FINAL JEE–MAIN EXAMINATION – JANUARY, 2020

(Held On Thursday 09th JANUARY, 2020) TIME : 2 : 30 PM to 5 : 30 PM

MATHEMATICS — TEST PAPER WITH ANSWER & SOLUTION

1. Let [t] denote the greatest integer ≤ t and $\lim_{x\to 0} x\left[\frac{4}{x}\right] = A$. Then the function, $f(x) = [x^2]\sin(\pi x)$ is discontinuous, when x is equal to :

(1) $\sqrt{A+5}$ (2) $\sqrt{A+1}$

(3) $\sqrt{A}$ (4) $\sqrt{A+21}$

NTA Ans. (2)

Sol. $A = \lim_{x\to 0} x\left[\frac{4}{x}\right] = \lim_{x\to 0} x\left(\frac{4}{x}\right) - x\left\{\frac{4}{x}\right\} = 4$

$f(x) = [x^2]\sin(\pi x)$ will be discontinuous at nonintegers

$\therefore\ x = \sqrt{A+1}$ i.e. $\sqrt{5}$

2. The following system of linear equations

$7x + 6y - 2z = 0$

$3x + 4y + 2z = 0$

$x - 2y - 6z = 0$, has

(1) infinitely many solutions, (x, y, z) satisfying $x = 2z$

(2) no solution

(3) only the trivial solution

(4) infinitely many solutions, (x, y, z) satisfying $y = 2z$

NTA Ans. (1)

Sol. $7x + 6y - 2z = 0$ .... (1)

$3x + 4y + 2z = 0$ .... (2)

$x - 2y - 6z = 0$ .... (3)

$$\Delta = \begin{vmatrix} 7 & 6 & -2 \\ 3 & 4 & 2 \\ 1 & -2 & -6 \end{vmatrix} = 0 \Rightarrow \text{infinite solutions}$$

Now (1) + (2) ⇒ y = –x put in (1), (2) & (3) all will lead to x = 2z

3. If $x = 2\sin\theta - \sin 2\theta$ and $y = 2\cos\theta - \cos 2\theta$, $\theta \in [0, 2\pi]$, then $\frac{d^2y}{dx^2}$ at $\theta = \pi$ is :

3 −3 3 −3

Sol. $x = 2\sin\theta - \sin 2\theta$

$$\Rightarrow \frac{dx}{d\theta} = 2\cos\theta - 2\cos 2\theta = 4\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{3\theta}{2}\right)$$

$y = 2\cos\theta - \cos 2\theta$

$$\Rightarrow \frac{dy}{d\theta} = -2\sin\theta + 2\sin 2\theta = 4\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\Rightarrow \frac{dy}{dx} = \cot\left(\frac{3\theta}{2}\right) \Rightarrow \frac{d^2y}{dx^2} = \frac{-\frac{3}{2}\operatorname{cosec}^2\left(\frac{3\theta}{2}\right)}{4\sin\left(\frac{\theta}{2}\right)\sin\frac{3\theta}{2}}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{\theta=\pi} = \frac{3}{8}$$

Alternate :-

$$\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-2\sin\theta + 2\sin 2\theta}{2\cos\theta - 2\cos 2\theta} = \frac{\sin\theta - \sin 2\theta}{-\cos\theta + \cos 2\theta}$$

$$\frac{d^2y}{dx^2}\cdot\frac{dx}{d\theta} = \frac{(-\cos\theta + \cos 2\theta)(\cos\theta - 2\cos 2\theta) - (\sin\theta - \sin 2\theta)(\sin\theta - 2\sin 2\theta)}{(-\cos\theta + \cos 2\theta)^2}$$

$$\frac{d^2y}{dx^2}.(-2-2) = \frac{(+1+1)(-1-2)-(0)}{(1+1)^2}$$

$$\frac{d^2y}{dx^2}(-4) = \frac{2\times -3}{4} = -\frac{3}{2}$$

$$\frac{d^2y}{dx^2} = \frac{3}{8}$$

Answer should be $\frac{3}{8}$. No options is correct.

