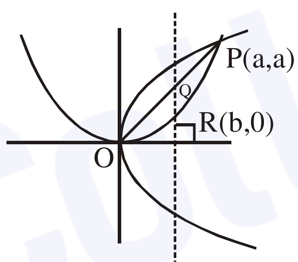
(2) $x^6 - 12x^3 - 4 = 0$

(3) $x^6 + 6x^3 - 4 = 0$

(4) $x^6 - 6x^3 + 4 = 0$

NTA Ans. (1)

Sol.



$$\int_0^b \left(\sqrt{ax} - \frac{x^2}{a}\right) dx = \frac{1}{2} \times \frac{16\left(\frac{a}{4}\right)\left(\frac{a}{4}\right)}{3}$$

$$\Rightarrow \left[\frac{2\sqrt{a}}{3} x^{3/2} - \frac{x^3}{3a}\right]_0^b = \frac{a^2}{6}$$

$$\Rightarrow \frac{2\sqrt{a}}{3} b^{3/2} - \frac{b^3}{3a} = \frac{a^2}{6} \quad \text{...(i)}$$

Also, $\frac{1}{2} \times b^2 = \frac{1}{2} \Rightarrow b = 1$

so, $\frac{2\sqrt{a}}{\ } - \frac{1}{\ } = \frac{a^2}{\ } \Rightarrow a^3 - 4a^{3/2} + 2 = 0$

12. Which one of the following is a tautology ?

(1) $P \wedge (P \vee Q)$

(2) $P \vee (P \wedge Q)$

(3) $Q \rightarrow (P \wedge (P \rightarrow Q))$

(4) $(P \wedge (P \rightarrow Q)) \rightarrow Q$

NTA Ans. (4)

Sol. (1) $P \wedge (P \vee Q) \equiv P$

(2) $P \vee (P \wedge Q) \equiv P$

(3) $Q \rightarrow (P \wedge (P \rightarrow Q))$

$\equiv Q \rightarrow (P \wedge (\sim P \vee Q)) \equiv Q \rightarrow (P \wedge Q)$

$\equiv (\sim Q) \vee (P \wedge Q) \equiv (P \vee (\sim Q))$

(4) $(P \wedge (P \rightarrow Q)) \rightarrow Q$

$\equiv (P \wedge (\sim P \vee Q)) \rightarrow Q \equiv (P \wedge Q) \rightarrow Q$

$\equiv ((\sim P) \vee (\sim Q)) \vee Q \equiv (\sim P) \vee t \equiv t$

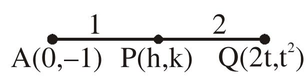
13. The locus of a point which divides the line segment joining the point (0,–1) and a point on the parabola, $x^2 = 4y$, internally in the ratio 1 : 2, is-

(1) $9x^2 - 3y = 2$ (2) $9x^2 - 12y = 8$

(3) $x^2 - 3y = 2$ (4) $4x^2 - 3y = 2$

NTA Ans. (2)

Sol.



$\Rightarrow 3h = 2t$ and $3k = t^2 - 2$

$$\Rightarrow 3y = \left(\frac{3x}{2}\right)^2 - 2 \Rightarrow 12y = 9x^2 - 8$$

14. If c is a point at which Rolle's theorem holds for the function, $f(x) = \log_e\left(\frac{x^2 + \alpha}{7x}\right)$ in the interval [3,4], where $\alpha \in R$, then $f''(c)$ is equal to

(1) $\frac{\sqrt{3}}{7}$ (2) $\frac{1}{12}$ (3) $-\frac{1}{24}$ (4) $-\frac{1}{12}$

Sol. $\frac{9+\alpha}{21}=\frac{16+\alpha}{28}\Rightarrow\alpha=12$

Also, $f'(x)=\frac{7x}{x^2+12}\times\frac{x^2-12}{7x^2}=\frac{x^2-12}{x(x^2+12)}$

Hence, $c=2\sqrt{3}$

Now, $f''(c)=\frac{1}{12}$

15. For which of the following ordered pairs (μ,δ), the system of linear equations

$x + 2y + 3z = 1$

$3x + 4y + 5z = \mu$

$4x + 4y + 4z = \delta$

is inconsistent ?