4. The length of the minor axis (along y-axis) of an ellipse in the standard form is $\frac{4}{\sqrt{3}}$. If this ellipse touches the line, x + 6y = 8; then its eccentricity is :

(1) $\sqrt{\frac{5}{6}}$ (2) $\frac{1}{2}\sqrt{\frac{11}{3}}$ (3) $\frac{1}{3}\sqrt{\frac{11}{3}}$ (4) $\frac{1}{2}\sqrt{\frac{5}{3}}$

Sol. Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; a > b;

$2b=\frac{4}{\sqrt{3}} \Rightarrow b=\frac{2}{\sqrt{3}} \Rightarrow b^2=\frac{4}{3}$

tangent $y=\frac{-x}{6}+\frac{4}{3}$ compare with

$y=mx\pm\sqrt{a^2m^2+b^2}$

$\Rightarrow m=\frac{-1}{6}\Rightarrow\sqrt{\frac{a^2}{36}+\frac{4}{3}}=\frac{4}{3} \Rightarrow a=4;$

$e=\sqrt{1-\frac{b^2}{a^2}}=\frac{1}{2}\sqrt{\frac{11}{3}}$

5. Let a, b ∈ R, a ≠ 0 be such that the equation, $ax^2-2bx+5=0$ has a repeated root α, which is also a root of the equation, $x^2-2bx-10=0$. If β is the other root of this equation, then $\alpha^2+\beta^2$ is equal to :

(1) 26 (2) 25 (3) 28 (4) 24

NTA Ans. (2)

Sol. $ax^2-2bx+5=0 \begin{cases}\alpha\\ \alpha\end{cases}$

$\Rightarrow \alpha=\frac{b}{a};\ \alpha^2=\frac{5}{a} \Rightarrow b^2=5a$

$x^2-2bx-10=0 \begin{cases}\alpha\\ \beta\end{cases} \Rightarrow \alpha^2-2b\alpha-10=0$

$\Rightarrow a=\frac{1}{4} \Rightarrow \alpha^2=20;\ \alpha\beta=-10 \Rightarrow \beta^2=5$

$\Rightarrow \alpha^2+\beta^2=25$

6. Given : $f(x)=\begin{cases}x, & 0\le x<\frac{1}{2}\\ \frac{1}{2}, & x=\frac{1}{2}\\ 1-x, & \frac{1}{2}<x\le 1\end{cases}$

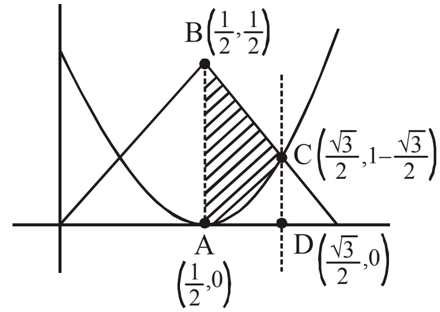
and $g(x)=\left(x-\frac{1}{2}\right)^2$, x ∈ R. Then the area (in sq. units) of the region bounded by the curves, $y=f(x)$ and $y=g(x)$ between the lines, $2x=1$ and $2x=\sqrt{3}$, is :

(1) $\frac{1}{3}+\frac{\sqrt{3}}{4}$ (2) $\frac{\sqrt{3}}{4}-\frac{1}{3}$

$\sqrt{}$ $\sqrt{}$

NTA Ans. (2)

Sol.



Required area = Area of trepezium ABCD – Area of parabola between $x=\frac{1}{2}$ & $x=\frac{\sqrt{3}}{2}$

$A=\frac{1}{2}\left(\frac{\sqrt{3}}{2}-\frac{1}{2}\right)\left(\frac{1}{2}+1-\frac{\sqrt{3}}{2}\right)-\int_{1/2}^{\sqrt{3}/2}\left(x-\frac{1}{2}\right)^2dx=\frac{\sqrt{3}}{4}-\frac{1}{3}$

7. A random variable X has the following probability distribution :

X	:	1	2	3	4	5
P(X)	:	K^2	2K	K	2K	$5K^2$

Then P(X > 2) is equal to :

(1) $\frac{7}{12}$ (2) $\frac{23}{36}$ (3) $\frac{1}{36}$ (4) $\frac{1}{6}$

NTA Ans. (2)

Sol. $\sum P(X)=1 \Rightarrow K^2+2K+K+2K+5K^2=1$

$\Rightarrow 6K^2+5K-1=0 \Rightarrow (6K-1)(K+1)=0$

$\Rightarrow K=-1$ (rejected) $\Rightarrow K=\frac{1}{6}$

$P(X>2)=K+2K+5K^2=\frac{23}{36}$

8. If $x=\sum_{n=0}^{\infty}(-1)^n\tan^{2n}\theta$ and $y=\sum_{n=0}^{\infty}\cos^{2n}\theta$, for $0<\theta<\frac{\pi}{4}$, then :

(1) y(1 + x) = 1 (2) x(1 + y) = 1

(3) y(1 – x) = 1 (4) x(1 – y) = 1

NTA Ans. (3)

Sol. $x=\sum_{n=0}^{\infty}(-1)^n\tan^{2n}\theta=1-\tan^2\theta+\tan^4\theta+....$

$\Rightarrow x=\cos^2\theta$

$y=\sum_{n=0}^{\infty}\cos^{2n}\theta \Rightarrow y=1+\cos^2\theta+\cos^4\theta+.....$

$\Rightarrow y=\frac{1}{\sin^2\theta}\Rightarrow y=\frac{1}{1-x}$

$\Rightarrow y(1-x)=1$

9. Let a function $f : [0, 5] \to \mathbf{R}$ be continuous, $f(1) = 3$ and F be defined as :

$$F(x)=\int_1^x t^2 g(t)\,dt \text{, where } g(t)=\int_1^t f(u)\,du .$$

Then for the function F, the point x = 1 is :

(1) a point of local minima.

(2) not a critical point.

(3) a point of inflection.

(4) a point of local maxima.

NTA Ans. (1)

Sol. $F'(x) = x^2\, g(x) = x^2 \int_1^x f(u)\,du \Rightarrow F'(1) = 0$

$F''(x) = x^2 f(x) - 2x \int_1^x f(u)\,du$

$F''(1) = 1.f(1) - 2 \times 0$

$F''(1) = 3$

$F'(1) = 0$ and $F''(1) = 3 > 0$ So, Minima

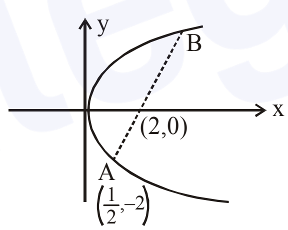
10. If one end of a focal chord AB of the parabola $y^2 = 8x$ is at $A\left(\frac{1}{2},-2\right)$, then the equation of the tangent to it at B is :

(1) $2x + y - 24 = 0$ (2) $x - 2y + 8 = 0$

(3) $2x - y - 24 = 0$ (4) $x + 2y + 8 = 0$

NTA Ans. (2)

Sol. $y^2 = 8x$



$4t_1 = -2 \Rightarrow t_1 = -\frac{1}{2}$,

$t_1.t_2 = -1$

$t_2 = -\frac{1}{t_1}$

$\Rightarrow t_2 = 2$

So coordinate of B is (8, 8)

$\therefore$ Equation of tangent at B is

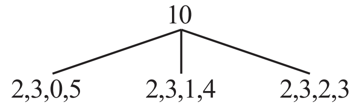
$8y = 4(x + 8) \Rightarrow 2y = x + 8$

11. If 10 different balls are to be placed in 4 distinct boxes at random, then the probability that two of these boxes contain exactly 2 and 3 balls is :

(1) $\frac{945}{2^{11}}$ (2) $\frac{965}{2^{11}}$ (3) $\frac{945}{2^{10}}$ (4) $\frac{965}{2^{10}}$

Ans. (BONUS)

Sol. 10 different balls in 4 different boxes.



$$\frac{1}{4^{10}}\left(4!\times\frac{10!}{2!\times3!\times0!\times5!}+4!\times\frac{10!}{2!\times3!\times1!\times4!}+4!\times\frac{10!}{(2!)^2\times2!\times(3!)^2\times2!}\right)$$

$$=\frac{17\times945}{2^{15}}$$

12. If $A = \{x \in \mathbf{R} : |x| < 2\}$ and $B = \{x \in \mathbf{R} : |x - 2| \ge 3\}$; then :

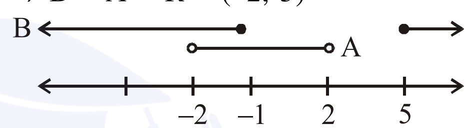
(1) $A \cup B = \mathbf{R} - (2, 5)$ (2) $A \cap B = (-2, -1)$