(1) (1,0) (2) (4,6) (3) (3,4) (4) (4,3)

NTA Ans. (4)

Sol. $2\times(ii)-2\times(i)-(iii)$: -

$0 = 2\mu - 2 - \delta$

$\Rightarrow \delta = 2(\mu - 1)$

16. Let A and B be two independent events such that $P(A)=\frac{1}{3}$ and $P(B)=\frac{1}{6}$. Then, which of the following is TRUE ?

(1) $P(A/B)=\frac{2}{3}$ (2) $P(A/(A\cup B))=\frac{1}{4}$

(3) $P(A/B')=\frac{1}{3}$ (4) $P(A'/B')=\frac{1}{3}$

NTA Ans. (3)

Sol. (1) $P(A/B)=P(A)=\frac{1}{3}$

(2) $P(A/(A\cup B))=\frac{P(A\cap(A\cup B))}{P(A\cup B)}=\frac{P(A)}{P(A\cup B)}$

$$=\frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{6}-\frac{1}{18}}=\frac{3}{4}$$

(3) $P(A/B')=P(A)=\frac{1}{3}$

(4) $P(A'/B')=P(A')=\frac{2}{3}$

17. The inverse function of $f(x)=\frac{8^{2x}-8^{-2x}}{8^{2x}+8^{-2x}}, x\in(-1,1)$, is

(1) $\frac{1}{4}(\log_8 e)\log_e\left(\frac{1-x}{1+x}\right)$

(2) $\frac{1}{4}\log_e\left(\frac{1-x}{1+x}\right)$

(3) $\frac{1}{4}(\log_8 e)\log_e\left(\frac{1+x}{1-x}\right)$

(4) $\frac{1}{4}\log_e\left(\frac{1+x}{1-x}\right)$

NTA Ans. (3)

Sol. $f(x)=y=\frac{8^{4x}-1}{8^{4x}+1}=1-\frac{2}{8^{4x}+1}$

so, $8^{4x}+1=\frac{2}{1-y}\Rightarrow 8^{4x}=\frac{1+y}{1-y}$

$\Rightarrow x=\ell n\left(\frac{1+y}{1-y}\right)\times\frac{1}{4\ell n8}=f^{-1}(y)$

Hence, $f^{-1}(x)=\frac{1}{4}\log_8 e\ell n\left(\frac{1+x}{1-x}\right)$

18. If $\int\frac{\cos x\,dx}{\sin^3 x(1+\sin^6 x)^{2/3}}=f(x)(1+\sin^6 x)^{1/\lambda}+c$ where c is a constant of integration, then $\lambda f\left(\frac{\pi}{3}\right)$ is equal to

(1) –2 (2) $-\frac{9}{8}$ (3) 2 (4) $\frac{9}{8}$

NTA Ans. (1)

Sol. $$\int\frac{\cos x\,dx}{\sin^3 x(1+\sin^6 x)^{2/3}}=\frac{-6}{-6}\int\frac{\cos x\,dx}{\sin^7 x\left(\frac{1}{\sin^6 x}+1\right)^{2/3}}$$

$$=-\frac{1}{6}\times 3\left(\frac{1}{\sin^6 x}+1\right)^{\frac{1}{3}}+c$$

$$=-\frac{1}{2}\frac{(1+\sin^6 x)^{\frac{1}{3}}}{\sin^2 x}+c$$

Hence, $\lambda = 3$ and $f(x)=-\frac{1}{2\sin^2 x}$

$\left(\pi\right)$

REMARK : Technically, this question should be marked as bonus. Because $f(x)$ and λ cannot be found uniquely.
For example, another such $f(x)$ and λ can be

$-\dfrac{(1+\sin^6 x)^{\frac{1}{6}}}{2\sin^2 x}$ and 6 respectively.

19. The shortest distance between the lines

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1} \text{ and}$$

$$\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4} \text{ is}$$

(1) $\frac{7}{2}\sqrt{30}$ (2) $3\sqrt{30}$ (3) 3 (4) $2\sqrt{30}$

NTA Ans. (2)

Sol. Shortest distance $=\dfrac{\left\|\begin{matrix}6 & 15 & -3\\ 3 & -1 & 1\\ -3 & 2 & 4\end{matrix}\right\|}{\sqrt{11\times29-49}}=\dfrac{270}{\sqrt{270}}$

$=\sqrt{270}=3\sqrt{30}$

20. Let $f(x) = x\cos^{-1}(-\sin|x|)$, $x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then which of the following is true ?