(3) $B - A = \mathbf{R} - (-2, 5)$ (4) $A - B = [-1, 2)$

NTA Ans. (3)

Sol. $A : x \in (-2, 2)$; $B : x \in (-\infty, -1] \cup [5, \infty)$

$\Rightarrow B - A = R - (-2, 5)$



13. If $\frac{dy}{dx}=\frac{xy}{x^2+y^2}$; $y(1) = 1$; then a value of x satisfying $y(x) = e$ is :

(1) $\sqrt{2}e$ (2) $\frac{e}{\sqrt{2}}$ (3) $\frac{1}{2}\sqrt{3}e$ (4) $\sqrt{3}e$

NTA Ans. (4)

Sol. $\frac{dy}{dx}=\frac{xy}{x^2+y^2}$

Let $y = vx$

$\frac{dy}{dx}=v+x.\frac{dv}{dx}$

$v+x\frac{dv}{dx}=\frac{xvx}{x^2+v^2x^2}=\frac{v}{1+v^2}$

$x\frac{dv}{dx}=\frac{v}{1+v^2}-v=\frac{v-v-v^3}{1+v^2}=-\frac{v^3}{1+v^2}$

$\int\frac{1+v^2}{v^3}.dv=\int-\frac{dx}{x}$

$\Rightarrow \int v^{-3}.dv+\int\frac{1}{v}\,dv=-\int\frac{dx}{x}$

$\Rightarrow \frac{v^{-2}}{-2}+\ell n\, v=-\ell n\, x+\lambda$

$\Rightarrow -\frac{1}{2v^2}+\ell n\left(\frac{y}{x}\right)=-\ell n\, x+\lambda$

$1 \; x^2$

$$\Rightarrow -\frac{1}{2}+0=\lambda \Rightarrow \lambda=-\frac{1}{2}$$

$$\Rightarrow -\frac{1}{2}\frac{x^2}{y^2}+\ell n\, y+\frac{1}{2}=0 \text{ at } y=e$$

$$\Rightarrow -\frac{1}{2}\frac{x^2}{e^2}+1+\frac{1}{2}=0 \Rightarrow \frac{x^2}{2e^2}=\frac{3}{2} \Rightarrow x^2=3e^2$$

$$\therefore x=\sqrt{3}e$$

14. If $\int \frac{d\theta}{\cos^2\theta(\tan 2\theta+\sec 2\theta)}=\lambda\tan\theta+2\log_e|f(\theta)|+C$ where C is a constant of integration, then the ordered pair (λ, $f(\theta)$) is equal to :

(1) (–1, 1 + tanθ) (2) (–1, 1 – tanθ)
(3) (1, 1 – tanθ) (4) (1, 1 + tanθ)

NTA Ans. (1)

Sol. $I=\int \frac{d\theta}{\cos^2\theta(\tan 2\theta+\sec 2\theta)}$

$$=\int \frac{\sec^2\theta\, d\theta}{\frac{2\tan\theta}{1-\tan^2\theta}+\frac{1+\tan^2\theta}{1-\tan^2\theta}}=\int \frac{(1-\tan^2\theta)\sec^2\theta\, d\theta}{(1+\tan\theta)^2}$$

$\tan\theta = t \Rightarrow \sec^2\theta\, d\theta = dt$

$$I=\int \frac{1-t^2}{(1+t)^2}dt=\int \frac{(1-t)(1+t)}{(1+t)^2}dt$$

$$=\int \frac{1}{1+t}-\frac{t}{1+t}dt$$

$$=\ell n|1+t|-\int\left(\frac{1+t}{1+t}-\frac{1}{1+t}\right)dt$$

$= \ell n|1 + t| - t + \ell n|1 + t|$

$= 2\ell n|1 + t| - t + C$

$= 2\ell n|1 + \tan\theta| - \tan\theta + C$

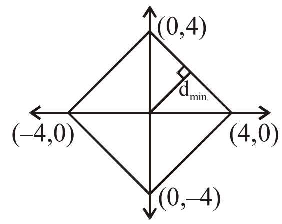
$\lambda = -1$, $f(\theta) = 1 + \tan\theta$

15. If z be a complex number satisfying |Re(z)| + |Im(z)| = 4, then |z| cannot be

(1) $\sqrt{\frac{17}{2}}$ (2) $\sqrt{10}$ (3) $\sqrt{8}$ (4) $\sqrt{7}$

NTA Ans. (4)

Sol.



z = x + iy |x| + |y| = 4

$$|z|=\sqrt{x^2+y^2} \Rightarrow |z|_{min}=\sqrt{8} \;\&\; |z|_{max}=4=\sqrt{16}$$

So |z| cannot be $\sqrt{7}$

16. If p → (p ∧ ~q) is false, then the truth values of p and q are respectively :

(1) F, T (2) T, T (3) F, F (4) T, F

NTA Ans. (2)

Sol. p → (p ∧ ~q) is F ⇒ p is T & p ∧ ~q is F ⇒ q is T

∴ p is T, q is T

17. Let a – 2b + c = 1.

If $f(x)=\begin{vmatrix} x+a & x+2 & x+1 \\ x+b & x+3 & x+2 \\ x+c & x+4 & x+3 \end{vmatrix}$, then :

(1) $f(-50) = 501$ (2) $f(-50) = -1$
(3) $f(50) = 1$ (4) $f(50) = -501$

NTA Ans. (3)

Sol. $R_1 \to R_1 + R_3 - 2R_2$

$$f(x)=\begin{vmatrix} a+c-2b & 0 & 0 \\ x+b & x+3 & x+2 \\ x+c & x+4 & x+3 \end{vmatrix}$$

$= (a + c - 2b)\,((x + 3)^2 - (x + 2)(x + 4))$

$= x^2 + 6x + 9 - x^2 - 6x - 8 = 1$

$\Rightarrow f(x) = 1 \quad \Rightarrow f(50) = 1$

18. In the expansion of $\left(\frac{x}{\cos\theta}+\frac{1}{x\sin\theta}\right)^{16}$, if ℓ_1 is the least value of the term independent of x when $\frac{\pi}{8}\le\theta\le\frac{\pi}{4}$ and ℓ_2 is the least value of the term independent of x when $\frac{\pi}{16}\le\theta\le\frac{\pi}{8}$, then the ratio $\ell_2 : \ell_1$ is equal to :

(1) 1 : 8 (2) 1 : 16

(3) 8 : 1 (4) 16 : 1

NTA Ans. (4)

Sol. $T_{r+1} = {}^{16}C_r\left(\frac{x}{\cos\theta}\right)^{16-r}\left(\frac{1}{x\sin\theta}\right)^r$

$= {}^{16}C_r(x)^{16-2r}\times\frac{1}{(\cos\theta)^{16-r}(\sin\theta)^r}$

For independent of x; $16 - 2r = 0 \Rightarrow r = 8$

$\Rightarrow T_9 = {}^{16}C_8\frac{1}{\cos^8\theta\sin^8\theta}$

$= {}^{16}C_8\frac{2^8}{(\sin 2\theta)^8}$

for $\theta\in\left[\frac{\pi}{8},\frac{\pi}{4}\right]$ ℓ_1 is least for $\theta_1=\frac{\pi}{4}$

for $\theta\in\left[\frac{\pi}{16},\frac{\pi}{8}\right]$ ℓ_2 is least for $\theta_2=\frac{\pi}{8}$

$\frac{\ell_2}{\ell_1}=\frac{(\sin 2\theta_1)^8}{(\sin 2\theta_2)^8}=(\sqrt{2})^8=\frac{16}{1}$

19. Let a_n be the n^{th} term of a G.P. of positive terms.

If $\sum_{n=1}^{100}a_{2n+1}=200$ and $\sum_{n=1}^{100}a_{2n}=100$, then $\sum_{n=1}^{200}a_n$ is equal to :

(1) 225 (2) 175 (3) 300 (4) 150

NTA Ans. (4)