(1) f' is decreasing in $\left(-\frac{\pi}{2},0\right)$ and increasing in $\left(0,\frac{\pi}{2}\right)$

(2) f is not differentiable at x = 0

(3) $f'(0)=-\frac{\pi}{2}$

(4) f' is increasing in $\left(-\frac{\pi}{2},0\right)$ and decreasing in $\left(0,\frac{\pi}{2}\right)$

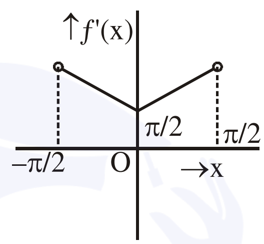
NTA Ans. (1)

Sol. $f(x)$ is an odd function.
Now, if $x \geq 0$, then $f(x) = x\cos^{-1}(-\sin x)$

$$=x\left(\frac{\pi}{2}-\sin^{-1}(-\sin x)\right)=x\left(\frac{\pi}{2}+x\right)$$

Hence, $f(x)=\begin{cases}x\left(\frac{\pi}{2}+x\right) & ; \quad x\in\left[0,\frac{\pi}{2}\right]\\ x\left(\frac{\pi}{2}-x\right) & ; \quad x\in\left[-\frac{\pi}{2},0\right)\end{cases}$

so, $f'(x)=\begin{cases}\frac{\pi}{2}+2x & ; \quad x\in\left[0,\frac{\pi}{2}\right)\\ \frac{\pi}{2}-2x & ; \quad x\in\left(-\frac{\pi}{2},0\right)\end{cases}$



21. The number of all 3 × 3 matrices A, with enteries from the set {–1,0,1} such that the sum of the diagonal elements of AA^T is 3, is

NTA Ans. (672.00)

Sol. trace $(AA^T) = \Sigma a_{ij}^2 = 3$

Hence, number of such matrices

$= {}^9C_3 \times 2^3 = 672.00$

22. The least positive value of 'a' for which the equation $2x^2 + (a - 10)x + \frac{33}{2} = 2a$ has real roots is

NTA Ans. (8.00)

Sol. $D \geq 0 \Rightarrow (a-10)^2 - 4\times 2\times\left(\frac{33}{2}-2a\right)\geq 0$

$\Rightarrow a^2 - 4a - 32 \geq 0$

$\Rightarrow a \in (-\infty, 4] \cup [8,\infty)$

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23. Let the normal at a point P on the curve $y^2 - 3x^2 + y + 10 = 0$ intersect the y-axis at $\left(0, \frac{3}{2}\right)$. If m is the slope of the tangent at P to the curve, then |m| is equal to

NTA Ans. (4.00)

Sol. Let $P(\alpha,\beta)$

so, $\beta^2 - 3\alpha^2 + \beta + 10 = 0$ ...(i)

Now, $2yy' - 6x + y' = 0$

$$\Rightarrow m = \frac{6\alpha}{2\beta+1} \quad ....(ii)$$

Also, $\frac{\beta - \frac{3}{2}}{\alpha} = -\frac{1}{m}$

$$\Rightarrow \frac{2\beta - 3}{2\alpha} = -\frac{(2\beta+1)}{6\alpha} \text{ (from (ii))}$$

$\Rightarrow \beta = 1 \Rightarrow \alpha^2 = 4$ (from (1))

Hence, $|m| = \frac{12}{3} = 4.00$

24. The sum $\sum_{k=1}^{20}(1+2+3+...+k)$ is ....

NTA Ans. (1540.00)

Sol. $\sum_{k=1}^{20}\frac{k(k+1)}{2} = \frac{1}{2}\sum_{k=1}^{20}\frac{k(k+1)(k+2)-(k-1)k(k+1)}{3}$

$= \frac{1}{6} \times 20 \times 21 \times 22 = 1540.00$

25. An urn contains 5 red marbles, 4 black marbles and 3 white marbles. Then the number of ways in which 4 marbles can be drawn so that at the most three of them are red is ....

NTA Ans. (490.00)

Ans. (490.00 OR 13.00)

Sol. The question does not mention that whether same coloured marbles are distinct or identical. So, assuming they are distinct our required answer = ${}^{12}C_4 - {}^{5}C_4 = 490$

And, if same coloured marbles are identical then required answer = (2 + 3 + 4 + 4) = 13