Sol. $\sum_{n=1}^{100}a_{2n+1}=200 \Rightarrow a_3 + a_5 + a_7 + ... + a_{201} = 200$

$\Rightarrow ar^2\frac{(r^{200}-1)}{(r^2-1)}=200$

$\sum_{n=1}^{100}a_{2n}=100 \Rightarrow a_2 + a_4 + a_6 + ... + a_{200} = 100$

$\Rightarrow \frac{ar(r^{200}-1)}{(r^2-1)}=100$

On dividing r = 2

on adding $a_2 + a_3 + a_4 + a_5 + ... + a_{200} + a_{201} = 300$

$\Rightarrow r(a_1 + a_2 + a_3 + ... + a_{200}) = 300$

$\Rightarrow \sum_{n=1}^{200}a_n=150$

20. Let f and g be differentiable functions on **R** such that fog is the identity function. If for some a, b ∈ **R**, g'(a) = 5 and g(a) = b, then $f'(b)$ is equal to :

(1) $\frac{2}{5}$ (2) 1 (3) $\frac{1}{5}$ (4) 5

NTA Ans. (3)

Sol. $f(g(x)) = x$

$f'(g(x))\, g'(x) = 1$

put x = a

$\Rightarrow f'(b)\, g'(a) = 1$

$f'(b) = \frac{1}{5}$

21. The number of terms common to the two A.P.'s 3, 7, 11, ..., 407 and 2, 9, 16, ..., 709 is ______.

NTA Ans. (14)

Sol. Common term are : 23, 51, 79, ... T_n

$T_n \le 407 \Rightarrow 23 + (n - 1)28 \le 407$

$\Rightarrow n \le 14.71$

n = 14

22. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 5$, $\vec{b}.\vec{c} = 10$ and the angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$. If $\vec{a}$ is perpendicular to the vector $\vec{b} \times \vec{c}$, then $\left|\vec{a} \times (\vec{b} \times \vec{c})\right|$ is equal to _____.

NTA Ans. (30)

Sol. $\vec{b}.\vec{c} = 10 \Rightarrow 5|\vec{c}|\cos\frac{\pi}{3} = 10 \Rightarrow |\vec{c}| = 4$

$\left|\vec{a} \times (\vec{b} \times \vec{c})\right| = |\vec{a}|\,|\vec{b} \times \vec{c}|$

$= \sqrt{3}\,.\,5\,.\,4\,.\,\sin\frac{\pi}{4} = 30$

23. If the distance between the plane, $23x - 10y - 2z + 48 = 0$ and the plane containing the lines $\frac{x+1}{2} = \frac{y-3}{4} = \frac{z+1}{3}$ and $\frac{x+3}{2} = \frac{y+2}{6} = \frac{z-1}{\lambda} (\lambda \in R)$ is equal to $\frac{k}{\sqrt{633}}$, then k is equal to ______.

NTA Ans. (3)

Sol. If $\lambda = -7$, then planes will be parallel & distance between them will be $\frac{3}{\sqrt{633}} \Rightarrow k = 3$

But if $\lambda \neq -7$, then planes will be intersecting & distance between them will be 0

24. If $C_r \equiv {}^{25}C_r$ and $C_0 + 5.C_1 + 9.C_2 + ... + (101).C_{25} = 2^{25}.k$, then k is equal to _____.

NTA Ans. (51)

Sol. $S = 1.{}^{25}C_0 + 5.{}^{25}C_1 + 9.{}^{25}C_2 + ... + (101){}^{25}C_{25}$

$S = 101{}^{25}C_{25} + 97{}^{25}C_1 + ... + 1{}^{25}C_{25}$

$2S = (102)(2^{25})$

$S = 51(2^{25})$

25. If the curves, $x^2 - 6x + y^2 + 8 = 0$ and $x^2 - 8y + y^2 + 16 - k = 0$, $(k > 0)$ touch each other at a point, then the largest value of k is ______.

NTA Ans. (36)

Sol. Common tangent is $S_1 - S_2 = 0$

$\Rightarrow -6x + 8y - 8 + k = 0$

Use p = r for I[st] circle

$\Rightarrow \frac{|-18-8+k|}{10} = 1$

$\Rightarrow k = 36$ or $16 \Rightarrow k_{max} = 36$